Generating Functions for Markov Renewal Processes

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Summary

A general matrix representation is given for the multivariate transition probability generating functions of a Markov Renewal Process with a finite number of states. It is indicated how numerous derived probability distributions can be obtained by simple substitutions. Finally an application is made to the distribution of the maximum length of an M/M/1 queue.

Let $N(t) = (N_1(t), \ldots N_m(t))$ denote a Markov Renewal Process with a finite number m of states and with a matrix of transition probability distributions $Q = Q_{ij}$ [2,3]. The $Q_{ij}(t)$ are non-decreasing left-continuous functions satisfying

(1) $$Q_{ij}(0) = 0 \quad \text{for } i, j = 1, \ldots, m$$
$$\sum_{j=1}^{m} Q_{ij}(\infty) = 1 \text{ for } i = 1, \ldots, m$$

The random variable $N_i(t)$ is equal to the number of visits to state i during the time interval $[0,t]$. The stochastic process Z_t is referred to as the semi-Markov process (S-M P) associated with the Markov renewal process. $Z_t = i$ when state i is being visited at time t. We assume that

(2) $$P\left\{Z_o = i\right\} = p_i^{o} \quad \text{with} \quad \sum_{i=1}^{m} p_i^{o} = 1$$

Let $k = (k_1, \ldots, k_m)$ denote an m-tuple of non-negative integers and define $T(k)$ as follows:

$$(3) \qquad T(k) = \inf \left\{ t : N_1(t) = k_1, \ldots, N_m(t) = k_m \right\} .$$

i.e., $T(k)$ is the random time at which the Markov renewal process enters the state $k = (k_1, \ldots, k_m)$.

Let $Z(k) = Z_{T(k)}$ and $T'(k)$ denote the time instant at which the S-M P leaves the state $Z(k)$. We define the following transition probabilities for the Markov renewal process.

$$(4) \qquad C_j(k,t) = P\left\{ T(k) \leq t \text{ and } Z(k) = j \right\}$$

and

$$(5) \qquad D_j(k,t) = P\left\{ T(k) \leq t < T'(k) \text{ and } Z_t = j \right\}$$

The probabilities defined in (4) and (5) satisfy the following relations.

$$(6) \qquad \begin{aligned} C_j(e_i,t) &= \delta_{ij} p_j^{\,o} \, I(t) \\ C_j(k,t) &= \sum_{\nu=1}^{m} C_\nu(k-e_j,t) * Q_{\nu j}(t) \quad \text{for } k \neq e \\ C_\nu(k-e_j,t) &\equiv 0 \quad \text{if } k_j = 0 \\ D_j(k,t) &= [1 - H_j(t)] * C_j(k,t) \end{aligned}$$

where $H_j(t) = \sum_{\nu=1}^{m} Q_{j\nu}(t)$ and $I(t)$ is the distribution degenerate at zero. The m-tuple e_i has all components but the ith equal to zero and the ith equal to one.

We introduce the following notations for the Laplace transforms.

$$(7) \qquad C_j^{\,*}(k,s) = \int_0^\infty e^{-st} dC_j(k,t) \qquad D_j^{\,*}(k,s) = \int_0^\infty e^{-st} dD_j(k,t)$$

$$Q_{ij}^{\,*}(s) = \int_0^\infty e^{-st} dQ_{ij}(t) \qquad H_j^{\,*}(s) = \int_0^\infty e^{-st} dH_j(t)$$

The formulae (6) become:

$$(8)\qquad C_j^*(e_i,s) = \delta_{ij}p_j^{\circ}$$

$$C_j^*(k,s) = \sum_{\nu=1}^{m} C_\nu^*(k-e_j,s)\, Q_{\nu j}^*(s) \quad \text{for } k \neq e_r$$

$$D_j^*(k,s) = [1 - H_j^*(s)]\, C_j^*(k,s)$$

We now introduce the multivariate probability generating functions.

$$(9)\quad G_j^*(z,s) = G_j^*(z_1, \ldots, z_m,s) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} C_j^*(k,s)\, z_1^{k_1} \cdots z_m^{k_m}$$

and

$$K_j^*(z,s) = K_j^*(z_1, \ldots, z_m,s) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} D_j^*(k,s)\, z_1^{k_1} \cdots z_m^{k_m}$$

and the columnvectors

$$G^* = [G_1^*(z,s), \ldots, G_m^*(z,s)] \quad K^* = [K_1^*(z,s), \ldots, K_m^*(z,s)]$$

The column-vectors G^* and K^* now have the following matrix representation.

<u>Theorem</u>

$$(10)\qquad K^*(z,s) = [I - \Delta(H^*)]\, G^*(z,s)$$

$$= [I - \Delta(H^*)][K - \Delta(z)\, Q^{*\prime}]^{-1}\Delta(z)p^{\circ}$$

in which

$$\Delta(H^*) = \text{Diag}\,(H_1^*(s), \ldots, H_m^*(s))$$

$$\Delta(z) = \text{Diag}\,(z_1, \ldots, z_m) \text{ and } |z_i| < 1 \quad i = 1, \ldots, m$$

$$Q^* = \{ Q_{ij}^* \}$$

and p° is the column-vector $[p_1^{\circ}, \ldots, p_m^{\circ}]$

<u>Proof</u>

The first equality is equivalent to the third in (8). Moreover it follows from the other equalities in (8) that

$$G_j^*(z,s) = p_j^{\circ}z_j + z_j \sum_{\nu=1}^{m} Q_{\nu j}^*(s)\, G_\nu^*(z,s)$$

which implies the second equality in (10) if the inverse of $I - \Delta(z)\, Q^{*\prime}$ exists. This is easily seen to be so in view of

(11) $$|z_j Q_{\nu j}^{*}(s)| \leq Q_{\nu j}(\infty)$$

The numbers $Q_{j\nu}(\infty)$ form an m x m stochastic matrix which therefore has spectral radius equal to one. A theorem of Wielandt [4] to the effect that (11) implies that the spectral radius of $\Delta(z)Q^{*\prime}$ is not less than one, now implies the result. QED.

A Particular Case: Discrete time finite Markov chains.

If $C_j(k)$ denotes the probability that in $k_1 + \ldots + k_m - 1$ transitions a discrete Markov chain reaches state j and has visited state ν exactly k_ν times ($\nu = 1, \ldots m$) then

$$C_j^{*}(k,s) = C_j(k)\, e^{-s(k_1 + \ldots + k_m - 1)}$$

$Q^{*} = e^{-s}P \qquad G_j^{*}(z,s) = e^{s}G_j(ze^{-s})$ in which $G_j(\xi)$ is the generating function of the $C_j(k)$. After setting $z_i e^{-s} = \xi_i$ we obtain

$$G(\xi) = [I - \Delta(z)P']^{-1}\Delta(z)p^{\circ}$$

which was proved earlier by Neuts [1].

Generating Functions derivable from formula (10)

Generating functions for many related probabilities can be derived from K(z,s) by an appropriate choice of the variables z_i.

If we set some of the z_i equal to a same variable u we find the transition probabilities of the S-M P which specify only the number of visits to certain but not all states. If we set certain variables z_i in (10) equal to zero, we find generating functions for taboo-probabilities, i.e. transition probabilities of events in which one specifies that certain states should not be visited.

Finally if we perform the substitutions

$$Z_\nu = \rho\, e^{\alpha_\nu i v} \qquad\qquad 0 < \rho < 1$$

for all or some of the variables z in which α_i is equal to zero, plus or minus one we obtain generating functions for events defined with respect to algebraic sums of the random variables $N_i(t)$.

Some detailed examples of these substitutions have been worked out in the case of finite Markov chains. Neuts [1].

An application

We consider a single server Poisson queue with input rate λ and service rate μ. We wish to evaluate the probabilities $\pi^n_{ij}(t)$ that in the time-interval $[0,t]$ there have been n transitions in the queue, the queue length at time t is j and neither of the queue lengths zero and b have been attained, given that the initial queue length was i. $0 < i,\ j < b$.

Let us consider the b + 1 state S-M P in which

$$Q_{ij}(t) = \begin{cases} 1 - e^{-\lambda t} & i = 0 \quad j = 1 \\ \frac{\lambda}{\lambda+\mu}\,[1 - e^{-(\lambda+\mu)t}] & j = i+1 \quad i = 1, \ldots, b-1 \\ \frac{\mu}{\lambda+\mu}\,[1 - e^{-(\lambda+\mu)t}] & j = i-1 \quad i = 1, \ldots, b-1 \\ I(t) & i = b \qquad j = b-1 \\ 0 & \text{elsewhere.} \end{cases}$$

If we substitute Q into formula (10) and set $p^o = e_i$ and $z_o = z_b = 0$ $z_1 = \ldots z_{b-1} = u$ we obtain

$$K_j{*}\,(0,u, \ldots u, 0, s) = u \sum_{n=0}^{\infty} u^n \int_0^\infty e^{-st}\, d\pi_{ij}{}^n(t)$$

for $j = 1, \ldots, b-1$

Set
$$\sum_{n=0}^{\infty} u^n \int_0^{\infty} e^{-st}\, d\pi_{ij}^{\,n}(t) = P_{ij}^*(u,s)$$

then
$$P_{ij}^*(u,s) = \frac{1}{u} K^*(o,u,\ldots,u,o,s) = [1 - H_j^*(s)][(I - \Delta Q^{*\prime})^{-1}]_{ji}$$

where Δ = Diag $(o,u,\ldots,u,o)$

After inversion of the jacobi-matrix $I - \Delta Q^{*\prime}$ we find

$$P_{ij}^*(u,s) = \frac{s}{s+\lambda+\mu}\left(\frac{\lambda u}{s+\lambda+\mu}\right)^{j-i} \frac{(\xi_1^{\,i} - \xi_2^{\,i})(\xi_1^{\,b-j} - \xi_2^{\,b-j})}{(\xi_1 - \xi_2)(\xi_1^{\,b} - \xi_2^{\,b})} \quad \text{for } j \geq i$$

$$= \frac{s}{s+\lambda+\mu}\left(\frac{\mu u}{s+\lambda+\mu}\right)^{i-j} \frac{(\xi_1^{\,j} - \xi_2^{\,j})(\xi_1^{\,b-i} - \xi_2^{\,b-i})}{(\xi_1 - \xi_2)(\xi_1^{\,b} - \xi_2^{\,b})} \quad \text{for } j \leq i$$

where
$$\xi_{1,2} = \frac{1}{2}\left\{1 \pm \left[1 - \frac{4\lambda\mu u^2}{(s+\lambda+\mu)^2}\right]^{1/2}\right\}$$

If we set $\frac{2\sqrt{\lambda\mu}}{s+\lambda+\mu} = \frac{1}{\cos\alpha}$, we find for $i \leq j$ ($j \leq i$ is analogous)

$$P_{ij}^*(u,s) = s\,\lambda^{\frac{j-i-1}{2}}\,\mu^{\frac{i-j-1}{2}}\,u^{-1}\,\frac{\sin i\alpha\ \sin(b-j)\alpha}{\sin\alpha\ \sin b\alpha}$$

$u^{-1}\,\frac{\sin i\alpha\ \sin(b-j)\alpha}{\sin\alpha\ \sin b\alpha}$ is a rational function of u with $b-1$ distinct poles at

$u_\rho = \cos\frac{\rho\pi}{b} \qquad \rho = 1, \ldots, b-1.$

Partial fraction expansion yields:

$$\frac{1}{s} P_{ij}^{*}(u,s) = \left(\frac{\lambda}{\mu}\right)^{\frac{j-i}{2}} \sum_{\rho=1}^{b-1} \frac{\frac{2}{b} \sin i\rho\pi/b \, \sin(b-j)\rho\pi/b}{s+\lambda+\mu-2\mu\sqrt{\lambda\mu}\cos\frac{\rho\pi}{b}}$$

whence

$$\pi_{ij}^{n}(t) = 2^{n} \lambda^{\frac{n+j-i}{2}} \mu^{\frac{n+i-j}{2}} \frac{1}{n!} t^{n} e^{-(\lambda+\mu)t} \sum_{\rho=1}^{b-1} \frac{2}{b} \sin\frac{i\rho\pi}{b} \sin\frac{(b-j)\rho\pi}{b} \left(\cos\frac{\rho\pi}{b}\right)^{n}$$

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