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Moments of Randomly Stopped Sums

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1. Introduction. Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $x_1, x_2, \ldots$ be a sequence of random variables on Ω, and let $\mathcal{F}_n$ be the σ-algebra generated by $x_1, \ldots, x_n$, with $\mathcal{F}_0 = (\phi, \Omega)$. A stopping variable (of the sequence $x_1, x_2, \ldots$) is a random variable t on Ω with positive integer values such that the event $[t=n] \in \mathcal{F}_n$ for every $n \geq 1$. Let $S_n = \sum_1^n x_i$; then

$$S_t = S_{t(\omega)}(\omega) = \sum_1^t x_i$$

is a randomly stopped sum. We shall always assume that

(1) $$E|x_n| < \infty, \qquad E(x_{n+1} | \mathcal{F}_n) = 0, \qquad (n \geq 1) .$$

The moments of S_t have been investigated since the advent of Sequential Analysis, beginning with Wald [9], whose theorem states that for independent, identically distributed (i.i.d.) x_i with $Ex_i = 0$, $Et < \infty$ implies that $ES_t = 0$. For higher moments of S_t, the known results [1,3,4,5,10] are not entirely satisfactory. We shall obtain theorems for ES_t^r $(r = 2,3,4)$; the case $r = 2$ is of special interest in applications. In the case of i.i.d. x_i with $Ex_i = 0$ and $Ex_i^2 = \sigma^2 < \infty$, we shall show that $Et < \infty$ implies $ES_t^2 = \sigma^2 Et$.

2. The second moment. It follows from assumption (1) that $(S_n, \mathcal{F}_n; n \geq 1)$ is a martingale; i.e., that

(2) $$E|S_n| < \infty, \qquad E(S_{n+1} | \mathcal{F}_n) = S_n \qquad (n \geq 1).$$

The following well-known fact [3, p. 302] will be stated as

Lemma 1. Let $(S_n, \mathcal{F}_n;\ n \geq 1)$ be a martingale and let t be any stopping variable such that

$$(3) \qquad E|S_t| < \infty, \qquad \liminf \int_{[t > n]} |S_n| = 0;$$

then

$$(4) \qquad E(S_t \mid \mathcal{F}_n) = S_n \text{ if } t \geq n \qquad (n \geq 1)$$

and hence $\qquad ES_t = ES_1$.

Lemma 2. If $E \sum_1^t |x_i| < \infty$, then (3) holds.

Proof. $|S_t| \leq \sum_1^t |x_i|$, so that $E|S_t| < \infty$, and

$$\lim \int_{[t > n]} |S_n| \leq \lim \int_{[t > n]} \sum_1^t |x_i| = 0.$$

In this section we shall suppose, in addition to (1) that

$$(5) \qquad Ex_n^2 < \infty \qquad (n \geq 1)$$

and we define for $n \geq 1$

$$(6) \qquad Z_n = S_n^2 - \sum_1^n x_i^2 .$$

The sequence $(Z_n, \mathcal{F}_n;\ n \geq 1)$ is also a martingale, with $EZ_1 = 0$.

For any stopping variable t, let $t(n) = \min(n, t)$; then Lemma 1 applies to Z_n and $t(n)$, so that $EZ_{t(n)} = 0$, and hence

$$(7) \qquad ES_{t(n)}^2 = E \sum_1^{t(n)} x_i^2 .$$

Letting $n \to \infty$ we have a.e. $S_{t(n)}^2 \to S_t^2$ and $\sum_1^{t(n)} x_i^2 \uparrow \sum_1^t x_i^2$.

Hence, by Fatou's lemma and (7),

(8) $$ES_t^2 \leq \lim ES_{t(n)}^2 = \lim E \sum_1^{t(n)} x_i^2 = E \sum_1^t x_i^2 .$$

The question now arises under what circumstances equality holds in (8). (By Lemma 1 this will be the case if (3) holds with S replaced by Z, but, as we shall see, this requirement is unnecessarily stringent.) According to (8), we need only consider the case in which $ES_t^2 < \infty$, and it will suffice to prove that

(9) $$ES_t^2 \geq ES_{t(n)}^2 \qquad (n \geq 1) .$$

Lemma 3. If

(10) $$\lim\inf \int_{[t > n]} |S_n| = 0,$$

then $ES_t^2 = E \sum_1^t x_i^2$.

Proof. We may suppose that $ES_t^2 < \infty$ whence, by (10) and Lemma 1, (4) holds. Hence

$$ES_t^2 = \int_{[t \leq n]} S_t^2 + \int_{[t > n]} (S_n + (S_t - S_n))^2$$

$$\geq \int_{[t \leq n]} S_t^2 + \int_{[t > n]} S_n^2 + 2\int_{[t > n]} S_n E(S_t - S_n | \mathcal{F}_n) = ES_{t(n)}^2 .$$

Lemma 4. If

(11) $$\lim\inf \int_{[t > n]} S_n^2 < \infty,$$

then (10) holds.

Proof. Suppose (10) does not hold; then

$$\lim\inf \int_{[t > n]} |S_n| = \epsilon > 0.$$

Hence for any constant $0 < a < \infty$,

$$\liminf \int_{[t > n]} S_n^2 \geq a \liminf \int_{[t > n, |S_n| > a]} |S_n| = a\epsilon$$

which contradicts (11), since a may be arbitrarily large.

Lemma 5. If $E \sum_1^t x_i^2 < \infty$, then (11) holds.

Proof. Setting $S_0 = 0$ we have

$$\int_{[t > n]} S_n^2 = \sum_{i=1}^{n} \left(\int_{[t > i]} S_i^2 - \int_{[t > i-1]} S_{i-1}^2\right)$$

$$\leq \sum_{i=1}^{n} \int_{[t \geq i]} (S_i^2 - S_{i-1}^2) \leq \sum_1^{\infty} \int_{[t \geq i]} x_i^2 = E \sum_1^t x_i^2 < \infty .$$

From Lemmas 1-5 we have

Theorem 1. Let $(S_n, \mathcal{F}_n; n \geq 1)$ be a martingale with $ES_n^2 < \infty$ and let t be any stopping variable. Set $x_1 = S_1$, $x_{n+1} = S_{n+1} - S_n$. Then

(12) $$ES_t^2 \leq E \sum_1^t x_i^2 .$$

If any one of the four conditions

(13) $$\liminf \int_{[t > n]} |S_n| = 0, \ \liminf \int_{[t > n]} S_n^2 < \infty, \ E \sum_1^t |x_i| < \infty, \ E\sum_i x_i^2 < \infty$$

holds, then

(14) $$ES_t^2 = E \sum_1^t x_i^2 .$$

If $E \sum_1^t x_i^2 < \infty$, then (3) and (4) hold.

Theorem 1 generalizes (a) and (b) of Theorem II of [1]. In order to apply it, we first verify

Lemma 6. For any stopping variable t and any $r > 0$,

$$E \sum_1^t |x_i|^r = E \sum_1^t E(|x_i|^r | \mathcal{F}_{i-1}) .$$

Proof. $E\sum_1^t |x_i|^r = \sum_{j=1}^\infty \int_{[t=j]} \sum_{i=1}^j |x_i|^r = \sum_{i=1}^\infty \int_{[t \geq i]} |x_i|^r$

$$= \sum_{i=1}^\infty \int_{[t \geq i]} E(|x_i|^r | \mathcal{F}_{i-1}) = E \sum_1^t E(|x_i|^r | \mathcal{F}_{i-1}) .$$

For independent x_n, we have from Theorem 1 and Lemma 6

Theorem 2. Let $x_1, x_2, \ldots$ be independent with $Ex_n = 0$, $E|x_n| = a_n$, $Ex_n^2 = \sigma_n^2 < \infty$ $(n \geq 1)$ and let $S_n = \sum_1^n x_i$. Then either of the two relations

(15) $$E \sum_1^t a_i < \infty, \qquad E \sum_1^t \sigma_n^2 < \infty$$

implies

(16) $$ES_t^2 = E \sum_1^t x_i^2 = E \sum_1^t \sigma_n^2 .$$

If $\sigma_n^2 = \sigma^2 < \infty$, then $Et < \infty$ implies

(17) $$ES_t^2 = E \sum_1^t x_i^2 = \sigma^2 \, Et.$$

Some stronger sufficient conditions for (16) have been given in [10,1,5,3 (p. 351), 4].

Corollary 1. Let $x_1, x_2, \ldots$ be independent with $Ex_n = 0$, $Ex_n^2 = 1$, and define

t^*(resp. t_*) = $1^{\underline{st}}$ $n \geq 1$ such that $|S_n| > n^{1/2}$(resp. $<$)(= ∞ otherwise).

Then $Et^* = Et_* = \infty$.

Proof. If $Et^* < \infty$, then t^* is a genuine stopping variable (i.e., $P(t^* < \infty) = 1$ and by the definition of t^* and (17),

$$Et^* = ES_{t^*}^2 > Et^* ,$$

a contradiction; similarly for t_* .

We note that t^* is a genuine stopping variable if the law of the iterated logarithm holds for $x_1, x_2, \ldots$.)

The example $P[x_n=1] = P[x_n= -1] = 1/2$ shows that the $>$ ($<$) cannot be replaced by $\geq$ ($\leq$), since $Ex_n = 0$, $Ex_n^2 = 1$, and $t^* = t_* = 1$. On the other hand, if t^* is redefined as the first $n > 1$ for which $|S_n| \geq n^{1/2}$, Et^* is again infinite; similarly for t_*.

Corollary 1 is a generalization of Theorem 1 of [2]. The following corollary generalizes Theorem 2 of [2].

Corollary 2. Let $x_1, x_2, \ldots$ be independent with $Ex_n = 0$, $Ex_n^2 = 1$, $P[|x_n| \leq a < \infty] = 1$. For $0 < c < 1$ and $m = 1, 2, \ldots,$ define

$$t = \text{first } n \geq m \text{ such that } |S_n| > c\, n^{1/2}.$$

Then $Et < \infty$.

Proof. For $k = m, m+1, \ldots,$ put $t' = \min(t,k)$. Then t' is a stopping variable and by Theorem 2,

$$\int_{[t \leq k]} t + kP[t > k] = Et' = ES_{t'}^2 \leq \int_{[t > k]} S_k^2 + \int_{[t \leq k]} (ct^{1/2} + a)^2$$

$$\leq c^2 kP[t > k] + c^2 \int_{[t \leq k]} t + 2ac\Big(\int_{[t \leq k]} t^{1/2}\Big) + a^2 .$$

Hence

$$(1-c^2)\Big(kP[t > k] + \int_{[t \leq k]} t\Big) \leq 2ac\Big(\int_{[t \leq k]} t^{1/2}\Big) + a^2 .$$

Therefore as $k \to \infty$, $\int_{[t \leq k]} t = O(1)$ and $P[t > k] = O(k^{-1}) = o(1)$, so that t is a genuine stopping variable and $Et < \infty$.

Corollary 3. If $x_1, x_2, \ldots,$ are i.i.d. with $Ex_n = 0$, $Ex_n^2 = \sigma^2$, $P[|x_n| \leq a < \infty] = 1$, and if $ES_t^2 < \infty$ for a stopping variable t, then $Et < \infty$ if and only if

(18) $$\liminf nP[t > n] = 0.$$

Proof. The ''only if'' part is obvious. Now suppose (18) holds. Then since

$$\int_{[t > n]} |S_n| \leq anP[t > n] ,$$

the first condition of (13) holds and hence $\sigma^2 Et = ES_t^2 < \infty$, so that $Et < \infty$ if $\sigma^2 > 0$. (If $\sigma^2 = 0$, then $P[x_n = 0] = 1$ and hence t is equal a.e. to a fixed positive integer, so $Et < \infty$ in this case too.)

Applied to the case $P[x_i = 1] = P[x_i = -1] = 1/2$, with t = first $n \geq 1$ such that $S_t = 1$, we have by Wald's theorem $Et = \infty$, but by Corollary 3 the stronger result $\liminf nP[t > n] > 0$.

Corollary 4. Let $(x_n, n \geq 1)$ satisfy $E(x_{n+1} | \mathcal{F}_n) = 0$ and let $E(x_{n+1}^2 | \mathcal{F}_n) = \sigma_{n+1}^2 < \infty$ be constant for $n \geq 1$. Then for $\epsilon > 0$,

$$P[\max_{n \leq m} |S_n| \geq \epsilon] \leq \epsilon^{-2} \sum_1^m \sigma_n^2 .$$

If moreover $\sup_{n \geq 1} |x_n| = z$ with $Ez < \infty$, then

$$P[\max_{n \leq m} |S_n| \geq \epsilon] \geq 1 - [E(\epsilon+z)^2 \Big/ \sum_1^m \sigma_n^2] . \tag{19}$$

Proof. Define t = first $n \geq 1$ such that $|S_n| \geq \epsilon$. Then $t' = \min(t,m)$ is a bounded stopping variable. Hence, by (14) and Lemma 6,

$$\epsilon^2 P[\max_{n \leq m} |S_n| \geq \epsilon] = \epsilon^2 P[t \leq m] \leq ES_{t'}^2 = E \sum_1^{t'} \sigma_n^2 \leq \sum_1^m \sigma_n^2 .$$

If $Ez < \infty$, then

$$E(\epsilon+z)^2 \geq ES_{t'}^2 = E \sum_1^{t'} \sigma_n^2 \geq \sum_{k=1}^m \int_{[t=k]} \sum_{j=1}^k \sigma_j^2 = \sum_{j=1}^m \sigma_j^2 P[t \geq j]$$

$$\geq \left(\sum_{j=1}^m \sigma_j^2 \right) P[t \geq m]$$

and (19) holds.

The first part of Corollary 4 is a special case of submartingale inequalities [6, p. 391], and the second part generalizes slightly one of the Kolmogorov inequalities [6, p. 235] which requires that z be constant.

3. The Fourth Moment.

The analysis in the case of the fourth moment of S_t is somewhat easier than that of the third moment and consequently is presented first. In this section Ex_n^4 will be supposed finite. Define for r = 1,2,3,4, and n = 1,2, ...

$$u_{r,n} = E(x_n^r \mid \mathcal{F}_{n-1}) , \qquad U_{r,n} = \sum_1^n u_{r,j} ,$$

(20) $$v_{r,n} = E(|x_n|^r \mid \mathcal{F}_{n-1}) , \qquad V_{r,n} = \sum_1^n v_{r,j} ,$$

$$T_{r,n} = \sum_1^n |x_j|^r , \qquad T_{1,n} = T_n .$$

In these terms, Lemma 6 asserts that $ET_{r,t} = EV_{r,t}$.

Lemma 7. If $ES_t^2 < \infty$ and $\liminf \int_{\{t > n\}} |S_n| = 0$, then

$$E(S_t^2 \mid \mathcal{F}_n) \geq S_n^2 \text{ and } E(|S_t| \mid \mathcal{F}_n) \geq |S_n| \text{ for } t > n.$$

Proof. For any $A \in \mathcal{F}_n$, by Lemma 1

$$\int_{A[t > n]} S_t^2 = \int_{A[t > n]} [S_n^2 + 2S_n(S_t - S_n) + (S_t - S_n)^2] \geq \int_{A[t > n]} S_n^2 .$$

Hence the first inequality of the lemma holds, and the second inequality follows immediately from Lemma 1 and the fact that $E(|S_t| \mid \mathcal{F}_n) \geq |E(S_t \mid \mathcal{F}_n)|$.

Theorem 3. If t is a stopping variable such that $E[t \sum_1^t E(x_j^4 \mid \mathcal{F}_{j-1})] < \infty$, then $ES_t^4 < \infty$ and

(21) $$ES_t^4 = EU_{4,t} + 4ES_tU_{3,t} + 6ES_t^2U_{2,t} - 6E\sum_1^t u_{2,j}U_{2,j} .$$

Proof. Set $Y_n = S_n^4 - 6S_n^2U_{2,n} - 4S_nU_{3,n} - U_{4,n} + 6\sum_{j=1}^n u_{2,j}U_{2,j}$

and $t' = \min(t,k)$. Since $\{Y_n, \mathcal{F}_n; n \geq 1\}$ is a martingale with $EY_1 = 0$, by Lemma 1,

$$ES^4_{t'} = 6ES^2_{t'}U_{2,t'} + 4ES_{t'}U_{3,t'} + EU_{4,t'} - 6E\left(\sum_{j=1}^{t'} u_{2,j}U_{2,j}\right)$$

$$\leq 6(E^{1/2}S^4_{t'})(E^{1/2}U^2_{2,t'}) + 4(E^{1/4}S^4_{t'})(E^{3/4}V^{4/3}_{3,t'}) + EU_{4,t'} ,$$

whence, if $ES^4_{t'} > 0$,

(22) $$E^{1/2}S^4_{t'} \leq 6E^{1/2}U^2_{2,t'} + 4(E^{3/4}V^{4/3}_{3,t'})(ES^4_{t'})^{-1/4} + (EU_{4,t'})(ES^4_{t'})^{-1/2}.$$

Now if $p > 1$, $r > 0$,

(23) $$V_{r,n} = \sum_{j=1}^{n} E\{|y_j|^r | \mathcal{F}_{j-1}\} \leq n^{\frac{p-1}{p}} \left(\sum_{j=1}^{n} E^p\{|y_j|^r| \mathcal{F}_{j-1}\}\right)^{1/p}$$
$$\leq n^{\frac{p-1}{p}} \left(\sum_{j=1}^{n} E\{|y_j|^{pr}| \mathcal{F}_{j-1}\}\right)^{1/p} = n^{\frac{p-1}{p}} V^{1/p}_{pr,n}$$

and thus setting $p = 2$, $r = 2$ and then $p = 4/3$, $r = 3$,

(24) $$EU^2_{2,t} = EV^2_{2,t} \leq EtV_{4,t} < \infty, \quad EV^{4/3}_{3,t} \leq Et^{1/3}\, V_{4,t} < \infty .$$

Moreover, $EU_{4,t} \leq E(tU_{4,t}) < \infty$ and $E\left(\sum_{j=1}^{t} u_{2,j}\, U_{2,j}\right) \leq EU^2_{2,t} < \infty$. Thus, the R.H.S. of (22) is a bounded function of k, implying via Fatou's lemma that $ES^4_t < \infty$.

Since

$$|Y_n| \leq S^4_n + 6S^2_n\, U_{2,n} + 4|S_n|\, V_{3,n} + U_{4,n} + 6\sum_{j=1}^{n} u_{2,j}\, U_{2,j} = Y'_n \text{ (say)},$$

it follows from the preceding that

$$E|Y_t| \leq EY'_t \leq ES^4_t + 6(E^{1/2}S^4_t)(E^{1/2}U^2_{2,t}) + 4(E^{1/4}S^4_t)(E^{3/4}V^{4/3}_{3,t})$$
$$+ EU_{4,t} + 6EU^2_{2,t} < \infty .$$

From (24), $ET_{2,t} = EU_{2,t} < \infty$. Thus, (8) of section 2 and Lemmas 4 and 5 are valid, whence by Lemma 7, $E\{S_t^2 | \mathcal{F}_k\} \geq S_k^2$ for $t > k$, $k = 1,2, \ldots$. Consequently,

$$\int_{[t>n]} S_t^4 = \int_{[t>n]} [S_n^4 + 2S_n^2(S_t^2 - S_n^2) + (S_t^2 - S_n^2)^2] \geq \int_{[t>n]} S_n^4$$

$$+ 2\int_{[t>n]} S_n^2 E\{S_t^2 - S_n^2 | \mathcal{F}_n\} \geq \int_{[t>n]} S_n^4$$

implying $\int_{[t>n]} S_n^4 = o(1)$ and concomitantly

$$\int_{[t>n]} S_n^2 U_{2,n} \leq \left(\int_{[t>n]} S_n^4\right)^{1/2} \left(\int_{[t>n]} U_{2,t}^2\right)^{1/2} = o(1)$$

(25)

$$\int_{[t>n]} |S_n| V_{3,n} \leq \left(\int_{[t>n]} S_n^4\right)^{1/4} \left(\int_{[t>n]} V_{3,t}^{4/3}\right)^{3/4} = o(1)$$

$$\int_{[t>n]} U_{4,n} \leq \int_{[t>n]} U_{4,t} = o(1)$$

$$\int_{[t>n]} \sum_{j=1}^{n} u_{2,j} U_{2,j} \leq \int_{[t>n]} U_{2,n}^2 \leq \int_{[t>n]} U_{2,t}^2 = o(1) .$$

Thus, $\int_{[t>n]} |Y_n| \leq \int_{[t>n]} Y_n' = o(1)$ and by Lemma 1 $EY_t = EY_1 = 0$.

Alternative expressions for ES_t^4 are possible as indicated in

__Theorem 4__. If $E(t \sum_{j=1}^{t} E\{x_j^4 | \mathcal{F}_{j-1}\}) < \infty$, then setting $S_0 = 0$,

$$ES_t^4 = 6E\sum_{j=1}^{t} S_{j-1}^2\, u_{2,j} + 4E\sum_{j=1}^{t} S_{j-1}\, u_{3,j} + EU_{4,t} .$$

The proof of Theorem 4 is similar to that of Theorem 3 and will be omitted.

__Corollary__. If $E(t\, U_{4,t}) < \infty$, then

$$E\left(6\sum_{j=2}^{t} S_{j-1}^2 u_{2,j} + 4\sum_{j=2}^{t} S_{j-1}\, u_{3,j}\right) = 6ES_t^2 U_{2,t} + 4ES_t U_{3,t} - 6E\left(\sum_{j=1}^{t} u_{2,j} U_{2,j}\right)$$

It is intuitively clear that terms with like coefficients are equal, and indeed we have

Lemma 8. If $E(t\, U_{4,t}) < \infty$, then $ES_t U_{3,t} = E(\sum_{j=2}^{t} S_{j-1} u_{3,j})$ and $E(S_t^2 U_{2,t}) = E(\sum_{j=2}^{t} S_{j-1}^2 u_{2,j}) + E(\sum_{j=1}^{t} u_{2,j} U_{2,j})$.

Proof. It suffices to verify the first of the two relationships since the second will then follow from the corollary to Theorem 4. Suppose first that

$$E(\sum_{n=1}^{t} |x_j|\ V_{r,j}) < \infty \tag{26}$$

Then

$$\sum_{k=1}^{\infty} \int_{[t=k]} \sum_{j=1}^{k} x_j U_{r,j} = \sum_{j=1}^{\infty} \int_{[t \geq j]} x_j U_{r,j} = \sum_{j=1}^{\infty} \int_{[t \geq j]} E(x_j \mid \mathcal{F}_{j-1}) U_{r,j} = 0,$$

whence

$$\begin{aligned} E(\sum_{n=1}^{t} S_{j-1} u_{r,j}) &= \sum_{k=1}^{\infty} \int_{[t=k]} [\sum_{j=2}^{k} S_{j-1} u_{r,j} + \sum_{j=1}^{k} x_j U_{r,j}] = \sum_{k=1}^{\infty} \int_{[t=k]} S_k U_{r,k} \\ &= ES_t U_{r,t} \quad . \end{aligned} \tag{27}$$

Thus, if $t' = \min(t,N)$, (27) holds with t replaced by t' irrespective of (26). However,

$$\begin{aligned} ES_t U_{3,t} &= \sum_{k=1}^{N} \int_{[t=k]} S_k U_{3,k} + \int_{[t > N]} S_t U_{3,t} \\ &= ES_{t'} U_{3,t'} - \int_{[t > N]} S_N U_{3,N} + \int_{[t > N]} S_t U_{3,t} , \end{aligned} \tag{28}$$

and analogously

$$\begin{aligned} E(\sum_{j=2}^{t} S_{j-1} u_{3,j}) &= E(\sum_{j=2}^{t'} S_{j-1} u_{3,j}) - \int_{[t > N]} \sum_{j=2}^{N} S_{j-1} u_{3,j} \\ &\quad + \int_{[t > N]} \sum_{j=2}^{t} S_{j-1} u_{3,j} . \end{aligned} \tag{29}$$

Now $E|S_t U_{3,t}| \leq EY_t' < \infty$, and employing Lemma 7,

$$E\sum_1^t |S_{j-1}u_{3,j}| = \sum_{k=1}^{\infty} \int_{[t=k]} \sum_1^k |S_{j-1}u_{3,j}| = \sum_{j=1}^{\infty} \int_{[t \geq j]} |S_{j-1}u_{3,j}|$$

$$\leq \sum_1^{\infty} \int_{[t \geq j]} |S_t u_{3,j}| \leq E|S_t|V_{3,t} \leq EY_t' < \infty .$$

These facts plus (25) imply that all unwanted terms of (28) and (29) are o(1) and the result follows.

Identities and inequalities analogous to (27) abound and several of these will be catalogued as

<u>Lemma 9</u>. $E(\sum_{n=1}^{t} S_n^2) \leq EtS_t^2$ under the conditions of Lemma 7.

$$E(\sum_{n=1}^{t} S_n) = EtS_t \qquad \text{if } EtT_t < \infty .$$

$$E(\sum_{n=1}^{t} T_n) \leq EtT_t \qquad \text{if } EtT_t < \infty .$$

<u>Proof</u>.

$$E\sum_{n=1}^{t} S_n^2 = \sum_{k=1}^{\infty} \int_{[t=k]} \sum_{n=1}^{k} S_n^2 = \sum_{n=1}^{\infty} \int_{[t \geq n]} S_n^2 \leq \sum_{n=1}^{\infty} \int_{[t \geq n]} E(S_t^2 | \mathcal{F}_n)$$

$$= \sum_{n=1}^{\infty} \int_{[t \geq n]} S_t^2 = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \int_{[t=k]} S_t^2 = \sum_{k=1}^{\infty} k \int_{[t=k]} S_t^2 = EtS_t^2$$

employing Lemma 7. Similarly,

$$E(\sum_{n=1}^{t} T_n) = \sum_{n=1}^{\infty} \int_{[t \geq n]} T_n \leq \sum_{n=1}^{\infty} \int_{[t \geq n]} T_t = EtT_t .$$

Finally,

$$E(\sum_{n=1}^{t} S_n) = \sum_{n=1}^{\infty} \int_{[t \geq n]} S_n = \sum_{n=1}^{\infty} \int_{[t \geq n]} E(S_t | \mathcal{F}_n) = \sum_{n=1}^{\infty} \int_{[t \geq n]} S_t$$

$$= EtS_t$$

in view of Lemmas 1 and 2 and the validity of interchanging the order of summation and integration.

4. The Third Moment

In this section $E(|x_n|^3)$ will be supposed finite. Define

$$Y_n = S_n^3 - 3S_n U_{2,n} - U_{3,n},$$

(30) $$W_n = S_n^3 - \sum_{j=1}^{n} S_{j-1} u_{2,j} - U_{3,n},$$

$$Z_n = S_n^3 - 3\sum_{j=1}^{n} S_j u_{2,j} - U_{3,n} .$$

It is readily checked that $(Y_n, \mathcal{F}_n;\ n \geq 1)$, $(W_n, \mathcal{F}_n;\ n > 1)$, $(Z_n, \mathcal{F}_n;\ n > 1)$ are all martingales and that $EY_1 = EW_1 = EZ_1 = 0$.

Theorem 5. If $EV_{3,t} < \infty$ and $EV_{1,t}^3 < \infty$, or equivalently if $ET_3^3 < \infty$, then $E|S_t|^3 < \infty$ and $ES_t^3 = 3E(\sum_{j=1}^{t} S_{j-1} u_{2,j}) + EU_{3,t}$.

Proof. Suppose that $EV_{3,t} < \infty$, $EV_{1,t}^3 < \infty$ (Their equivalence with $ET_t^3 < \infty$ will be deferred to Lemma 10). Then

$$E|S_t|^3 = \sum_{k=1}^{\infty} \int_{[t=k]} \sum_{n=1}^{k} (|S_n|^3 - |S_{n-1}|^3) \leq \sum_{k=1}^{\infty} \sum_{n=1}^{k} \int_{[t=k]} (|x_n|^3$$

(31) $$+ 3|S_{n-1}|x_n^2 + 3S_{n-1}^2|x_n|) \leq 6 \sum_{k=1}^{\infty} \sum_{n=1}^{k} \int_{[t=k]} (|x_n|^3 + S_{n-1}^2|x_n|)$$

$$= 6[E(\sum_{n=1}^{t} |x_n|^3) + E(\sum_{n=1}^{t} S_{n-1}^2 |x_n|)] .$$

By Lemma 6,

(32) $$E(\sum_{n=1}^{t} |x_n|^3) = EV_{3,t} < \infty .$$

On the other hand, $ES_t^2 \leq ET_t^2 \leq 1 + ET_t^3 < \infty$ and

$$\int_{[t > k]} |S_k| \leq \int_{[t > k]} T_k \leq \int_{[t > k]} T_t \leq \int_{[t > k]} (1 + T_t^3) = o(1)$$

in view of the asserted equivalence. Thus, Lemma 7 holds, whence

$$E(\sum_{n=1}^{t} S_{n-1}^2 |x_n|) = \sum_{k=1}^{\infty} \sum_{n=1}^{k} \int_{[t=k]} S_{n-1}^2 |x_n| = \sum_{n=1}^{\infty} \int_{[t \geq n]} S_{n-1}^2 v_{1,n}$$

(33)
$$\leq \sum_{n=1}^{\infty} \int_{[t \geq n]} E(S_t^2 | \mathcal{F}_{n-1}) v_{1,n} = \sum_{n=1}^{\infty} \int_{[t \geq n]} S_t^2 v_{1,n}$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{k} \int_{[t=k]} S_t^2 v_{1,n} = ES_t^2 V_{1,t} \leq (E^{2/3}|S_t|^3)(E^{1/3} V_{1,t}^3) .$$

Replace t by $t' = \min(t,k)$ in (31). Then from (32) and (33),

$$E|S_{t'}|^3 \leq 6EV_{3,t'} + 6(E^{2/3}|S_{t'}|^3)(E^{1/3}V_{1,t'}^3) = O(1) + O(1)E^{2/3}|S_{t'}|^3$$

whence, by Fatou's lemma,

(34)
$$E|S_t|^3 < \infty .$$

Next, (34) implies that the expectation in the L.H.S. of (33) is finite whence,

(35)
$$E(\sum_{n=1}^{t} |S_{n-1}| u_{2,n}) = \sum_{n=1}^{\infty} \int_{[t \geq n]} |S_{n-1}| x_n^2 = E(\sum_{n=1}^{t} |S_{n-1}| x_n^2)$$

$$\leq E[\sum_{n=1}^{t} (|x_n|^3 + |S_{n-1}|^2 |x_n|)] < \infty .$$

Combining (33), (34) and (35), $E|W_t| < \infty$. Since, paralleling (31),

$$\int_{[t > k]} |S_k|^3 \leq 6 \int_{[t > k]} \sum_{n=1}^{k} (|x_n|^3 + S_{n-1}^2 |x_n|) = o(1),$$

$\int_{[t > k]} |W_k| = o(1)$ and the theorem follows from Lemma 1.

<u>Corollary</u>. Under the same hypothesis, $E(\sum_{n=1}^{t} x_j u_{2,j}) = 0$.

<u>Proof</u>. Analogously, $EZ_t = 0$, whence $E(W_t - Z_t) = 0$.

Lemma 10. $EV_{3,t} < \infty$ and $EV_{1,t}^3 < \infty$ if and only if $ET_t^3 < \infty$.

Proof. Suppose $EV_{3,t} < \infty$ and $EV_{1,t}^3 < \infty$. The argument of (31) with T_t replacing S_t yields

$$ET_t^3 \leq 6 \sum_{k=1}^{\infty} \sum_{n=1}^{k} \int_{[t=k]} (|x_n|^3 + T_{n-1}^2 |x_n|).$$

The inequality of (33) also obtains with T replacing S in view of the fact that $T_t \geq T_{n-1}$ on the set $[t \geq n]$. Thus, analogously, $ET_{t'}^3 \leq O(1) + O(1)\, E^{2/3}\, T_{t'}^3$, implying $ET_t^3 < \infty$.

Conversely, if $ET_t^3 < \infty$, clearly $EV_{3,t} = ET_{3,t} \leq ET_t^3 < \infty$. Moreover,

$$EV_{1,t}^3 = \sum_{k=1}^{\infty} \int_{[t=k]} \sum_{n=1}^{k} (V_{1,n}^3 - V_{1,n-1}^3) \leq 6 \sum_{k=1}^{\infty} \sum_{n=1}^{k} \int_{[t=k]} (v_{1,n}^3 + 3V_{1,n-1}^2 v_{1,n} + 3V_{1,n-1} v_{1,n}^2)$$

$$\leq O(1) + 6 \sum_{k=1}^{\infty} \sum_{n=1}^{k} \int_{[t=k]} V_{1,n-1}^2 v_{1,n} = O(1) + 6 \sum_{n=1}^{\infty} \int_{[t \geq n]} |x_n| V_{1,n-1}^2$$

$$\leq O(1) + 6 \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \int_{[t=k]} |x_n| V_{1,t}^2 \leq O(1) + 6ET_t\, V_{1,t}^2$$

$$\leq O(1) + O(1)\, E^{2/3}\, V_{1,t}^3$$

which implies, as earlier, that $EV_{1,t}^3 < \infty$ and completes the proof.

Theorem 6. If $ET_t^3 < \infty$ and $E\, t^{1/2} V_{3,t} < \infty$, $ES_t^3 = 3ES_t U_{2,t} + EU_{3,t} < \infty$.

Proof. As in Theorem 3, after setting $p = 3/2$, $r = 2$ in (23) of section 3 to obtain

$$ES_t\, U_{2,t} \leq (E^{1/3} S_t^3)(E^{2/3} U_{2,t}^{3/2}) \leq (E^{1/3} S_t^3)(E^{2/3} t^{1/2} V_{3,t}) .$$

Corollary. Under the conditions of Theorem 6, $ES_t U_{2,t} = E(\sum_{j=1}^{t} S_{j-1} u_{2,j})$.

The single requirement $ET_t^3 < \infty$, although equivalent to the two conditions of Theorem 5, is difficult to check. The following single condition is easily seen to imply all those of Theorems 5 and 6:

(36) $$E(t^2 V_{3,t}) < \infty ,$$

and in addition yields

$$ET_t^3 = 3ET_t^2 V_{1,t} + 3ET_t(V_{2,t} - 2\sum_{j=1}^{t} V_{1,j} v_{1,j}) + EV_{3,t} - 3E(\sum_{j=1}^{t} V_{1,j} v_{2,j})$$
$$- 3E(\sum_{j=1}^{t} V_{2,j} v_{1,j}) + 6E(\sum_{j=1}^{t} v_{1,j} \sum_{i=1}^{j} V_{1i} v_{1i}) .$$

5. Sums of Independent Random Variables

In this section, the random variables $x_1, x_2, \ldots$ will be supposed independent. If $Ex_n = 0$, all prior theorems are, of course, applicable but may be reformulated in especially simple terms with conditions that are susceptible of immediate verification. For example, from Theorems 3 and 6, we obtain:

Theorem 7. If $x_1, x_2, \ldots$ are independent with $Ex_n = 0$, $Ex_n^2 = \sigma^2$, $Ex_n^3 = \gamma$, $Ex_n^4 = \beta < \infty$ and t is a stopping rule with $Et^2 < \infty$, then $ES_t^4 < \infty$ and

$$ES_t^4 = 6\,\sigma^2\, E\, t\, S_t^2 + 4\,\gamma\, E\, t\, S_t + \beta\, E\, t - 3\sigma^4\, Et(t+1).$$

Theorem 8. If $x_1, x_2, \ldots$ are independent with $Ex_n = 0$, $Ex_n^2 = \sigma^2$, $Ex_n^3 = \gamma$, $E|x_n|^3 \leq C < \infty$, and if t is a stopping variable with $Et^3 < \infty$, then $ES_t^3 = \gamma\, E\, t + 3\sigma^2\, E\, t\, S_t < \infty$.

Proof. According to Theorem 6 and Lemma 10, it suffices to verify that

$$EV_{3,t} \leq E(t^{1/2}V_{3,t}) \leq C\ E\ t^{3/2} < \infty,$$

$$EV_{1,t}^3 \leq E[t\ (1+C)]^3 < \infty\ .$$

In the final theorem, the requirement of Theorem 8 that $Et^3 < \infty$ will be relaxed at the expense of increasing the moment assumptions on x_n.

Theorem 9. If $x_1, x_2, \ldots$ are independent with $Ex_n = 0$, $Ex_n^2 = \sigma^2$, $Ex_n^3 = \gamma$, $Ex_n^4 \leq C < \infty$, and if t is a stopping variable with $Et^2 < \infty$, then $ES_t^3 = \gamma\ Et + 3\sigma^2\ EtS_t$.

Proof. Here, the martingale Y_n of (30) simplifies to $Y_n = S_n^3 - 3\sigma^2 nS_n - n\gamma$. The theorem will follow from Lemmas 1 and 2 once it is established that

$$E\sum_1^t |Y_{n+1}-Y_n| = E\sum_1^t E(|Y_{n+1}-Y_n|\,|\,\mathcal{F}_n) < \infty\ .$$

Now

$$E(|S_{n+1}^3-S_n^3|\,|\,\mathcal{F}_n) \leq 6\ E(|x_{n+1}|^3 + S_n^2|x_{n+1}|\,|\,\mathcal{F}_n) = O(1)S_n^2 + O(1),$$

$$E(|(n+1)S_{n+1}-nS_n|\,|\,\mathcal{F}_n) = E(|S_n + (n+1)x_{n+1}|\,|\,\mathcal{F}_n) \leq S_n^2 + n\ O(1),$$

whence

$$E(|Y_{n+1}-Y_n|\,|\,\mathcal{F}_n) = O(1)S_n^2 + n\ O(1).$$

Next, Lemma 9 is applicable below since (17) insures $ES_t^2 < \infty$ while Lemmas 6 and 2 guarantee (10). Consequently,

$$E \sum_{1}^{t} E(|Y_{n+1}-Y_n| \mid \mathcal{F}_n) \leq O(1)\ E(\sum_{1}^{t} S_n^2) + O(1)\ Et$$

$$\leq O(1)\ Et\ S_t^2 + O(1)\ Et$$

$$\leq O(1)\ (E^{1/2}\ t^2)(E^{1/2}\ S_t^4) + O(1)\ E\ t < \infty.$$

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