On the moments of the second elementary symmetric function of the roots of a matrix.

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1. Summary and Introduction: Distribution problems in multivariate analysis are often related to the joint distribution of the characteristic roots of a matrix derived from sample observations taken from multivariate normal populations. This joint distribution (under certain null hypotheses) of s non-null characteristic roots given by Fisher [4], Girshick [5], Hsu [6], and Roy [14] can be expressed in the form

$$(1.1)\qquad f(\theta_1,\ldots,\theta_s) = C(s,m,n)\prod_{i=1}^{s}\theta_i^m(1-\theta_i)^n\prod_{i>j}(\theta_i-\theta_j)$$

$$0 < \theta_1 \leq \ldots \leq \theta_s < 1$$

where

$$(1.2)\qquad C(s,m,n) = \pi^{s/2}\prod_{i=1}^{s}\Gamma\left(\frac{2m+2n+s+i+2}{2}\right)\Big/\left[\Gamma\left(\frac{2m+i+1}{2}\right)\Gamma\left(\frac{2n+i+1}{2}\right)\Gamma(i/2)\right]$$

and m and n are defined differently for various situations described in [9], [11]. Nanda [7] has shown that if $\xi_i = n\theta_i$ $(i=1,\ldots,s)$, then the limiting distribution of ξ_i's as n tends to infinity is given by

$$(1.3) \qquad f_1(\xi_1,\xi_2,\ldots,\xi_s) = K(s,m) \prod_{i=1}^{s} \xi_i^m e^{-\xi_i} \prod_{i>j} (\xi_i - \xi_j)$$

$$0 < \xi_1 \leq \cdots \leq \xi_s < \infty$$

where

$$(1.4) \qquad K(s,m) = \pi^{s/2} \Big/ \left[\prod_{i=1}^{s} \Gamma\left(\frac{2m+i+1}{2}\right) \Gamma(i/2) \right].$$

The distribution (1.3) can also be arrived at as that of $\xi_i = \frac{1}{2}\gamma_i$ $(i=1,2,\ldots,s)$ where γ_i's are the roots of the equation $|S - \gamma\,\Sigma| = 0$ where S is the variance-covariance matrix computed from a sample taken from an s-variate normal population with dispersion matrix Σ. In this paper, the first four moments of $W_2^{(s)}$, the second elementary symmetric function (esf) in s ξ's, have been obtained.

2. Formulae for the first four moments of $W_2^{(s)}$.

The joint distribution (1.3) can be thrown into a determinantal form of the Vandermonde type and integrated over the range R, $0 < \xi_1 \leq \cdots \leq \xi_s < \infty$, giving

$$(2.1) \int_R f_1(\xi_1,\xi_2,\ldots,\xi_s) \prod_{i=1}^{s} d\xi_i = K(s,m) \begin{vmatrix} \int_0^\infty \xi_s^{m+s-1} e^{-\xi_s}\, d\xi_s \ldots \int_0^\infty \xi_s^{m} e^{-\xi_s}\, d\xi_s \\ \\ \int_0^{\xi_2} \xi_1^{m+s-1} e^{-\xi_1}\, d\xi_1 \ldots \int_0^{\xi_2} \xi_1^{m} e^{-\xi_1}\, d\xi_1 \end{vmatrix}.$$

Now denote by $W(s-1,s-2,\ldots,1,0)$ the determinant on the right side of (2.1). Using Lemma 1 of [12], the first four moments of $W_2^{(s)}$ can be obtained as follows: (denoting $E(W_2^{(s)})^r$ by μ_r').

(2.2)
$$\mu_1' = K(s,m)\, W(s,s-1,s-3,\ldots,1,0)$$

(2.3)
$$\mu_2' = K(s,m)\, [W(s+1,s,s-3,\ldots,1,0) + W(s+1,s-1,s-2,s-4,\ldots,1,0) + W(s,s-1,s-2,s-3,s-5,\ldots,1,0)]$$

(2.4)
$$\begin{aligned}\mu_3' = K(s,m)\, [&W(s+2,s+1,s-3,\ldots,1,0) + 2\, W(s+2,s,s-2,s-4,\ldots,1,0) \\ &+ 3\, W(s+1,s,s-2,s-3,s-5,\ldots,1,0) + W(s+2,s-1,s-2,s-3,s-5,\ldots,1,0) \\ &+ W(s+1,s,s-1,s-4,\ldots,1,0) + 2W(s+1,s-1,s-2,s-3,s-4,s-6,\ldots,1,0) \\ &+ W(s,s-1,s-2,s-3,s-4,s-5,s-7,\ldots,1,0)]\end{aligned}$$

and

(2.5)
$$\begin{aligned}\mu_4' = K(s,m)[&W(s+3,s+2,s-3,\ldots,1,0) + 3W(s+3,s+1,s-2,s-4,\ldots,1,0) \\ &+ 6W(s+2,s+1,s-2,s-3,s-5,\ldots,1,0) + 2W(s+3,s,s-1,s-4,\ldots,1,0) \\ &+ 3W(s+3,s,s-2,s-3,s-5,\ldots,1,0) + 3W(s+2,s+1,s-1,s-4,\ldots,1,0) \\ &+ 7W(s+2,s,s-1,s-3,s-5,\ldots,1,0) + 8W(s+2,s,s-2,s-3,s-4,s-6,\ldots,1,0) \\ &+ 3W(s+1,s,s-1,s-2,s-5,\ldots,1,0) + 6W(s+1,s,s-1,s-3,s-4,s-6,\ldots,1,0) \\ &+ 6W(s+1,s,s-2,s-3,s-4,s-5,s-7,\ldots,1,0) \\ &+ W(s+3,s-1,s-2,s-3,s-4,s-6,\ldots,1,0) \\ &+ 3W(s+2,s-1,s-2,s-3,s-4,s-5,s-7,\ldots,1,0) \\ &+ 3W(s+1,s-1,s-2,s-3,s-4,s-5,s-6,s-8,\ldots,1,0) \\ &+ W(s,s-1,s-2,s-3,s-4,s-5,s-6,s-7,s-9,\ldots,1,0)].\end{aligned}$$

3. A method of evaluation of the W-determinants. Let us denote by $V(q_s,q_{s-1},\dots,q_1)$ the determinant which could be obtained from $W(q_s,q_{s-1},\dots,q_1)$ by replacing ξ_i by θ_i in (1.1), $e^{-\xi_i}$ by $(1-\theta_i)^n$ and the range of integration by that in (1.1). Pillai [8], [10] has given a method of reducing the sth order determinant $V(q_s,q_{s-1},\dots,q_1)$ in terms of $(s-2)$th order determinants and an sth order determinant with q_s changed to q_s-1, the last one being zero if $q_s-1 = q_{s-1}$. The method of reduction for $W(q_s,\dots,q_1)$ can be deduced from that for $W(q_s,\dots,q_1)$ in [10] and we obtain the following:

$$(3.1)\quad W(q_s,q_{s-1},\dots,q_1) = 2\sum_{j=s-1}^{1}(-1)^{s-j-1}I(q_s+q_j;2)\,W(q_{s-1},\dots,q_{j+1},q_{j-1},\dots,q_1)$$
$$+ (m+q_s)\,W(q_s-1,q_{s-1},\dots,q_1)$$

where $I(p;2) = \int_0^\infty x^p e^{-2x}\,dx = \Gamma(p+1)/2^{p+1}$.

The values of the W-determinants involved in (2.2)-(2.5) are obtained using (3.1) and presented in the following section.

4. Values of the W-determinants. Let us set

$$(4.1)\qquad (2g+a)(2g+b)\dots = G(a,b,\dots).$$

Then for the first moment

$$(4.2)\qquad K(s,m)\,W(s,s-1,s-3,\dots,1,0) = s(s-1)\,M(s,s+1)/2^3.$$

In fact, in general

$$(4.3)\quad K(s,m)\, W(s,s-1,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0) = \binom{s}{i} M(s-i+2,\ldots,s+1)/2^{i}.$$

For the <u>second raw moment</u>, we get

$$(4.4)\quad K(s,m)W(s+1,s,s-3,\ldots,1,0)$$

$$= \left[\binom{s}{2} M(s,s+1)/2^{4}3!\right]\left[4s(s+1)m^{2}+ 2s(2s^{2}+ 5s+9)m+s^{4}+4s^{3}+11s^{2}+ 8s+12\right].$$

$$(4.5)\quad K(s,m)\, W(s+1,s-1,s-2,s-4,\ldots,1,0)$$

$$= \left[\binom{s}{3} M(s-1,s,s+1)/2^{6}\right]\left[2(3s-1)m+3s^{2}+s+10\right]$$

$$+ (m+s+1)\, K(s,m)\, W(s,s-1,s-2,s-4,\ldots,1,0)$$

The last determinant on the right side of (4.5) is evaluated by putting $i=3$ in (4.3). In general

$$(4.6)\quad K(s,m)\, W(s+1,s-1,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0)$$

$$= \left[i \binom{s}{i} M(s-i+2,\ldots,s+1)/2^{i+1}(i+1)\right]\left[2(s+1)m + (s+1)(s+2)+i+1\right].$$

$K(s,m)W(s,s-1,s-2,s-3,s-5,\ldots,1,0)$ is obtained from (4.3) by putting $i=4$. Now using the results (4.2)-(4.6) we get μ_2, the <u>second central moment</u> of $W_2^{(s)}$, as

(4.7) $$\mu_2 = [\binom{s}{2} M(s,s+1)/2^3][4(s-1)m + 2s^2 - 2s+3].$$

For the <u>third raw moment</u>, we get

(4.8) $$K(s,m)\ W(s+2,s+1,s-3,\ldots,1,0)$$

$$= [\binom{s+2}{4} M(s,s+1,s+2,s+3)/2^6\cdot 3!][4s(s+1)m^2+2s(2s^2+5s+21)m+s^4+4s^3 + 23s^2+20s+72].$$

In fact, in general

(4.9) $$K(s,m)\ W(s+2,s+1,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0)$$

$$= [i(i-1)\binom{s+2}{i+2} M(s-i+2,\ldots,s+3)/2^{i+5}3!][4s(s+1)m^2+2s(2s^2+5s+4i+13)m +s(s+1)(s^2+3s+4i+12)+6(i+1)(i+2)].$$

(4.10) $$K(s,m)W(s+2,s,s-2,s-4,\ldots,1,0)$$

$$= [\binom{s+1}{4} M(s-1,s,s+1,s+2)/2^5\cdot 15][2s(8s+1)m^2+s(16s^2+19s+109)m + 4s^4+9s^3+59s^2+54s+180]$$
$$+(m+s+2)\ K(s,m)\ W(s+1,s,s-2,s-4,\ldots,1,0).$$

The value of the determinant in the last term on the right side of (4.10) is obtained by putting $i=3$ in the following general result:

(4.11) $K(s,m)\ W(s+1,s,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0)$

$$= [(i-1)\binom{s}{i}M(s-i+2,\ldots,s+1)/2^{i+3}(i+1)][4s(s+1)m^2+2s(2s^2+5s+2i+5)m$$
$$+s^4+4s^3+(2i+7)s^2+(2i+4)s+2i(i+1)]$$

Now $K(s,m)\ W(s+1,s,s-2,s-3,s-5,\ldots,1,0)$ is obtained from (4.11) by putting $i=4$.

(4.12) $K(s,m)\ W(s+2,s-1,s-2,s-3,s-5,\ldots,1,0)$

$$= [\binom{s+1}{5}M(s-2,\ldots,s+2)/2^5 3!][2(5s-2)m+5s^2+s+42]$$
$$+ (m+s+2)\ K(s,m)\ W(s+1,s-1,s-2,s-3,s-5,\ldots,1,0).$$

In fact, in general

(4.13) $K(s,m)\ W(s+j,s-1,s-2,\ldots,s-i+1,s-i-1,\ldots 1,0)$

$$= [\binom{i+j-2}{j-1}\binom{s+j-1}{i+j-1}\ M(s-i+2,\ldots,s+j)/2^{i+j}j(i+j)]\ \times$$
$$\times\ [2\{(i+j-1)s-j\}\ m+(i+j-1)s^2+(i-j-1)s+j(i-1)(i+j+1)]$$
$$+ (m+s+j)\ K(s,m)\ W(s+j-1,s-1,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0).$$

The value of the last determinant on the right side of (4.12) is obtained easily from (4.6) by putting $i=4$.

(4.14) $K(s,m)\ W(s+1,s,s-1,s-4,\dots,1,0)$

$$= [\binom{s}{3}M(s-1,s,s+1)/2^6 4!][8(s-1)s(s+1)m^3+12(s-1)s(s^2+2s+5)m^2$$
$$+ 2(s-1)(3s^4+9s^3+32s^2+14s+72)m$$
$$+ s(s-1)(s^4+4s^3+17s^2+14s+72) + 144]$$

Now, $K(s,m)\ W(s+1,s-1,s-2,s-3,s-4,s-6,\dots,1,0)$ is obtained from (4.6) by putting $i=5$ and $K(s,m)\ W(s,s-1,s-2,s-3,s-4,s-5,s-7,\dots,1,0)$ from (4.3) with $i=6$.

For the <u>fourth raw moment</u> we get

(4.15) $K(s,m)\ W(s+3,s+2,s-3,\dots,1,0)$

$$= [\binom{s+2}{4}\ M(s,s+1,s+2,s+3)/2^8\ 5!]\ x$$
$$x\ [16s(s+1)(s+2)(s+3)m^4+8s(s+2)(s+3)(4s^2+14s+46)m^3$$
$$+ 4(s+1)(s+2)(6s^4+48s^3+233s^2+609s+720)m^2$$
$$+ 2(s+2)(4s^6+46s^5+310s^4+1320s^3+3542s^2+5802s+6480)m$$
$$+ (s+2)(s+3)(s^6+11s^5+81s^4+373s^3+1118s^2+2256s+ 4320)+2880].$$

(4.16) $K(s,m)\ W(s+3,s+1,s-2,s-4,\dots,1,0)$

$$= [\binom{s+2}{5}\ M(s-1,\dots,s+3)/2^8 4!][8s(s+1)(5s+3)m^3+4s(15s^3+47s^2+189s+213)m^2$$
$$+ 2(15s^5+70s^4+403s^3+966s^2+1842s+1440)m$$
$$+ 5s^6+31s^5+217s^4+769s^3+2210s^2+4512s+5760]$$
$$+ (m+s+3)\ K(s,m)\ W(s+2,s+1,s-2,s-4,\dots,1,0).$$

The value of the determinant in the last term of the right side of (4.16) is obtained from (4.9) by putting $i=3$. $K(s,m)W(s+2,s+1,s-2,s-3,s-5,\ldots,1,0)$ is deduced from (4.9) with $i=4$.

(4.17) $K(s,m)W(s+3,s,s-1,s-4,\ldots,1,0)$

$$= [\binom{s+2}{5} M(s-1,\ldots,s+3)/2^8\, 3!] \times$$

$$\times[8s^2(s-1)m^3 + 4s(s-1)(3s^2+2s+24)m^2+2(s-1)(3s^4+4s^3+49s^2+24s+180)m$$

$$+ s(s-1)(s^4+2s^3+25s^2+24s+180)+360]$$

$$+ (m+s+3)\, K(s,m)\, W(s+2,s,s-1,s-4,\ldots,1,0).$$

The value of the determinant in the last term on the right side of (4.17) is obtained from the following result by putting $i=3$.

(4.18) $K(s,m)\, W(s+2,s,s-1,s-3,\ldots,s-i+1,s-i-1,\ldots,1,0)$

$$= [(i-2)\binom{s+1}{i+1} M(s-i+2,\ldots,s+2)/2^{i+4}(i+2)4!] \times$$

$$\times [8s(s-1)\{3(i+1)s+2(i-1)\}m^3+4s(s-1)\{9(i+1)s^2+2(7i-1)s$$

$$+ 2(6i^2+20i+4)\}m^2$$

$$+ 2(s-1)\{9(i+1)s^4+2(11i+1)s^3+(24i^2+85i+17)s^2+2(6i^2+19i+5)s$$

$$+ 24i(i+1)(i+2)\}m$$

$$+ s(s-1)\{3(i+1)s^4+2(5i+1)s^3+3(4i^2+15i+3)s^2+2(6i^2+19i+5)s$$

$$+ 24i(i+1)(i+2)\} + 24(i-1)i(i+1)(i+2)]$$

$$+ (m+s+2)\, K(s,m)W(s+1,s,s-1,s-3,\ldots,s-i+1,s-i-1,\ldots,1,0),$$

where the values of the last determinant on the right side of (4.18) is obtained from the following:

$$(4.19) \quad K(s,m)\ W(s+1,s,s-1,s-3,\ldots,s-i+1,s-i-1,\ldots,1,0)$$

$$= [(i-2)\tbinom{s}{i}\ M(s-i+2,\ldots,s+1)/2^{i+1}(i+1)\ 4!]\ \times$$

$$\times[8(s+1)s(s-1)m^3+12s(s-1)(s^2+2s+i+2)m^2$$

$$+\ 2(s-1)\{3s^4+9s^3+(6i+14)s^2+(3i+5)s+6i(i+1)\}m$$

$$+\ s(s-1)\{s^4+4s^3+(3i+8)s^2+(3i+5)s+6i(i+1)\}\ +\ 6(i-1)i(i+1)].$$

$K(s,m)W(s+3,s,s-2,s-3,s-5,\ldots,1,0)$ is deduced from the following result by putting $i=4$.

$$(4.20) \quad K(s,m)\ W(s+3,s,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0)$$

$$= [(i^2-1)\tbinom{s+2}{i+2}M(s-i+2,\ldots,s+3)/2^{i+4}(i+3)\ 4!]\ \times$$

$$\times\ [4s\{3(i+2)s+(i-6)\}m^2+2s\{6(i+2)s^2+3(3i-2)s+12i^2+49i-6\}m$$

$$+\ s(s+1)\{3(i+2)s^2+(5i-6)s+12i(i+4)\}+12i(i+2)(i+3)]$$

$$+\ (m+s+3)K(s,m)W(s+2,s,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0)$$

where the value of the determinant in the last term on the right side of (4.20) is given by

(4.21) $K(s,m)W(s+2,s,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0)$

$$= [(i-1)\binom{s+1}{i+1} M(s-i+2,\ldots,s+2)/2^{i+3}(i+2)\ 3!] \times$$

$$\times [4s\{2(i+1)s+i-2\}m^2+2s\{4(i+1)s^2+(7i-2)s+(6i^2+19i-2)\}m$$

$$+ s(s+1)\{2(i+1)s^2+2(2i-1)s+6i(i+3)+6i(i+1)(i+2)]$$

$$+ (m+s+2)K(s,m)W(s+1,s,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0),$$

where the value of the determinant in the last term on the right side of (4.21) is obtained from (4.11).

(4.22) $K(s,m)\ W(s+2,s+1,s-1,s-4,\ldots,1,0)$

$$= [\binom{s+2}{5}M(s-1,\ldots,s+3)/2^{11}] \times$$

$$\times [8(s+1)s(s-1)m^3+4s(s-1)(3s^2+6s+31)m^2$$

$$+ 2(s-1)(3s^4+9s^3+64s^2+30s+240)m+s(s-1)(s^4+4s^3+33s^2+30s+240)+480].$$

$K(s,m)W(s+2,s,s-1,s-3,s-5,\ldots,1,0)$ is obtained from (4.18) by putting $i=4$ and $K(s,m)W(s+2,s,s-2,s-3,s-4,s-6,\ldots,1,0)$ from (4.21) with $i=5$.

(4.23) $K(s,m)W(s+1,s,s-1,s-2,s-5,\ldots,1,0)$

$$= [\binom{s}{4}M(s-2,\ldots,s+1)/2^8\ 5!] \times$$

$$\times [16(s+1)(s)(s-1)(s-2)m^4+16s(s-1)(s-2)(2s^2+3s+11)m^3$$

$$+ 4(s-1)(s-2)(6s^4+12s^3+65s^2-s+240)m^2$$

$$+ 2(s-2)(4s^6+6s^5+54s^4-68s^3+462s^2-698s+1680)m$$

$$+s^8+2s^7+15s^6+109s^5-28886s^4+542313s^3-4156490s^2+14607096s-19408320].$$

(The eighth degree (last) polynomial in s in [ ] on the right side of (4.23) is true only for $s=4,5,\ldots,9$.)

$K(s,m)W(s+1,s,s-1,s-3,s-4,s-6,\ldots,1,0)$ is obtained from (4.19) by putting $i=5$; $K(s,m)W(s+1,s,s-2,s-3,s-4,s-5,s-7,\ldots,1,0)$ from (4.11) with $i=6$; $K(s,m)W(s+3,s-1,s-2,s-3,s-4,s-6,\ldots,1,0)$ from (4.13) by putting $j=3$ and $i=5$; $K(s,m)W(s+2,s-1,s-2,s-3,s-4,s-5,s-7,\ldots,1,0)$ from (4.13) by putting $j=2$ and $i=6$; $K(s,m)W(s+1,s-1,s-2,s-3,s-4,s-5,s-6,s-8,\ldots,1,0)$ from (4.6) with $i=7$; and $K(s,m)W(s,s-1,s-2,s-3,s-4,s-5,s-6,s-7,s-9,\ldots,1,0)$ from (4.3) by putting $i=8$.

5. Some remarks. It may be pointed out that $2\sum_{i=1}^{s}\xi_i$ is distributed [8] as a chi-square with $s(2m+s+1)$ degrees of freedom and hence the distribution problem in this case is very simple. However, the moments of the second esf presented in this paper show that the distribution problem herein is no longer so. However, methods similar to those employed by Pillai [11], [13] in obtaining (through the use of moment quotients) approximate percentage points for the test criterion $V(s) = \sum_{i=1}^{s}\theta_i$, where θ's follow the distribution (1.1), can be successfully utilized here also. It might also be pointed out that several tests based on the esf's of the characteristic roots have been observed to have monotonicity property of power [1], [2], [3].

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