Prob.

The Single Server Queue with Poisson Input and semi-Markov Service Times III

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Summary

In (I) and (II) we considered a queue with Poisson input and semi-Markov service times and studied its virtual waiting time process, its busy periods, the queue length and the output process.

In this paper we apply some of these results to a queue in which the successive customer types vary cyclically. A particular case of this is the queue with bulk service and fixed bulk size. We show how the equations simplify greatly in the latter case and we obtain slightly more detailed results for this queue, than the ones that were previously known.

We refer to (I) and (II) for definitions and notations and we will number the equations continuously in the three papers.

11. The Queue with Cyclic Customer Types

An important example of a queue with semi-Markov service times is the queue in which there are m types of customers, arriving in cyclic order. The matrix of transition probability distributions $Q(x)$ takes on the following special form:

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$$Q(x) = \begin{pmatrix} 0 & Q_1(x) & 0 & ,\ldots\ldots & 0 \\ 0 & 0 & Q_2(x) & \ldots\ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots\ldots & Q_{m-1}(x) \\ Q_m(x) & 0 & 0 & \ldots\ldots & 0 \end{pmatrix} \cdot$$

in which $Q_j(x)$, $j = 1,\ldots, m$ denotes the service time distribution of the j-th type of customer.

The bulk queue

We obtain a queueing process whose relevant features are identical to those of the queue in which the customers are served in groups of size m only, by setting $Q_1(x),\ldots,Q_{m-1}(x)$ equal to a distribution degenerate at zero and $Q_m(x)$ equal to $Q(x)$, the service time distribution of the groups.

This can be done as follows. We define the customer type of the n-th customer as the residue class, modulo m in which his order of arrival n, falls. We label $\bar{0}$, $\bar{1}$, $\bar{2},\ldots,\overline{m-1}$ as $m,1,2,\ldots,m-1$ and assign a service time equal to zero to all customers who are not of type m. The customers of type m are served during a length of time equal to that of a group of size m.

The random variable J_t, defined in (I) as the type of the last customer to arrive before time t is here also equal to the queue-length, modulo m. This random variable was introduced earlier by Takács [1] to make the process of the virtual waiting time Markovian.

We now recall from (I) and (II) that the characteristics of a semi-Markov queue may be expressed in terms of the roots of the determinantal equation (6)

$$\det\left[z\ I - w\ \Psi(s+\lambda-\lambda z)\right] = 0$$

which lie in the unit circle $|z| \leq 1$ and the eigenvectors of the matrix $\Psi(s+\lambda-\lambda z)$.

We now calculate these roots and the eigenvectors. Let $\psi_j(s)$ be the L.S. transform of $Q_j(x)$; $j = 1,\ldots,m$ then equation (6) can be written as

$$z^m = w^m \prod_{\nu=1}^{m} \psi_\nu(s+\lambda-\lambda z) \tag{128}$$

In order that we may consider the m-th roots of (128) and still be dealing with analytic functions, we must impose the weak, but somewhat intractable, non-degeneracy condition that none of the functions $\psi_j(s)$, $j = 1,\ldots,m$ vanishes in the right half-plane $\mathrm{Re}\ s > 0$.

This is satisfied for a large class of distribution functions. If it is not satisfied we may make a careful investigation of the Riemann surface for

$$\left[\prod_{\nu=1}^{m} \psi_\nu(s+\lambda-\lambda z)\right]^{1/m} \tag{129}$$

and show that it is possible to choose the cuts in such a way that the end-results of our calculations, such as the distributions of the busy periods have transforms, analytic in the open right half-plane. However, this is very difficult to do in general and hence the non-degeneracy conditions, which appear in (I) and (II). As before we will assume throughout this paper that the root (129) takes on m distinct values, defined as analytic functions of $s+\lambda-\lambda z$ in the region $\mathrm{Re}\ s > 0$, $|z| \leq 1$.

We then write the m equations

$$z = we^{\frac{2\Pi i\rho}{m}} \eta_m(s+\lambda-\lambda z), \quad \rho = 1,\ldots,m \tag{130}$$

in which $\eta_m(s+\lambda-\lambda z)$ is the principle value of (129). Let $\gamma_\rho(s,w)$ be the root with smallest absolute value of equation (130). We note that we have

$$\gamma_\rho(s,w) = \gamma_m\left[s, w\, e^{\frac{2\Pi i\rho}{m}}\right] \quad \text{for} \quad \rho = 1,\ldots,m. \tag{131}$$

so that it suffices to solve the equation (130) corresponding to $\rho = m$.

Next we calculate the left and right eigenvectors of the matrix $\Psi(s)$ corresponding to the m eigenvalues $e^{2\Pi i\rho/m}\eta_m(s)$. We obtain for the right eigenvectors $\bar{\alpha}_\rho(s)$:

$$\begin{aligned} \psi_m(s)\,\alpha_{1\rho}(s) &= e^{\frac{2\Pi i\rho}{m}}\eta_m(s)\,\alpha_{m\rho}(s) \\ \psi_1(s)\,\alpha_{2\rho}(s) &= e^{\frac{2\Pi i\rho}{m}}\eta_m(s)\,\alpha_{1\rho}(s) \\ &\vdots \\ \psi_{m-1}(s)\alpha_{m\rho}(s) &= e^{\frac{2\Pi i\rho}{m}}\eta_m(s)\alpha_{m-1\rho}(s) \end{aligned} \tag{132}$$

which yields if we set $\alpha_{m\rho}(s) \equiv 1$:

$$\alpha_{k\rho}(s) = \frac{\left[e^{\frac{2\Pi i\rho}{m}}\eta_m(s)\right]^k}{\psi_m(s)\psi_1(s)\ldots\psi_{k-1}(s)} \qquad \begin{aligned} k &= 1,\ldots,m \\ \rho &= 1,\ldots,m. \end{aligned} \tag{133}$$

We note that this normalization is a desirable one, since

$$\alpha_{km}(0+) = 1$$

Solving the corresponding equations for the left hand eigenvectors we find the row-vectors $\bar{\beta}_\rho(s)$ with

$$\beta_\rho = \frac{1}{m}\left[e^{\frac{2\Pi i\rho}{m}} \eta_m(s)\right]^{-\ell} \psi_1(s) \ldots \psi_{\ell-1}(s)\, \psi_m(s) \tag{134}$$

for $\rho = 1,\ldots,m$; $\ell = 1,\ldots,m$.

It is easy to verify that the matrix with the $\bar{\alpha}_\rho$ as columns and the matrix with the $\bar{\beta}_\rho$ as rows are each others inverse. The normalization used in $\bar{\beta}_\rho$ is again the proper one, since $\beta_{m\ell}(0+) = 1/m$.

12. The successive busy periods

We first calculate the critical quantity $1 + \lambda\, \eta_m'(0+)$ whose sign determines the stability or instability of the queue, as shown in (I). We obtain:

$$1 + \lambda\, \eta_m'(0+) = 1 - \lambda/m \sum_{j=1}^{m} \alpha_j \tag{135}$$

where α_j is the mean service time of the j-th customer.

We conclude that the queue will be transient if and only if $\frac{1}{\lambda}$ is less than $\frac{1}{m} \sum_{j=1}^{m} \alpha_j$, unstable when equality holds and transient when the reversed inequality holds.

Let now $E_{ij}(w,s)$ be the generating function - Laplace transform of the joint distribution of the number of customers served in a busy period of length not exceeding x - and such that the type of the first customer of the busy period is i and the type of the first customer of the next busy period is j. We have shown in (I) that the transforms $E_{ij}(w,s)$ are obtained as the solutions to a system of linear equations (30) given by:

$$(136) \qquad \sum_{\nu=1}^{m} E_{i\nu}(w,s)\ \alpha_{\nu\rho}\left[s+\lambda-\lambda\gamma_{\rho}(s,w)\right] =$$

$$\gamma_{\rho}(s,w)\ \alpha_{i\rho}\left[s+\lambda-\lambda\gamma_{\rho}(s,w)\right], \qquad \rho,\ i = 1,\dots,\ m.$$

If we substitute the expressions found in (133) and keep account of equation (130) we find:

$$(137) \qquad \sum_{\nu=1}^{m} E_{i\nu}(w,s)\ w^{-\nu}\ \gamma_{\rho}^{\nu}(s,w)\ \psi_{1}^{-1}\left[s+\lambda-\lambda\gamma_{\rho}(s,w)\right]\dots\ \psi_{\nu-1}^{-1}\left[s+\lambda-\lambda\gamma_{\rho}(s,w)\right] =$$

$$\gamma_{\rho}^{i+1}\ (s,w)\ w^{-i}\ \psi_{1}^{-1}\left[s+\lambda-\lambda\gamma_{\rho}\right]\dots\ \psi_{i-1}^{-1}\left[s+\lambda-\lambda\gamma_{\rho}\right]$$

In the case of the bulk queue these equations simplify further, since $\psi_1(s) = \dots = \psi_{m-1}(s) = 1$ and $\psi_m(s) = \psi(s)$. We obtain:

$$(132) \qquad \sum_{\nu=1}^{m} E_{i\nu}(w,s)\ w^{-\nu}\ \gamma_{\rho}^{\nu}\ (s,w) = w^{-i}\ \gamma_{\rho}^{i+1}\ (s,w)$$

For $i = 1,\dots,m-1$ we obtain the expected result:

$$E_{i,i+1}(w,s) = w \qquad E_{ij}(w,s) = 0 \qquad \text{for} \quad j \neq i + 1$$

This corresponds to the zero service length of customers other than those of type m.

For $i = m$ we have:

$$(139) \qquad \sum_{\nu=1}^{m} E_{m\nu}(w,s)\, w^{-\nu}\, \gamma_{\rho}^{\nu-1}(s,w) = w^{-m}\, \gamma_{\rho}^{m}(s,w)$$

The solution to this system may be written symbolically

$$(140) \qquad E_{mk}(w,s) = w^{k-m} \; \frac{\| \, 1 \;\; \gamma_{\rho} \; \cdots\cdots \; \overset{k}{\overset{\downarrow}{\gamma_{\rho}^{km}}} \; \cdots \; \gamma_{\rho}^{m-1} \, \|}{\| \, 1 \;\; \gamma_{\rho} \; \cdots\cdots \; \gamma_{\rho}^{k} \; \cdots \; \gamma_{\rho}^{m-1} \, \|}$$

By a direct argument Takács [1] has found an expression for the sum on k of $E_{mk}(w,s)$.

Analogous expressions may be obtained for the initial busy period.

13. The Virtual Waiting time

Following the results of (I), we first calculate the probabilities of idleness. We substitute the expressions (133) into the equation (51). The transforms $W^{*}_{ij}(s)$ are given by the following equations:

$$(14)\qquad \sum_{\nu=1}^{m} W^*_{i\nu}(s)\ \gamma_\rho^{\nu-1}(s,1)\ \psi_1^{-1}\left[\zeta_\rho(s)\right] \cdots \psi_{\nu-1}^{-1}\left[\zeta_\rho(s)\right] =$$

$$\Omega_i\left[\zeta_\rho(s)\right]\ \zeta_\rho^{-1}(s)\ \gamma_\rho^{i-1}(s,1)\ \psi_1^{-1}\left[\zeta_\rho(s)\right] \cdots \psi_{i-1}^{-1}\left[\zeta_\rho(s)\right],$$

in which

$$\zeta_\rho(s) = s + \lambda - \lambda\gamma_\rho(s,1)$$

In the bulk queue we should take the initial virtual waiting time to be the total service time, that remains for the complete batches that are in the system at $t = 0$.

The equations (141) simplify further to:

$$(142)\qquad \sum_{\nu=1}^{m} W^*_{i\nu}(s)\ \gamma_\rho^{\nu-1}(s,1) = \Omega_i\left[\zeta_\rho(s)\right]\ \zeta_\rho^{-1}(s)\ \gamma_\rho^{i-1}(s)$$

and the calculation of the $W^*_{i\nu}(s)$ again involves the inversion of a Van der Monde matrix in terms of the roots $\gamma_\rho(s,1)$ $\rho = 1,\ldots,m$.

The system of differential equations for the transforms $\Omega_{ij}(t,s)$ is also greatly simplified in the case of a cyclic queue. We find, from equation (60):

$$(143)\qquad \frac{\partial}{\partial t}\ \Omega_{ij}(t,s) =$$

$$(s-\lambda)\ \Omega_{ij}(t,s) + \lambda\ \Omega_{i,j-1}(t,s)\ \psi_{j-1,j}(s) - s\ W_{ij}(t,0)$$

for $j = 2,\ldots, m$.

and

$$\frac{\partial}{\partial t}\ \Omega_{il}(t,s) = (s-\lambda)\Omega_{il}(t,s) + \lambda\Omega_{im}(t,s)\ \psi_{ml}(s) - s\ W_{il}(t,0)$$

Concluding Remark

Further results on the cyclic queue and the special case of bulk service may be obtained by substituting the expressions for the roots $\gamma_\rho(s,w)$ and for the eigenvectors of the matrix ψ. In the case of the bulk queue these lead to a direct derivation of Takács' results.

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A more complete bibliography is given at the end of (I).