A System of Inequalities for the Incomplete Gamma Function and the Normal Integral

by

Shanti S. Gupta and Mrudulla N. Waknis

Purdue University

Department of Statistics

Division of Mathematical Sciences

Mimeograph Series No. 19

July 1964

A System of Inequalities for the Incomplete Gamma Function and the Normal Integral*

by

Shanti S. Gupta and Mrudulla N. Waknis

Purdue University

1. Introduction and Summary

In this paper a new set of inequalities and bounds for the incomplete gamma function are obtained. These inequalities and bounds are based on continued fraction expansions of the incomplete gamma function (Sections 2 and 3).

Comparisons between the two sets of inequalities and some other known inequalities are made (Section 4).

Bounds are also obtained for the Mill's ratio for the normal integral (Section 5) and an analogue of Mill's ratio (Section 6) for the gamma distribution. Some other applications of these bounds to distribution theory problems arising in multiple decision theory are described (Section 6).

2. System of inequalities for $\gamma(a,x)$ based on the continued fraction expansion

Let $\gamma(a,x) = \int_0^x x^{a-1} e^{-x} dx$. Various authors (see, Khovanskii (1956)) have derived the following continued fraction expansion

$$(2.1) \qquad x^{-a} e^x \gamma(a,x) = \frac{1}{a-} \frac{ax}{1+a+x-} \frac{(1+a)x}{2+a+x-} \cdots \frac{(n-1+a)x}{-n+a+x-} \cdots$$

where the more commonly used notation has been employed for the representation

*Research supported in part by Contract AF 33(657-11737) with the Aerospace Research Laboratories.

of the continued fraction.

The terminating continued fraction

$$\frac{P_n(a,x)}{Q_n(a,x)} = \frac{1}{a-}\;\frac{ax}{1+a+x-}\;\frac{(1+a)x}{2+a+x-}\cdots\frac{(n-1+a)x}{-n+a+x} \tag{2.2}$$

is called the nth convergent (approximant) of the continued fraction (2.1). Using certain well-known recurrence relations, it can be shown that

$$\begin{aligned} P_n(a,x) &= \sum_{j=0}^{n-1} (n-1+a)_j \, x^{n-j-1} \\ Q_n(a,x) &= (n-1+a)_n \end{aligned} \tag{2.3}$$

where

$$(n)_r = n(n-1) \;\ldots\; (n-r+1), \quad r \geq 1, \text{ and } (n)_0 \doteq 1.$$

Since

$$\frac{P_n(a,x)}{Q_n(a,x)} - \frac{P_{n-1}(a,x)}{Q_{n-1}(a,x)} = \frac{x^{n-1}}{(n-1+a)_n} > 0, \quad x > 0,\; a > 0,\; n \geq 1, \tag{2.4}$$

$\frac{P_n(a,x)}{Q_n(a,x)}$ is a monotonically increasing sequence converging to $e^x x^{-a}\gamma(a,x)$.

Again from (2.4),

$$\sum_{k=n}^{\infty}\left(\frac{P_k(a,x)}{Q_k(a,x)} - \frac{P_{k-1}(a,x)}{Q_{k-1}(a,x)}\right) = \sum_{k=n}^{\infty} \frac{x^{k-1}}{(k-1+a)_k} < \frac{x^{n-1}(n+a)}{(n-1+a)_n\,(n+a-x)}\,,\; x < (n+a). \tag{2.5}$$

From (2.5),

$$(2.6) \qquad \lim_{k \to \infty} \frac{P_k(a,x)}{Q_k(a,x)} - \frac{P_{n-1}(a,x)}{Q_{n-1}(a,x)} < \frac{x^{n-1}(n+a)}{(n-1+a)_n(n+a-x)}, \quad x < n+a$$

This leads to the following system of inequalities

$$(2.7) \qquad \frac{P_n(a,x)}{Q_n(a,x)} < e^x x^{-a} \gamma(a,x) < \frac{P_n(a,x)}{Q_n(a,x)} + \frac{x^n(n+1+a)}{(n+a)_{n+1}(n+1+a-x)}, \quad x < n+a+1$$

$$n = 1,2,3,\ldots$$

where $x < n+a+1$ is a necessary restriction only on the inequalities on the right hand side of (2.7) and where

$$(2.8) \qquad \frac{P_n(a,x)}{Q_n(a,x)} = \frac{1}{a}\left[1+\frac{x}{1+a}+\frac{x^2}{(1+a)(2+a)}+\ldots\ldots+\frac{x^{n-1}}{(1+a)(2+a)\ldots(n-1+a)}\right]$$

It should be noted that the length of the interval between the two bounds in (2.7) is $< \frac{x^n(n+1+a)}{(n+a)_{n+1}(n+1+a-x)}$, $x < n+a+1$. It follows from (2.8) that for fixed x and n, if we let

$$(2.9) \qquad \varphi_n(a,x) = \frac{P_n(a,x)}{Q_n(a,x)},$$

then for fixed x, $\varphi_n(a,x)$ decreases monotonically as a increases. Hence

$$(2.10) \qquad \varphi_n(a_1,x) \ge \varphi_n(a,x) \ge \varphi_n(a_2,x), \quad a_1 \le a \le a_2$$

$$(2.11) \qquad x^{-a_2} \gamma(a_2,x) \le x^{-a} \gamma(a,x) \le x^{-a_1} \gamma(a_1,x), \quad a_1 \le a \le a_2.$$

In particular, if $a_1 = p$ and $a_2 = p+1$, $p \le a \le p+1$, where p is a positive integer,

$$(2.12)\qquad x^{-(p+1)}\,\Gamma(p+1)\Big(\sum_{j=p+1}^{\infty}\frac{e^{-x}x^j}{j!}\Big) \le x^{-a}\,\gamma(a,x) \le x^{-p}\Gamma(p)\Big(\sum_{j=p}^{\infty}\frac{e^{-x}x^j}{j!}\Big).$$

The two bounds in (2.12) can be computed easily from a table of Poisson cumulative distribution.

The tables at the end of the paper illustrate the sharpness of the inequalities (bounds) in (2.7).

It appears that the inequalities (2.7) give close estimates of the function for x small $(x \le 1)$. The bounds improve in precision as a increases. For fixed a, the relative error appears to increase with x. For fixed x, the relative error decreases as a increases.

3. System of inequalities for $\Gamma(a+p,x)(0 < a < 1,\ p = 0,1,2,\ldots)$ based on a different continued fraction expansion

Let $\Gamma(a,x) = \int_x^{\infty} e^{-x}\, x^{a-1}\, dx$, $0 < a < 1$ then a continued fraction expansion for $\Gamma(a,x)$ (see, for example Wall (1948), Khovanskii (1956)) is

$$(3.1)\qquad x^{-a}\, e^{x}\, \Gamma(a,x) = \frac{1}{x+}\,\frac{1-a}{1+}\,\frac{1}{x+}\,\frac{2-a}{1+}\,\frac{2}{x+}\,\frac{3-a}{1+}\cdots + \frac{n-1}{x+}\,\frac{n-a}{1+}\cdots\cdots$$

The odd and even convergents (approximants) of the above continued fraction are

$$(3.2)\qquad \frac{P'_{2n+1}(a,x)}{Q'_{2n+1}(a,x)} = \frac{1}{Q'_1(a,x)} - \frac{1(1-a)}{Q'_1(a,x)Q'_2(a,x)} + \frac{1(1-a)}{Q'_2(a,x)Q'_3(a,x)} - \cdots$$

$$+ \frac{(n!)(n-a)_n}{Q'_{2n}(a,x)Q'_{2n+1}(a,x)}\ ,\ n \ge 0\ ,$$

$$(\pm t)_0 \equiv 1\ .$$

$$(3.3)\quad \frac{P'_{2n}(a,x)}{Q'_{2n}(a,x)} = \frac{1}{Q'_1(a,x)} - \frac{1(1-a)}{Q'_1(a,x)Q'_2(a,x)} + \frac{1(1-a)}{Q'_2(a,x)Q'_3(a,x)} - \cdots - \frac{(n-1)!(n-a)_n}{Q'_{2n-1}(a,x)Q'_{2n}(a,x)}, \quad n \geq 1$$

where

$$(3.4)\quad Q'_{2n}(a,x) = \sum_{j=0}^{n} x^{n-j}(n-a)_j \binom{n}{j} \text{and} \quad Q'_{2n+1}(a,x) = \sum_{j=0}^{n} x^{n+1-j}(n+1-a)_j \binom{n}{j} .$$

For $0 < a < 1$, the coefficients of the continued fraction in (3.1) are positive; hence the even order convergents $\frac{P'_{2n}(a,x)}{Q'_{2n}(a,x)}$ $(n = 1,2,\ldots)$ generate a monotonically increasing sequence and the odd order convergents $\frac{P'_{2n+1}(a,x)}{Q'_{2n+1}(a,x)}$ $(n = 0,1,2,\ldots)$ generate a monotonically decreasing sequence. Both sequences converge to the function $e^x x^{-a} \Gamma(a,x)$. Thus, the following system of inequalities (bounds) is obtained,

$$(3.5)\quad \frac{P'_{2n}(a,x)}{Q'_{2n}(a,x)} < e^x x^{-a} \Gamma(a,x) < \frac{P'_{2n+1}(a,x)}{Q'_{2n+1}(a,x)}, \quad a < 1,\ n = 1,2,3,\ldots .$$

The first two sets of these inequalities are , for any $x > 0$, $1 > a$

(3.5a)

$$\frac{1}{x+1-a} < e^x x^{-a} \Gamma(a,x) < \frac{1+x}{x^2+2x-ax}$$

$$\frac{x+3-a}{x^2+2(2-a)x+(2-a)(1-a)} < e^x x^{-a} \Gamma(a,x) < \frac{x(x+5-a)+2}{x^3+2x^2(3-a)+x(2-a)(3-a)}$$

Again, using the recurrence relation

$$\Gamma(\nu,x) = (\nu-1)\,\Gamma(\nu-1,x) + e^{-x}\,x^{\nu-1}, \quad \nu > 1 \tag{3.6}$$

the following bounds are obtained. For any positive integer p,

$$(a+p-1)_p \frac{P'_{2n}(a,x)}{Q'_{2n}(a,x)} + x^{p-1}\sum_{j=0}^{p-1} \frac{(a+p-1)_j}{x^j} < e^x x^{-a}\,\Gamma(a+p,x) \tag{3.7}$$

$$< (a+p-1)_p \frac{P'_{2n\pm1}(a,x)}{Q'_{2n\pm1}(a,x)} + x^{p-1}\sum_{j=0}^{p-1} \frac{(a+p-1)_j}{x^j}$$

$$n = 1,2,3,\ldots.$$

It can be shown that the length of the interval i.e. the distance $d=d(n,p,x)$ between the lower and upper bounds in (3.7) satisfies the inequality

$$\frac{(a+p-1)_p(n-a)_n(n!)}{x(n-a+x)^n(n+1-a+x)^n} < d(n,p,x) < \frac{(a+p-1)_p(n-a)_n(n!)}{x(1-a+x)^n(2-a+x)^n} \tag{3.8}$$

The table at the end of the paper illustrates the sharpness of the bounds in (3.7).

4. Comparison of Inequalities and Bounds for the Gamma Integral

In Sections 2 and 3, two different sets of bounds have been obtained for the integrals $\gamma(a,x)$ and $\Gamma(a,x)$, respectively. The lower bounds for $\gamma(a,x)$ for given a and x as given in (2.7) form a monotonically increasing sequence converging to the true value. The upper bounds in (2.7) form a monotonically decreasing sequence converging to the true value for $n \geq n_0$, where n_0 is the smallest positive integer which satisfies $x < n_0 + 1 + a$. The proof of this latter statement is straightforward and hence has been omitted. If x is small as compared to $(1 + a)$, then the lower bounds obtained in (2.7) are very good since the successive terms in the series for this bound rapidly converge to zero. The series for the lower bound was also obtained by Pearson (1922) by a method different from ours. Pearson (1922) used this series expansion for computing the tables of the incomplete gamma function. Pearson and his collaborators (1922) did not obtain explicit expressions for upper and lower bounds.

For selected values of n, x and a, upper and lower bounds in (2.7) were computed. For $n = 2,3,4,7$, $x = .3,.5,1.0,1.5$ and $a = .5,1.5,2.5,5.5$, the values have been included in Table I at the end of the paper. A glance at this table confirms the earlier assertion that the bounds are very good in the range of values n and a for which $x/(1+a)$ is very small. For example, if $x = .5$ and $a = 5.5$ so that $x/(1+a) = 1/13$, the upper bound is accurate to 6 decimal places for n as small as 2. For the same case the lower bound is accurate to within one unit in the sixth decimal place for n as small as 3.

The system of bounds for $\Gamma(a,x)$ as given in (3.5) and (3.7) are monotone. The lower bounds form a monotone increasing sequence and the upper bounds form a monotone decreasing sequence. Empirical evidence as illustrated in Table I shows that for small x, the bounds on $\gamma(a,x)$ (as obtained from (3.5) and (3.7) and following from the relation that $\gamma(a,x) + \Gamma(a,x) = \Gamma(a)$) are much worse than the bounds using (2.7). The bounds on $\Gamma(a,x)$ improve as x increases. For selected values of n,x and a, the bounds on $\Gamma(a,x)$ were computed and an excerpt from these values is given in Table II for $n = 2(1)7$, $x = 1.5,2.0,5.0,10.0$ and $a = .5, 1.5,2.5,5.5$. Of course the lower and upper bounds are formed by the pairs of convergents corresponding to $n = (2,3),(4,5)$ and $(6,7)$. A glance at this table shows that for $x \geq 10$, and $n \geq 2$, there is agreement to 6 decimal places for $a \leq 2.5$ and the values agree to 5 decimal places for $a \leq 5.5$. It would be interesting to find out the behaviour of these bounds with respect to a and also with respect to x/a.

Some other bounds on the incomplete gamma function, $\Gamma(a,x)$, $a < 1$, have been derived by Gautschi (1959). Gautschi's inequality gives a lower and an upper bound for fixed a and x. For $a = .5$ and $x = 1,2$ the following table compares Gautschi's result with our bounds of Sections 2 and 3.

n	Method	Lower Bound		Upper Bound		Length of the Interval	
		x=1	x=2	x=1	x=2	x=1	x=2
4	Section 2	.2787	.0747	.2939	.2377	.0152	.1630
	Section 3	.2711	.0804	.2943	.0963	.0232	.0159
	Gautschi	.2693	.0793	.2924	.0840	.0231	.0047
6	Section 2	.2788	.0801	.2792	.0820	.0004	.0019
	Section 3	.2760	.0805	.2830	.0809	.0070	.0004
	Gautschi	.2693	.0793	.2924	.0840	.0231	.0047

As x increases, Gautschi's bounds improve. From the above table and Table II, it appears that for x large, the bounds of Section 3 of this paper are as good as Gautschi's and seem to be better for n moderately small $(n \leq 6)$ and $x = 2$ and for $a = .5$.

Wilk, Gnanadesikan and Huyett (1962) have discussed the approximation of the incomplete gamma function. These authors studied the truncation error in using the series earlier also given by Pearson (1922) and the partial sum of which forms the left hand side of (2.7) derived in this paper by the method of continued fraction. Thus the right hand side of equation (5) of their paper gives the upper bound on truncation error at n terms as

$$\frac{x^n}{a(a+1)\ldots(a+n)(1-x/(n+a))}, \qquad x < n+a \tag{4.1}$$

which is greater than the corresponding bound i.e.

$$\frac{x^n}{a(a+1)\ldots(a+n)(1-x/(n+1+a))}, \qquad x < n+1+a. \tag{4.2}$$

Clearly (4.1) is sharper than (4.2). Finally, the upper bound in (2.7) of this paper has been shown to be monotonically decreasing.

Reference should be made to Whittlesey (1963) who gave brief details of some subroutines for computing the incomplete gamma function.

5. System of Inequalities for the Mill's Ratio and the Cumulative Distribution Function of the Normal

Let $\Phi(\cdot)$ and $\varphi(\cdot)$ denote the cumulative distribution function and the density function of the standard normal random variable. Then the Mill's Ratio is defined as

(5.1)
$$R(x) = (1 - \Phi(x))/\varphi(x)$$

Laplace (1802) gave the following (by now well-known) expansion for $R(x)$

(5.2)
$$R(x) = \frac{1}{x+}\,\frac{1}{x+}\,\frac{2}{x+}\,\frac{3}{x+}\cdots \qquad (x > 0).$$

For more recent work on the Mill's Ratio for the normal reference should be made to Ruben (1963) and the references contained therein.

Now using the fact that

(5.3)
$$\Gamma(1/2,x) = 2\sqrt{\pi}\,(1 - \Phi(\sqrt{2x})),$$

we have from (3.5)

(5.4)
$$\frac{t}{2}\,\frac{P'_{2n}(1/2,\, t^2/2)}{Q'_{2n}(1/2,\, t^2/2)} < R(t) < \frac{t}{2}\,\frac{P'_{2n+1}(1/2,\, t^2/2)}{Q'_{2n+1}(1/2,\, t^2/2)}, \quad t > 0,\ n = 1,2,\ldots$$

where the $\dfrac{P'_n(1/2,\, t^2/2)}{Q'_n(1/2,\, t^2/2)}$ are defined in (3.2 and (3.4). The first few of these convergents are,

$$\frac{P'_1(1/2,\, t^2/2)}{Q'_1(1/2,\, t^2/2)} = \frac{2}{t^2},\quad \frac{P'_2(1/2,\, t^2/2)}{Q'_2(1/2,\, t^2/2)} = \frac{1}{t^2+1},\quad \frac{P'_3(1/2,\, t^2/2)}{Q'_3(1/2,\, t^2/2)} = \frac{2(2+t^2)}{t^2(3+t^2)}.$$

It should be pointed out that these inequalities in (5.4) are the same as obtained by using the successive convergents of Laplace's continued fraction expansion (5.2) as derived by Murty (1952). However, our method of derivation is different.

The bounds in (5.4) are reasonably good for large t as illustrated by Shenton (1954). For $t = 4.0$ and $t = 5.0$, the value of n required to achieve an accuracy of the order of 2.5×10^{-7} is 5 and 4 respectively.

We now give a new set of inequalities for the Mill's ratio and the normal integral. Using the fact that

$$\gamma(1/2, t^2/2) = 2\sqrt{\pi}\,[\Phi(t) - \tfrac{1}{2}], \quad t > 0, \tag{5.5}$$

we obtain, from (2.7),

$$\frac{\sqrt{2\pi}}{2}\, e^{t^2/2} - \left[t + \frac{t^3}{1.3} + \frac{t^5}{1.3.5} + \dots + \frac{t^{2n-1}}{1.3.5\dots(2n-1)}\right] - \frac{t^{2n+1}(2n+3)}{2^{n+1}(2n+3-t^2)(n+\frac{1}{2})_{n+1}} < R(t), \; t>0, \; t^2 < 2n+3. \tag{5.6}$$

$$R(t) < \frac{\sqrt{2\pi}}{2}\, e^{t^2/2} - \left[t + \frac{t^3}{1.3} + \frac{t^5}{1.3.5} + \dots + \frac{t^{2n-1}}{1.3.5\dots(2n-1)}\right], \; t > 0 .$$

It is interesting to note that the expression within the square brackets in (5.6) represents the first n terms in Pólya series (see formula (2.8) of Pólya (1949)) for $(\Phi(t) - 1/2)/\varphi(t)$.

To illustrate the bounds in (5.6) we give the following brief table for $\bar{R}(t) = (\sqrt{2\pi}\, e^{t^2/2})/2 - R(t)$.

Table 1 illustrating the bounds on $\bar{R}(t)$

as obtained from (5.6)

t	n	Lower Bound	Upper Bound	Exact Value
.1	3	.100,334,000,000	.100,334,000,953	.100,334,000,953
.5	5	.543,826,47	.543,826,52	.543,826,52
1.0	5	1.410,582,01	1.410,686,23	1.410,686,13
3.0	8	105.776,010,93	112.896,452,14	112.515,153,2

The entries in the above table for $t = .1$ and $t = 3.0$ can be compared with the values given by Shenton (see Table I of Shenton (1954)) which are based on a different continued fraction expansion. These values are

$t = .1$; $n = 3$, $\bar{R}(t) \simeq .100,334,001,3$; $n = 8$, $\bar{R}(t) \simeq .100,334,000,953$

$t = 3.0$; $n = 8$, $\bar{R}(t) \simeq 98.7$; $n = 15$, $\bar{R}(t) \simeq 112.515,2$

The above table indicates that the bounds given in (5.6) are sufficiently close (agree to 7 decimal places with the true value for $n = 5$, $t \leq .5$) to the true value. As t increases, the value of n has to be increased to achieve the same degree of accuracy. It should be noted that the value of n is subject to the condition $n \geq \left[\frac{t^2-3}{2}\right] + 1$. Comparison with Shenton's results seems to indicate that the bounds in (5.6) are better. It should be pointed out that the upper bound in the above table is very close (much closer than lower bound) to the true value.

6. Applications of the bounds

A. Analogue of Mill's Ratio and the Hazard Rate Function

Inequalities and bounds have been obtained for the Mill's ratio which is

$(1 - \Phi(x))/\varphi(x)$ where $\Phi(\cdot)$ and $\varphi(\cdot)$ refer to the cumulative distribution function and the density function of the standard normal distribution. An analogue of this function for the gamma distribution is

$$(6.1) \qquad R(a,x) = e^{x} x^{-a+1} \int_{x}^{\infty} e^{-t} t^{a-1} dt.$$

It should be noted that the reciprocal of the function $R(a,x)$ in (6.1) is the hazard rate (failure rate) which is important in the study of statistical reliability. Barlow, Marshall and Proschan (1963) have studied the properties of distributions with monotone hazard rate. From the results proved in the above paper, it is seen that

(i) $R(a,x)$ is an increasing function of x for $a < 1$

(ii) $R(a,x)$ is a decreasing function of x for $a > 1$

(for $a = 1$, $R(a,x)$ is constant for all x) .

The first set of inequalities for $R(a,x)$ is

$$(6.2) \qquad e^{+x} x^{-a+1}\Big[\Gamma(a) - e^{-x} x^{a} \sum_{j=0}^{n-1} \frac{x^{j}}{(j+a)_{j+1}} - \frac{e^{-x} x^{n+a}(n+a+1)}{(n+a)_{n+1}(n+a+1-x)}\Big] <$$

$$R(a,x) < e^{+x} x^{-a+1}\Big[\Gamma(a) - e^{-x} x^{a} \sum_{j=0}^{n-1} \frac{x^{j}}{(j+a)_{j+1}}\Big], \text{ for the first part } x < n+1+a ,$$

the second being obvious from (3.5), (3.7).

B. Applications to Multiple Decision

For the problems of selecting a subset containing the best of several gamma populations as discussed by Gupta and Sobel (1962) and Gupta (1963) it is found that the following integrals have to be evaluated

$$(6.3) \qquad [\Gamma(a)]^{-(p+1)} \int_0^\infty [\gamma(a,cx)]^p e^{-x} x^{a-1} dx$$

$$(6.4) \qquad [\Gamma(a)]^{-(p+1)} \int_0^\infty [\Gamma(a,dx)]^p e^{-x} x^{a-1} dx.$$

Bounds on the above integrals can be obtained by using the results of Sections 2 and 3.

It should be noted that if we equate the integrals in (6.3) to α, then c is the αth percentile of the statistic $F_{max} = \max(\frac{\chi_1^2}{\chi_0^2}, \ldots, \frac{\chi_p^2}{\chi_0^2})$ where $\chi_0^2, \chi_1^2, \ldots, \chi_p^2$ are $(p + 1)$ independent chi-square random variables with 2a degrees of freedom. Similarly the integral in (6.4) represents the probability integral of $F_{min} = \min(\frac{\chi_1^2}{\chi_0^2}, \ldots, \frac{\chi_p^2}{\chi_0^2})$.

We now derive explicitly lower bounds for (6.3) for the special case $p = 1$.

$$(6.5) \qquad \frac{1}{(\Gamma(a))^2} \int_0^\infty \gamma(a,cx) e^{-x} x^{a-1} dx > \frac{c^a}{(1+c)^{2a}(\Gamma(a))^2} \sum_{j=0}^{n-1} \frac{\Gamma(2a+j)\ c^j}{(1+c)^j (a+j)_{j+1}}$$

It should be pointed out that (6.5) represents the probability that the random variable F with $2a$, $2a$ degrees of freedom does not exceed c.

In order to obtain explicit upper bounds on (6.4), one proceeds in a similar manner as above. It should be pointed out that lower (upper) bounds on the moments of the smallest (largest) order statistic from a gamma distribution can also be obtained in the manner outlined above.

Armitage and Krishnaiah (1964) have been interested in the distribution of the Studentized largest chi-square. The inequalities of Section 2 of the present paper can be used to obtain the bounds on this distribution function and to approximate it.

7. Acknowledgments

The authors wish to acknowledge the assistance of Mrs. Shirley Wolfe of Purdue University for programming and carrying out some of the computations.

Table I

Lower and upper bounds on $\gamma(a,x)$ based on (2.7)

n	a	x = .3	x = .5	x = 1.0	x = 1.5
2	.5	.973831, .995133	1.143685, 1.210400	1.226265, 1.500948	1.093110, 1.666993
3		.993307, .995096	1.200869, 1.210060	1.422467, 1.494542	1.421043, 1.631857
5		.995088, .995095	1.209946, 1.210036	1.490982, 1.493659	1.608433, 1.625043
7		.995095, .995095	1.210035, 1.210036	1.493596, 1.493648	1.624158, 1.624874
		.944381,* 1.060733*	1.195682,* 1.230708*	1.492251,* 1.496013*	1.624643,* 1.625304*
		.995095**	1.210036**	1.493648**	1.624871**
2	1.5	.090891, .091785	.171533, .176148	.343354, .379391	.437244, .542651
3		.091726, .091784	.175637, .176137	.371383, .378996	.507515, .539723
5		.091784, .091784	.176132, .176136	.378744, .378945	.537327, .539170
7		.091784, .091784	.176136, .176136	.378942, .378945	.539097, .539158
		.066428,* .124604*	.168959,* .186472*	.378246,* .380127*	.539044,* .539375*
		.091784**	.176136**	.378945**	.539158**
2	2.5	.015860, .015948	.049015, .049764	.189195, .200614	.351357, .399668
3		.015943, .015948	.049696, .049763	.198538, .200546	.386492, .398950
5		.015948, .015948	.049762, .049763	.200498, .200538	.398286, .398823
7		.015948, .015948	.049763, .049763	.200537, .200538	.398807, .398821
		0* .065176*	.038997,* .065267*	.199490,* .202311*	.398650,* .399146*
		.015948**	.049763**	.200538**	.398821**
2	5.5	.000188, .000188	.002624, .002638	.077177, .078732	.464381, .485527
3		.000188, .000188	.002637, .002638	.078549, .078730	.481795, .485444
5		.000188, .000188	.002638, .002638	.078728, .078730	.485353, .485433
7		.000188, .000188	.002638, .002638	.078730, .078730	.485432, .485433
		0* .019386*	0* .612123,*	.037462,* .148553*	.478701,* .498222*
		.000188**	.002638**	.078730**	.485433**

* Lower and upper bounds for $\gamma(a,x)$ obtained from the results of Section 3 using the 6th and 7th convergents. Negative lower bounds have been replaced by zero.

** Exact value of $\gamma(a,x)$

Table II

Lower and upper bounds on $\Gamma(a,x)$ based on the results of Section 3

n	a	x = 1.5		x = 2.0		x = 5.0		x = 10.00	
2,3	.5	.136639,	.151821	.076557,	.082026	.002739,	.002782	.000014,	.000014
4,5		.145748,	.148447	.080118,	.080870	.002773,	.002775	.000014,	.000014
6,7		.147149,	.147810	.080546,	.080695	.002774,	.002775	.000014,	.000014
		.147579,*	.148295*	.080612,*	.084658*	0,*	.322277*	0,*	1.467725*
		.147583**		.080647**		.002775**		.000014*	
2,3	1.5	.341597,	.349188	.229672,	.232406	.016436,	.016457	.000150,	.000150
4,5		.346152,	.347501	.231452,	.231828	.016453,	.016454	.000150,	.000150
6,7		.346852,	.347183	.231666,	.231741	.016454,	.016454	.000150,	.000150
		.347069,*	.347130*	.231714,*	.232175*	.009574,*	.101877*	0,*	.643348*
		.347069**		.231717**		.016454**		.000150*	
2,3	2.5	.922312,	.933698	.727293,	.731395	.099987,	.100018	.001661,	.001661
4,5		.929144,	.931168	.729964,	.730528	.100012,	.100014	.001661,	.001661
6,7		.930195,	.930690	.730285,	.730397	.100013,	.100013	.001661,	.001661
		.930519,*	.930534*	.730360,*	.730505*	.096667,*	.162564*	0,*	.806473*
		.930519**		.730361**		.100013**		.001661**	
2,3	5.5	51.5342,	51.9825	50.6474,	50.8089	27.7609,	27.7621	2.37326,	2.37326
4,5		51.8032,	51.8829	50.7525,	50.7747	27.7619,	27.7619	2.37326,	2.37326
6,7		51.8446,	51.8641	50.7652,	50.7696	27.7619,	27.7619	2.37326,	2.37326
		51.8573,*	51.8573*	50.7682,*	50.7682*	27.7584,*	27.9371*	0,*	15.6251*
		51.8573**		50.7682**		27.7619**		2.37326**	

The index n in column one gives the indices of the convergents used to get the lower and upper bounds respectively.

* Lower and upper bounds for $\Gamma(a,x)$ obtained from Section 2, when $n = 7$. Negative lower bounds replaced by 0.

** Exact value

References

1. Armitage, J.V. and Krishnaiah, P. (1964). Tables for the Studentized largest chi-square distribution and their applications. Submitted for publication.

2. Barlow, R.E., Marshall, A.W. and Proschan, F. (1963). Properties of probability distributions with monotone hazard rate. Ann. Math. Statist. 34, 375-389.

3. Gautschi, W. (1959). Some elementary inequalities relating to the gamma and incomplete gamma function. J. Math. Phys. 38, 77-81.

4. Gupta, S.S. and Sobel, M. (1962). On selecting a subset containing the population with the smallest variance. Biometrika 49, 495-507.

5. Gupta, S.S. (1963). On a selection and ranking procedure for gamma populations. Ann. Inst. Statist. Math. Tokyo. 14, 199-216.

6. Khovanskii, A.N. (1956). The application of continued fractions. (Translated by P. Wynn (1962)). P. Noordhoff, Ltd. Groningen.

7. Laplace, P.S. (1802). Traité de mécanique céleste. t3, livre 10. Paris.

8. Murty, V.N. (1952). On a result of Birnbaum regarding the skewness of X in a bivariate normal population. J. Indian Soc. Agric. Stat. 4, 85-87.

9. Pearson, Karl (ed.) (1922). Tables of the incomplete Γ-function. Cambridge University Press, Cambridge, England.

10. Pólya, G. (1949). Remarks on computing the probability integral in one and two dimensions. Proc. Berkeley Symposium on Mathematical Statistics and Probability. Berkeley: Univ. of California Press 1949.

11. Ruben, Harold (1963). A convergent asymptotic expansion for Mill's ratio and the normal probability integral in terms of rational functions. Math. Ann. 151, 355-364.

12. Shenton, L.R. (1954). Inequalities for the normal integral including a new continued fraction. Biometrika 41, 177-189.

13. Wall, H.S. (1948). Analytic theory of continued fractions. D. Van Nostrand Co., New York.

14. Whittlesey, J.R.B. (1963). Incomplete gamma functions for evaluating Erlang process probabilities. Mathematics of Computation, 17, 11-17.

15. Wilk, M.B., Gnanadesikan, R. and Huyett, M. (1962). Probability plots for the gamma distribution. Technometrics, 4, 1-20.