On the Distribution of Linear Functions and Ratios of Linear Functions of Ordered Correlated Normal Random Variables with Emphasis on Range

by

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0. Summary

This paper deals with the distribution of linear functions and ratios of linear functions of order statistics from an equally correlated set of normal random variables. Some special cases of this family of linear functions are considered. The case of range is studied in great detail both for the case of equal correlations as well as in the general case. For the general case, formulae for the distribution of range are obtained for sample sizes of two, three and four. In the equally correlated case, expressions are obtained for the probability integral, percentage points and moments of the linear functions in terms of the corresponding expressions for the uncorrelated cases. It is also known that ratios of certain linear functions of the order statistics have a distribution independent of this common correlation.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be jointly normally distributed random variables with $EX_i = 0$, $EX_i^2 = 1$, $EX_iX_j = \rho_{ij}$, $(i \neq j = 1, 2, \ldots, n)$. Let $X_{(k)}$ be the kth order statistic when X_i's are arranged in an increasing order as follows:

$$X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(1)} \leq \ldots \leq X_{(n)}. \qquad (1.1)$$

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The distribution of these order statistics is of interest, for instance, in problems of multiple decisions and life testing. Owen and Steck (1962) have obtained the moments of $X_{(k)}$ when $\rho_{ij} = \rho$, all i, j. Gupta (1963a) discussed the evaluation of the probability integral of $X_{(n)}$, and has given tables of this integral for selected values of $\rho_{ij} = \rho$. In a separate bibliography Gupta (1963b) gives references to other papers in this area. The papers by Stuart (1958), Ruben (1961), Steck and Owen (1962), Steck (1962), and Thigpen (1962) are of more direct interest.

In this paper, we shall consider a class of linear functions of $X_{(i)}$ given by $Y = \sum_{i=1}^{n} a_i X_{(i)}$. This class covers a number of useful statistics that arise in various problems of statistical inference. For instance, when $a_1 = -1$, $a_n = 1$ and $a_2 = a_3 = \dots = a_{n-1} = 0$, the above linear function of order statistics reduces to the range of the n random variables X_i. We discuss the latter case in great detail. Another special case is that of the Nair statistic (1948) which is discussed briefly. When $\rho_{ij} = \rho$, we use the reduction first given by Dunnett and Sobel (1955) [see also Stuart (1958)] to obtain the moments of Y and to derive formulae for the probability integral and the percentage points of Y. We also consider statistics of the form Y_1/Y_2 which are used in rejection of outliers and as a substitute for the F-statistic in testing equality of variances.

In Section 2, we discuss the distribution of Y for the equicorrelated case. Section 3 deals with moments of Y. In Section 4, we discuss the Studentized case when the variables have a common unknown variance. In Section 5, we discuss the distribution of certain ratios of Y's. In Section 6, we discuss the distribution of the range, a particular case

of Y. For $n = 2, 3$, and 4, the distribution of the range is obtained in the general case when the correlations are not necessarily equal.

2. Distribution of Y when X_j's are equally correlated

If X_i's are equally correlated, then it is well-known that this common $\rho \geq -\frac{1}{n-1}$. Let us consider the cases $\rho \geq 0$ and $-\frac{1}{n-1} \leq \rho < 0$, separately. Following Owen and Steck (1962b), we generate the random variables $\{X_i\}$ from the random variables $\{Z_i\}$ as follows.

$$(2.1) \qquad X_i = (\rho)^{1/2} Z_0 + (1 - \rho)^{1/2} Z_i, \quad \rho \geq 0, \ i = 1,2, \ldots, n$$

where $Z_0, \ldots, Z_n$ are ind. $N(0,1)$ and

$$(2.2)\qquad X_i = (-\rho)^{1/2} Z_0 + (1-\rho)^{1/2} Z_i \;, \qquad \rho<0,\ i = 1,2,\ldots, n$$

where Z_i ($i = 1, 2, \ldots, n$) are ind. $N(0,1)$, Z_0 is $N(0,1)$ and

$E(Z_0 Z_i) = -(-\rho)^{1/2} / (1-\rho)^{1/2}$.

It follows from (2.1) and (2.2) that the linear function $Y = \Sigma a_i X_{(i)}$ can be expressed in terms of the corresponding linear function $\Sigma a_i Z_{(i)}$ (Z_i's uncorrelated) and a term involving Z_0. Hence, we have for $\rho \geq 0$,

$$(2.3)\qquad P\{Y \leq y\} = P\left\{\sum_{i=1}^{n} a_i Z_{(i)} \leq -[\tfrac{\rho}{1-\rho}]^{1/2} \left(\sum_{i=1}^{n} a_i\right) Z_0 + \frac{y}{(1-\rho)^{1/2}}\right\}$$

and for $\rho < 0$

$$(2.4)\qquad P\{Y \leq y\} = P\left\{\sum_{i=1}^{n} a_i Z_{(i)} \leq -\left[\frac{-\rho}{1-\rho}\right]^{1/2} \left(\sum_{i=1}^{n} a_i\right) Z_0 + \frac{y}{(1-\rho)^{1/2}}\right\}.$$

From (2.3) we obtain by putting $\left(\frac{\rho}{1-\rho}\right)^{1/2} = c$,

$$(2.5)\qquad P\{Y \leq y\} = H(y;\rho) = \int_{-\infty}^{\infty} H\left(-cx(\Sigma a_i) + \frac{y}{(1-\rho)^{1/2}} \;;\; 0\right) \phi(x)dx$$

where $\phi(x)$ denotes the density of a standard normal random variable.

It is interesting to note that if $\sum_{i=1}^{n} a_i = 0$, then for both negative and positive ρ, we have one result, namely,

$$(2.6)\qquad H(y;\rho) = H\left(\frac{y}{(1-\rho)^{1/2}} \;;\; 0\right).$$

Using (2.6) we can evaluate the cdf of $\Sigma a_i X_{(i)}$ when $\Sigma a_i = 0$, a case of common practical interest. We mention the following four cases.

(a) $a_1 = -1$; $a_n = 1$; $a_i = 0$, $i = 2, 3, \ldots, n-1$.

In this case the cdf of the range of correlated random variables reduces to the already well-studied case (see Pearson and Hartley (1942), Gumbel (1949), Pillai (1952), Harter (1960)) of the range of uncorrelated variables. Letting $y_\alpha(n;\rho)$ denote the α percentage points of the range in the correlated case, we see that

(2.7) $$y_\alpha(n;\rho) = y_\alpha(n;0)/(1-\rho)^{1/2} .$$

(b) Quasi Range, a particular case when a_i and a_{n-i+1} $(i = 2, 3, \ldots, [\frac{n}{2}])$ are equal to -1 and +1, respectively, and the remaining a_i's are equal to zero. The formulae (2.6) and (2.7), again, give the distribution of the correlated case.

(c) Nair's statistic (Nair, 1948) is obtained by putting $a_n = 1 - \frac{1}{n}$ and $a_1 = a_2 = \ldots = a_{n-1} = -\frac{1}{n}$. Thus formulae (2.6) and (2.7) are valid for reducing the distribution of the Nair Statistic from the correlated to the uncorrelated case, the latter being tabulated by Nair (1948), Pillai (1959) and by others.

(d) Tests for outliers based on $X_{(n)} - X_{(n-1)}$ (see Grubbs (1950)) or similar differences of successive order statistics. These are again special cases of $Y = \sum_{i=1}^{n} a_i X_{(i)}$ and can be solved by the same formulae (2.6) and (2.7).

3. Moments of Y

Owen and Steck (1962) have given formulae and tables for the moments and cumulants of $X_{(k)}$ in terms of the corresponding moments and cumulants of $X_{(k)}$ and Z_0. More directly, from (2.1) and (2.2) we can write

(3.1) $$X_{(k)} = (\pm\rho)^{1/2} Z_0 + (1-\rho)^{1/2} Z_{(k)}$$

where in the first term on the right in (3.1) we take + or - sign according as $\rho \geq 0$ or $\rho < 0$.

It follows then

(3.2) $$E(Y^j) = E\left[(\Sigma a_i)(\pm\rho)^{1/2} Z_0 + (1-\rho)^{1/2} \sum_{i=1}^{n} a_i Z_{(i)}\right]^j$$

$$= \sum_{p=0}^{j} (\Sigma a_i)^p (\pm\rho)^{p/2} (1-\rho)^{(j-p)/2} \; E\left[Z_0{}^p \left(\Sigma a_i Z_{(i)}\right)^{j-p}\right] \binom{j}{p}.$$

If $\rho \geq 0$, then we get

$$(3.3) \quad E(Y^j) = \sum_{\alpha=0}^{[\frac{j}{2}]} (\Sigma a_i)^{2\alpha} (\rho)^{\alpha} (1-\rho)^{(j-2\alpha)/2} \frac{(2\alpha)!}{\alpha! 2^{2\alpha}} E\left(\Sigma a_i Z_{(i)}\right)^{j-2\alpha} \binom{j}{2\alpha}$$

where $[x]$ is the largest integer less than or equal to x.

In (3.3), one can substitute from the tables (Ruben (1954), Teichroew (1956), Harter (1961)) the expected values of the powers of the i^{th} order statistic in a random sample of size n from a $N(0,1)$. Note that if $\sum_{i=1}^{n} a_i = 0$,

$$(3.4) \quad E(Y^j) = (1-\rho)^{j/2} E\left(\sum_{i=1}^{n} a_i Z_{(i)}\right)^j ; \quad \Sigma a_i = 0; \quad \text{all } \rho.$$

In particular, for the moments of range W_n, we get

$$(3.5) \quad E(W_n) = 2(1-\rho)^{1/2} EZ_{(n)}$$

and, generally,

$$(3.6) \quad E(W_n^j) = (1-\rho)^{j/2} E[Z_{(n)} - Z_{(1)}]^j$$

Note that

$$(3.7) \quad EY = (1-\rho)^{1/2} \sum_{i=1}^{n} a_i EZ_{(i)}, \quad \text{all } \rho.$$

4. Studentized Case

If X_i are assumed to be equally correlated normal random variables with $EX_i = 0$, $EX_i^2 = \sigma^2$ $(i = 1, 2, \ldots, n)$, and if s_ν^2 is an estimate of σ^2 which is distributed as $\sigma^2 \chi^2/\nu$ and which is independent of $\sum_{i=1}^{n} a_i X_{(i)}$, then we can write

$$(4.1) \quad P\left\{\frac{\sum_{i=1}^{n} a_i X_{(i)}}{s_\nu} \leq y'\right\} = \int_0^\infty H(y'x; \rho)\, g_\nu(x)dx$$

where $g_\nu(x)$ is the density function for the random variable $\chi_\nu/\sqrt{\nu}$ and where $H(y; \rho)$ is the integral defined by (2.5). If $\Sigma a_i = 0$, then we have

$$(4.2)\quad P\left\{\frac{\sum_{i=1}^{n} a_i X_{(i)}}{s_\nu} \le y'\right\} = \int_0^\infty H\left(\frac{y' x}{\sqrt{1-\rho}} ; 0\right) g_\nu(x)dx , \qquad \Sigma a_i = 0.$$

It follows from (4.2) that the percentage point $y'_\alpha\ (n, \nu ; \rho)$ of the studentized correlated case is related to the percentage point $y'_\alpha\ (n, \nu ; 0)$ of the studentized uncorrelated case by

$$(4.3)\quad y'_\alpha\ (n, \nu ; \rho) = y'_\alpha\ (n, \nu ; 0)/(1-\rho\ )^{1/2} ,$$

which is analogous result to (2.7).

We remark that the moments of $\Sigma a_i X_{(i)}/s_\nu$ will exist only if the negative moments of s_ν exist. We have

$$\begin{aligned}(4.4)\qquad E(\Sigma a_i X_{(i)}/s_\nu\ )^j &= E(\Sigma a_i X_{(i)}/\sigma)^j\ E(s_\nu/\sigma)^{-j} \quad \text{if} \quad j < \nu \\ &= E(\Sigma a_i X_{(i)}/\sigma)^j\ \frac{\Gamma(\frac{\nu - j}{2})(\frac{\nu}{2})^{j/2}}{\Gamma(\frac{\nu}{2})} , \quad j < \nu.\end{aligned}$$

It should be pointed out that in (4.4) the first factor in the second line of (4.4) can be evaluated by using formulae derived in Section 3. In particular we can obtain the moments of the studentized range, studentized Nair statistic and studentized quasi ranges by using the formula (4.4) and the formulae of Section 3. The probability integral and percentage points for these particular cases can be evaluated by using (4.1) - (4.3).

5. Distribution of the range of correlated normal variables

The joint distribution of $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ defined in (1.1) can be written in the form

$$(5.1)\quad f(X_{(1)}, X_{(2)}, \ldots, X_{(n)}) = \frac{n!\ e^{-X' A^{-1} X}}{(2\pi)^{n/2}\ |A|^{1/2}}$$

where X is the colum vector of the $X_{(i)}$'s and A, the matrix of simple correlation coefficients between pairs of unordered X's and $|A|$, the determinant of A.

5. Distribution of Y_1/Y_2 when X_j's are equally correlated

Let $W = X_{(n)} - X_{(1)}$, and let $W_1, W_2, \ldots, W_k$ be independently, identically distributed variables with the same distribution as W.

We shall show that the random variables

$R = (X_{(h)} - X_{(i)})/(X_{(j)} - X_{(k)})$, $(h>i, j>k)$

$S = \max W_i/(W_1 + \ldots + W_k)$,

$T = W_1/W_2$

all have distributions independent of ρ. We shall also find the distribution of T in the more general case where $W_i (i = 1,2)$ is the range in an equicorrelated sample of size n_i with $\rho = \rho_i$.

In the case $\rho = 0$, these random variables have been discussed by several authors. Statistics of the form of R are discussed and their percentage points tabulated by Dixon (1951) in connection with testing the consistency of suspected observations with the whole sample. The statistic S is suggested and the 5% points tabulated by Bliss, Cochran and Tukey (1956) for testing the same hypothesis in a different situation. For more discussion of these statistics see Dixon (1962). Link (1950) suggested the use of T as an alternative to the usual F-test for equality of variances and partially tabulated its percentage points. More recently, Pillai and Buenaventura (1961) and Harter (1963) have provided more extensive tabulation of the percentage points.

The distribution of R is independent of ρ since

$$P(R \le r|\rho) = P(X_{(h)} - X_{(i)} - rX_{(j)} + rX_{(j)} + rX_{(k)} + \le 0|\rho),$$

$$= P(\{X_{(h)} - X_{(i)} - rX_{(j)} + rX_{(k)} \}\sqrt{(1 - \rho)} \le 0|0) = P(R \le r|0)$$

by (2.6).

The distribution of S is determined by the distribution of the maximum of k identically distributed variables of the form $U_k = W_1/(W_2 + \ldots + W_k)$ (See Bliss, Cochran, and Tukey (1956)), and we now show that the distribution of U_k is independent of ρ. The characteristic function of W_i is $\varphi_{W_i}(t|\rho) = \varphi_{W_i}(t\sqrt{(1 - \rho)}|0)$ as seen above. Consequently, the joint characteristic function of the $\{W_i\}$ is

$$\varphi_{W_1,W_2,\ldots, W_n}(t_1, t_2,\ldots, t_n|\rho) = \prod \varphi_{W_i}(t_i|\rho) = \prod \varphi_{W_i\sqrt{(1-\rho)}}(t_i|0). \tag{5.1}$$

Since the distribution of $W_1/(W_2 + \ldots + W_k)$ and $aW_1/(aW_2 + \ldots + aW_k)$ are the same it follows from (5.1) that the distribution of U_k is independent of ρ.

The distribution of T in the more general case is argued as follows. The joint characteristic function of W_1 and W_2 is

$$\varphi_{W_1,W_2}(t, u|\rho_1, \rho_2, n_1, n_2) = \varphi_{W_1\sqrt{(1-\rho_1)},\ W_2\sqrt{(1-\rho_2)}}(t,u|0,0,n_1,n_2).$$

This implies that

$$P(W_1/W_2 \le r|\rho_1,\rho_2,n_1,n_2) = P(W_1/W_2 \le r\sqrt{((1-\rho_2)/(1-\rho_1))}|0,0,n_1,n_2).$$

Thus the distribution of W_1/W_2 in equicorrelated samples can be found easily from the distribution in independent samples.

6. Distribution of the range of correlated normal random variables

Let X_1, X_2, X_3, X_4 be normally distributed variables with zero means, unit variances and $EX_iX_j = \rho_{ij}$ $(i \neq j)$. Also, let

$$W = \max X_i - \min X_i,$$

$$a_{ij} = \sqrt{2(1 - \rho_{ij})},$$

$$\Delta_{ijk} = 1 - \rho_{ij}^2 - \rho_{ik}^2 - \rho_{jk}^2 + 2\, \rho_{ij}\rho_{jk}\rho_{ik}.$$

The distribution of W is given by

$$(6.1)\quad P(W \leq w) = P(W \leq w \text{ and } X_1 < X_2 < X_3 < X_4) + \ldots \text{ (24 terms)}.$$

The other 23 terms are obtained from the first by permuting the indices of the $\{X_i\}$. The first term, call it P_1, is

$$(6.2)\quad P_1 = P(X_1 - X_2 \leq 0,\ X_2 - X_3 \leq 0,\ X_3 - X_4 \leq 0,\ 0 \leq X_4 - X_1 \leq w)$$

$$= P(U_1 \leq 0,\ U_2 \leq 0,\ U_3 \leq 0,\ 0 \leq U_4 \leq w/a_{14}),$$

where

$$U_1 = (X_1 - X_2)/a_{12},\quad U_2 = (X_2 - X_3)/a_{23},\quad U_3 = (X_3 - X_4)/a_{34},\quad U_4 = (X_4 - X_1)/a_{14}.$$

The $\{U_i\}$ have zero means and unit variances, and if $\xi_{ij} = EU_iU_j$, then

$$\xi_{12} = -(1 + \rho_{13} - \rho_{12} - \rho_{23})/a_{12}a_{23}$$

$$\xi_{13} = -(\rho_{23} + \rho_{14} - \rho_{13} - \rho_{24})/a_{12}a_{34}$$

$$\xi_{14} = -(1 + \rho_{24} - \rho_{12} - \rho_{14})/a_{12}a_{14}$$

$$\xi_{23} = -(1 + \rho_{24} - \rho_{23} - \rho_{34})/a_{23}a_{34}$$

$$\xi_{24} = -(\rho_{12} + \rho_{34} - \rho_{13} - \rho_{24})/a_{23}a_{14}$$

$$\xi_{34} = -(1 + \rho_{13} - \rho_{14} - \rho_{34})/a_{14}a_{34}.$$

Note that $a_{12}U_1 + a_{23}U_2 + a_{34}U_3 + a_{14}U_4 = 0$. This implies, since the $\{a_{ij}\}$ are nonnegative, that $P(U_1 \le 0, U_2 \le 0, U_3 \le 0, U_4 \le 0) = 0$. Noting this, and substituting for U_4 in (6.2) gives

$$P_1 = P(U_1 \le 0,\ U_2 \le 0,\ U_3 \le 0,\ a_{12}U_1 + a_{23}U_2 + a_{34}U_3 \le - w).$$

The variables $\{U_i\}$ can be generated from independent $N(0,1)$ variables $\{X_i\}$ by

$$U_1 = X_1$$

$$U_2 = \xi_{12}X_1 + \sqrt{1 - \xi_{12}^2}\ X_2$$

$$U_3 = \xi_{13}X_1 + \left((\xi_{23} - \xi_{12}\xi_{13})/\ \sqrt{1 - \xi_{12}^2}\right) X_2 + \sqrt{\Delta_{123}/(1 - \xi_{12}^2)}\ X_3,$$

and in terms of the $\{X_1\}$ the plane $a_{12}U_1 + a_{23}U_2 + a_{34}U_3 + w = 0$ can be shown to be

$$(6.3)\quad - \xi_{14}X_1 - \left((\xi_{24} - \xi_{12}\ \xi_{14})/\ \sqrt{1 - \xi_{12}^2}\right) X_2 + \sqrt{\Delta_{124}/(1 - \xi_{12}^2)}\ X_3 + w/a_{14} = 0.$$

The distance from the origin to this plane is w/a_{14}.

In terms of the $\{X_i\}$ the volume of interest is that of the tetrahedron with vertices at

$$A_o = (0,0,0)$$

$$A_1 = \left(0,0,\ - \frac{w}{a_{34}} \sqrt{\frac{1 - \xi_{12}^2}{\Delta_{123}}}\right)$$

$$A_2 = \left(0, - \frac{w}{a_{23} \sqrt{1 - \xi_{12}^2}}, \; \frac{w}{a_{23}} \; \frac{\xi_{23} - \xi_{12}\xi_{13}}{\sqrt{\Delta_{123}(1 - \xi_{12}^2)}} \right)$$

$$A_3 = \left(- \frac{w}{a_{12}}, \; \frac{w\xi_{12}}{a_{12} \sqrt{1 - \xi_{12}^2}}, \; \frac{w}{a_{12}} \; \frac{\xi_{13} - \xi_{12}\xi_{23}}{\sqrt{\Delta_{123} (1 - \xi_{12}^2)}} \right),$$

where A_1, A_2, A_3 lie on the plane whose equation is given by (6.3).

The perpendicular from the origin to this plane intersects the plane at

$$A_4 = \left(\frac{w\xi_{14}}{a_{14}}, \; \frac{w}{a_{14}} \; \frac{\xi_{24} - \xi_{12}\xi_{14}}{\sqrt{1 - \xi_{12}^2}}, \; - \frac{w}{a_{14}} \sqrt{\frac{\Delta_{124}}{1 - \xi_{12}^2}} \right)$$

and the coordinates of A_4 are given by the following linear combination of the coordinates of the A_i

$$A_4 = - \left(\frac{a_{34}\xi_{34}}{a_{14}} A_1 + \frac{a_{23}\xi_{24}}{a_{14}} A_2 + \frac{a_{12}\xi_{14}}{a_{14}} A_3 \right),$$

where the sum of the weights is unity.

Imagine first that A_4 lies within the triangle $A_1A_2A_3$ (such will be the case if the weights are positive). If a perpendicular is drawn from A_4 to each of the sides of $A_1A_2A_3$ then the tetrahedron $A_0A_1A_2A_3$ is divided into six regions and the probability that (X_1, X_2, X_3) lies in one of these regions is expressible in terms of the S-function

described by Steck (1958). Consequently, the probability content of the tetrahedron, P_1, is obtained by adding the contents of the six regions. If A_4 lies outside $A_1A_2A_3$ then P_1 is obtained as a linear combination of the contents of the regions, the weights being ±1.

The process of finding the requisite lengths is a tedious but straight-forward process, though it helps to note equalities of the following type

$$a_{14} + a_{12}\xi_{14} + a_{23}\xi_{24} + a_{34}\xi_{34} = 0$$

$$a_{14}^2 \Delta_{124} = a_{34}^2 \Delta_{123}.$$

Let

$$h = \frac{w}{a_{14}},$$

$$a_1 = \frac{\xi_{14}}{\sqrt{1 - \xi_{14}^2}} \qquad a_2 = \frac{\xi_{24}}{\sqrt{1 - \xi_{24}^2}} \qquad a_3 = \frac{\xi_{34}}{\sqrt{1 - \xi_{34}^2}},$$

$$b_{11} = \frac{\xi_{34} - \xi_{13}\xi_{14}}{\xi_{14} \sqrt{\Delta_{134}}} \qquad b_{21} = \frac{\xi_{14} - \xi_{12}\xi_{24}}{\xi_{24} \sqrt{\Delta_{124}}} \qquad b_{31} = \frac{\xi_{14} - \xi_{13}\xi_{34}}{\xi_{34} \sqrt{\Delta_{134}}},$$

$$b_{12} = \frac{\xi_{24} - \xi_{12}\xi_{14}}{\xi_{14} \sqrt{\Delta_{124}}} \qquad b_{22} = \frac{\xi_{34} - \xi_{24}\xi_{23}}{\xi_{24} \sqrt{\Delta_{234}}} \qquad b_{32} = \frac{\xi_{24} - \xi_{23}\xi_{34}}{\xi_{34} \sqrt{\Delta_{234}}},$$

and let

$$W(h,a,b) = \frac{\arctan b}{2\pi}\left(G(h) - \tfrac{1}{2}\right) + \frac{1}{4\pi}\arctan\frac{b}{\sqrt{(1 + a^2 + a^2b^2)}} - S(h,a,b).$$

Then $W(h,a,b)$ is the probability content of a region of the tetrahedron and

$$P_1 = \sum_{k=1}^{3}\sum_{j=1}^{2} W(h,a_k,b_{kj}).$$

Corresponding results for the trivariate normal distribution can be obtained by letting $X_3 \equiv X_4$. Then

$$\rho_{14} = \rho_{13} \quad a_{14} = a_{13} \quad \xi_{12} = -(1 + \rho_{13} - \rho_{12} - \rho_{23})/a_{12}a_{23} = \theta_{12} \quad \text{(say)}$$

$$\rho_{24} = \rho_{23} \quad a_{24} = a_{23} \quad \xi_{14} = -(1 + \rho_{23} - \rho_{12} - \rho_{13})/a_{12}a_{13} = \theta_{13} \quad \text{(say)}$$

$$\rho_{34} = 1 \quad a_{34} = 0 \quad \xi_{24} = -(1 + \rho_{12} - \rho_{13} - \rho_{23})/a_{13}a_{23} = \theta_{23} \quad \text{(say)}$$

$$\xi_{13} = \xi_{23} = \xi_{34} = 0/0 = 0,$$

and it can be shown that each θ is no larger than the product of the other two and that no more than one θ can be positive.

Also,

$$\Delta_{134} = 1 - \theta_{13}^2 \qquad \Delta_{124} = 0 \qquad \Delta_{234} = 1 - \theta_{23}^2$$

$$a_1 = \frac{\theta_{13}}{\sqrt{(1 - \theta_{13}^2)}} \qquad a_2 = \frac{\theta_{23}}{\sqrt{(1 - \theta_{23}^2)}} \qquad a_3 = 0$$

$$b_{11} = 0 \qquad b_{21} = -\infty \operatorname{sgn} \theta_{23} \qquad b_{31} = \pm\infty$$

$$b_{12} = -\infty \operatorname{sgn} \theta_{13} \qquad b_{22} = 0 \qquad b_{32} = \pm\infty.$$

Further,

$$W(h,a,0) = 0$$

$$W(h,0,b) = 0$$

$$W(h,a,\infty) = -W(h,a,-\infty)$$

$$W(h,a,\infty) = \tfrac{1}{2} \; V(h,|a|h)$$

$$V(h,a) = V(h,|a|h) \text{ sgn } a$$

$$W(h,a,-\infty \text{ sgn } a) = -\text{sgn } a \; W(h,a,\infty).$$

where $V(a,b)$ is the V-function described by Nicholson (1943).

Consequently,

$$P_1 = -\tfrac{1}{2}\left[V\left(\frac{w}{a_{13}}, \frac{w\,\theta_{13}}{a_{13}\sqrt{(1-\theta_{13}^2)}}\right) + V\left(\frac{w}{a_{13}}, \frac{w\,\theta_{23}}{a_{13}\sqrt{(1-\theta_{23}^2)}}\right)\right]$$

Any permutation of the integers 1,2,3,4 which does not have 3 and 4 adjacent will have the corresponding $P_i = 0$, since $P(X_3 \leq X_1 \leq X_4) = 0$ if $X_3 \equiv X_4$. There are twelve permutations with 3 and 4 adjacent and these yield three sets of four equal P_i.

Thus, in the trivariate case,

$$\begin{aligned} P(W \leq w) = -2 \Bigg[& V\left(\frac{w}{a_{12}}, \frac{w\,\theta_{12}}{a_{12}\sqrt{(1-\theta_{12}^2)}}\right) + V\left(\frac{w}{a_{12}}, \frac{w\,\theta_{13}}{a_{12}\sqrt{(1-\theta_{13}^2)}}\right) \\ & + V\left(\frac{w}{a_{13}}, \frac{w\theta_{13}}{a_{13}\sqrt{(1-\theta_{13}^2)}}\right) + V\left(\frac{w}{a_{13}}, \frac{w\theta_{23}}{a_{13}\sqrt{(1-\theta_{23}^2)}}\right) \\ & + V\left(\frac{w}{a_{23}}, \frac{w\theta_{12}}{a_{23}\sqrt{(1-\theta_{12}^2)}}\right) + V\left(\frac{w}{a_{23}}, \frac{w\theta_{23}}{a_{23}\sqrt{(1-\theta_{23}^2)}}\right) \Bigg]. \end{aligned}$$

Finally, in the bivariate case one has

$$P(W \le w) = 2\Phi(w/\sqrt{\{2(1-\rho)\}}) - 1, \quad (0 \le w \le \infty),$$

where $\Phi(x)$ is the distribution function of a $N(0,1)$ random variable. This follows since for $n=2$ the general case and the equally correlated case coincide and (2.6) applies.

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#3

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On linear functions of ordered correlated normal random variables

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1. INTRODUCTION AND SUMMARY

Let $X_1, X_2, \ldots, X_n$ be jointly normally distributed random variables with $EX_i = 0$, $EX_i^2 = 1$ and $EX_iX_j = \rho_{ij}\,(i \neq j = 1, 2, \ldots, n)$. Let $X_{(k)}$ be the *kth* order statistic when X_i's are arranged in an increasing order as follows:

$$X_{(1)} \leqslant X_{(2)} \leqslant \ldots \leqslant X_{(k)} \leqslant \ldots \leqslant X_{(n)}. \tag{1·1}$$

Many problems of statistical inference, notably the ones in multiple decisions and life testing involve the use of ordered X_i's. In an earlier paper Gupta, Pillai & Steck (1964) considered the distribution of linear functions of $X_{(i)}$'s and gave closed form results for the case when the random variables are equally correlated. Also in the same paper closed form expressions for the distribution of the range $W = X_{(n)} - X_{(1)}$ were obtained for $n = 3$ and 4 for the general case. In this paper we study the characteristic functions of individual order statistics and also the linear functions for the bivariate and trivariate cases for the general correlation matrix. Formulae for the expected values of $X_{(i)}$, $X_{(i)}^2$ and $X_{(i)}X_{(j)}$ and the first and second moments of a linear function of the $X_{(i)}$'s are obtained. The joint distribution of the range and the mid-range is given in a closed form in the trivariate case, from which the distributions of the mid-range and the mid-range/range ratio are derived, again in closed forms. Best linear unbiased estimators of the common mean of three correlated normal variables have been obtained and tabulation of the coefficients made for different sets of values of ρ_{ij}'s. Applications in the fields of life testing and time series analysis are discussed.

2. THE BIVARIATE CASE

In the bivariate case, the characteristic function is given by

$$\phi_{X_{(1)}\,X_{(2)}}(t_1, t_2) = E\{\exp(it_1X_{(1)} + it_2X_{(2)})\} = 2E\{\exp\{it_1X_1 + it_2X_2\}/X_1 \leqslant X_2\}. \tag{2·1}$$

Hence

$$\phi_{X_{(1)},\,X_{(2)}}(t_1, t_2) = \frac{1}{\pi\sqrt{(1-\rho^2)}}\int_{-\infty}^{\infty}\int_{-\infty}^{X_2} \exp(it_1X_1 + it_2X_2) \times \exp\left[-\frac{X_1^2 + X_2^2 - 2\rho X_1X_2}{2(1-\rho^2)}\right] dX_1\,dX_2. \tag{2·2}$$

Now we state a lemma which simplifies (2·2).

LEMMA 1. *For any real numbers* α *and* β

$$\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \Phi(\alpha X + \beta)\, e^{-\frac{1}{2}X^2}\, dX = \Phi\left(\frac{\beta}{\sqrt{(1+\alpha^2)}}\right) \tag{2·3}$$

where $\Phi(\cdot)$ *denotes the* C.D.F. *of the random variable* $N(0, 1)$.

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After completing squares, performing integrations on X_1 and using the lemma (2·3), one obtains from the moment generating function corresponding to (2·2), the following explicit closed form for the characteristic function.

$$\phi_{X_{(1)},\,X_{(2)}}(t_1, t_2) = 2\exp\{-\tfrac{1}{2}(t_i^2+2\rho t_1 t_2+t_2^2)\}\,\Phi(\tfrac{1}{2}\sqrt{(1-\rho)}\,i(t_2-t_1)\}. \tag{2·3a}$$

(The function $\Phi(z)$ is well defined and analytic in the complex plane.) Note that the formula (2·3a) gives the answer for the characteristic function in an explicit form and has been obtained directly. The formula (5) in the paper by Owen & Steck (1962) after evaluation of the characteristic function of the two ordered random variables (arising from independent case) reduces to the right hand side of (2·3a).

It follows from (2·3a) that

$$\phi_{X_{(2)}}(t_2) = 2\exp(-\tfrac{1}{2}t_2^2)\,\Phi(\tfrac{1}{2}\sqrt{(1-c)}\,it_2) \tag{2·4}$$

and by substituting $t_2 = t_1$, we get the characteristic function for $X_{(1)}$. Note that the characteristic function of a linear function of $X_{(1)}$ and $X_{(2)}$ can be obtained from (2.3a) by substituting $t_i = a_i t\,(i = 1, 2)$ and is given by

$$\phi_{a_1 X_{(1)}+a_2 X_{(2)}}(t) = 2\exp\{-\tfrac{1}{2}t^2(a_1^2+2\rho a_1 a_2+a_2^2)\}\,\Phi(\tfrac{1}{2}\sqrt{(1-\rho)}\,it(a_2-a_1)). \tag{2·5}$$

From (2·5) one can obtain all the moments of $a_1 X_{(1)}+a_2 X_{(2)}$ and verify the usual formulae for $a_1 = a_2 = 1$, i.e. the case of the mid-range. The characteristic function for $a_1 = a_2 = 1$ is

$$\phi_{X_{(1)}}+{}_{X_{(2)}}(t) = \exp\{-t^2(1+\rho)\}, \tag{2·6}$$

and hence mid-range follows the normal distribution $N(0, 2(1+\rho))$.

From (2·5) we obtain the density function $f(y)$ of $Y = a_1 X_{(1)}+a_2 X_{(2)}$ as

$$f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_y(t)\,e^{-ity}\,dy \tag{2·7}$$

which after some simplification gives

$$f(y) = \sqrt{\frac{2}{\pi\xi}}\exp\left(-\frac{y^2}{2\xi}\right)\Phi(\eta y) \tag{2·8}$$

where $\quad \xi = a_1^2+a_2^2+2\rho a_1 a_2 \quad$ and $\quad \eta = [\sqrt{\{2(1-\rho)\}}\,(a_2-a_1)]/[(1+\rho)\,(a_2+a_1)^2]$.

By substituting $a_1 = 0$ and $a_2 = 1$ in (2·8), we obtain the density function for $X_{(2)}$, which is,

$$f(X_{(2)}) = \sqrt{\frac{2}{\pi}}\exp(-\tfrac{1}{2}X_{(2)}^2)\,\Phi\left(\frac{\sqrt{\{2(1-\rho)\}}}{(1+\rho)}X_{(2)}\right) \tag{2·9}$$

which is a special case of the distribution of the maximum of several equally correlated normal random variables and is discussed and tabulated by Gupta (1963). Now we discuss the evaluation of the density function of the statistic $a_1 X_{(1)}+a_2 X_{(2)} = X_0$ where X_0 is $N(0, 1)$ distributed independently of X_1 and X_2. This statistic arises in connexion with some multiple decision problems (Gupta & Sobel, 1957).

A convolution of the earlier derived distribution of $a_1 X_{(1)}+a_2 X_{(2)}$ with X_0 gives the following result for the density, $g(Z)$, of $Z = a_1 X_{(1)}+a_2 X_{(2)}-X_0$

$$g(Z) = \frac{2}{\sqrt{\{2\pi(1+\xi)\}}}\exp\left[-\frac{Z^2}{2(1+\xi)}\right]\Phi\left(\frac{Z}{\{(1+\xi)\,[1+\xi(1+\eta^2)]\}^{\frac{1}{2}}}\right). \tag{2·10}$$

3. TRIVARIATE CASE

$$E[\exp\{i(t_1 X_{(1)}+t_2 X_{(2)}+t_3 X_{(3)})\}]$$

$$= \sum_{i,j,k} \iiint\limits_{X_i\leqslant X_j\leqslant X_k} \exp\{i(t_1 X_1+t_2 X_2+t_3 X_3)\} \frac{\exp -\frac{1}{2}(X'A^{-1}X)}{|A|^{\frac{1}{2}}(2\pi)^{\frac{3}{2}}}\, dX_1\, dX_2\, dX_3 \qquad (3\cdot 1)$$

where (i,j,k) represents a permutation of the positive integers $(1,2,3)$ and where the summation extends over all the six permutations of (i,j,k). We shall evaluate the one term of (3·1) corresponding $X_1 \leqslant X_2 \leqslant X_3$. We transform from X_1, X_2, X_3 $(X_1 \leqslant X_2 \leqslant X_3)$ to

$$\left.\begin{aligned} W &= X_3-X_1,\\ M &= X_3+X_1,\\ U &= X_2+(A^{32}X_3+A^{12}X_1)/A^{22}, \end{aligned}\right\} \qquad (3\cdot 1a)$$

where A^{12}, A^{22}, etc., are elements of A^{-1} (a 3×3 matrix) whose positions are denoted by respective upper suffixes. Then

$$I = \iiint\limits_{X_1\leqslant X_2\leqslant X_3} \exp\{i(t_1 X_1+t_2 X_2+t_3 X_3)\} \frac{\{\exp -\frac{1}{2}(X'A^{-1}X)\}}{(2\pi)^{\frac{3}{2}}|A|^{\frac{1}{2}}}\, dX_1\, dX_2\, dX_3$$

$$= \frac{1}{2|A|^{\frac{1}{2}}(2\pi)^{\frac{3}{2}}} \int_0^\infty dW \int_{-\infty}^{\infty} dM \int_{C_0M+d_1W}^{C_0M+d_2W} \exp[it_1\tfrac{1}{2}(M-W)+it_3\tfrac{1}{2}(M-W)$$

$$+it_2(U+a_0M+bW)]\exp\left[-\frac{W^2}{4(1-\rho_{13})}-\frac{M^2}{4(1+\rho_{13})}-\frac{A^{22}U^2}{2}\right] dU \qquad (3\cdot 2)$$

where

$$\left.\begin{aligned} &a_0 = -\frac{A^{12}+A^{32}}{2A^{22}} = +\frac{\rho_{12}-\rho_{23}}{2(1+\rho_{13})}, \quad b = \frac{A^{12}-A^{32}}{2A^{22}} = -\frac{\rho_{12}+\rho_{23}}{2(1-\rho_{13})},\\ &\left.\begin{aligned} &C_0 = \tfrac{1}{2}-a_0, && d_1 = -\tfrac{1}{2}-b, && d_2 = \tfrac{1}{2}-b,\\ &\quad = \frac{1+\rho_{13}-\rho_{12}-\rho_{23}}{2(1+\rho_{13})}, && \quad = \frac{-1+\rho_{13}+\rho_{12}-\rho_{23}}{2(1-\rho_{13})} && \quad = \frac{1-\rho_{13}+\rho_{12}-\rho_{23}}{2(1-\rho_{13})}, \end{aligned}\right\}\\ &|A| = 1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{23}\rho_{13}. \end{aligned}\right\} \qquad (3\cdot 3)$$

After some simplification (completing the squares) and integrating out U, we get

$$I = \frac{\exp[-t_2^2/(2A^{22})}{4\pi|A|^{\frac{1}{2}}\sqrt{A^{22}}} \int_0^\infty \Psi_2(W)\, dW \int_{-\infty}^{\infty} \left[\Phi\left(\sqrt{A^{22}}(C_0M+d_2W)-\frac{it_2}{\sqrt{A^{22}}}\right)\right.$$

$$\left.-\Phi\left(\sqrt{A^{22}}(C_0M+d_1W)-\frac{it_2}{\sqrt{A^{22}}}\right)\right]\Psi_1(M)\, dM, \qquad (3\cdot 4)$$

where

$$\left.\begin{aligned} \Psi_1(M) &= \exp\left[-\frac{M^2}{4(1+\rho_{13})}+\tfrac{1}{2}iM(t_1+t_3+2a_0t_2)\right],\\ \Psi_2(W) &= \exp\left[-\frac{W^2}{4(1-\rho_{13})}+\tfrac{1}{2}iW(-t_1+t_3+2bt_2)\right]. \end{aligned}\right\} \qquad (3\cdot 5)$$

Now integrating out M by applying the Lemma (2·3), we obtain

$$I = \frac{1}{2}\sqrt{\left(\frac{1}{(1-\rho_{13})\pi}\right)} \exp\left[-\frac{t_2^2}{2A^{22}}-\frac{(1+\rho_{13})}{4}(t_1+t_3+2a_0t_2)^2\right] I_1, \qquad (3\cdot 6)$$

where

$$I_1 = \int_0^\infty \exp\left[-\frac{W^2}{4(1-\rho_{13})}+\tfrac{1}{2}iW(-t_1+t_3+2bt_2)\right][\Phi(d_2)-\Phi(d_1)]\, dW \qquad (3\cdot 7)$$

and

$$\Phi(d_i) = \Phi\left(\frac{C_0\sqrt{A^{22}}(1+\rho_{13})\, i(t_1+t_3+2a_0t_2)-(it_2/\sqrt{A^{22}})+\sqrt{A^{22}}\, d_iW}{\sqrt{\{1+2A^{22}(1+\rho_{13})C_0^2\}}}\right). \qquad (3\cdot 8)$$

Thus, formula (3·6) provides the characteristic function $\phi(t_1, t_2, t_3)$ for $X_1 \leqslant X_2 \leqslant X_3$. By permuting the indices 1, 2, 3 in (3·6), we can obtain the other cases. Note (3·6) involves an integral I_1.

It should be pointed out that the characteristic function for the linear function

$$a_1 X_{(1)} + a_2 X_{(2)} + a_3 X_{(3)}$$

is given by

$$\phi_{a_1 X_{(1)}+a_2 X_{(2)}+a_3 X_{(3)}} = \phi_{X_1, X_2, X_3}(a_1 t, a_2 t, a_3 t). \tag{3·8a}$$

Special case

If we are interested in $\phi_{X_{(1)}}(t_1)$ we can find this by substituting $t_2 = t_3 = 0$ in the six terms of $\phi_{X_{(1)}, X_{(2)}, X_{(3)}}(t_1, t_2, t_3)$. Thus

$$\phi_{X_{(1)}}(t_1) = \sum_{i, j, k} I^{i, j, k}(t_1, 0, 0), \tag{3·8b}$$

where

$$I^{i, j, k}(t_1, t_2, t_3) = \text{integral similar to } I \text{ of (3·6)} \tag{3·9}$$

which corresponds to the permutation (i, j, k) instead of $(1, 2, 3)$ appearing there, the subscript 1 in I_1 remaining unchanged. Now we write one term in (3·8b), viz.

$$I^{1, 2, 3}(t_1, 0, 0,) = \frac{1}{2}\sqrt{\left(\frac{1}{(1-\rho_{13})\pi}\right)} \exp\left[-\tfrac{1}{4}(1+\rho_{13}) t_1^2\right] I_1^{1, 2, 3}(t_1, 0, 0), \tag{3·10}$$

where

$$I_1^{1, 2, 3}(t_1, 0, 0) = \int_0^\infty \exp\left[-\frac{W^2}{4(1-\rho_{13})} - \frac{it_1 W}{2}\right]\left[\Phi\left(\frac{iC_0\sqrt{A^{22}(1+\rho_{13})}\,t_1 + \sqrt{A^{22}}\,d_2\,\omega}{\sqrt{\{1+2A^{22}(1+\rho_{13})\,C_0^2\}}}\right)\right.$$
$$\left. - \Phi\left(\frac{iC_0\sqrt{A^{22}(1+\rho_{13})}\,t_1 + \sqrt{(A^{22})}\,d_1\,W}{\sqrt{\{1+2A^{22}(1+\rho_{13}\,C_0^2\}}}\right)\right] dW. \tag{3·11}$$

From (3·11), we obtain

$$E(X_1 | X_1 \leqslant X_2 \leqslant X_3) = \left.\frac{\partial I^{1, 2, 3}(t_{1, 0, 0})}{i\,\partial t_1}\right|_{t_1=0}, \tag{3·12}$$

$$E(X_1 | X_1 \leqslant X_2 \leqslant X_3) = -\frac{1}{2}\sqrt{\frac{(1-\rho_{13})}{\pi}}\, I_3^{1, 2, 3} + \frac{1}{2}\sqrt{\left(\frac{1}{\pi(1-\rho_{13})}\right)}\, I_4^{1, 2, 3}, \tag{3·13}$$

$$I_3^{1, 2, 3} = -\tfrac{1}{2}\theta(d_1)\left[\frac{1}{2(1-\rho_{13})} + \{\theta(d_1)\}^2\right]^{-\frac{1}{2}} + \tfrac{1}{2}\theta(d_2)\left[\frac{1}{2(1-\rho_{13})} + \{\theta(d_2)\}^2\right]^{-\frac{1}{2}} \tag{3·14}$$

$$\theta(d_i) = d_i\sqrt{\left(\frac{A^{22}}{1+2A^{22}(1+\rho_{13})\,C_0^2}\right)} \quad (i = 1, 2),$$

and

$$I_4^{1, 2, 3} = \frac{C_0(1+\rho_{13})}{2}\left[\frac{\theta(d_2)}{d_2}\left\{\{\theta(d_2)\}^2 + \frac{1}{2(1-\rho_{13})}\right\}^{-\frac{1}{2}}\right.$$
$$\left. - \frac{\theta(d_1)}{d_1}\left\{\{\theta(d_1)\}^2 + \frac{1}{2(1-\rho_{13})}\right\}^{-\frac{1}{2}}\right]. \tag{3·15}$$

Again from (3·13), by permuting the indices 1, 2, 3 of ρ_{ij}, one obtains the other five terms.

Special case

If $\rho_{ij} = \rho$, $i \neq j = 1, 2, 3$, we have

$$\left.\begin{array}{l} d_1 = -d_2 = -\tfrac{1}{2}, \quad C_0 = \dfrac{1-\rho}{2(1+\rho)}; \\ A^{11} = A^{22} = A^{33} = (1+\rho)/[(1-\rho)(1+2\rho)]; \; A^{12} = A^{23} = A^{13} = -\rho/[(1-\rho)(1+2p)] \end{array}\right\} \tag{3·16}$$

so that
$$I_4^{i,j,k} = 0 \quad \text{and} \quad I_3^{i,j,k} = \tfrac{1}{2} \quad \text{for all } i,j,k, \tag{3·17}$$

$$EX_{(1)} = 6E(X_1|X_1 \leqslant X_2 \leqslant X_3) = -\frac{3}{2}\sqrt{\left(\frac{1-\rho}{\pi}\right)}, \tag{3·18}$$

which checks with the result obtained directly for this case and further for $\rho = 0$ checks with the well-known values.

From (3·8a), following similar methods as for obtaining the $E(X_1|X_1 \leqslant X_2 \leqslant X_3)$, we can obtain

$$E(a_1X_1+a_2X_2+a_3X_3|X_1 \leqslant X_2 \leqslant X_3) = \frac{1}{2}\sqrt{\left(\frac{1-\rho_{13}}{\pi}\right)}\, I_5^{1,2,3} + \frac{1}{2}\sqrt{\left(\frac{1}{\pi(1-\rho_{13})}\right)}\, I_6^{1,2,3}, \tag{3·19}$$

where
$$I_5^{1,2,3} = (-a_1+a_3+2ba_2)\, I_3^{1,2,3}$$

and
$$I_6^{1,2,3} = \left(a_1+a_3+2a_0a_2 - \frac{a_2}{C_0(1+\rho_{13})A^{22}}\right) I_4^{1,2,3}.$$

From (3·19) we obtain

$$\begin{aligned} &E(X_2|X_1 \leqslant X_2 \leqslant X_3) \\ &\qquad = b\sqrt{\left(\frac{1-\rho_{13}}{\pi}\right)}\, I_1^{1,2,3} + \frac{1}{2}\sqrt{\left(\frac{1}{\pi(1-\rho_{13})}\right)} \left(2a_0 - \frac{1}{C_0(1+\rho_{13})A^{22}}\right) I_4^{1,2,3} \end{aligned} \tag{3·20}$$

and
$$E(X_3|X_1 \leqslant X_2 \leqslant X_3) = \frac{1}{2}\sqrt{\left(\frac{1-\rho_{13}}{\pi}\right)}\, I_3^{1,2,3} + \frac{1}{2}\sqrt{\left(\frac{1}{\pi(1-\rho_{13})}\right)}\, I_4^{1,2,3}. \tag{3·21}$$

Again, putting $-a_1 = a_3 = 1$ and $a_2 = 0$, we get

$$E(X_3-X_1|X_1 \leqslant X_2 \leqslant X_3) = \sqrt{\left(\frac{1-\rho_{13}}{\pi}\right)}\, I_3^{1,2,3}. \tag{3·22}$$

Similarly,
$$E(X_3+X_1|X_1 \leqslant X_2 \leqslant X_3) = \sqrt{\left(\frac{1}{\pi(1-\rho_{13})}\right)}\, I_4^{1,2,3}. \tag{3·23}$$

Covariance between $X_{(j)}$ and $X_{(2)}$, $j = 1,3$

From (3·8) proceeding as usual we obtain

$$E(X_jX_2|X_1 \leqslant X_2 \leqslant X_{3;}j = 1,3) = \Psi_j, \tag{3·24}$$

where

$$\begin{aligned} \psi_j = {} & \frac{1}{2\pi}[a_0(1+\rho_{13})+b\gamma_j(1-\rho_{13})] \arctan\left[\frac{\theta(d_2)-\theta(d_1)}{\sqrt{\{2(1-\rho_{13})\}}\left[\dfrac{1}{2(1-\rho_{13})}+\theta(d_2)\,\theta(d_1)\right]}\right] \\ & + \frac{1}{2\pi\sqrt{\{2(1-\rho_{13})\}}\left(\dfrac{1}{2(1-\rho_{13})}+[\theta(d_2)]^2\right)}\left[b\gamma_j(1-\rho_{13}) + \frac{bC_0(1+\rho_{13})\,\theta(d_2)}{d_2}\right. \\ & \qquad \left. + \frac{\gamma_j\lambda_0}{2} - \frac{C_0[\theta(d_2)]^2(1+\rho_{13})\lambda_0}{d_2}\right] \\ & - \frac{1}{2\pi\sqrt{\{2(1-\rho_{13}\}}\left(\dfrac{1}{2(1-\rho_{13})}+[\theta(d_1)]^2\right)}\left[b\gamma_j(1-\rho_{13}) + bC_0(1+\rho_{13})\frac{\theta(d_1)}{d_1}\right. \\ & \qquad \left. + \frac{\gamma_j\lambda_0}{2} - \frac{C_0[\theta(d_1)]^2(1+\rho_{13})\lambda_0}{d_1}\right], \end{aligned} \tag{3·25}$$

where λ_0 is obtained by putting $a_1 = a_3 = 0$ and $a_2 = 1$ in λ defined in (3·28) below. It should be noted that $\gamma_j = -1$ or $+1$ according as $j = 1$ or 3. Further, in the case of equal correlation, we obtain from (3·25)

$$6E(X_1 X_2|X_1 \leqslant X_2 \leqslant X_3) = E(X_3 X_2|X_1 \leqslant X_2 \leqslant X_3) = \rho + (\sqrt{3}/2\pi)(1-\rho). \quad (3\cdot26)$$

Now we obtain $E[(a_1 X_1 + a_2 X_2 + a_3 X_3)^2|X_1 \leqslant X_2 \leqslant X_3]$. Starting from (3·8$a$) and differentiating the function $E[\exp\{it(a_1 X_1 + a_2 X_2 + a_3 X_3)\}]$ twice partially with respect to t and putting $t = 0$, we obtain after simplification (details omitted)

$$\begin{aligned} &E[(a_1 X_1 + a_2 X_2 + a_3 X_3)^2|X_1 \leqslant X_2 \leqslant X_3] \\ &\quad = \frac{1}{\pi}[\beta + \delta^2(1-\rho_{13})] \arctan\left[\frac{\theta(d_2) - \theta(d_1)}{\sqrt{\{2(1-\rho_{13})\}}\left[\dfrac{1}{2(1-\rho_{13})} + \theta(d_2)\,\theta(d_1)\right]}\right] \\ &\qquad + \frac{1}{\pi}\sqrt{\left(\frac{1-\rho_{13}}{2}\right)}\frac{1}{[2\{\theta(d_2)\}^2(1-\rho_{13})+1]}[2(1-\rho_{13})\,\delta^2\,\theta(d_2) + \lambda(2\delta - \lambda\theta(d_2))] \\ &\qquad - \frac{1}{\pi}\sqrt{\left(\frac{1-\rho_{13}}{2}\right)}\frac{1}{[2\{\theta(d_1)\}^2(1-\rho_{13})+1]}[2(1-\rho_{13})\,\delta^2\theta(d_1) + \lambda(2\delta - \lambda\theta(d_1))], \end{aligned} \quad (3\cdot27)$$

where
$$\left.\begin{aligned} \beta &= \frac{2a_2^2 + (1+\rho_{13})\,A^{22}(a_1 + a_3 + 2a_0\,a_2)^2}{4A^{22}}, \\ \delta &= \tfrac{1}{2}(-a_1 + a_3 + 2ba_2) \\ \lambda &= \frac{C_0\,\sqrt{A^{22}}(1+\rho_{13})\,(a_1 + a_3 + 2a_0\,a_2) - (a_2/\sqrt{A^{22}})}{\sqrt{\{1 + 2A^{22}(1+\rho_{13})\,C_0^2\}}}, \end{aligned}\right\} \quad (3\cdot28)$$
and

where a_0, C_0, b, d_1, d_2 are defined in (3·3), $\theta(d_1)$ and $\theta(d_2)$ are defined by (3·14) and where $A^{22} = (1-\rho_{13}^2)/|A|$. As a particular case take $a_2 = a_3 = 0$, $a_1 = 1$ and $\rho_{ij} = \rho$ (all $i, j, i \neq j$), then we obtain

$$EX_{(1)}^2 = 6E(X_1^2|X_1 \leqslant X_2 \leqslant X_3) = 1 + (1-\rho)[1 + \sqrt{(3)}/(2\pi)] \quad (3\cdot9)$$

which agrees with the well-known result for this case (see, for example, Owen & Steck, 1962).

4. DISTRIBUTIONS OF THE RANGE, MID-RANGE AND THE RATIO OF MID-RANGE TO THE RANGE

The distribution of the range for $n = 3$ and 4 normal random variables for the general case has been obtained by Gupta, *et al.* (1964). Now we shall obtain closed-form expressions for the distribution of the ratio of mid-range to the range.

For the case of $n = 3$ correlated normal random variables, it is easy to show that the joint density function of W, M, U, defined in (3·1a), is given by

$$f(W, M, U) = \frac{\exp\left[-\dfrac{W^2}{4(1-\rho_{13})} - \dfrac{M^2}{4(1+\rho_{13})} - \dfrac{A^{22}}{2}U^2\right]}{2(2\pi)^{\frac{3}{2}}\,|A|^{\frac{1}{2}}}. \quad (4\cdot1)$$

Integrating out U in (4·1) and writing

$$w_1 = \tfrac{1}{2}\sqrt{A^{22}}W \quad \text{and} \quad z_1 = \tfrac{1}{2}\sqrt{A^{22}}M(1+\rho_{13}-\rho_{12}-\rho_{23})/(1+\rho_{13})$$

we obtain

$$f(W_1, z) = C\exp\left[-\frac{W_1^2}{(1-\rho_{13})\,A^{22}} - \frac{Dz_1^2}{2}\right][\Phi(z_1 + W_1(a+1)) - \Phi(z_1 + W_1(a-1))] \quad (4\cdot2)$$

where
$$\left.\begin{aligned} a = (\rho_{12}-\rho_{23})/(1-\rho_{13}) \quad D &= 2\left[A^{22}(1+\rho_{13})\left(1-\frac{\rho_{12}+\rho_{23}}{1+\rho_{13}}\right)^2\right]^{-1}, \\ C &= 1/\pi\,\surd\{D/(2A^{22}(1-\rho_{13}))\}. \end{aligned}\right\} \tag{4·3}$$

From (4·2) we obtain (by putting $z_1/W_1 = U_1$ and $W_1 = W_1$ and integrating out W_1), the density function of U_1 as

$$f(U_1) = \frac{C}{\lambda_1}\left\{\frac{\Phi\left(\dfrac{U_1}{\surd\{1+\lambda_1/(a+1)^2\}}\right)}{\surd 1+\lambda_1/(a+1)^2}\exp\left[-\frac{U_1^2}{2}\left(1-\frac{1}{1+\lambda_1/(a+1)^2}\right)\right]\right.$$
$$\left. -\frac{\Phi\left(\dfrac{-U_1}{\surd\{1+\lambda_1/(a-1)^2\}}\right)}{\surd\{1+\lambda_1/(a-1)^2\}}\exp\left[-\frac{U_1^2}{2}\left(1-\frac{1}{1+\lambda_1/(a-1)^2}\right)\right]\right\}, \tag{4·4}$$

where
$$\lambda_1 = DU_i^2+2/\{A^{22}(1-\rho_{13})\}.$$

Equation (4·4) provides the density function of

$$U_1 = \frac{M}{W}\frac{(1+\rho_{13}-\rho_{12}-\rho_{23})}{(1+\rho_{13})}. \tag{4·5}$$

Again from (4·2), we obtain the distribution of z_1 by making use of the following lemma.

LEMMA 2. *For any real numbers α_1 and β_1*

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty \Phi(\alpha_1+\beta_1 x)\,e^{-\frac{1}{2}x^2}\,dx = \tfrac{1}{2}\Phi(\alpha_1/\surd(1+\beta_1^2))$$
$$+(\arctan\beta_1)/(2\pi) - V(\alpha_1/\surd(1+\beta_1^2), \alpha_1\beta_1/\surd(1+\beta_1^2)), \tag{4·6}$$

where $V(h,q)$ is the V-function of Nicholson (1943). *Applying Lemma* 2 *to* (4·2) *for integration with respect to W_1, we get*

$$f_2(z_1) = \{\surd D\,e^{-\frac{1}{2}Dz_1^2}/\surd(8\pi)\}\{\Phi(z_1/\surd(1+B_1^2)) - \Phi(z_1/\surd(1+B_2^2)) + [(\arctan B_1)-(\arctan B_2)]/\pi$$
$$-2V(z_1/\surd(1+B_1^2), z_1B_1/\surd(1+B_1^2)) + 2V(z_1/\surd(1+B_2^2), z_1B_2/\surd(1+B_2^2))\}, \tag{4·7}$$

where $B_1 = (a+1)\{\frac{1}{2}(1-\rho_{13})A^{22}\}^{\frac{1}{2}}$ *and* $B_2 = (a-1)\{\frac{1}{2}(1-\rho_{13})A^{22}\}^{\frac{1}{2}}$.

5. BEST LINEAR UNBIASED ESTIMATORS OF THE COMMON MEAN OF THE THREE CORRELATED NORMAL RANDOM VARIABLES

Now we consider the use of the results in the previous sections for the construction of the best linear unbiased estimators of the common mean μ of three correlated normal random variables. Let X_1', X_2', X_3' be normal random variables with the common mean μ, common variance unity and $E(X_i'X_j') = \rho_{ij}+\mu^2$. Then we are interested in finding the linear function based on $X_{(i)}'$ which is an unbiased estimator of μ and which has the smallest variance in the class of all linear unbiased estimators. Let $\Sigma a_i X_{(i)}'$ be the estimator that we are looking for. Then the condition for unbiasedness implies

$$\Sigma a_i = 1, \quad \Sigma a_i EX_{(i)} = 0. \tag{5·1}$$

It should be pointed out that (5·1) gives two necessary conditions for the unbiasedness of an estimator based on any number of random variables. Now we have to minimize the variance of $\Sigma a_i X_{(i)}'$ which is the same as the variance of $\Sigma a_i X_{(i)}$ and can be computed by

using the formula (3·27) for all the six permutations of the subscripts 1, 2, 3. Thus one minimizes the variance of $\Sigma a_i X_{(i)}$ subject to the condition $\Sigma a_i = 1$, and the condition that the sum of the terms obtained by interchanging the subscripts of ρ_{ij} in (3·19) equals zero. The equations that determine the coefficients corresponding to the minimum value are given in the Appendix. Table 1 gives the values of a_i's for selected set of values of $\rho_{12}, \rho_{13}, \rho_{23}$. The computations were carried out on IBM 7094. It should be pointed out that the selected values of ρ_{ij} have to satisfy certain linear restrictions.

6. APPLICATIONS

(i) *Test of equality of the means of a multivariate population having a common ρ and a common standard deviation*

Let $X^{(j)}_{(1)} \leqslant X^{(j)}_{(2)} \leqslant \ldots \leqslant X^{(j)}_{(n)} (j = 1, 2, \ldots, k)$, be the n ordered random variables which have come from a multivariate normal population with mean vector $(\mu_1, \mu_2, \ldots, \mu_n)$ and a common correlation coefficient ρ and a common standard deviation, σ. Suppose we are interested in testing the hypothesis H against the alternative A, where

$$H: \mu_1 = \mu_2 = \ldots = \mu_n, A: \quad \text{not} \quad H, \tag{6·1}$$

then the following simple test can be applied. Let $W_j = X^{(j)}_{(n)} - X^{(j)}_{(1)}$ and $W_{\max} = \max_j W_j$. *Test:* reject H if

$$W_{\max} \Big/ \sum_{j=1}^{k} W_j \geqslant C(\alpha, n, k),$$

where $C(\alpha, n, k)$ is a constant which depends on the size of the test and on n and k. In fact, $C(\alpha, n, k)$ is such that

$$P\left\{W_{\max} \Big/ \sum_{j=1}^{k} W_j \geqslant C(\alpha, n, k) \,|\, \mu_1 = \ldots = \mu_k\right\} = \alpha. \tag{6·2}$$

It should be pointed out that the fact that the distribution of $W_{\max} \Big/ \left(\sum_{j=1}^{k} W_j\right)$ is independent of ρ has already been shown in the paper by Gupta *et al.* (1964). The upper 5 % points for this test are tabulated by Bliss, Cochran & Tukey (1956).

It should be noted that the above has an important application to the situations in life testing where the observations are ordered and we can perform the test without knowing the unordered random samples from the multivariate normal population for which the common correlation coefficient and the common standard deviation are unknown.

(ii) *Confidence interval for the common ρ in a multivariate normal population with common mean and common known standard deviation*

It is clear that if $X_{(n)}$ and $X_{(1)}$ are the largest and smallest of n equally correlated random variables with a common mean and common standard deviation, say unity, then one- or two-sided $100\alpha\%$ confidence bounds for ρ are obtained from

$$\frac{X_{(n)} - X_{(1)}}{C_2} \leqslant \sqrt{(1-\rho)} \leqslant \frac{X_{(n)} - X_{(1)}}{C_1}, \tag{6·3}$$

where C_1 and C_2 are the percentiles of the distribution of the range of n independent and identically distributed normal random variables. It may be pointed out that in life testing where the observations are naturally ordered, use of (6·3) gives a confidence interval statement for ρ even when the unordered sample is not known.

(iii) *Test of equality of two oscillations in non-overlapping intervals of equal length in time series*

Tests based on the ratio of differences between peaks and troughs from two non-overlapping intervals of equal length might be indicative of the nature of the oscillations involved in the time series. If we assume the variables to follow a multivariate normal distribution within intervals of equal length, this test would amount to the one based on the ratio of two independent ranges in the correlated case. (The distribution of this ratio for common mean and common standard deviation can be obtained as a series of beta functions using the distribution of range developed by the authors as a series of gamma functions which is available in Mimeograph series no. 13 Department of Statistics, Purdue University. For the case of means not necessarily equal the distribution has not been worked out.) To establish the equality of two oscillations, in addition, tests based on the differences of mid-ranges may have to be carried out. However, since tests based on the differences of mid-ranges cannot be easily performed, the ratio of mid-range to the range can be used to test any assigned value of the average of two population means (corresponding to the peak and trough) and this test can be repeated for each interval. In all the discussions above it is assumed that the variables have a common standard deviation, otherwise adjustments have to be made at various stages. Also the assumption of a multivariate normal distribution means that the set of correlation coefficients are known or otherwise have to be estimated, possibly through the use of serial correlation coefficients. However, if the correlation coefficients are neither known nor evaluated, some simulation process could be adopted for the performance of the tests.

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APPENDIX

The condition $\Sigma a_i EX_{(i)} = 0$ given in (5·1) reduces to the following equation by using (3·19) and $\Sigma a_i = 1$.

$$k_1 a_1 = k_2 a_2 + k_3, \tag{A 1}$$

where

$$k_1 = \sqrt{\left(\frac{1-\rho_{13}}{\pi}\right)} I_3^{(1)} + \sqrt{\left(\frac{1-\rho_{12}}{\pi}\right)} I_3^{(2)} + \sqrt{\left(\frac{1-\rho_{23}}{\pi}\right)} I_3^{(3)},$$

$$\begin{aligned} k_2 = -\Bigg\{ & \sqrt{\left(\frac{1-\rho_{13}}{\pi}\right)} d_{21} I_3^{(1)} + \sqrt{\left(\frac{1-\rho_{12}}{\pi}\right)} d_{22} I_3^{(2)} + \sqrt{\left(\frac{1-\rho_{23}}{\pi}\right)} d_{23} I_3^{(3)} \\ & + [\pi(1-\rho_{13})]^{-\frac{1}{2}} + [C_{01} + (2A^{22}C_{01}(1+\rho_{13}))^{-1}] I_4^{(1)} \\ & + [\pi(1-\rho_{12})]^{-\frac{1}{2}} [C_{02} + (2A^{33}C_{02}(1+\rho_{12}))^{-1}] I_4^{(2)} \\ & + [\pi(1-\rho_{23})]^{-\frac{1}{2}} [C_{03} + ((2A^{11}C_{03}(1+\rho_{23}))^{-1}] I_4^{(3)} \Bigg\}, \end{aligned}$$

$$\begin{aligned} k_3 = \frac{1}{2} \Bigg[& \sqrt{\left(\frac{1-\rho_{13}}{-\pi}\right)} I_3^{(1)} + \sqrt{\left(\frac{1-\rho_{12}}{\pi}\right)} I_3^{(2)} + \sqrt{\left(\frac{1-\rho_{23}}{\pi}\right)} I_3^{(3)} \\ & + [\pi(1-\rho_{13})]^{-\frac{1}{2}} I_4^{(1)} + [\pi(1-\rho_{12})]^{-\frac{1}{2}} I_4^{(2)} + [\pi(1-\rho_{23})]^{-\frac{1}{2}} I_4^{(3)} \Bigg]. \end{aligned}$$

where $I_3^{(1)}$ is a simpler notation for $I_3^{1,2,3}$ in (3·14), $d_{11}, d_{21}, d_{21}, C_{01}, \theta(d_{11})$ and $\theta(d_{21})$ being the same as $d_1, d_2, C_0, \theta(d_1)$ and $\theta(d_2)$ respectively in (3·14). $I_3^{(2)}$ is obtained from $I_3^{(1)}$ by interchanging in the latter ρ_{13} and ρ_{12} and so also d_{12} from d_{11}, d_{22} from d_{21}, C_{02} from C_{01}, etc. Similarly, $I_4^{(1)}$ is a briefer notation for

$I_4^{1,2,3}$ in (3·15). $I_4^{(2)}$ is obtained from $I_4^{(1)}$ by interchange of ρ_{13} and ρ_{12}. Further, $I_3^{(3)}$ is obtained from $I_3^{(1)}$ by interchanging in the latter ρ_{13} and ρ_{23} and similarly $I_4^{(3)}$ from $I_4^{(1)}$. d_{13}, d_{23}, C_{03}, etc., are obtained in a similar manner.

The value of a_2 corresponding to the minimum variance is given by the equation

$$l_1 a_2 = l_2, \tag{A 2}$$

where

$$\begin{aligned}
l_1 = 2\Bigg[& A_1\left(\frac{(A^{22})^{-1}}{2}+C_{01}^2(1+\rho_{13})\right)+A_2\left(\frac{(A^{33})^{-1}}{2}+C_{02}^2(1+\rho_{12})\right) \\
& +A_3\left(\frac{(A^{11})^{-1}}{2}+C_{03}^2(1+\rho_{23})\right)+B_1'\left(d_{21}+\frac{k_2}{k_1}\right)^2+B_2'\left(d_{22}+\frac{k_2}{k_1}\right)^2+B_3'\left(d_{23}+\frac{k_2}{k_1}\right)^2 \\
& +C_1\frac{\theta(d_{11})}{d_{11}}\left(2C_{01}^2(1+\rho_{13})+\frac{1}{A^{22}}\right)\left(d_{21}+\frac{k_2}{k_1}\right)+C_2\frac{\theta(d_{12})}{d_{12}}\left(2C_{02}^2(1+\rho_{12})+\frac{1}{A^{33}}\right)x \\
& \quad \times\left(d_{22}+\frac{k_2}{k_1}\right)+C_3\frac{\theta(d_{13})}{d_{13}}\left(2C_{03}^2(1+\rho_{23})+\frac{1}{A^{11}}\right)\left(d_{23}+\frac{k_2}{k_1}\right) \\
& +D_1\frac{\theta^2(d_{11})}{d_{11}^2}\left(2C_{01}^2(1+\rho_{13})+\frac{1}{A^{22}}\right)^2+D_2\frac{\theta^2(d_{12})}{d_{12}^2}\left(2C_{02}^2(1+\rho_{12})+\frac{1}{A^{33}}\right)^2 \\
& +D_3\frac{\theta^2(d_{13})}{d_{13}^2}(2C_{03}^2(1+\rho_{23})+\frac{1}{A^{11}}\Big)^2\Bigg],
\end{aligned}$$

$$\begin{aligned}
l_2 = {} & A_1(1+\rho_{13})C_{01}+A_2(1+\rho_{12})C_{02}+A_3(1+\rho_{23})C_{03} \\
& +\left(1-\frac{2k_3}{k_1}\right)\left\{B_1'\left(d_{21}+\frac{k_2}{k_1}\right)+B_2'\left(d_{22}+\frac{k_2}{k_1}\right)+B_3'\left(d_{23}+\frac{k_2}{k_1}\right)\right\} \\
& +C_1\left\{C_{01}\left(d_{21}+\frac{k_2}{k_1}\right)(1+\rho_{13})\frac{\theta(d_{11})}{d_{11}}+\frac{1}{2}\left(1-\frac{2k_3}{k_1}\right)(2C_{01}^2(1+\rho_{13})\frac{\theta(d_{11})\,\theta(d_{11})}{d_{11}\,d_{11}\,A^{22}}\Big)\Big)\right\} \\
& +C_2\left\{(1+\rho_{12})C_{02}\left(d_{22}+\frac{k_2}{k_1}\right)\frac{\theta(d_{12})}{d_{12}}+\frac{1}{2}\left(1-\frac{2k_3}{k_1}\right)\left(2C_{02}^2(1+\rho_{12})\frac{\theta(d_{12})}{d_{12}}+\frac{\theta(d_{12})}{d_{12}A^{33}}\right)\right\} \\
& +C_3\left\{(1+\rho_{23})C_{03}\left(d_{23}+\frac{k_2}{k_1}\right)\frac{\theta(d_{13})}{d_{13}}+\frac{1}{2}\left(1-\frac{2k_3}{k_1}\right)\left(2C_{03}^2(1+\rho_{23})\frac{\theta(d_{13})}{d_{13}}+\frac{\theta(d_{13})}{d_{13}A^{11}}\right)\right\} \\
& +2\Bigg\{D_1C_{01}\frac{\theta^2(d_{11})}{d_{11}^2}(1+\rho_{13})\left(2C_{01}^2(1+\rho_{13})+\frac{1}{A^{22}}\right)+D_2C_{02}\frac{\theta^2(d_{12})}{d_{12}^2}(1+\rho_{12}) \\
& \quad \times\left(2C_{02}^2(1+\rho_{12})+\frac{1}{A^{33}}\right)+D_3C_{03}\frac{\theta^2(d_{13})}{d_2^{13}}(1+\rho_{23})\left(2C_{03}^2(1+\rho_{23})+\frac{1}{A^{11}}\right)\Bigg\},
\end{aligned}$$

and where

$$A_1 = \frac{1}{\pi}\arctan\left[\frac{\theta(d_{21})-\theta(d_{11})}{\left[\dfrac{1}{2(1-\rho_{13})}+\theta(d_{21})\,\theta(d_{11})\right]\sqrt{\{2(1-\rho_{13})\}}}\right],$$

$$B_1' = \left\{(1-\rho_{13})A_1+\frac{\sqrt{2}}{\pi}(1-\rho_{13})^{\frac{3}{2}}\left[\frac{\theta(d_{21})}{\{2\theta^2(d_{21})(1-\rho_{13})+1\}}-\frac{\theta(d_{11})}{\{2\theta^2(d_{11})(1-\rho_{13})+1\}}\right]\right\},$$

$$C_1 = \frac{1}{\pi}\sqrt{\{2(1-\rho_{13})\}}\left\{\frac{1}{\{2\theta^2(d_{21})(1-\rho_{13})+1\}}-\frac{1}{\{2\theta^2(d_{11})(1-\rho_{13})+1\}}\right\},$$

$$D_1 = \frac{1}{\pi}\sqrt{\left(\frac{1-\rho_{13}}{2}\right)}\left\{\frac{\theta(d_{11})}{2\theta^2(d_{11})(1-\rho_{13})+1}-\frac{\theta(d_{21})}{2\theta^2(d_{21})(1-\rho_{13})+1}\right\}.$$

Finally, interchange ρ_{13} and ρ_{12} in A_1 to get A_2 and ρ_{13} and ρ_{23} in A_1 to get A_3. B_2' and B_3' are obtained in a similar manner from B_1', C_2 and C_3 from C_1, and D_2 and D_3 from D_1.

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Table 1. *Values* of a_i's $(i=1,2,3)$ corresponding to the minimum variance for given values of ρ_{ij}'s*

ρ_{12}	ρ_{13}	ρ_{23}	a_1	a_2	a_3
−·9	−·5	·1	0·4299	0·2873	0·2828
−·9	−·5	·7	·6175	−·1348	·5173
−·9	−·3	−·1	·4259	·2745	·2996
−·9	−·3	·3	·4874	·2140	·2986
−·9	−·1	−·3	·4283	·2468	·3249
−·9	−·1	·5	0·7431	−0·3505	0·6074
−·9	·1	−·5	·4371	·2034	·3594
−·9	·1	−·1	·5131	·1316	·3553
−·9	·1	·3	·7587	−·3260	·5673
−·9	·3	−·5	·4846	·1244	·3909
−·9	·3	·1	0·7884	−0·3451	0·5567
−·9	·5	−·7	·4982	·0639	·4379
−·9	·5	−·1	·8467	−·4186	·5719
−·9	·7	−·9	·4996	·0009	·4996
−·9	·7	−·5	·7397	−·2730	·5333
−·9	·9	−·7	1·1658	−1·0081	0·8423
−·7	−·7	·1	0·4060	0·2985	·2955
−·7	−·7	·7	·4723	·2512	·2765
−·7	−·7	·9	·4917	·2422	·2661
−·7	−·5	−·1	·3862	·3177	·2961
−·7	−·5	·9	0·6187	−0·3001	0·6814
−·7	−·3	−·3	·3750	·3145	·3105
−·7	−·3	·5	·4819	·2361	·2820
−·7	−·3	·7	·6257	−·1436	·5179
−·7	−·1	−·5	·3736	·2877	·3387
−·7	−·1	−·1	0·4184	0·2718	0·3098
−·7	−·1	·7	·7707	−·5040	·7334
−·7	·1	−·7	·3834	·2331	·3835
−·7	·1	·1	·5013	·1610	·3377
−·7	·3	−·7	·4154	·1692	·4154
−·7	·3	−·1	0·5215	·01034	0·3751
−·7	·3	·3	·7979	−·3773	·5795
−·7	·5	−·9	·4379	·0638	·4982
−·7	·5	−·3	·5433	·0400	·4167
−·7	·7	−·9	·4659	−·0227	·5568
−·7	·7	−·1	1·0204	−0·6742	0·6538
−·7	·9	−·9	0·8423	−1·0081	1·1658
−·5	−·9	·1	·4145	0·2426	0·3429
−·5	−·9	·5	·4759	·1479	·3762
−·5	−·7	−·1	·3761	·3030	·3208
−·5	−·7	·7	0·4819	0·1803	0·3377
−·5	−·5	−·3	·3501	·3301	·3198
−·5	−·5	·3	·4179	·2901	·2921
−·5	−·5	·9	·4905	·2424	·2670
−·5	−·3	−·5	·3367	·3266	·3367
−·5	−·3	·1	0·3957	0·3126	0·2917
−·5	−·3	·9	·6287	−·3132	·6845
−·5	−·1	−·7	·3387	·2877	·3736
−·5	−·1	·1	·4109	·2933	·2958
−·5	−·1	·5	·4803	·2322	·2874
−·5	·1	−·9	0·3594	0·2034	0·4371
−·5	·1	−·3	·3851	·2714	·3435
−·5	·1	·3	·4893	·1931	·3175
−·5	·3	−·9	·3909	·1244	·4846
−·5	·3	−·1	·4564	·1873	·3563
−·5	·3	·5	0·8163	−0·4455	0·6291
−·5	·5	−·9	·4279	·0204	·5517
−·5	·5	·5	0	0	1
−·5	·7	−·9	·5333	−·2730	0·7397
−·5	·7	−·3	·5483	−·0003	·4520
−·5	·7	·1	1·0259	−0·6835	0·6576
−·5	·9	−·7	0·8136	−·9542	1·1406
−·5	·9	−·1	−5·1259	9·0862	−2·9603
−·3	−·9	−·1	0·3929	0·2379	0·3692
−·3	−·9	·3	·4644	·1286	·4069

* For $\rho_{ij} = \rho(i \neq j = 1,2,3)$, $a_1 = a_2 = a_3 = \frac{1}{3}$.

Table 1 (*cont.*)

ρ_{12}	ρ_{13}	ρ_{23}	a_1	a_2	a_3
−·3	−·7	−·3	0·3479	0·3041	0·3479
−·3	−·7	·5	·4713	·1465	·3821
−·3	−·5	−·5	·3198	·3301	·3501
−·3	−·5	·1	·3848	·2930	·3221
−·3	−·5	·7	·4785	·1789	·3426
−·3	−·3	−·7	0·3105	0·3145	0·3750
−·3	−·3	·1	·3770	·3173	·3057
−·3	−·3	·9	·4890	·2429	·2682
−·3	−·1	−·9	·3249	·2468	·4283
−·3	−·1	·1	·3802	·3197	·3000
−·3	−·1	·9	0·6415	−0·3300	0·6885
−·3	·1	−·9	·3519	·1783	·4698
−·3	·1	−·1	·3715	·3011	·3274
−·3	·1	·5	·4787	·2261	·2951
−·3	·3	−·9	·3905	·7625	·5332
−·3	·3	−·1	0·4036	0·2445	0·3520
−·3	·3	·3	·4925	·1739	·3336
−·3	·5	−·9	·4972	−·1915	·6943
−·3	·5	−·1	·4576	·1504	·3920
−·3	·5	·5	·8932	−·5492	·6560
−·3	·7	−·7	0·5286	−0·2680	0·7394
−·3	·7	−·1	·5405	·0185	·4410
−·3	·7	·3	1·0421	·7180	·6758
−·3	·9	−·5	0·7838	−·8989	·1115
−·3	·9	−·1	1·0896	−·8427	·7531
−·1	−·9	−·3	0·3692	0·2379	0·3929
−·1	−·9	·5	·6403	−·3088	·6685
−·1	−·7	−·5	·3208	·3030	·3761
−·1	−·7	·3	·4570	·1290	·4140
−·1	−·5	−·7	·2961	·3177	·3862
−·1	−·5	·1	0·3845	0·2617	0·3538
−·1	−·5	·9	·8430	−1·4005	1·5575
−·1	−·3	−·9	·2996	0·2745	0·4259
−·1	−·3	−·1	·3387	·3226	·3387
−·1	−·3	·7	·4738	·1778	·3484
−·1	−·1	−·9	0·3201	0·2209	0·4590
−·1	−·1	·1	·3570	·3275	·3155
−·1	−·1	·9	·4867	·2436	·2696
−·1	·1	−·9	·3553	·1316	·5131
−·1	·1	−·1	·3398	·3205	·3398
−·1	·1	·9	0·6587	−0·3527	0·6940
−·1	·3	−·7	·3751	·1034	·5215
−·1	·3	−·1	·3617	·2766	·3617
−·1	·3	·5	·4774	·2157	·3069
−·1	·5	−·9	·5719	−·4186	·8467
−·1	·5	·1	0·4369	0·1951	0·3680
−·1	·5	·7	·9636	−·7794	·8158
−·1	·7	−·7	·6538	−·6742	1·0204
−·1	·7	−·1	·4616	·0768	0·4616
−·1	·7	·5	1·0843	−·8164	·7321
−·1	·9	−·5	−2·9603	9·0862	−5·1259
−·1	·9	·3	8·1657	−11·5549	4·3892
·1	−·9	−·5	0·3429	0·2426	0·4145
·1	−·9	·3	·6347	−·2756	·6409
·1	−·7	−·7	·2955	·2985	·4061
·1	−·7	−·1	0·3875	0·2000	0·4125
·1	−·7	·5	·6568	−·3395	·6827
·1	−·5	−·9	·2828	·2873	·4299
·1	−·5	−·1	·3538	·2617	·3845
·1	−·5	·7	·7003	−·4985	·7982
·1	−·3	−·9	0·2954	0·2505	0·4541
·1	−·3	·1	·3624	·2752	·3624
·1	−·3	·9	·9105	−1·5755	1·6650
·1	−·1	−·9	·3243	0·1784	0·4973
·1	−·1	·1	·3412	·3176	·3412
·1	−·1	·7	0·4668	0·1778	0·3555
·1	·1	−·9	·4772	−·1353	·6581
·1	·1	−·1	·3145	·3237	·3619
·1	·1	·9	·4834	·2450	·2715
·1	·3	−·9	·5567	−·3450	·7884
·1	·3	−·1	0·3304	0·2839	0·3856
·1	·3	·7	·4670	−·2546	·2785
·1	·5	−·7	·5819	−·4358	·8539
·1	·5	·1	·3775	·2450	·3775
·1	·5	·9	1·5250	−2·9382	2·4133
·1	·7	−·5	0·6576	−0·6835	1·0259
·1	·7	·1	·4407	·1185	0·4407
·1	·7	·7	1·2195	−1·1603	·9408
·1	·9	−·3	−22·5518	64·0420	−40·4902
·1	·9	·5	4·5413	−5·9373	2·3960
·3	−·9	−·5	0·3528	0·2039	0·4433
·3	−·9	·1	·6409	−·2756	·6347
·3	−·7	−·7	·2892	·2825	·4283
·3	−·7	·3	·6576	−·3152	·6576
·3	−·5	−·9	·2776	·2673	·4551
·3	−·5	−·1	0·3631	0·2156	0·4214
·3	−·5	·5	·6839	−·3872	·7032
·3	−·3	−·9	·2986	·2140	·4874
·3	−·3	−·1	·3256	·2769	·3976
·3	−·3	·7	·7380	−·5675	·8295
·3	−·1	−·7	0·3066	0·2055	0·4879
·3	−·1	·1	·3332	·2910	·3759
·3	−·1	·9	1·0315	−1·8878	1·8563
·3	·1	−·5	0·3175	0·1931	0·4893
·3	·1	·1	·3121	·3247	·3632

Table 1 (*cont.*)

ρ_{12}	ρ_{13}	ρ_{23}	a_1	a_2	a_3
·3	·1	·7	0·4556	0·1800	0·3644
·3	·3	−·7	·5795	−·3773	·7979
·3	·3	−·1	·3106	·2656	·4238
·3	·3	·9	·4781	·2478	·2741
·3	·5	−·5	·6040	−·4708	·8668
·3	·5	·1	0·3390	0·2482	0·4128
·3	·5	·9	·7239	−·4400	·7161
·3	·7	·1	·4024	·0879	·5098
·3	·7	·7	·7517	−·2960	·5442
·3	·9	−·1	4·3892	−11·5549	8·1657
·3	·9	·5	1·0234	−0·6864	0·6630
·5	−·9	−·7	0·3176	·2234	·4590
·5	−·9	−·1	·6685	−·3088	·6402
·5	−·7	−·9	·2679	·2704	·4617
·5	−·7	·1	·6827	−·3395	·6568
·5	−·5	−·9	0·2779	0·2388	0·4833
·5	−·5	−·3	·3244	·2310	·4446
·5	−·5	·5	0	0	1
·5	−·3	−·9	·4877	−·1190	0·6313
·5	−·3	−·1	·3289	·2376	·4335
·5	−·3	·5	0·7372	−0·4743	0·7372
·5	−·1	−·5	·2874	·2322	·4803
·5	−·1	−·1	·2914	·2884	·4202
·5	−·1	·3	·4071	·1506	·4422
·5	·1	−·7	·6156	−·3880	·7725
·5	·1	·1	0·2970	0·3018	0·4012
·5	·1	·9	1·3231	−2·6337	2·3105
·5	·3	−·5	0·6291	−0·4455	·8163
·5	·3	·1	·2892	·2973	·4135
·5	·3	·9	·6491	−·4010	·7519
·5	·5	−·5	0	0	1
·5	·5	·1	0·3270	0·1958	0·4773
·5	·5	·9	·4681	·2539	·2780
·5	·7	−·1	·7321	−·8164	1·0842
·5	·9	·1	2·3960	−5·9373	4·5413
·5	·9	·7	1·0259	−0·6835	0·6576
·7	−·9	−·9	0·2752	·2493	·4755
·7	−·9	−·5	·5526	−·1218	·5692
·7	−·7	−·9	·2624	·2532	·4843
·7	−·7	−·1	·7777	−·4547	·6770
·7	−·5	−·9	0·5173	−0·1348	0·6175
·7	−·5	·1	·7982	−·4985	·7003
·7	−·3	−·7	·5179	−·1436	·6257
·7	−·3	·3	·8295	−·5675	·7380
·7	−·1	−·7	·7334	−·5040	·7707
·7	−·1	−·1	0·2822	0·2631	0·4547
·7	−·1	·5	·8854	−·6973	·8119
·7	·1	−·5	·7478	−·5553	·8075
·7	·1	·1	·2853	·2714	·4433
·7	·1	·7	1·0264	−1·0527	1·0264
·7	·3	−·3	0·7711	−0·6346	0·8635
·7	·3	·7	·6203	−·2405	·6203
·7	·5	−·1	·8158	−·7794	·9636
·7	·5	·9	·7003	−·4985	·7982
·7	·7	·3	·5442	−·2960	·7517
·7	·7	·9	0·4433	0·2714	0·2853
·7	·9	·9	1·2195	−1·1603	·9407
·9	−·9	−·9	0·2652	·2421	·4927
·9	−·7	−·7	·2660	·2422	·4917
·9	−·5	−·5	·2670	·2424	·4905
·9	−·3	−·3	0·2682	0·2429	0·4890
·9	−·1	−·5	1·5950	−1·5320	·9370
·9	−·1	−·1	0·2696	0·2436	·4867
·9	−·1	·3	1·8563	−1·8878	1·0315
·9	·1	·5	2·3105	−2·6337	1·3231
·9	·3	−·1	1·9207	−2·0968	1·1761
·9	·3	·3	0·2741	0·2478	0·4781
·9	·5	·5	·2780	·2539	·4681
·9	·7	·9	1·0264	−1·0527	1·0264
·9	·9	·7	0·9408	−1·1603	1·2195