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No. 2

Department of Statistics

AN INVENTORY MODEL WITH AN OPTIMAL TIME LAG

by

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1. Introduction and Summary

At various, equally spaced, time instants orders can be placed to replenish a supply which is being depleted by demands during the successive periods between reorderings. Two types of orders are allowed, a first type with immediate delivery at a unit cost of k and a second type at a unit cost of ℓ with delivery at the end of one period after the order is placed. At each reordering point orders of both types are allowed.

Assuming a linear penalty for understorage and a convex increasing storage cost, we prove that the optimal reordering policy, guaranteeing overall minimum discounted cost for a process of unlimited duration, has the following structure: " At the beginning of each period, if the stock at hand x is less than a first critical level x^* the stock should be replenished up to the level x^* by immediate delivery and an amount u^*, independent of x should be ordered with one period time lag. If the initial stock x lies between x^* and a second critical level $\tilde{x}$ only an amount $u(\tilde{x})$ should be ordered with one period time lag. Finally if x is larger than $\tilde{x}$ no order should be placed. Moreover the amount $\tilde{u}(x)$ is a decreasing continuous function of x with $\tilde{u}(x^*) = u^*$ and $\tilde{u}(\tilde{x}) = 0$ ".

Various degenerate forms of this policy are also of interest.

2. The functional equation for the optimal discounted cost

We introduce the following notations:

$\varphi(s)ds$: the probability that the demand in a given period lies between s and $s + ds$ $\int_0^\infty s\,\varphi(s)ds < \infty$.

x : the initial supply

$(y-x)k$: the cost of ordering an amount $y - x \geq 0$ with immediate delivery.

$u\ell$: the cost of ordering an amount $u \geq 0$ with delivery after one period.

$z.p$: the penalty if the demand in a given period exceeds the available supply by $z \geq 0$.

$h(y)$: the storage cost for an amount $y \geq 0$. $h(\,.\,)$ is assumed to be a convex increasing, twice differentiable function of y with $h(0) = 0$.

a : a discount factor applied to future costs. $0 < a < 1$.

$f(x)$: the minimum overall discounted cost, starting the process with an initial amount x.

A direct application of Bellman's Principle of Optimality shows that $f(x)$ must satisfy the following functional equation.

(1) $$f(x) = \min_{y \geq x} \min_{u \geq 0} \left\{ k(y-x) + \ell u + L(y) + af(u) \cdot \int_y^\infty \varphi(s)ds + a \int_0^y f(y+u-s)\,\varphi(s)\,ds \right\}$$

in which

(2) $$L(y) = \int_0^y h(y-s)\,\varphi(s)ds + \int_y^\infty (s-y)p\,\varphi(s)ds.$$

3. Structure of the solution to equation (1).

We first observe that $\ell \geq k$ implies that there is no advantage to ordering with time lag. In this case $u = 0$ in the optimal policy and the problem reduces to finding the optimal ordering policy under the assumption of instantaneous delivery. This case has been studied in great detail [1,2] so we can concentrate on the case $\ell < k$. We now prove the following theorem:

<u>Theorem</u>

The optimal reordering policy corresponding to equation (1) is given by the following rule:

$$\begin{array}{lll} y = x^* & u = u^* & \text{for } 0 \leq x \leq x^* \\ y = x & u = \tilde{u}(x) & \text{for } x^* \leq x \leq \tilde{x} \\ y = x & u = 0 & \text{for } x \geq \tilde{x} \end{array}$$

with $\tilde{u}(x) \geq 0$ continuous and monotone decreasing in x.

<u>Proof.</u> The proof proceeds by induction based upon a sequence of successive approximations to $f(x)$. We define $f_o(x)$ by the equation

$$(3) \qquad f_o(x) = L(x) + af_o(0) \int_x^\infty \varphi(s)ds + a \int_0^x f_o(x-s)\, \varphi(s)ds$$

and for $n = 0,1,\ldots$ we define $f_{n+1}(x)$ by:

$$(4) \qquad f_{n+1}(x) = \underset{y \geq x}{\text{Min}} \ \underset{u \geq 0}{\text{Min}} \left\{ k(y-x) + \ell u + L(y) + af_n(u) \int_y^\infty \varphi(s)ds + a \int_0^y f_n(y+u-s)\, \varphi(s)ds \right\}$$

We denote the expression contained within the braces in (4) by $T(y,u,x,f_n)$. We consider $T(y,u,x,f_o)$ as a function $M_1(y,u)$ and obtain the following partial derivatives of $M_1(y,u)$.

(5) $$\frac{\partial}{\partial y} M_1(y,u) = k + L'(y) + a \int_0^y f_o'(y + u - s)\varphi(s)ds$$

(6) $$\frac{\partial^2}{\partial y^2} M_1(y, u) = L''(y) + af_o'(u)\varphi(y) + a\int_o^y f_o''(y + u - s)\varphi(s)ds$$

(7) $$\frac{\partial}{\partial u} M_1(y, u) = \ell + af_o'(u)\int_y^\infty \varphi(s)ds + a\int_o^y f_o'(y + u - s)\varphi(s)ds$$

(8) $$\frac{\partial^2}{\partial u^2} M_1(y, u) = af_o''(u)\int_y^\alpha \varphi(s)ds + a\int_o^y f_o''(y + u - s)\varphi(s)ds$$

(9) $$\frac{\partial^2}{\partial u \partial y} M_1(y, u) = a\int_o^y f_o''(y + u - s)\varphi(s)ds$$

Now $f_o''(x) > 0$ by a property of the renewal equation, so it follows that

$$\frac{\partial^2}{\partial y^2} M_1(y,u) > 0 \quad \frac{\partial^2}{\partial u^2} M_1(y,u) > 0 \quad \frac{\partial^2}{\partial u \partial y} M_1(y,u) > 0$$

and

$$\frac{\partial^2 M_1}{\partial y^2} \cdot \frac{\partial^2 M_1}{\partial u^2} - \left(\frac{\partial^2 M_1}{\partial y \partial u}\right)^2 > 0$$

for all $y \geq x$ and $u \geq 0$.

For $x = 0$ and under conditions of non-degeneracy to be specified in paragraph 4 there will be a unique point x_1^*, u_1^*, <u>interior</u> to the domain $y \geq 0$, $u \geq 0$ where the function $M_1(y, u)$ attains its minimum. We note however that the equations $\frac{\partial}{\partial y} M_1 = \frac{\partial}{\partial u} M_1 = 0$ which determine the interior minimum, donnot depend on x. Therefore for all x in the interval $[0, x_1^*)$ the point (x_1^*, u_1^*) is the unique interior minimum.

Now for some range of values x, greater than x_1^* the unique minimum in the domain $y \geq x$, $u \geq 0$ will be attained for $y = x$ and for $u = \tilde{u}_1(x)$. The value $\tilde{u}_1(x)$ is the unique point for which $\frac{\partial}{\partial u} M_1(x, u)$ vanishes. Since $\frac{\partial^2}{\partial u \partial y} M_1(y, u)$ is greater than zero, we conclude that $\tilde{u}_1(x)$ is a

decreasing function of x. Let $\tilde{x}_1$ be the value of x, if any, for which $\tilde{u}_1(x)$ reaches zero. It is clear that for values of x larger than $\tilde{x}_1$ the function $M_1(y, u)$ attains its minimum for $y = x$ and $u = 0$.

Summarizing we can state that the minimum points of $T(y,u,x, f_o)$ are located as follows:

$$\begin{array}{llll} y = x_1^* & u = u_1^* & \text{for} & 0 \leq x \leq x_1^* \\ y = x & u = \tilde{u}_1(x)\ \tilde{u}_1(x)\downarrow & \text{for} & x_1^* \leq x \leq \tilde{x}_1 \\ y = x & u = 0 & \text{for} & x \geq \tilde{x}_1 \end{array}$$

and hence

$$(10)\qquad f_1(x) = \begin{cases} T(x_1^*, u_1^*, x, f_o) & \text{for } 0 \leq x \leq x_1^* \\ T(x, \tilde{u}_1(x), x, f_o) & \text{for } x_1^* \leq x \leq \tilde{x}_1 \\ T(x, 0, x, f_o) & \text{for } x \geq \tilde{x}_1 \end{cases}$$

We now supply the essential tools of a proof by induction by showing that $f_1(x)$ has all these properties which enabled us to describe the set of minima for $f_o(x)$.

Lemma 1: $f_1''(x) \geq 0$

In the interval $[0, x_1^*]$ we have $f_1'(x) = -k$, $f_1''(x) = 0$. In the interval $[x_1^*, \tilde{x}_1]$ we have:

$$(11)\qquad f_1'(x) = L'(x) + a \int_0^x f_o'(x + \tilde{u}_1(x) - s)\,\varphi(s)\,ds$$

But $\frac{\partial}{\partial y} M_1(y, u) > 0$ for $x_1^* < y \leq \tilde{x}_1$ so $f_1'(x) \geq -k$ and furthermore

(12) $f_1''(x) = L''(x) + af_o'(\tilde{u}_1)\varphi(x) + \left[a\int_0^x f_o''(x+\tilde{u}_1(x)-s)\varphi(s)ds \right]\left(1+\frac{d\tilde{u}_1}{dx}\right)$

We have $f_o'(0) = -p$ so $p + af_o'(\tilde{u}_1)$ will be strictly positive for $\tilde{u}_1 > 0$ since $f_o'(u)$ is strictly increasing in u. So, $f_1''(x)$ will be strictly positive if we can show that $1 + \frac{d\tilde{u}_1}{dx} \geq 0$. In order to show this, differentiate $\frac{\partial}{\partial u} M_1(x, u) = 0$ with respect to x. We obtain:

(13) $\left[af_o''(\tilde{u}_1)\int_x^{\infty}\varphi(s)ds + a\int_0^x f_o''(x+\tilde{u}_1-s)\varphi(s)ds \right]\frac{d\tilde{u}_1}{dx} +$

$$a\int_0^x f_o''(x+\tilde{u}_1-s)\varphi(s)ds = 0$$

Equation (13) and $f_o''(x) > 0$ imply that

(14) $\frac{d\tilde{u}_1}{dx} \leq 0 \quad 1 + \frac{d\tilde{u}_1}{dx} \geq 0$

Finally in $[\tilde{x}_1, \infty)$ we have

(15) $f_1''(x) = L''(x) + af_o'(0)\varphi(x) + a\int_0^x f_o''(x-s)\varphi(s)ds$

but this implies that $f_1''(x) > 0$ since $f_1(x) = f_o(x)$

<u>Lemma 2:</u> $f_1'(x) \geq f_o'(x)$

In $[0, x_1^*]$ we have $\frac{\partial}{\partial y} M_1(y, u) < 0$ so

$$-k > L'(x) + a\int_0^x f_o'(x+u-s)\varphi(s)ds \geq f_o'(x)$$

since $f_o'(x)$ is monotone increasing

In $[x_1^*, \tilde{x}_1]$ we have:

$$f_1'(x) = L'(x) + a\int_0^x f_o'(x+\tilde{u}_1-s)\varphi(s)ds \geq f_o'(x)$$

and finally in $[\tilde{x}_1, \infty)$ we have $f_1(x) = f_o(x)$.

We have now shown that $f_1(x)$ has all the properties of $f_o(x)$ which were essential in the proof of the structure of the set of minima of $T(y,u,x,f_o)$. The set of minima of $T(y,u,x,f_1)$ will have the same structure and $x_2^* \leq x_1^*$, $u_2^* \leq u_1^*$, $\tilde{x}_2 \leq \tilde{x}_1$, $\tilde{u}_2(x) \leq \tilde{u}_1(x)$ since $f_1'(x) \geq f_o'(x)$.

A direct induction argument will now establish the existence of sequences x_n^*, u_n^*, $\tilde{x}_n$ and non-increasing differentiable functions $\tilde{u}_n(x)$ defined on the intervals $\left[x_n^*, \tilde{x}_n \right]$. The sequences x_n^*, u_n^*, $\tilde{x}_n$ and $\tilde{u}_n(x)$ will be non-increasing in n and therefore passage to the limit is in order. In the limit we obtain the structure of the optimal ordering policy, which is as described in the statement of the theorem.

4. Some particular cases.

We have already argued that $u^* = 0$ and $x^* = x$ when $\ell \geq k$.

Consider

$$\left[\frac{\partial}{\partial y} M_1(y, u) \right]_{y=x=0} = k - p$$

and

$$\left[\frac{\partial}{\partial u} M_1(y, u) \right]_{y=x=0} = \ell + af_o'(u) \geq \ell + af_o'(0) = \ell - ap \,.$$

then it follows that:

a. if $k \geq p$ and $\ell a^{-1} \geq p$ then $x^* = \tilde{x} = 0$ $u^* = 0$ i.e. nothing is ordered.

b. if $k \geq p > \ell a^{-1}$ then $x^* = 0$ i.e. one should not order with immediate delivery.

c. if $\ell a^{-1} \geq p > k$ then $u^* = 0$, $x^* = \tilde{x}$ i.e. no delayed order should be placed.

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2

J. Soc. Indust. Appl. Math.
Vol. 12, No. 1, March, 1964
Printed in U.S.A.

AN INVENTORY MODEL WITH AN OPTIONAL TIME LAG*

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1. Summary. This paper deals with the following model in Inventory Control. At various, equally spaced, time instants orders can be placed to replenish a supply which is being depleted by random demands during the successive periods between reorderings. Two types of orders are allowed, a first type with immediate delivery at a unit cost of k and a second type at a unit cost of l with delivery at the end of one period after the order is placed. At each reordering point orders of both types are allowed.

Assuming a linear penalty for understorage and a convex increasing storage cost, we prove that the optimal reordering policy, which guarantees over-all minimum discounted cost for a process of unlimited duration, has the following structure. *At the beginning of each period, if the stock at hand* x *is less than a first critical level* x^*, *the stock should be replenished up to the level* x^* *by immediate delivery and an amount* u^*, *independent of* x, *should be ordered with one period lag. If the initial stock* x *lies between* x^* *and a second critical level* $\bar{x}$, *only an amount* $\tilde{u}(x)$ *should be ordered with one period lag. Finally if* x *is larger than* $\bar{x}$ *no order should be placed. Moreover the amount* $\tilde{u}(x)$ *is a decreasing continuous function of* x *with* $\tilde{u}(x^*) = u^*$ *and* $\tilde{u}(\bar{x}) = 0$.

Various degenerate forms of this policy are also of interest.

This model is essentially a merger of the ordinary Arrow-Harris-Marshak dynamic model, discussed in [2], and the Karlin-Scarf model with a time lag, discussed in [3]. Our arguments parallel those of [2] and [3] but are slightly more involved due to the higher dimensionality of the problem.

2. The functional equation for the optimal discounted cost. We introduce the following notations:

$\varphi(s)\,ds$	the probability that the demand in a given period lies between s and $s + ds$,
x	the initial supply,
$(y - x)k$	the cost of ordering an amount $y - x \geqq 0$ with immediate delivery,
ul	the cost of ordering an amount $u \geqq 0$ with delivery one period hence,
zp	the penalty if the demand in a given period exceeds the available supply by an amount $z \geqq 0$,
$h(y)$	the storage cost for an amount $y \geqq 0$,

* Received by the editors April 18, 1963, and in revised form August 5, 1963.

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$$f_0''(x) = \varphi(x)\,[h'(0) + (1-a)p] + \int_0^x L''(x-s)\varphi(s)\,ds$$
$$+\, a\int_0^x f_0''(x-s)\varphi(s)\,ds.$$

This is an equation of the renewal type with $\varphi(s) \geqq 0$, and

$$\varphi(x)\,[h'(0) + (1-a)p] + \int_0^x h''(x-s)\varphi(s)\,ds \geqq 0.$$

This implies (see [1, pp. 177–178]) that $f_0''(x) \geqq 0$. Direct inspection of the renewal equation shows that $f_0''(x)$ cannot vanish under the proviso of the Remark. The fact that $f_0''(x) > 0$ implies now that

$$\frac{\partial^2}{\partial u^2} M_1(y, u) > 0$$

and

$$\frac{\partial^2}{\partial u\,\partial y} M_1(y, u) > 0$$

and

$$\frac{\partial^2 M_1}{\partial y^2}\frac{\partial^2 M_1}{\partial u^2} - \left(\frac{\partial^2 M_1}{\partial y\,\partial u}\right)^2 > 0.$$

The strict positivity of $\dfrac{\partial^2}{\partial y^2} M_1(y, u)$ will be established if we can show that

$$L''(y) + af_0'(u)\varphi(y) > 0.$$

However, since $f_0''(x) > 0$, we have

$$L''(y) + af_0'(u)\varphi(y) > L''(y) + af_0'(0)\varphi(y)$$
$$= h'(0)\varphi(y) + \int_0^y h''(y-s)\varphi(s)\,ds + (1-a)p\,\varphi(y)$$
$$\geqq 0.$$

For $x = 0$ and excluding the degenerate cases to be discussed in §4, there will be a unique point x_1^*, u_1^*, *interior* to the domain $y \geqq 0$, $u \geqq 0$, where the function $M_1(y, u)$ attains its minimum. We note however that the equations

$$\frac{\partial}{\partial y} M_1(y, u) = 0, \qquad \frac{\partial}{\partial u} M_1(y, u) = 0,$$

which determine the interior minimum do not depend on x. Therefore for all x in the interval $[0, x_1^*)$ the point (x_1^*, u_1^*) is the unique interior minimum point.

Now for some range of values x, greater than x_1^*, the unique minimum in the domain $y \geqq x, u \geqq 0$ will be attained for $y = x$ and for $u = \tilde{u}_1(x)$. The value $\tilde{u}_1(x)$ is the unique point for which $\frac{\partial}{\partial u} M_1(x, u)$ vanishes.

Since $\frac{\partial^2}{\partial u \, \partial y} M_1(y, u)$ is strictly positive it follows that $\tilde{u}_1(x)$ is a decreasing function of x. Considering (7) it follows that for some value $\bar{x}_1$ of x, $\tilde{u}_1(x)$ must reach zero. It is clear that for values of x larger than $\bar{x}_1$ the function $M_1(y, u)$ must attain its minimum for $y = x$ and $u = 0$. Summarizing we can state that the minimum points of $T(y, u, x, f_0)$ with respect to the variables y and u are located as follows.

$$
\begin{aligned}
&y = x_1^*, && u = u_1^*, && \text{for } 0 \leqq x \leqq x_1^*, \\
&y = x, && u = \tilde{u}_1(x), && \text{for } x_1^* \leqq x \leqq \bar{x}_1 \text{ with} \\
& && && \tilde{u}_1(x) \text{ decreasing and} \\
& && && \tilde{u}_1(x_1^*) = u_1^* \text{ and } \tilde{u}_1(\bar{x}_1) = 0 \\
&y = x, && u = 0, && \text{for } x \geqq \bar{x}_1 .
\end{aligned}
$$

We now supply the essential tools of a proof by induction by showing that $f_1(x)$ has all these properties which enabled us to describe the set of minima of $T(y, u, x, f_0)$. It will follow that $T(y, u, x, f_1)$ has a similar set of minima and so on.

LEMMA 2. $f_1''(x) \geqq 0$.

Proof. In the interval $[0, x_1^*]$ we have $f_1'(x) = -k$, $f_1''(x) = 0$. In the interval $(x_1^*, x_1]$ we have, by (7) and the fact that $\frac{\partial}{\partial u} M_1(x, \tilde{u}_1(x)) = 0$, that

$$
f_1'(x) = L'(x) + a \int_0^x f_0'(x + \tilde{u}(x) - s)\varphi(s)\, ds. \tag{10}
$$

But $\frac{\partial}{\partial y} M_1(y, u) > 0$ for $x_1^* < y \leqq \bar{x}_1$, so $f_1'(x) \geqq -k$. Furthermore,

$$
\begin{aligned}
f_1''(x) = L''(x) + af_0'(\tilde{u}_1)\varphi(x) + \left(1 + \frac{d\tilde{u}_1}{dx}\right) \\
\cdot a \int_0^x f_0''(x + \tilde{u}_1(x) - s)\varphi(s)\, ds.
\end{aligned} \tag{11}
$$

We have $f_0'(0) = -p$, so $p + af_0'(\tilde{u}_1)$ will be strictly positive for $\tilde{u}_1 > 0$ since $f_0'(u)$ is strictly increasing in u. So $f_1''(x)$ will be strictly positive if we can show that

$$1 + \frac{d\tilde{u}_1}{dx} \geqq 0.$$

In order to show this, differentiate $\frac{\partial}{\partial u} M_1(x, u) = 0$ with respect to x. We obtain

$$\left[af_0''(\tilde{u}_1)\int_x^\infty \varphi(s)\, ds + a\int_0^x f_0''(x + \tilde{u}_1(x) - s)\varphi(s)\, ds\right]\frac{d\tilde{u}_1}{dx} + a\int_0^x f_0''(x + \tilde{u}_1(x) - s)\varphi(s)\, ds = 0. \tag{12}$$

Equation (12) and $f_0''(x) > 0$ imply that

$$\frac{d\tilde{u}_1}{dx} \leqq 0, \qquad 1 + \frac{d\tilde{u}_1}{dx} \geqq 0. \tag{13}$$

Finally in $[\tilde{x}_1, \infty)$ we have

$$f_1''(x) = L''(x) + af_0'(0)\varphi(x) + a\int_0^x f_0''(x - s)\varphi(s)\, ds, \tag{14}$$

but this implies that $f_1''(x) > 0$, since $f_1(x) = f_0(x)$ in this interval.

LEMMA 3. $f_1'(x) \geqq f_0'(x)$.

Proof. In $[0, x_1^*)$ we have $\frac{\partial}{dy} M_1(y, u) < 0$, so

$$-k > L'(x) + a\int_0^x f_0'(x + u - s)\varphi(s)\, ds \geqq f_0'(x),$$

since $f_0'(x)$ is monotone increasing.

In $[x_1^*, \tilde{x}_1]$ we have

$$f_1(x) = L'(x) + a\int_0^x f_0'(x + \tilde{u}_1(x) - s)\varphi(s)\, ds \geqq f_0'(x),$$

since $f_0'(x)$ is monotone increasing.

Finally in $[\tilde{x}_1, \infty)$ we have $f_1(x) = f_0(x)$, and hence $f_1'(x) = f_0'(x)$.

We have now shown that $f_1(x) = T(y, x, u, f_0)$ has all the properties of $f_0(x)$ which were essential in the proof of the structure of the set of minima of $T(y, x, u, f_0)$. It follows that the set of minima of $T(y, x, u, f_1)$ will have the same structure and moreover, in view of the analogues of (5) and (7) and Lemma 3, that

$$x_2^* \leqq x_1^*, \qquad u_2^* \leqq u_1^*, \qquad \bar{x}_2 \leqq \bar{x}_1, \qquad \tilde{u}_2(x) \leqq \tilde{u}_1(x).$$

A direct induction argument will establish the existence of sequences x_n^*, u_n^*, $\bar{x}_n$, and nonincreasing differentiable functions $\tilde{u}_n(x)$ defined on the intervals $[x_n^*, \bar{x}_n]$. The sequences x_n^*, u_n^*, $\bar{x}_n$ and $\tilde{u}_n(x)$ will be nonincreasing in n and therefore passage to the limit is in order. In the limit we obtain the structure of the optimal ordering policy, which is as described in the statement of the theorem.

4. Some degenerate cases. Under certain conditions on the parameters of the inventory problem various degenerate forms of the policy described in Theorem 1 obtain. The following are some of the more tractable. We have already argued that $u^* = 0$ and $x^* = \bar{x}$ when $l \geqq k$. Consider

$$\left[\frac{\partial}{\partial y} M_1(y, u)\right]_{y=x=0} = k - p,$$

$$\left[\frac{\partial}{\partial u} M_1(y, u)\right]_{y=x=0} = l + af_0'(u) \geqq l + af_0'(0) = l - ap.$$

Then it follows that:

a. if $k \geqq p$ and $la^{-1} \geqq p$, then $x^* = \bar{x} = 0$, $u^* = 0$, i.e., nothing is ordered.
b. if $k \geqq p > la^{-1}$, then $x^* = 0$, i.e., one should not order with immediate delivery.
c. if $la^{-1} \geqq p > k$, then $u^* = 0$, $x^* = \bar{x}$, i.e., no delayed order should be placed.

5. Acknowledgments. The author is gratefully indebted to Professor Leonard D. Berkovitz of Purdue University and to the referee for numerous clarifying remarks.

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