On Optimal Stopping Rules for s_n/n

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1. Introduction.

Let

(1) $$x_1, x_2, \ldots$$

be a sequence of independent, identically distributed random variables on a probability space $(\Omega, \mathcal{F}, P)$ with

(2) $$P(x_1 = 1) = P(x_1 = -1) = 1/2,$$

and let $s_n = x_1 + \ldots + x_n$. Let $i = 0, \pm 1, \ldots$ and $j = 0, 1, \ldots$ be two fixed integers. Assume that we observe the sequence (1) term by term and can decide to stop at any point; if we stop with x_n we receive the reward $(i + s_n)/(j + n)$. What stopping rule will maximize our expected reward?

Formally, a stopping rule t of (1) is any positive integer valued random variable such that the event $t = n$ is in $\mathcal{F}_n$ $(n \geq 1)$ where $\mathcal{F}_n$ is the Borel field generated by $x_1, \ldots, x_n$. Let T denote the class of all stopping rules; for any t in T, s_t is a well-defined random variable, and we set

(3) $$v_j(i|t) = E\left(\frac{i + s_t}{j + t}\right), \quad v_j(i) = \sup_{t \in T} v_j(i|t).$$

It is by no means obvious that for given i and j there exists a stopping rule $\tau_j(i)$ in T such that

(4) $$v_j(i|\tau_j(i)) = v_j(i) = \max_{t \in T} v_j(i|t);$$

such a stopping rule of (1) will be called optimal for the reward sequence

(5) $$\frac{i+s_1}{j+1}, \quad \frac{i+s_2}{j+2}, \quad \ldots \quad .$$

Theorem 1 below asserts that for every $i = 0, \pm 1, \ldots$ and $j = 0, 1, \ldots$ there exists an optimal stopping rule $\tau_j(i)$ for the reward sequence (5).

We remark that for any t in T and any $i = 0, \pm 1, \ldots$ and $j = 0, 1,$ the random variable

(6) $$t' = \begin{cases} t & \text{if } i + s_t \geq 1, \\ \text{first } n > t \text{ such that } i+s_n = 1 & \text{if } i+s_t \leq 0 \end{cases}$$

is in T and

(7) $$i+s_{t'} \geq 1, \; 0 < E\left(\frac{i+s_{t'}}{j+t'}\right) \geq E\left(\frac{i+s_t}{j+t}\right) .$$

It follows that

(8) $$v_j(i) = \sup_{t \in T} E\left[\frac{(i+s_t)}{j+t}^+\right],$$

where by definition $a^+ = \max(0, a)$.

2. Reduction of the problem to the study of bounded stopping rules. For any fixed $N = 1, 2, \ldots$ let T_N denote the class of all t in T such $t \leq N$. By the usual backward induction (see e.g. [1]) it may be shown that in T_N there exists a minimal optimal stopping rule of (1) for the reward sequence

(1) $$\frac{(i+s_1)^+}{j+1}, \quad \frac{(i+s_2)^+}{j+2}, \ldots \quad ;$$

that is, an element $\tau_j^N(i)$ of T_N such that, setting

(2) $$w_j\ (i|t) = E\ [\frac{(i+s_t)^+}{j+t}],$$

we have

(3) $$w_j\ (i|\ \tau_j^N\ (i)) = \max_{t\in T_N}\ w_j(i|t),$$

and such that if $\tilde{t}$ is any element of T_N for which

(4) $$w_j(i|\tilde{t}) = \max_{t\in T_N}\ w_j(i|t),$$

then $\tau_j^N\ (i) \leq t$. The sequence $\tau_j^1\ (i)$, $\tau_j^2(i)$, ... is such that as $N \longrightarrow \infty$,

(5)
$$1 \leq \tau_j^1\ (i) \leq \tau_j^2\ (i) \leq \cdots \longrightarrow \tau_j^*\ (i) \leq \infty,$$
$$0 \leq w_j(i|\ \tau_j^1(i)) \leq w_j(i|\ \tau_j^2(i)) \leq \cdots \longrightarrow \sup_{t\in T} w_j(i|t) = v_j(i),$$

the last equality following from (1.8). It is shown in [1] that there exists an optimal element in T for the reward sequence (1.5) if and only if

(6) $$\tau_j^*\ (i) = \lim_{N\longrightarrow\infty}\ \tau_j^N\ (i)$$

is in T that is, if and only if

(7) $$P(\ \tau_j^*\ (i) < \infty) = 1$$

and when (7) holds $\tau_j^*(i)$ is the minimal element of T which satisfies (1.4). The remainder of the present paper is devoted to proving that (7) holds.

3. The constants $a_n^N(i)$ and $a_n(i)$. In order to study the nature of the optimal bounded stopping rules $\tau_j^N(i)$ of Section 2 we proceed as follows. Define for $n = 1, 2, \ldots$ and $i = 0, \pm 1, \ldots$ the constants

$$b_N^N(i) = \frac{i+}{N},$$

(1)

$$b_N^N(i) = \max\left(\frac{i+}{n}, \frac{b_{n+1}^N(i+1) + b_{n+1}^N(i-1)}{2}\right) \quad (n = 1, 2, \ldots, N-1).$$

Then

(2) $$b_n^N(i) = \max\left(\frac{i+}{n}, \sup_{t\in T_{N-n}} E\left[\frac{(i+s_t)^+}{n+t}\right]\right) \quad (n = 1, 2, \ldots, N-1),$$

(3) $$\tau_j^N(i) = \text{first } n \geq 1 \text{ such that } b_{j+n}^{j+N}(i+s_n) = \frac{(i+s_n)^+}{j+n},$$

and

(4) $$\sup_{t\in T_N} E\left[\frac{(i+s_t)^+}{j+t}\right] = \frac{1}{2}\left[b_{j+1}^{j+N}(i+1) + b_{j+1}^{j+N}(i-1)\right].$$

In view of (2) and (3) it is convenient to introduce the constants $a_n^N(i)$ defined for $N = 1, 2, \ldots ; i = 0, \pm 1, \ldots; \quad n = 1, 2, \ldots, N$ by

(5) $$a_n^N(i) = b_n^N(i) - \frac{i+}{n};$$

then (3) becomes

(6) $$\tau_j^N(i) = \text{first } n \geq 1 \text{ such that } a_{j+n}^{j+N}(i+s_n) = 0.$$

From (5) and (1) it follows that the constants $a_n^N(i)$ satisfy the recursion relations

$$a_N^N(i) = 0 \qquad (\text{all } i),$$

(7)
$$a_n^N(i) = \left[\frac{a_{n+1}^N(i+1) + a_{n+1}^N(i-1)}{2} + \frac{(i+1)^+ + (i-1)^+}{2\,(n+1)} - \frac{i+}{n} \right]^+$$

$$(n = 1, 2, \ldots, N-1)$$

from which they may be successively computed for $n = N, N-1, \ldots, 1$. Moreover, from (2) and (4) we have

(8)
$$a_n^N(i) = \sup_{t \in T_{N-n}} E^+ \left[\frac{(i+s_t)^+}{n+t} - \frac{i+}{n} \right] \quad (n = 1, 2, \ldots, N-1)$$

and

(9)
$$\sup_{t \in T_N} E\left[\frac{(i+s_t)^+}{j+t} \right] = \frac{1}{2} \left[a_{j+1}^{j+N}(i+1) + a_{j+1}^{j+n}(i-1) + \frac{(i+1)^+ + (i-1)^+}{j+1} \right].$$

For any $i = 0, \pm 1, \ldots$ and $n = 1, 2, \ldots$ we have $0 = a_n^n(i) \le a_n^{n+1}(i) \le \ldots$, and letting $N \to \infty$ we obtain constants $a_n(i) = \lim_{N \to \infty} a_n^N(i)$ such that

(10)
$$a_n^N(i) \uparrow a_n(i) = \sup_{t \in T} E^+ \left[\frac{(i+s_t)^+}{n+t} - \frac{i+}{n} \right],$$

while for $j = 0, 1, \ldots$

(11)
$$\sup_{t \in T} E\left[\frac{(i+s_t)^+}{j+t} \right] = \sup_{t \in T} E\left(\frac{i+s_t}{j+t} \right) = v_j(i) =$$

$$= \frac{1}{2} \left[\frac{(i+1)^+ + (i-1)^+}{j+1} + a_{j+1}(i+1) + a_{j+1}(i-1) \right];$$

moreover $\tau_j^N(i) \uparrow \tau_j^*(i)$ where

$$(12)\qquad \tau_j^*(i) = \begin{cases} \text{first } n \geq 1 \text{ such that } a_{j+n}(i+s_n) = 0, \\ \infty \text{ if no such } n \text{ exists} \end{cases}$$

Thus (2.7) holds if and only if

$$(13)\qquad P(a_{j+n}(i+s_n) = 0 \text{ for some } n \geq 1) = 1.$$

In the next section we shall prove (lemma 4) that there exists a positive integer n_o such that $n \geq n_o$ and $i > 13\sqrt{n}$ together imply that $a_n(i) = 0$. Hence

$$(14)\qquad P(a_{j+n}(i+s_n) = 0 \text{ for some } n \geq 1) \geq P(s_n > 13\sqrt{j+n-i} \text{ for some } n \geq n_o).$$

The law of the iterated logarithm implies that the latter probability is 1 and this establishes (13); hence $\tau_j^*(i)$ defined by (12) is in T and is optimal for the reward sequence (1.5). We thus have the following

<u>Theorem 1</u>. For the sequence (1.1) with the distribution (1.2) and the reward sequence (1.5) there exists an optimal stopping rule $\tau_j^*(i)$ defined by (12); the expected reward in using $\tau_j^*(i)$ is

$$(15)\qquad v_j(i) = \max_{t \in T} E\left(\frac{i+s_t}{j+t}\right) = \frac{1}{2}\left[\frac{(i+1)^+ + (i-1)^+}{j+1} + a_{j+1}(i+1) + a_{j+1}(i-1)\right]$$

$(i = 0, \pm 1, \ldots;\ j = 0, 1, \ldots)$. The constants $a_n(i) = \lim_{n \to \infty} a_n^N(i)$ which occur in (12) and (15) are determined by (7).

4. Lemmas.

Lemma 1. $a_n(0) \leq \frac{1}{\sqrt{n}}$ $(n = 1, 2, \ldots).$

Proof. From (3.7) we have

$$(1) \quad a_n^N(i) = \begin{cases} \dfrac{a_{n+1}^N(i+1) + a_{n+1}^N(i-1)}{2} & (i \leq -1), \\[2ex] \dfrac{a_{n+1}^N(1) + a_{n+1}^N(-1)}{2} + \dfrac{1}{2(n+1)} & (i = 0), \\[2ex] \left[\dfrac{a_{n+1}^N(i+1) + a_{n+1}^N(i-1)}{2} - \dfrac{i}{n(n+1)}\right]^+ \leq \dfrac{a_{n+1}^N(i+1) + a_{n+1}^N(i-1)}{2} & (i \geq 1) \end{cases}$$

Hence

$$a_n^N(0) = \frac{a_{n+1}^N(1) + a_{n+1}^N(-1)}{2} + \frac{1}{2(n+1)} \leq \frac{1}{2^2}\left[a_{n+2}^N(2) + 2a_{n+2}^N(0) + a_{n+2}^N(-2)\right] + \frac{1}{2(n\cdot}$$

$$(2) \quad \leq \frac{1}{2^3}\left[a_{n+3}^N(3) + 3a_{n+3}^N(1) + 3a_{n+3}^N(-1) + a_{n+3}^N(-3)\right] + \frac{1}{2(n+1)} + \frac{\binom{2}{1}}{2^3(n+3)}$$

$$\leq \cdots \leq \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k+1}(n+2k+1)},$$

since $a_N^N(i) = 0$. By Stirling's formula

$$(3) \quad \binom{2k}{k} < \frac{2^{2k}}{\sqrt{k\pi}},$$

and

(4) $$\sum_{k=n}^{\infty} \frac{1}{2\sqrt{k\pi}\,(n+2k+1)} \le \frac{1}{2\sqrt{\pi}} \int_{r-\frac{1}{2}}^{\infty} \frac{x}{\sqrt{x}\,(n+2x+1)}\,dx = \frac{1}{\sqrt{2\pi(n+1)}}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{2r-1}{n+1}}\right).$$

Hence

(5) $$a_n(0) = \lim_{N\to\infty} a_n^N(0) \le \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k+1}(n+2k+1)} + \frac{1}{\sqrt{2\pi(n+1)}}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{2r-1}{n+1}}\right).$$

For $r = 1$ this gives

(6) $$a_n(0) \le \frac{1}{2(n+1)} + \frac{1}{\sqrt{2n}} \le \frac{1}{\sqrt{n}} .$$

Lemma 2. For $n = 1, 2, \ldots$

(7) $$0 < \ldots \le a_n(-2) \le a_n(-1) \le a_n(0) \ge a_n(1) \ge a_n(2) \ge \ldots \ge 0,$$

(8) $$a_{n+1}(i) \ge \frac{n+1}{n+2}\, a_n(i) \qquad \text{(all } i).$$

Proof. For $i \le 0$ we have from (3.10) and (1.7)

(9) $$a_n(i) = \sup_{t\in T} E\left[\frac{(i+s_t)^+}{n+t}\right] > 0;$$

hence

(10) $$a_n(i) \ge \sup_{t\in T} E\left[\frac{(i-1+s_t)^+}{n+t}\right] = a_n(i-1).$$

For $i \ge 0$ we have

$$(11) \qquad a_n(i) = \sup_{t\in T} E\left[\frac{i+s_t}{n+t} - \frac{i}{n}\right]^+ = \sup_{t\in T} E\left[\frac{ns_t - it}{n(n+t)}\right]^+$$

$$\geq \sup_{t\in T} E\left[\frac{ns_t-(i+1)t}{n(n+t)}\right]^+ = a_n(i+1) \geq 0.$$

(7) follows from (10) and (11). To prove (8) we shall show that for $n = 1,2, \ldots, N$,

$$(12) \qquad \frac{n+2}{n+1}\ a_{n+1}^{N+1}(i) \geq a_n^N(i) \qquad \text{(all } i\text{);}$$

(8) will follow from (12) on letting $N \to \infty$. (12) is true trivially for $n = N$ since $a_N^N(i) = 0$. Assume now that (12) holds; for $i \neq 0$ we have by (1),

$$(13) \qquad \frac{n+1}{n}\ a_n^{N+1}(i) = \frac{n+1}{n}\left[\frac{a_{n+1}^{N+1}(i+1) + a_{n+1}^{N+1}(i-1)}{2} - \frac{i^+}{n(n+1)}\right]^+$$

$$\geq \frac{n+1}{n}\left[\frac{n+1}{n+2}\ \frac{a_n^N(i+1) + a_n^N(i-1)}{2} - \frac{i^+}{n(n+1)}\right]^+$$

$$\geq \left[\frac{a_n^N(i+1) + a_n^N(i-1)}{2} - \frac{i^+}{(n-1)n}\right]^+ = a_{n-1}^N(i).$$

The case $i = 0$ is treated similarly. Thus (12) holds with n replaced by $n-1$, and hence (12) holds for all $n=N, N-1, \ldots, 2, 1$.

Lemma 3. Let i and j be non-negative integers such that $a_n(i+j) > 0$. Let τ_o denote the first integer $m \geq 1$ such that $s_m = j+1$. Then for any given t in T there exists a τ in T such that

$$(14) \qquad \tau \geq t, \quad \tau \geq \tau_o, \quad E\left(\frac{i+s_\tau}{n+\tau}\right) \geq E\left(\frac{i+s_t}{n+t}\right).$$

Proof. We have from (3.10) and (3.11) for $i \geq 0$,

(15) $$a_n(i) = \left[\sup_{t\in T} E\left(\frac{i+s_t}{n+t}\right) - \frac{i}{n}\right]^+ .$$

By (7) and (8) the inequality $a_n(i+j) > 0$ implies that for every positive integer m and every integer $k \leq j$,

(16) $$a_{n+m}(i+k) > 0,$$

and hence that there exists a stopping rule $t_{m,k}$ of the sequence $x_{m+1}, x_{m+2}, \ldots$ such that

(17) $$E\left(\frac{i+k+x_{m+1}+x_{m+2}+\ldots+x_{m+t_{m,k}}}{n+m+t_{m,k}}\right) > \frac{i+k}{n+m}$$

Let A be the event $\{t < Z_o\}$, and define

(18) $$t_1(\omega) = \begin{cases} t(\omega) & \text{if } \omega \notin A, \\ t(\omega) + t_{m,k}(\omega) & \text{if } \omega \in A,\ t(\omega) = m,\ s_{t(\omega)} = k \\ (m = 1, 2, \ldots;\ k \leq j). \end{cases}$$

Then t_1 is a stopping rule, $t_1 \geq t$, and $t_1(\omega) \geq t(\omega) + 1$ if $\omega \in A$. Moreover

(19) $$E\left(\frac{i+s_{t_1}}{n+t_1}\right) = \int_{\Omega-A} \frac{i+s_t}{n+t}\, dP + \sum_{m,k} \int_{\{t=m,s_t=k,t<Z_o\}} \frac{i+s_{t+t_{m,k}}}{n+t+t_{m,k}}\, dP$$

$$\geq \int_{\Omega-A} \frac{i+s_t}{n+t}\, dP + \sum_{m,k} \int_{\{t=m,s_t=k,t<Z_o\}} \frac{i+k}{n+m}\, dP = E\left(\frac{i+s_t}{n+t}\right) .$$

Set $t_o = t$ and $A_o = A$. By a repetition of the preceding argument we may define a sequence of stopping rules t_ℓ,

(20) $$t = t_o \leq t_1 \leq t_2 \leq \cdots$$

and events $A_\ell = \{t_\ell < \tau_o\}$ with

(21) $$A = A_o \supset A_1 \supset A_2 \supset \cdots$$

such that

(22) $$t_{\ell+1}(\omega) \begin{cases} = t_\ell(\omega) & \text{if } \omega \notin A_\ell , \\ \geq t_\ell(\omega) + 1 & \text{if } \omega \in A_\ell . \end{cases}$$

Set

(23) $$\tau = \lim_{\ell \to \infty} \tau_\ell ;$$

then $\{\tau = \infty\} = \{\tau_o = \infty\}$, so that τ is in T, and $\tau \geq \tau_o$, $\tau \geq t$. By the Lebesgue dominated convergence theorem,

(24) $$E\left(\frac{i+s_\tau}{n+\tau}\right) = \lim_{\ell \to \infty} E\left(\frac{i+s_{t_\ell}}{n+t_\ell}\right) \geq E\left(\frac{i+s_t}{n+t}\right),$$

and the proof is complete.

Lemma 4. There exists a positive integer n_o such that $n \geq n_o$ and $i > 13\sqrt{n}$ imply that $a_n(i) = 0$.

Proof. Let i be a positive integer such that $a_n(2i) > 0$, and let τ denote the first integer $m \geq 1$ such that $s_m = i$. Then [2; p. 87] as $i \to \infty$,

(25) $$P(\tau \geq i^2) \to \sqrt{\frac{2}{\pi}} \int_0^1 e^{-\frac{u^2}{2}} du > \sqrt{\frac{2}{\pi e}} > \frac{1}{3} .$$

Hence there exists $i_o > o$ such that

(26) $$E\left(\frac{\tau}{i^2+\tau}\right) > \frac{1}{6} \qquad (i \geq i_o),$$

and therefore

(27) $$E\left(\frac{\tau}{n+\tau}\right) > \frac{1}{6} \qquad (i \geq i_o,\ 1 \leq n \leq i^2)\ .$$

By (7), $a_n(i) > 0$, and hence by Lemma 3 (putting $j = i$) there exists a $t \in T$ such that $t \geq \tau$ and

(28) $$E\left(\frac{i+s_t}{n+t}\right) > \frac{i}{n}\ .$$

Hence by Lemma 1 and (11),

(29) $$\frac{1}{\sqrt{n}} \geq a_n\ (0) \geq E\left(\frac{s_t}{n+t}\right) = E\left(\frac{i+s_t}{n+t}\right) - E\left(\frac{i}{n+t}\right) > \frac{i}{n} - E\left(\frac{i}{n+t}\right) = \frac{i}{n} E\left(\frac{t}{n+t}\right)$$
$$\geq \frac{i}{n} E\left(\frac{\tau}{n+\tau}\right) > \frac{i}{6n} \qquad (i \geq i_o,\ 1 \leq n \leq i^2).$$

Assume now that $a_n(j) > 0$ for some $j > 13\sqrt{n}$ and $n \geq n_o = i_o^2$. Then by (7),

(30) $$a_n\left(2\left[\tfrac{j}{2}\right]\right) > 0, \quad \left[\tfrac{j}{2}\right]^2 \geq n \geq 1, \quad \left[\tfrac{j}{2}\right] \geq i_o\ .$$

Hence, setting $i = \left[\frac{j}{2}\right]$ in (29),

(31) $$\left[\tfrac{j}{2}\right] < 6\sqrt{n}\ ,$$

and therefore

(32) $$j < 12\sqrt{n} + 1 \leq 13\sqrt{n}\ ,$$

a contradiction. The proof of Lemma 4, and hence of Theorem 1, is complete.

5. Remarks.

1. If we define for $n = 1,2, \ldots$

(1) $$k_n = \text{smallest integer } k \text{ such that } a_n(k) = 0,$$

then from Lemma 2 it follows that

(2) $$0 < k_1 \leq k_2 \leq \cdots$$

and that

(3) $$a_n(i) = 0 \text{ if and only if } i \geq k_n .$$

It is easily seen that

(4) $$\begin{aligned} \tau_j^*(i) &= \text{first } n \geq 1 \text{ such that } a_{j+n}(i+s_n) = 0 \\ &= \text{first } n \geq 1 \text{ such that } i+s_n = k_{j+n} . \end{aligned}$$

Hence the stopping rules $\tau_j^*(i)$ are completely defined by the sequence of positive integers k_n. It is difficult to obtain an explicit formula for k_n; by Lemma 4 we know that $k_n = 0(\sqrt{n})$ as $n \to \infty$. We note also that

(5) $$\lim_{n \to \infty} k_n = \infty .$$

Otherwise we would have $k_n < M$ for some finite positive integer M and every $n = 1,2, \ldots$. If so, let t = first $m \geq 1$ such that $s_m = M$. Then since $a_n(M) = 0$,

(6) $$E\left(\frac{M+s_t}{n+t}\right) \leq \frac{M}{n} ,$$

and hence

(7) $$E\left(\frac{2M}{n+t}\right) \le \frac{M}{n}, \quad E\left(\frac{n}{n+t}\right) \le \frac{1}{2}.$$

But as $n \longrightarrow \infty$,

(8) $$E\left(\frac{n}{n+t}\right) \longrightarrow 1,$$

which contradicts (7).

2. We have from (3.15),

(9) $$v_o(0) = \max_{t \in T} E\left(\frac{s_t}{t}\right) = \frac{1}{2}\left[1 + a_1(1) + a_1(-1)\right].$$

Now by (4.15), since $s_t \le t$,

(10) $$a_1(1) = \left[\sup_{t \in T} E\left(\frac{1+s_t}{1+t}\right) - 1\right]^+ = 0,$$

and by (4.6) and (4.7),

(11) $$a_1(-1) \le a_1(0) \le \frac{1}{4} + \frac{1}{\sqrt{2}} < .96 .$$

Hence

(12) $$v_o(0) < .98 .$$

This inequality is very crude and can be greatly improved by a more detailed analysis of the term $a_1(-1)$, but it is interesting to note that even (12) is not easy to prove directly from the definition of $v_o(0)$.

3. In this connection let us define

(13)
$$v_N = \max_{t \in T_N} E\left[\frac{s_t^+}{t}\right];$$

then as $N \longrightarrow \infty$

(14)
$$v_N \uparrow v_o(0) = \max_{t \in T} E\left(\frac{s_t^+}{t}\right) = \max_{t \in T} E\left(\frac{s_t}{t}\right).$$

Now for any fixed $N = 1, 2, \ldots$ the value v_N can be computed by recursion; by (3.4) and (3.2),

(15)
$$v_N = \frac{1}{2}\left[b_1^N(1) + b_1^N(-1)\right] = \frac{1}{2}\left[1 + b_1^N(-1)\right],$$

where by (3.1)

(16)
$$b_N^N(i) = \frac{i^+}{N},$$
$$b_n^N(i) = \max\left(\frac{i^+}{n}, \frac{b_{n+1}^N(i+1) + b_{n+1}^N(i-1)}{2}\right) \quad (n = 1, 2, \ldots, N-1).$$

The computation of the $b_n^N(i)$ is easily programmed for a high speed computer; the following results were kindly supplied to us by R. Bellman and S. Dreyfus:

(17)
$$v_{100} = .5815$$
$$v_{200} = .5835$$
$$v_{500} = .5845$$
$$v_{1000} = .5850$$

4. It would be interesting to see whether the existence of an optimal stopping rule for s_n/n can be proved for sequences $x_1, x_2, \ldots$ with a more general distribution than (1.2). We have some preliminary extensions of Theorem 1 to more general cases but no definite results as yet.

References

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