# The Single Server Queue with Poisson Input and semi-Markov Service Times, II

by

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## Summary

In this paper the queue-length and the waiting time of the nth customer for a queueing process with Poisson input and semi-Markovian service times are discussed.

It is a sequel to a paper with identical title (I) in which the process of the busy periods and the virtual waiting time process were discussed.

We refer to (I) for notations and definitions and we will number the equations continuously in I and II.

## 8. The Distribution of the Queue-length.

Let us denote by $\xi_n$ the number of customers in the system at the time of departure of the nth customer. We assume here that customers are served in the order of their arrival and that there is a departure at time $t = o$. By $\xi_o$ we denote the queue length at time o+. $J_o$ is the type of the first customer to join the queue after $t = o$ and $J_n$ is the type of the $(n + 1)$st customer If several customers leave the queue at the same time, then several values of $(\xi_n, J_n)$ are associated with the same instant of time. Since the customers are served in a given order, there is no ambigiuity however about the value taken on by the random variables $(\xi_n, J_n)$. The random variables $\{(\xi_n, J_n), n = o,1,..$ form a homogeneous Markov chain, whose state space is the cartesian product $\{o,1,\ldots\} x \{1,\ldots, m\}$.

The transition probabilities

$$P(i,h;j,\ell) = P\{\xi_{n+1} = \ell, J_{n+1} = j | \xi_n = h, J_n = i\}$$

$n = o,1,\ldots; i,j = 1,\ldots, m; h, \ell = o,1,\ldots$ are given by

$$(79) \quad P(i,h;j,\ell) = p_{\ell-h+1}(i,j), \quad \ell \geq h-1;\ h = 1,2,\ldots;\ i,j = 1,\ldots, m.$$

$$= p_{\ell}(i,j), \quad \ell \geq o;\ h = o;\ i,j = 1,\ldots, m.$$

$$= o \quad \ell < h-1;\ h = 2,3,\ldots;\ i,j = 1,\ldots, m$$

where

$$(80) \quad p_k(i,j) = \int_o^{\infty} e^{-\lambda x} \frac{(\lambda x)^k}{k!} d\, Q_{ij}(x), \quad k = o,1,\ldots$$

Remark:

The Markov chain $\{(\xi_n, J_n);\ n = 0,1,\ldots\}$ may be reducible and may have periodic states. If the matrix $P = Q(\infty)$ is strictly positive, then all $p_k(i,j)$ in (80) are strictly positive and the chain is irreducible, because every state can be reached from every other state. The chain will then have the same period as the irreducible finite chain $P$.

We will now calculate the generating functions for the n-step transition probabilities

$$P^{(n)}(i,h;j,\ell) = P\left\{\xi_n = \ell,\ J_n = j \mid \xi_0 = h,\ J_0 = i\right\}$$

We introduce the following generating functions:

$$\text{(81)} \qquad U_{ij}^{(n)}(z) = \sum_{\ell=0}^{\infty} P^{(n)}(i,h;j,\ell) z^{\ell}, \qquad n = 0,1,\ldots$$

$$V_{ij}(z,w) = \sum_{n=0}^{\infty} U_{ij}^{(n)}(z) w^n, \qquad i,j = 1,\ldots, m$$

$$\Pi_{ij}(w) = \sum_{n=0}^{\infty} P^{(n)}(i,h;j,0) w^n, \qquad i,j = 1,\ldots, m$$

and prove the following theorem:

Theorem 9

Assuming that the non-degeneracy conditions, introduced in (I), are fulfilled, the matrix $V(z,w) = \{V_{ij}(z,w)\}$ is given

$$V(z,w) = \left[z^{h+1} I + (z-1)w \, \Pi(w) \, \Psi(\lambda-\lambda z)\right]\left[zI - w \, \Psi(\lambda-\lambda z)\right]^{-1},$$

in which the matrix $\Pi(w)$ is given as the unique solution to the system of linear equations:

$$\sum_{\nu=1}^{m} \Pi_{i\nu}(w) \, \alpha_{\nu\rho}\left[\lambda-\lambda \, \gamma_\rho(o+,w)\right] = \gamma_\rho^{h} \, (o+,w)\left[1-\gamma_\rho(o+,w)\right]^{-1} \alpha_{i\rho}\left[\lambda-\lambda \, \gamma_\rho(o+,w)\right],$$

for $i,\rho = 1,\ldots, m$. It is assumed that initially there are $h$ customers in the queue.

Proof:

The random variables $\xi_n$ and $\xi_{n+1}$ are related by the usual formula

$$\xi_{n+1} = [\xi_n - 1]^+ + \gamma_{n+1} \tag{82}$$

in which $\gamma_{n+1}$ denotes the number of customers, arriving during the (n+1)st service period.

It follows that:

(83) $P\left\{\xi_{n+1} = \ell \, ,J_{n+1} = j \,|\, \xi_o = h,J_o = i\right\} =$

$$\sum_{\nu=1}^{m} P\left\{\xi_n = o,J_n = \nu \,|\, \xi_o = h,J_o = i\right\} \int_o^\infty e^{-\lambda x} \frac{(\lambda x)^\ell}{\ell!} \, d \, Q_{\nu j}(x)$$

$$+ \sum_{\nu=1}^{m} \sum_{\alpha=1}^{\ell+1} P\left\{\xi_n = \alpha,J_n = \nu \,|\, \xi_o = h,J_o = i\right\} \int_o^\infty e^{-\lambda x} \frac{(\lambda x)^{\ell+1-\alpha}}{(\ell+1-\alpha)!} \, dQ_{\nu j}(x),$$

It now follows from (81) and (83) that:

(84) $$U_{ij}^{(o)}(z) = \delta_{ij}\, z^{h},$$

$$U_{ij}^{(n+1)}(z) = \sum_{\nu=1}^{m} \left[\frac{U_{i\nu}^{(n)}(z)-P^{(n)}(i,h;\nu,o)}{z} + P^{(n)}(i,h;\nu,o)\right] \psi_{\nu j}(\lambda-\lambda z),$$

and:

(85) $$zV_{ij}(z,w) = \delta_{ij}\, z^{h+1} + w \sum_{\nu=1}^{m} V_{i\nu}(z,w)\; \psi_{\nu j}(\lambda-\lambda z) +$$

$$(z-1)w \sum_{\nu=1}^{m} \Pi_{i\nu}(w)\; \psi_{\nu j}(\lambda-\lambda z), \quad (i,j = 1,\ldots, m)$$

or in matrix notation:

(86) $$V(z,w) = \left[z^{h+1}\, I + (z-1)w\, \Pi(w)\; \Psi\; (\lambda-\lambda z)\right]\left[z\; I - w\; \Psi(\lambda-\lambda z)\right]^{-1},$$

The functions $V_{ij}(z,w)$ are regular in $|z| < 1$, $|w| < 1$ and the determinant of $z\ I - w\ \Psi(\lambda-\lambda z)$ vanishes only at the points where

$$z = \gamma_\rho(o+,w) \qquad \rho = 1,\ldots, m$$

in which the functions $\gamma_\rho(o+,w)$ are the unique solutions of the equations

$$z = w\; \eta_\rho(\lambda-\lambda z) \quad \rho = 1,\ldots, m$$

in the open unit disk. The $\eta_\rho(\lambda-\lambda z)$ are the m (distinct) eigenvalues of the matrix $\Psi(\lambda-\lambda z)$.

Imposing that the $\gamma_\rho(o+,w)$ must also be zeros of the denominators of the expressions for $V_{ij}(z,w)$, derived from formula (86), we obtain:

$$(87) \qquad \sum_{\nu=1}^{m} \Pi_{i\nu}(w)\, \alpha_{\nu\rho}[\lambda-\lambda\gamma_\rho(o+,w)] = \gamma_\rho^h(o+,w)[1-\gamma_\rho(o+,w)]^{-1}\, \alpha_{i\rho}[\lambda-\lambda\, \gamma_\rho(o+,w)]$$

for $i, \rho = 1,\ldots, m$.

Using matrix notation the matrix $\Pi(w)$ may be written in terms of the matrix $T(w,o+) = \left\{ \alpha_{i\rho}[\lambda-\lambda\gamma_\rho(o+,w)] \right\}$ as follows:

$$(88) \qquad \Pi(w) = T(w,o+)\ \nabla(w)\ T^{-1}(w,o+),$$

in which

$$(89) \qquad \nabla(w) = \operatorname{diag}\left\{ \frac{\gamma_1^h(o+,w)}{1-\gamma_1(o+,w)}, \ldots, \frac{\gamma_1^h(o+,w)}{1-\gamma_m(o+,w)} \right\}$$

By using (86) and (88) the probabilities $P^{(n)}(i,h;j,\ell)$ may in principle be determined explicitly.

If the Markov chain is irreducible, we may obtain the generating function for its limiting probability distribution directly from formulae (86) and (87). The limiting probability distribution will exist only if $\lambda\eta_m'(o+) > -1$. In all other cases the Cesaro limits

$$\frac{1}{n} \sum_{\nu=0}^{n} P^{(\nu)}(i,h;j,\ell)$$

will be zero.

Let us denote the limits

$$\lim_{w \to 1-} \Pi_{i\nu}(w)(1-w) = A_\nu$$

These limits are independent of the initial state. The numbers $A_\nu$ are the solutions to the system of linear equations:

$$(90) \qquad \sum_{\nu=1}^{m} A_\nu \, \alpha_{\nu\rho}\left[\lambda-\lambda \; \gamma_\rho(o+,1)\right] = o, \qquad \text{for} \quad \rho = 1,\ldots, m-1$$

$$\sum_{\nu=1}^{m} A_\nu = \frac{1+\lambda\eta_m'(o+)}{\eta_m'(o+)}, \qquad \text{for} \quad \rho = m.$$

If $1 + \lambda\eta_m'(o+) = o$ then the limits $A_\nu$ are zero. If $1 + \lambda\eta_m'(o+)$ is strictl positive the system (90) has a unique solution. Let us now denote by $P(j, \ell)$ the limits of the Cesaro sums given above. It follows from the ergodic theorem of Markov chains and from Abel's theorem, that the generating functions

$$(91) \qquad V_j^*(z) = \sum_{=o}^{\infty} P(j, \ell) \, z^{\ell}$$

are given by the limits $\lim_{w \to 1-}(1-w) \, V_{ij}(z,w)$.

Taking this limit in formula (86) we obtain:

$$(92) \qquad V_j^*(z) = \sum_{\rho=1}^{m} \frac{\eta_\rho(\lambda-\lambda z)}{z-w\eta_\rho(\lambda-\lambda z)} \, \beta_{\rho j}(\lambda-\lambda z)(z-1) \sum_{\nu=1}^{m} A_\nu \, \alpha_{\nu\rho}(\lambda-\lambda z),$$

in which the numbers $A_\nu$ are the unique solutions to the system (90). If the $A_\nu$ are zero, it is clear that all $P(j, \ell)$ vanish. If the $A_\nu$ are different from zero, formula (91) gives the generating function for the $P(j, \ell)$.

## 9. The Queue-length in Continuous Time

In this section we investigate the process $\{\xi(t), J_t^*; t \geq o\}$, in which $\xi(t)$ denotes the number of customers in the queue at time $t$ and $J_t^*$ is the type of the customer who is being served at time $t$. The state space of the process is the cartesian product $\{o,1,\dots\} \times \{1,\dots, m\}$. As in the case of renewal service times - Takacs [2] - we need to derive the joint distribution of the random variables $\xi_n$, $\tau_n'$ and $J_n$ first. We recall that $\tau_n'$ is the time of departure of the nth customer. The process $\xi(t)$ may not be defined for certain values of $t$, since several customers may leave the queue in the same instant of time. This is not a major difficulty, since the process is continuous at almost all points and we may define the value of $\xi(t)$ by right continuity at all points where this difficulty occurs.

We again assume that the customers are served in the order of their arrival and that a departure occurs at $t = o$. We denote by $\xi_o$ the value of $\xi$ at $o+$ and set $\tau_o' = o$.

We first introduce the following notations:

$$(93) \qquad \tilde{U}_{ij}^{n}(s,z) = \sum_{\alpha=o}^{\infty} z^{\alpha} \int_o^{\infty} e^{-st}\, d\, P\left\{\tau_n' \leq t,\ \xi_n = \alpha,\ J_n = j \mid J_o = i\right\}$$

for $\mathrm{Re}\, s \geq o$, $|z| \leq 1$ and $n = o,1,\dots$ $(i,j = 1,\dots, m)$

and

(94) $$\tilde{V}_{ij}\ (s,z,w) = \sum_{n=o}^{\infty} \tilde{U}_{ij}\ (s,z)\ w^n, \quad |w| < 1.$$

We have the following theorem:

Theorem 10

The matrix $\tilde{V}(s,z,w) = \left\{\tilde{V}_{ij}(s,z,w)\right\}$ is given by:

(95) $$V(s,z,w) = \left[z\ \tilde{U}^o\ (s,z) - w\ \frac{s+\lambda-\lambda z}{s+\lambda}\ \tilde{V}\ (s,o,w)\ \Psi(s+\lambda-\lambda z)\right]\left[zI-w\ \Psi(s+\lambda-\lambda z)\right]^{-1}$$

in the region $\operatorname{Re} s \geq o,\ |z| \leq 1,\ |w| < 1.$ $\tilde{U}^o(s,z) = \left\{\ \tilde{U}^o_{ij}\ (s,z)\right\}$

The matrix $\tilde{V}(s,o,w)$ is the unique solution, under the usual non-degeneracy conditions, of the following system of linear equations:

(96) $$\left[s+\lambda-\lambda\gamma_\rho(s,w)\right]\ \sum_{\nu=1}^{m}\tilde{V}_{i\nu}(s,o,w)\ \alpha_{\nu\rho}\left[s+\lambda-\lambda\ \gamma_\rho(s,w)\right] =$$

$$(s+\lambda)\ U^o_{ii}\left[s,\gamma_\rho(s,w)\right]\ \alpha_{i\rho}\left[s+\lambda-\lambda\ \gamma_\rho(s,w)\right], \qquad (i,\rho = 1,\ldots,\ m)$$

in which the functions $\gamma_\rho(s,w)$ are the $m$ solutions of the equation

$$\det\left[z\ I - w\ \Psi(s+\lambda-\lambda z)\right] = o$$

which lie in the open disk $|z| < 1.$

Proof:

We have:

(97) $$\xi_{n+1} = [\xi_n - 1]^+ + \nu_{n+1}$$

and $$\tau'_{n+1} = \tau'_n + X_{n+1} + \begin{cases} o & \text{if } \xi_n \geq 1 \\ \theta^*_n & \text{if } \xi_n = o \end{cases}$$

in which $\nu_{n+1}$ denotes the number of customers to arrive during the (n+1)st service and where $\theta^*_n = \tau_{n+1} - \tau'_n$ is a negative exponential random variable, independent of $X_{n+1}$, the service time of the (n+1)st customer.

Furthermore:

(98) $$P\left\{\nu_n = \alpha,\ J_n = j \mid X_n = x,\ J_{n-1} = i\right\} = e^{-\lambda x}\frac{(\lambda x)^\alpha}{\alpha!}$$

for $\alpha = o,1,\ldots$ and $i,j = 1,\ldots, m$.

It now follows from formulae (93) and (97) that:

(99) $$z\,\tilde{U}^{n+1}_{ij}(s,z) = \sum_{\nu=1}^{m}\tilde{U}^{n}_{i\nu}(s,z)\ \psi_{\nu j}(s+\lambda-\lambda z) - \sum_{\nu=1}^{m}\tilde{U}^{n}_{i\nu}(s,o)\ \psi_{\nu j}(s+\lambda-\lambda z)$$

$$+\lambda z(\lambda+s)^{-1}\sum_{\nu=1}^{m}\tilde{U}^{n}_{i\nu}(s,o)\ \psi_{\nu j}(s+\lambda-\lambda z)$$

We note that it follows from the definition of $\tilde{U}^{o}_{ij}$ that it vanishes for $i \neq j$. Summing in (99) after multiplication of both sides by $z^n$ we obtain:

$$(100) \quad z\,\tilde{V}_{ij}(s,z,w) = z\,\delta_{ij}\,U^{o}_{ii}(s,z) + w \sum_{\nu=1}^{m} \tilde{V}_{i\nu}(s,z,w)\,\psi_{\nu j}(s+\lambda-\lambda z)$$

$$- w \sum_{\nu=1}^{m} \tilde{V}_{i\nu}(s,o,w)\,\psi_{\nu j}(s+\lambda-\lambda z)$$

$$+ w\,\lambda(\lambda+s)^{-1} \sum_{\nu=1}^{m} \tilde{V}_{i\nu}(s,o,w)\,\psi_{\nu j}(s+\lambda-\lambda z)$$

and in matrix notation:

$$(101) \quad \tilde{V}(s,z,w)\left[z\,I - w\,\Psi(s+\lambda-\lambda z)\right] = z\,U^{o}(s,z) - w\,\frac{s+\lambda-\lambda z}{s+\lambda}\,\tilde{V}(s,o,w)\,\Psi(s+\lambda-\lambda z)$$

The matrix $z\,I - w\,\Psi(s+\lambda-\lambda z)$ is nonsingular for $\text{Re } s \geq o$, $|w| < 1$, $|z| \leq 1$ except for

$$z = \gamma_{\rho}(s,w), \qquad \rho = 1,\ldots, m$$

where the $\gamma_{\rho}(s,w)$ are the $m$ roots of the equation $\det\left[z\,I - w\,\Psi(s+\lambda-\lambda z)\right] = c$ in the open unit disk. Taking the inverse in equation (101) and after diagonalization of the matrix $\Psi(s+\lambda-\lambda z)$ we obtain:

$$(102) \quad \tilde{V}_{ij}(s,z,w) = \sum_{\rho=1}^{m} \frac{\beta_{\rho j}(s+\lambda-\lambda z)}{z - w\eta_{\rho}(s+\lambda-\lambda z)} \Big\{ z\,U^{o}_{ii}(s,z)\,\alpha_{i\rho}(s+\lambda-\lambda z)$$

$$- w\,\frac{s+\lambda-\lambda z}{s+\lambda} \sum_{\nu=1}^{m} \tilde{V}_{i\nu}(s,o,w)\,\alpha_{\nu\rho}(s+\lambda-\lambda z)\,\eta_{\rho}(s+\lambda-\lambda z) \Big\}$$

This leads to the following system of linear equations for the unknown functions $\widetilde{V}_{i\nu}(s,o,w)$, since the right hand side of (102) must be regular in $|z| < 1$, $\text{Re } s \geq o$, $|w| < 1$

$$(103)\qquad (s+\lambda)\, U_{ii}^{o}\left[s,\gamma_\rho(s,w)\right]\, \alpha_{i\rho}\left[s+\lambda-\lambda\, \gamma_\rho(s,w)\right] =$$

$$\left[s+\lambda-\lambda\, \gamma_\rho(s,w)\right]\, \sum_{\nu=1}^{m} \widetilde{V}_{i\nu}(s,o,w)\, \alpha_{\nu\rho}\left[s+\lambda-\lambda\, \gamma_\rho(s,w)\right],$$

for $i,\ \rho = 1,\ldots, m$.

It should be noted, as in all previous instances, that if $m = 1$ our formulae reduce to Takacs' results. [2] We now study the probabilities

$$(104)\qquad P(i,h;j,\ell;t) = P\left\{\xi(t) = \ell\ ,\ J_t^* = j \,|\, \xi(o) = h,\ J_o^* = i\right\}$$

We introduce the following notations:

$$(105)\qquad \Pi\,(i,h;j,\ell;s) = \int_o^\infty e^{-st}\, P(i,h;j,\ell;t)dt, \qquad \text{Re } s > o$$

$$(106)\qquad \widetilde{\Pi}_{ij}(s,z) = \sum_{\ell=o}^{\infty} z^{\ell}\, \Pi(i,h;\ j,\ell;\ s), \qquad \text{Re } s > o,\ |z| \leq 1$$

Under the assumption $\xi(o+) = h$, let us denote by $M_{ij}^{\ell}(t)$ $(i,j = 1,\ldots, m$ and $\ell = o,1,\ldots)$ the expected number of times the queue length increases from $\ell$ to $\ell + 1$ in $(o,t]$, whereby the type of the arriving customer is $j$, given that the type of the first customer was $i$. Let $N_{ij}^{\ell}(t)$ be the expected number of times the queue length decreases from $\ell + 1$ to $\ell$ in the interval $(o,t]$, whereby the next customer to be served is of type $j$, given that the first customer in the process was $i$.

We have, under the assumption that $\xi(o) = h$, that:

$$M^{o}_{ij}(t) = \lambda \int_{o}^{t} P(i,h;j,o;u)\, du \tag{107}$$

The matrix $m^{o}(s)$ of Laplace-Stieltjes transforms of $M^{o}_{ij}(t)$ is given by formula (71), where $\widetilde{E}(1,s)$ may be calculated using the fact that there is a departure at time $t = o$ and that there are $h$ customers waiting at time $o+$. It suffices therefore to express the generating functions in formula (106) in terms of the matrix $m^{o}(s)$ and other matrices which may be calculated.

We prove the following theorem:

Theorem 11

We have for $\operatorname{Re} s > o$ and $|z| \leq 1$ that:

$$\widetilde{\Pi}_{ij}(s,z) = \frac{1-h_j(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\left[\widetilde{V}_{ij}(s,z,1) + m^{o}_{ij}(s) + z^{h}\delta_{ij}\right] + \frac{1}{\lambda}\, m^{o}_{ij}(s) \tag{108}$$

for $h > o$ and

$$\widetilde{\Pi}_{ij}(s,z) = \frac{1-h_j(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\left[\widetilde{V}_{ij}(s,z,1) + m^{o}_{ij}(s)\right] + \frac{m^{o}_{ij}(s)}{\lambda} \tag{109}$$

for $h = o$, in which

$$h_j(s+\lambda-\lambda z) = \sum_{\nu=1}^{m} \psi_{j\nu}(s+\lambda-\lambda z) \tag{110}$$

and $\widetilde{V}_{ij}(s,z,1)$ is given by formulae (95) and (96).

Proof:

We have:

$$N_{ij}^{\ell}(t) = \sum_{n=1}^{\infty} P\left\{\tau_n' \leq t,\ J_n = j,\ \xi_n = \ell \mid J_o = i\right\}$$

so that by formula (93), (94), (102) and (103), it follows that:

$$\sum_{\alpha=o}^{\infty} z^{\alpha} \int_o^{\infty} e^{-su}\, d\, N_{ij}^{\alpha}(u) = \sum_{\alpha=o}^{\infty} \widehat{U}_{ij}^{n}(s,z) = \widetilde{V}_{ij}(s,z,1)$$

We now know $M_{ij}^{o}(t)$ and $N_{ij}^{\alpha}(t)$ for all $t \geq o$, $\alpha = o,1,\ldots$ and $i,j = 1,\ldots,m$
We now express the probabilities $P(i,h;j,\ell;t)$ as follows:

(111) $$P(i,h;j,o;t) = \delta_{ho}^{*}\delta_{ij}e^{-\lambda t} + \int_o^t e^{-\lambda(t-u)}\, d\, N_{ij}^{o}(u),$$

and

(112) $$P(i,h;j,\ell;t) = \delta_{h\ell}^{*}\,\delta_{ij}\left[1-H_j(t)\right]e^{-\lambda t}\,\frac{(\lambda t)^{\ell-h}}{(\ell-h)!}$$

$$+ \sum_{\alpha=1}^{\ell} \int_o^t \left[1-H_j(t-u)\right]e^{-\lambda(t-u)}\,\frac{\left[\lambda(t-u)\right]^{\ell-\alpha}}{(\ell-\alpha)!}\, d\, N_{ij}^{\alpha}(u)$$

$$+ \int_o^t \left[1-H_j(t-u)\right]e^{-\lambda(t-u)}\,\frac{\left[\lambda(t-u)\right]^{\ell-1}}{(\ell-1)!}\, d\, M_{ij}^{o}(u)$$

in which

$$H_j(t) = \sum_{\nu=1}^{m} Q_{j\nu}(t)$$

and $\delta^*_{oo} = 1$, $\delta^*_{h\ell} = 1$ if $h = 1,2,\ldots,\ell$ and zero in all other cases.

The probabilistic argument underlying (111) and (112) is the following: The event $\xi(t) = o$, $J^*_t = j$ can occur as follows: Either there are no customers in the queue at $t = o$ and no new customers arrive, or the customer of type $j$ arrived at some time $t-u$ between $o$ and $t$ and no new customers have arrived since.

The event $\xi(t) = \ell > o$, $J^*_t = j$ can occur as follows: Initially there are $h$ customers, the service of the customer of type $j$, whose service started at $o+$ is not yet finished at time $t$ and $\ell - h$ new customers have arrived, or the customer of type $j$ started service at time $t - u$ and there were $\alpha > o$ customers present at $t - u+$, while between $t - u$ and $t$, $\ell - \alpha$ new customers have arrived. Finally the third term is due to the case $\alpha = o$, where no customers were waiting at $t - u+$ and $\ell - 1$ new customers arrive in $(t - u,t]$.

Taking Laplace transforms in (112) we obtain:

$$(113) \qquad \tilde{\Pi}_{ij}(s,z) = [1-h_j(s+\lambda-\lambda z)]\ (s+\lambda-\lambda z)^{-1}\left\{\tilde{V}_{ij}(s,z,1) + m^o_{ij}(s)\right\}$$

$$+ \frac{1}{\lambda+s}\int_o^\infty e^{-su} d\, N^o_{ij}(u) + \begin{cases} \dfrac{1}{\lambda+s}\,\delta_{ij} & \text{for } h = o \\ z^h(s+\lambda-\lambda z)^{-1}\,[1-h_j(s+\lambda-\lambda z)]\ \delta_{ij} & \text{for } h \neq o. \end{cases}$$

Finally we may express the Laplace-Stieltjes transform of $N^{o}_{ij}(t)$ in terms of $m^{o}_{ij}(s)$, using formulae (107) and (111). We obtain:

$$(114) \qquad \int_{o}^{\infty} e^{-su}\, d\, N^{o}_{ij}(u) = \frac{\lambda+s}{\lambda}\; m^{o}_{ij}(s) - \delta_{ij}\, \delta^{*}_{ho}$$

Substituting in (113) this leads immediately to (108) and (109).

Remark:

The study of the limiting behavior of the probabilities $P(i,h;j,\ell;t)$ for $t \to \infty$ appears to be very difficult. This is tied in with the fact that the ''imbedded'' Markov chain $\{\xi_n, J_n\}$ may be reducible and may have periodic states. At best we may obtain theorems on the behavior of

$$\frac{1}{t} \int_{o}^{t} P(i,h;j,\ell,u)\, du$$

using Tauberian arguments. We will not pursue this subject further in this paper.

The expected number of departures in the interval $(o,t]$ is also readily obtained from the previous discussion. We prove the following theorem:

Theorem 12

Let $\widetilde{N}_{ij}(t)$ denote the expected number of customers of type $j$ departing from the queue in the time interval $(o,t]$, given that the first customer is of type $i$. We denote by $N^{*}_{ij}(s)$ the Laplace-Stieltjes transform of $N_{ij}(t)$ and the matrix $\{N^{*}_{ij}(s)\}$ by $N^{*}(s)$. Then:

$$(115) \qquad N^{*}(s) = \left[I - \frac{s}{s+\lambda}\, \widetilde{V}(s,o,1)\, \Psi(s)\right]\left[I - \Psi(s)\right]^{-1} - I.$$

for $\operatorname{Re} s > o$. The matrix $\tilde{V}(s,o,1)$ is the unique solution to the system of linear equations:

$$(116) \qquad [s+\lambda-\lambda\, \gamma_\rho(s,1)] \sum_{\nu=1}^{m} \tilde{V}_{i\nu}(s,o,1)\, \alpha_{\nu\rho}[s+\lambda-\lambda\, \gamma_\rho(s,1)]$$

$$= (s+\lambda)\, U^{o}_{ii}[s,\gamma_\rho(s,1)]\, \alpha_{i\rho}[s+\lambda-\lambda\, \gamma_\rho(s,1)],$$

for $i, \rho = 1,\ldots, m$.

Proof:

We have:

$$(117) \qquad N_{ij}(t) = \sum_{n=1}^{\infty} P\left\{\tau'_n \leq t,\ J_n = j \mid J_o = i\right\}$$

and

$$\tilde{V}_{ij}(s,z,w) = \sum_{n=o}^{\infty} w^n \sum_{\alpha=o}^{\infty} z^\alpha \int_o^\infty e^{-st}\, d\, P\left\{\tau'_n \leq t,\ \xi_n = \alpha,\ J_n = j \mid J_o = i\right\}$$

hence, since $\tau'_o = o$ we have:

$$\int_o^\infty e^{-st}\, d\, N_{ij}(t) = \tilde{V}_{ij}(s,1-,1-) - \delta_{ij},$$

Setting $z$ and $w$ equal to $1-$, in formulae (95) and (96) the result follows.

10. The Waiting-time of the nth customer.

We now study the sequence of pairs of random variables $\{(\eta_n, J_{n-1})\ n=1,\dots$ in which $\eta_n$ denotes the length of time the nth customer will have to wait before entering service and $J_{n-1}$ is the type of the nth customer. The random variables $\eta_n$ are related to the virtual waiting time process, studied in (I) by

$$\eta_n = \eta(\tau_n - o)$$

We will now obtain expressions for the probability distribution

$$P\{\eta_n \leq x,\ J_{n-1} = j \mid J_o = i\}$$

We introduce the Laplace transforms

(118) $$\Omega_{ij}^{n}(s) = \int_o^{\infty} e^{-sx}\, d\, P\{\eta_n \leq x,\ J_{n-1} = j \mid J_o = i\},$$

and the generating functions

(119) $$\tilde{\Omega}_{ij}(s,w) = \sum_{n=1}^{\infty} \Omega_{ij}^{n}(s)\, w^n,$$

The matrix $\tilde{\Omega}(s,w)$ has the entries $\tilde{\Omega}_{ij}(s,w)$ for $i,j = 1,\dots, m$. We now prove the following theorem:

Theorem 13

Under the non-degeneracy conditions, stated in (I), the matrix $\tilde{\Omega}(s,w)$ is given by: $(\text{Re } s \geq o, |w| < 1)$

$$(120) \qquad \tilde{\Omega}(s,w) = [(\lambda-s)w\,\Omega^1(s) - s\,Z(w)][(\lambda-s)\,I - \lambda\,w\,\Psi(s)]^{-1}$$

in which

$$\Omega^1(s) = \{\Omega^1_{ii}(s)\,\delta_{ij}\} \quad \text{and} \quad Z_{ij}(w) = \sum_{n=1}^{\infty} P\{\eta_{n+1} = o,\ J_n = j \mid J_o = i\}\,w^{n+1}$$

The matrix $Z(w)$ is obtained as the unique solution of the following system of linear equations:

$$(121) \qquad \sum_{\nu=1}^{m} Z_{i\nu}(w)\,\alpha_{\nu\rho}[\lambda-\lambda\,\gamma_\rho(o+,w)]$$

$$= \gamma_\rho(o+,w)\,[1 - \gamma_\rho(o+,w)]^{-1} w\,\Omega^1_{ii}[\lambda-\lambda\,\gamma_\rho(o+,w)]\alpha_{i\rho}[\lambda-\lambda\gamma_\rho(o+,w)]$$

for $i,\rho = 1,\ldots, m$. The functions $\gamma_\rho(o+,w)$, $\rho = 1,\ldots, m$ are the roots in the open unit disk of the equation:

$$\det [z\,I - w\,\Psi(\lambda-\lambda z)] = o,$$

Proof:

We have the usual relation between the successive waiting times:

$$(122) \qquad \eta_{n+1} = [\eta_n + X_n - \theta_n]^+$$

in which $X_n$ is the service time of the nth customer and $\theta_n = \tau_{n+1} - \tau_n$. It follows from the independence assumptions that:

$$(123) \qquad (\lambda-s)\, \Omega^{n+1}_{ij}(s) = \lambda \sum_{\nu=1}^{m} \Omega^{n}_{i\nu}(s)\, \psi_{\nu j}(s) - s\, P\left\{\eta_{n+1} = o,\ J_n = j \mid J_o = i\right\}$$

for $i,j = 1,\ldots, m$.

In matrix notation we obtain:

$$(124) \qquad \widetilde{\Omega}(s,w)\left[(\lambda-s)\, I - \lambda\, w\, \Psi(s)\right] = (\lambda-s)\, w\, \Omega^{1}(s) - s\, Z\,(w)$$

In view of the non-degeneracy assumptions the matrix $\Psi(s)$ can be written in spectral form:

$$\Psi(s) = R(s)\ \ H\ (s)\ R^{-1}(s)$$

in which $R_{ij} = \alpha_{ij}(s)$.

We obtain:

$$(125) \qquad \widetilde{\Omega}_{ij}(s,w) = \sum_{\nu=1}^{m}\left[(\lambda-s)\, w\, \delta_{i\nu}\Omega^{1}_{ii}(s) - s\, Z_{i\nu}(w)\right] \sum_{\rho=1}^{m} \frac{\alpha_{\nu\rho}(s)\beta_{\rho j}(s)}{\lambda-s-\lambda w\eta_\rho(s)}$$

The denominators $\lambda-s-\lambda w\eta_\rho(s)$ have unique zeros in the domain $\mathrm{Re}\ s \geq o$, $|w| < 1$ given by

$$s = \lambda - \lambda\ \gamma_\rho(o+,w)$$

It follows from the fact that the left hand sides of (125) are analytic in this domain, that the unknown functions $Z_{i\nu}(w)$ must satisfy the equation (121). This concludes the proof of the theorem.

The following theorem states the main result on the existence of a limiting distribution of the waiting time.

<u>Theorem 14</u>

If $\lambda\eta_m'(o+) \leq -1$, then $\lim_{n \to \infty} P\{\eta_n \leq x, J_{n-1} = j | J_o = i\} = o$ for all $x \geq o$.

If $\lambda\eta_m'(o+) > -1$, then:

$$\lim_{n \to \infty} P\{\eta_{r+nd} \leq x, J_{r+nd-1} = j | J_o = i\} = W_j^o(x)$$

exists. The integer $r$ is equal to the minimum number of transitions required to go from $i$ to $j$ with some positive probability in the semi-Markov process of successive service times. The number $d$ is the period of this same semi-Markov process.

The Laplace-Stieltjes transforms of the $W_j^o(x)$ are given by:

$$\Omega^o(s) = s[(s-\lambda)\ I + \lambda\ \Psi(s)]^{-1}\ P^* \tag{126}$$

and the vector $P^*$ is given by the system of equations.

$$\sum_{\nu=1}^{m} \alpha_{\nu\rho}[\lambda-\lambda\ \gamma_\rho(o+,1)]\ P_\nu^* = o \qquad \text{for} \quad \rho \neq m \tag{127}$$

$$\sum_{\nu=1}^{m} P_\nu^* = 1 + \lambda\ \eta_m'(o+) \qquad \text{for} \quad \rho = m.$$

These limiting distributions are the same as those found in (I) for the virtual waiting time process.

Proof:

The statement concerning the existence of the limiting distribution is a consequence of a theorem of M.F. Neuts and M. Tata [1] which extends Lindley's theorem.

The expressions given in formulae (126) and (127) follow from formulae (120) and (121) upon application of Abel's theorem

$$\Omega_j^o(s) = \lim_{w \to 1} (1-w)\, \tilde{\Omega}_{ij}(s;w)$$

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