On a Class of Multivariate Test Criteria

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1. Introduction and Summary. The paper first presents a brief review of some aspects of the distribution problems under null hypotheses, of multivariate tests criteria belonging to a general class defined later in the section. Some new results are then given (Starting section 3) which extend the earlier work of the central case to the non-central (linear) case.

In multivariate analysis, we generally wish to test three hypotheses, namely,

(I) that of equality of the dispersion matrices of two p-variate normal populations;

(II) that of equality of the p-dimensional mean vectors for ℓ p-variate normal populations (which is mathematically identical with the general problem of multivariate analysis of variance of means); and

(III) that of independence between a p-set and a q-set of variates in a (p + q)-variate normal population, with $p \leq q$.

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All tests proposed so far for these hypotheses have been shown to depend, when the hypotheses to be tested are true, only on the characteristic roots of matrices based on sample observations. For example, in case (I), all the tests proposed so far are based on the characteristic roots of the matrix $S_1(S_1 + S_2)^{-1}$, where S_1 and S_2 denote the usual sum of product (S.P.) matrices and where both are almost everywhere positive definite (a.e.p.d.). Thus $S_1(S_1 + S_2)^{-1}$ is a.e.p.d., whence it follows that all the p characteristic roots are greater than zero and less than unity. In case (II), the matrix is $S^*(S^* + S)^{-1}$, where S^* denotes the "between" S.P. matrix of means weighted by the sample sizes and S denotes the "within" S.P. matrix (pooled from the S.P. matrices of ℓ samples). Then S is a.e.p.d., and S^* is a least positive semi-definite of rank $s = \min(p, \ell-1)$. Thus, a.e., s of the characteristic roots are greater than zero and less than unity and the p - s remaining roots are zero. In case (III), the matrix is $S_{11}^{-1} S_{12}^{-1} S_{12}'$, where S_{11} is the S.P. matrix of the sample of observations on the p-set of variates, S_{22} that on the q-set, and S_{12}, the S.P. matrix between the observations on the p-set and those on the q-set. If $p \leq q$ and $p + q < k$, where k is the sample size, then a.e. the p characteristic roots of this matrix are greater than zero and less than unity.

In each case, if the hypotheses to be tested is true, the $s \leq p$ non zero roots θ_i $(i = 1,2,\ldots,s)$ have the well known Fisher - Girshick - Hsu - Mood - Roy distribution of the form:

$$(1.1) \quad f(\theta_1, \theta_2, \ldots, \theta_s) = C(s,m,n) \prod_{i=1}^{s} \theta_i^m (1 - \theta_i)^n \prod_{i>j} (\theta_i - \theta_j)$$

$$0 < \theta_1 \leq \theta_2 \leq \ldots \leq \theta_2 < 1$$

where

$$(1.2) \quad C(s,m,n) = \pi^{s/2} \prod_{i=1}^{s} \Gamma[(2m + 2n + s + i + 2)/2] / \left[\Gamma\left(\frac{2m + i + 1}{2}\right) \Gamma\left(\frac{2n + i + 1}{2}\right) \Gamma(i/2)\right],$$

and m and n are defined differently for various situations described in [12], [14]. Now, if $\lambda_i = \theta_i/(1 - \theta_i)$ $(i = 1,2,\ldots,s)$, the joint distribution of the λ's is obtained from (1.1) as

$$(1.3) \qquad f_1(\lambda_1,\ldots,\lambda_s) = C(s,m,n)\left[\prod_{i=1}^{s} \lambda_i^m/(1+\lambda_i)^{m+n+s+1}\right]\prod_{i>j}(\lambda_i - \lambda_j)$$

$$0 < \lambda_1 \le \ldots \le \lambda_s < +\infty.$$

A class of test criteria may now be defined as:

(i) Symmetric functions of θ's, and similarly

(ii) Symmetric functions of λ's.

Although this class has not been studied in its generality, various special cases of this class have been considered by many authors.

Some of these special cases come under a subclass of this wider class, namely,

(ia) elementary symmetric functions (esf's) in the θ's studied by Pillai [11, 13, 14, 15, 16], Pillai and Mijares [18] and Mijares [9,10], and

(iia) esf's in the λ's studied by Pillai [11, 13, 14, 15, 16, 19].

An important special case which comes under (ia) is Pillai's $V^{(s)}$ criterion which is the sum of the θ's. Under (iia) comes Lawley-Hotelling criterion which is a constant times $U^{(s)}$, where $U^{(s)}$ is defined by Pillai [11, 12, 13] as the sum of the λ's. Wilks' likelihood ratio criterion which is the s^{th} esf in $(1 - \theta)$'s is a special case of the wider class (i) and is also a function of the esf's in the θ's. Similarly, the harmonic mean criteria for the θ's as well as the λ's proposed by Pillai [12] are special cases of (i) and (ii) respectively and are also functions of the esf's. Therefore, a lemma useful for obtaining the moments of the esf's in the θ's or λ's is stated in the following section [15].

2. Moments of esf's. In this section we state a lemma concerning esf's and show how the moments of the esf's in θ's or λ's can be obtained by applications of this lemma.

Lemma 1. Let $D(g_s, g_{s-1}, \ldots, g_1)$ $(g_j \geq 0,\ j = 1,2, \ldots, s)$ denote the determinant

$$D(g_s, g_{s-1}, \ldots, g_1) = \begin{vmatrix} x_s^{g_s} & x_s^{g_{s-1}} & x_s^{g_1} \\ \vdots & & \vdots \\ x_1^{g_s} & x_1^{g_{s-1}} & x_1^{g_1} \end{vmatrix} .$$

If $a_r (r \leq s)$ denotes the r^{th} esf in s x's, then

1) $$a_r D(g_s, g_{s-1}, \ldots, g_1) = \Sigma' \, D(g_s', g_{s-1}', \ldots, g_1')$$

where $g_j' = g_j + \delta$, $j = 1,2, \ldots, s$, $\delta = 0,1$ and Σ' denotes the sum over the $\binom{s}{r}$ combinations of s g_s' taken r at a time for which r indices $g_j' = g_j + 1$ such that $\delta = 1$ while for other indices $g_j' = g_j$ such that $\delta = 0$.

2) $(a_r)^k (a_h)^\ell D(g_s, g_{s-1}, \ldots, g_1)$ $(k, \ell \geq 0)$ can be expressed as a sum of $\binom{s}{r}^k \binom{s}{h}^\ell$ determinants obtained by performing on $D(g_s, g_{s-1}, \ldots, g_1)$ in any order 1) k times and 1) ℓ times with r = h.

The proof of this lemma is given in [15].

Now let

$$(2.1)\qquad V(m+s-1+q_s, \ldots, m+q_1 ; n)$$

$$= \begin{vmatrix} \int_0^1 \theta_s^{m+s-1+q_s} (1-\theta_s)^n \, d\theta_s & \cdots & \int_0^1 \theta_s^{m+q_1} (1-\theta_s)^n \, d\theta_s \\ \int_0^{\theta_2} \theta_1^{m+s-1+q_s} (1-\theta_1)^n \, d\theta_1 & \cdots & \int_0^{\theta_2} \theta_1^{m+q_1} (1-\theta_1)^n \, d\theta_1 \end{vmatrix}$$

and let

$$U(m+s-1+q_s, \ldots, m+q_1 ; r)$$

$$(2.2)\qquad = \begin{vmatrix} \int_0^\infty \frac{\lambda_s^{m+s-1+q_s}}{(1+\lambda_s)^r} \, d\lambda_s & \cdots & \int_0^\infty \frac{\lambda_s^{m+q_1}}{(1+\lambda_s)^r} \, d\lambda_s \\ \int_0^{\lambda_2} \frac{\lambda_1^{m+s-1+q_s}}{(1+\lambda_1)^r} \, d\lambda_1 & \cdots & \int_0^{\lambda_2} \frac{\lambda_1^{m+q_1}}{(1+\lambda_1)^r} \, d\lambda_1 \end{vmatrix}$$

$$q_j \geq 0 \qquad j = 1, 2, \ldots, s, \quad \text{and } r = m+n+s+1$$

Now, from lemma 1 and (1.1), it is easy to see that the k^{th} moment $\mu_k' \{V_{i;m,n}^{(s)}\}$ of $V_{i,m,n}^{(s)}$, the i^{th} esf in s θ's, can be expressed as a linear compound of determinants of the V type in (2.1) where $q_s, q_{s-1}, \ldots, q_1$ may take different sets of values in different terms. Further, the coefficients of the linear compound would involve as a common factor $C(s,m,n)$ but otherwise would be independent of m and n.

Similarly, $\mu_k'\{U^{(s)}_{i,m,n}\}$, the k^{th} moment of the esf in the λ's can be shown to be a linear compound of the determinants of the U-type in (2.2). Now we state a second lemma [15].

Lemma 2. $\mu_k'\{U^{(s)}_{i,m,n}\}$ is derivable for $\mu_k'\{V^{(s)}_{i,m,n}\}$ by making the following changes in the expression for the latter (obtained by evaluating the linear compound of V-type determinants): (a) Multiply by -1 all terms except the term in n in each linear factor involving n and (b) change n to $m + n + s + 1$ after performing (a).

A proof of the lemma is given in [15]. We may illustrate lemmas 1 and 2 by considering the first moments of $V^{(s)}_{i,m,n}$ and $U^{(s)}_{i,m,n}$. Using lemma 1 we get [15]

$$(2.3) \quad \mu_1'\{V^{(s)}_{i,m,n}\} = C(s,m,n)\, V\,(m+s,\ m+s-1,\ \ldots,\ m+s-i+1,\ m+s-i-1,\ldots,\ m+1,\ m\ ;\ n),$$

$$= \binom{s}{i} \prod_{j=1}^{i} [(2m+s-j+2)\,/\,(2m+2n+2s-j+3)]\ .$$

From (2.3) using lemma 2

$$(2.4) \quad \mu_1'\{U^{(s)}_{i,m,n}\} = \binom{s}{i} \prod_{j=1}^{i} [(2m+s-j+2)\,/\,(2n+j-1)]\ .$$

For further results see [15].

3. Non-Central multivariate beta distribution. Let $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ be two positive definite matrices of order p, $\underset{\sim}{A}_1$ having a Wishart distribution [2,20] with f_1 degrees of freedom and $\underset{\sim}{A}_2$ (pseudo) non-central (linear) Wishart distribution [1, 3, 4, 20, 21] with f_2 degrees of freedom. Now transform

$$\underset{\sim}{A}_2 = \underset{\sim}{C}\,\underset{\sim}{Y}\,\underset{\sim}{Y}'\,\underset{\sim}{C}'$$

where $\underset{\sim}{C}$ is a lower triangular matrix such that

$$\underset{\sim}{A}_1 + \underset{\sim}{A}_2 = \underset{\sim}{C}\,\underset{\sim}{C}'$$

and the density function of $\underset{\sim}{Y} : p \times f_2$ is given by

$$(3.1) \qquad k_1\, e^{-\lambda^2} \sum_{j=0}^{\infty} (2\lambda\, y_{11})^j \;\Gamma\!\left[\tfrac{1}{2}(f_1 + f_2 + j)\right] \left|\underset{\sim}{I}_p - \underset{\sim}{Y}\underset{\sim}{Y}'\right|^{\frac{1}{2}(f_1 - p - 1)} / j!$$

where $\underset{\sim}{I}_p$ is an identity matrix of order p,

$$k_1 = \prod_{i=2}^{p} \Gamma[\tfrac{1}{2}(f_1 + f_2 - i + 1)] / \, \Pi^{\frac{1}{2}pf_2} \prod_{i=1}^{p} \Gamma[(f_1 - i + 1)/2] \; ,$$

λ is the only non-centrality parameter in the linear case and y_{11} is the element in the top left corner of the $\underline{Y}$ matrix.

Now $V^{(s)}$ criterion suggested by Pillai and $U^{(s)}$ (a constant times Hotelling's T_0^2), are the sums of the non-zero characteristic roots of the matrix $\underset{\sim}{Y}\underset{\sim}{Y}'$ and $(\underset{\sim}{I}_p - \underset{\sim}{Y}\underset{\sim}{Y}')^{-1} - \underset{\sim}{I}_p$ respectively. Here s is minimum (f_2, p). Also we may note that $V^{(s)}$ = trace $\underset{\sim}{Y}\underset{\sim}{Y}'$ = trace $\underset{\sim}{Y}'\underset{\sim}{Y}$ and $U^{(s)} = \mathrm{tr}(\underset{\sim}{I}_p - \underset{\sim}{Y}\underset{\sim}{Y}')^{-1} - p = \mathrm{tr}(\underset{\sim}{I}_{f_2} - Y'Y)^{-1} - f_2$. It can be shown that the density function of $Y'Y$ for $f_2 \leq p$ can be obtained from the density function of $Y\,Y'$ for $f_2 > p$ if in the latter case the following changes are made [5,20]

$$(3.2) \qquad (f_1, f_2, p) \longrightarrow (f_1 - f_2 + p, p, f_2).$$

Hence, for the criterion $V^{(s)}$, (and similarly for $U^{(s)}$), we shall only consider the density function of $\underset{\sim}{L} = \underset{\sim}{Y}\,\underset{\sim}{Y}'$ for $f_2 \geq p$ which is given by [8]

$$(3.3)\qquad f(\underset{\sim}{L}) = k\, e^{-\lambda^2}\, {}_1F_1\{\tfrac{1}{2}(f_1+f_2),\ \tfrac{1}{2}f_2,\ \lambda^2\, \ell_{11}\}\, |\underset{\sim}{L}|^{(f_2-p-1)/2} \times |\underset{\sim}{I}_p-\underset{\sim}{L}|^{(f_1-p-1)/2}$$

where

$$k = \pi^{-p(p-1)/4} \prod_{i=1}^{p} \Gamma[\tfrac{1}{2}(f_1+f_2+1-i)]/\{\Gamma[\tfrac{1}{2}(f_1+1-i)]\ \Gamma[\tfrac{1}{2}(f_2+1-i)]\}\ ,$$

ℓ_{11} is the element in the top left corner of the matrix $\underset{\sim}{L}$ and ${}_1F_1$ denotes the confluent hypergeometric function. We shall call the distribution of L: pXp the non-central (linear) multivariate beta distribution with f_2 and f_1 degrees of freedom. It may be noted that m and n of the previous sections are given by $m = (f_2-p-1)/2$ and $n = (f_1-p-1)/2$.

Pillai [17] had noted that the elements of the matrix $\underset{\sim}{L}$ can be transformed into independent beta variables which he showed for p = 2,3, 4 and 5. In the following section a theorem is given [6] which proves the general case.

4. Independent beta variables. Let

$$L = \begin{array}{cc} \begin{pmatrix} \ell_{11} & \ell' \\ \ell & \underset{\sim}{L}_{11} \end{pmatrix} & \begin{array}{l} 1 \\ p-1, \end{array} \\ \begin{array}{cc} 1 & \quad p-1 \end{array} & \end{array} \qquad \underset{\sim}{L}_{22} = \underset{\sim}{L}_{11} - \ell\ell'/\ell_{11}\ ,$$

and we note that

$$|\underset{\sim}{L}| = \ell_{11}\, |\underset{\sim}{L}_{22}|$$

and

$$\left|\underset{\sim}{I}_p - \underset{\sim}{L}\right| = (1 - \ell_{11}) \left|\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22} - \underset{\sim}{\ell}\underset{\sim}{\ell}' / [\ell_{11}(1 - \ell_{11})]\right| .$$

Then it is easy to show that

$$\ell_{11} \text{ and } (\underset{\sim}{L}_{22}, \ \underset{\sim}{v} = \underset{\sim}{\ell}/\sqrt{\ell_{11}(1 - \ell_{11})}\ )$$

are independently distributed and their respective distributions are

$$(4.1) \quad f_1(\ell_{11}) = [\beta(\tfrac{1}{2}f_2,\tfrac{1}{2}f_1)]^{-1}\exp(-\lambda^2)\ell^{\frac{1}{2}f_2-1}(1-\ell_{11})^{\frac{1}{2}f_1-1}{}_1F_1[\tfrac{1}{2}(f_1+f_2),$$

$$\tfrac{1}{2}f_2,\lambda^2\ell_{11}]$$

and

$$(4.2) \quad f_2(\underset{\sim}{L}_{22},\underset{\sim}{v}) = k_2 \ \underset{\sim}{L}_{22}^{\ \frac{1}{2}[(f_2-1)-(p-1)-1]} \left|\underset{\sim}{I}_{p-1}-\underset{\sim}{L}_{22}-\underset{\sim}{v}\,\underset{\sim}{v}'\right|^{\frac{1}{2}(f_1-p-1)}$$

where

$$k_2 = k\ \beta(\tfrac{1}{2}f_2,\tfrac{1}{2}f_1).$$

For further independence, we can use two types of transformations given by

$$(4.3) \quad \underset{\sim}{u} = (\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22})^{-\frac{1}{2}}\underset{\sim}{v} \quad \text{or} \quad \underset{\sim}{w} = \underset{\sim}{T}^{-1}\underset{\sim}{v}$$

where $\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22} = \underset{\sim}{T}\,\underset{\sim}{T}'$ and $\underset{\sim}{T}:(p-1) \times (p-1)$ is a lower triangular matrix. It is easy to show that $\underset{\sim}{u}$ (or $\underset{\sim}{w}$) and $\underset{\sim}{L}_{22}$ are independently distributed and their respective distributions are

$$(4.4)\quad f_3(\underset{\sim}{u}) = \Pi^{-\frac{1}{2}(p-1)} \frac{\Gamma(\frac{1}{2} f_1)}{\Gamma(\frac{f_1-p+1}{2})} (1-u'u)^{\frac{1}{2}(f_1-p-1)} \quad [\text{or } f_3(\underset{\sim}{w})]$$

and

$$(4.5)\quad f_4(\underset{\sim}{L}_{22}) = k_3 \, \underset{\sim}{L}_{22}^{\frac{1}{2}[f_2-1)-(p-1)-1]} \left| I-L_{22} \right|^{\frac{1}{2}[f_1-(p-1)-1]}$$

where $k_3 = \Pi^{\frac{1}{2}(p-1)} \left\{\Gamma(\frac{f_1-p+1}{2}) \Big/ \Gamma(\frac{f_1}{2})\right\} k_2$. We may note that the distribution of $\underset{\sim}{L}_{22}:(p-1)x(p-1)$ is central multivariate beta distribution with (f_2-1) and f_1 degrees of freedom, and the similar reduction from $\underset{\sim}{L}_{22}$ can be carried successively. We may also note that the transformation

$$(4.6)\qquad x_i = u_i^2/(1-u_1^2-\ldots-u_{i-1}^2), \quad i = 1,2,\ldots,p-1 \quad , u_o = 0$$

in (4.4) gives us the independent beta-variates and their density functions are given by

$$(4.7)\qquad g_i(x_i) = \{\beta[\tfrac{1}{2}, \tfrac{1}{2}(f_1-i)]\}^{-1} \, x_i^{\frac{1}{2}-1} (1-x_i)^{\frac{1}{2}(f_1-i)-1}$$

From the foregone, we have the following theorem:

<u>Theorem I</u>: If the distribution of $\underset{\sim}{L} = \begin{pmatrix} \ell_{11} & \underset{\sim}{\ell}' \\ \underset{\sim}{\ell} & \underset{\sim}{L}_{11} \end{pmatrix}$ is given by (3.3),

then ℓ_{11}, $\underset{\sim}{L}_{22} = \underset{\sim}{L}_{11} - \underset{\sim}{\ell}\,\underset{\sim}{\ell}'/\ell_{11}$ and $\underset{\sim}{u} = (\underset{\sim}{I}_{p-1}-\underset{\sim}{L}_{22})^{\frac{1}{2}} \underset{\sim}{\ell}/\sqrt{\ell_{11}(1-\ell_{11})}$

[or $\underset{\sim}{w} = \underset{\sim}{T}^{-1} \underset{\sim}{\ell}/\sqrt{\ell_{11}(1-\ell_{11})}$ where $\underset{\sim}{T}\,\underset{\sim}{T}' = \underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22}$ and $\underset{\sim}{T}$ is a lower triangular matrix] are independently distributed and their respective distributions are defined in (4.1), (4.5) and (4.4).

It can be verified for p=3 that from the variates ℓ_{11}, $\underset{\sim}{w}$ and $\underset{\sim}{L}_{22}$, we can obtain the independent beta-variates exactly the same as given by Pillai [17] but the use of ℓ_{11}, $\underset{\sim}{u}$ and $\underset{\sim}{L}_{22}$ will give independent beta variables different from those of Pillai [17] in spite of the identical beta distributions.

5. <u>The moments of $V^{(p)}$ in the linear case.</u> It is easy to see that

$$(5.1) \qquad V^{(p)} = \ell_{11} + (1 - \ell_{11})\,\underset{\sim}{u}'\,(\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22})\underset{\sim}{u} + \mathrm{tr}\,\underset{\sim}{L}_{22}$$

$$= -(1 - \ell_{11})(1 - \underset{\sim}{u}'\underset{\sim}{u} + \underset{\sim}{u}'\underset{\sim}{L}_{22}\underset{\sim}{u}) + \mathrm{tr}\,\underset{\sim}{L}_{22} + 1.$$

Now let θ_i' $(i=1,\ldots,p-1)$ be the characteristic roots of $\underset{\sim}{L}_{22}$. Then, from (5.1)

$$(5.2) \qquad V^{(p)} - 1 = \sum_{i=1}^{p-1} \theta_i' - (1-\ell_{11})\,[\,(1-\underset{\sim}{u}'\underset{\sim}{u}) + \sum_{i=1}^{p-1} \theta_i' u_i^2\,].$$

Again, let $\ell_{11,0}$ be a variate whose distribution is the same as that of ℓ_{11} when $\lambda = 0$ and independently distributed of $\underset{\sim}{u}$ and $\underset{\sim}{L}_{22}$. Let $V_o^{(p)}$ be the $V^{(p)}$ statistic when $\lambda = 0$. Then we may note that

$$(5.3) \qquad x_1 = E(1-\ell_{11;0}) - E(1-\ell_{11}) = f_1\,\delta(\nu)$$

$$(5.4) \qquad x_2 = E(1-\ell_{11.0})^2 - E(1-\ell_{11})^2 = f_1(f_1+2)\,\Delta_1/2$$

$$(5.5)\qquad x_3 = E(1-\ell_{11.0})^3 - E(1-\ell_{11})^3 = f_1(f_1+2)(f_1+4)\ \Delta_2/8$$

and

$$(5.6)\qquad x_4 = E(1-\ell_{11.0})^4 - E(1-\ell_{11})^4 = f_1(f_1+2)(f_1+4)(f_1+6)\ \Delta_3/48$$

where $\nu = f_1 + f_2$

$$(5.7)\qquad \delta(\nu) = \exp(-\lambda^2) \sum_{i=0}^{\infty} (\lambda^2)^{i+1} / [i!\ (\tfrac{1}{2}\nu + i + 1)\nu] \ ,$$

$$\Delta_1 = \delta(\nu) - \delta(\nu+2)\ , \quad \Delta_2 = \delta(\nu) - 2\delta(\nu+2) + \delta(\nu+4)$$

and $$\Delta_3 = \delta(\nu) - 3\delta(\nu+2) + 3\delta(\nu+4) - \delta(\nu+6)\ .$$

The results (5.3) -- (5.6) are obtained by using partial fractions for $1/\nu(\nu+2)(\nu+4)\ \ldots$.

Now using (5.2) -- (5.6)

$$(5.8)\qquad E[V^{(p)} - 1] = E[V_o^{(p)} - 1] + x_1\ E(\beta)$$

$$(5.9)\qquad E[V^{(p)} - 1]^2 = E[V_o^{(p)} - 1]^2 - x_2\ E(\beta^2) + 2x_1\ E(\alpha\beta)$$

$$(5.10)\qquad E[V^{(p)} - 1]^3 = E[V_o^{(p)} - 1]^3 + x_3E(\beta^3) - 3x_2E(\alpha\beta^2) + 3x_1E(\alpha^2\beta)$$

and

$$(5.11)\qquad E[V^{(p)} - 1]^4 = E[V_o^{(p)} - 1]^4 - x_4E(\beta^4) + 4x_3E(\alpha\beta^3) - 6x_2E(\alpha^2\beta^2) + 4x_1E(\alpha^3\beta),$$

where

$$\alpha = \mathrm{tr}\underset{\sim}{L}_{22} \quad \text{and} \quad \beta = 1 - \underset{\sim}{u}'\underset{\sim}{u} + \underset{\sim}{u}'\underset{\sim}{L}_{22}\underset{\sim}{u}.$$

Using (4.4) and the fact that θ_i's are the characteristic roots of $\underset{\sim}{L}_{22}$, after some lengthy computations [7] we can show that

$$(5.12) \quad E(\alpha^i\beta) = \frac{1}{f_1} \; E[\mathrm{tr}\underset{\sim}{L}_{22})^i f^{(1)} + (\mathrm{tr}\underset{\sim}{L}_{22})^{i+1}] = \frac{1}{f_1} r_i^{(1)}; \quad \text{for } i = 0,1,2,3 \; ;$$

$$(5.13) \quad E(\alpha^i\beta^2) = \frac{1}{f_1(f_1+2)} \; E(\mathrm{tr}\underset{\sim}{L}_{22})^i[f^{(2)} + 2f^{(1)}(\mathrm{tr}\underset{\sim}{L}_{22}) + 3(\mathrm{tr}L_{22})^2 - 4(\mathrm{tr}_2\underset{\sim}{L}_{22})] = \frac{1}{f_1(f_1+2)} \; r_i^{(2)} \text{ for } i = 0,1,2 \; ;$$

$$(5.14) \quad E(\alpha^i\beta^3) = \frac{1}{f_1(f_1+2)(f_1+4)} \; E(\mathrm{tr}\underset{\sim}{L}_{22})^i[f^{(3)} + 3f^{(2)}(\mathrm{tr}\underset{\sim}{L}_{22}) + 3f^{(1)}\{3(\mathrm{tr}\underset{\sim}{L}_{22})^2 - 4(\mathrm{tr}_2\underset{\sim}{L}_{22})\} + \{15(\mathrm{tr}\underset{\sim}{L}_{22})^3 - 36(\mathrm{tr}\underset{\sim}{L}_{22})(\mathrm{tr}_2\underset{\sim}{L}_{22}) + 24 \; \mathrm{tr}_3\underset{\sim}{L}_{22}] = \frac{1}{f_1(f_1+2)(f_1+4)} \; r_i^{(3)}$$

for $i = 0,1,$

$$(5.15) \quad E(\beta^4) = \frac{1}{f_1(f_1+2)(f_1+4)(f_1+6)} \quad E[f^{(4)} + 4f^{(3)}(\mathrm{tr}\underset{\sim}{L}_{22}) + 6f^{(2)}\{3(\mathrm{tr}\underset{\sim}{L}_{22})^2 - 4(\mathrm{tr}_2\underset{\sim}{L}_{22})\} + 4f^{(1)}\{15(\mathrm{tr}\underset{\sim}{L}_{22})^3 - 36(\mathrm{tr}\underset{\sim}{L}_{22})(\mathrm{tr}_2\underset{\sim}{L}_{22}) + 24(\mathrm{tr}_3\underset{\sim}{L}_{22})\} + 105(\mathrm{tr}\underset{\sim}{L}_{22})^4$$

$$-360(\mathrm{tr}\underset{\sim}{L}_{22})^2(\mathrm{tr}_2\underset{\sim}{L}_{22})+288(\mathrm{tr}\underset{\sim}{L}_{22})(\mathrm{tr}_3\underset{\sim}{L}_{22})-192(\mathrm{tr}_4\underset{\sim}{L}_{22})$$

$$+144(\mathrm{tr}_2\underset{\sim}{L}_{22})^2] = \frac{1}{f_1(f_1+2)(f_1+4)(f_1+6)} \quad r_0^{(4)} \ ,$$

where

$$(5.16) \qquad f^{(i)} = \prod_{j=1}^{i} (f_1-p-1+2j) \ .$$

Hence, we have

$$(5.17) \qquad E(V^{(p)}-1) = E(V_0^{(p)}-1) + \delta(\nu)\ r_0^{(1)} \ ,$$

$$(5.18) \qquad E(V^{(p)}-1)^2 = E(V_0^{(p)}-1)^2+\delta(\nu)[2r_1^{(1)}-\tfrac{1}{2}r_0^{(2)}]+(\tfrac{1}{2}r_0^{(2)})\delta(\nu+2)$$

$$(5.19) \qquad E(V^{(p)}-1)^3 = E(V_0^{(p)}-1)^3+\delta(\nu)[3r_2^{(1)}-\tfrac{3}{2}r_1^{(2)}+\tfrac{1}{8}r_0^{(3)}]$$

$$+\delta(\nu+2)[\tfrac{3}{2}r_1^{(2)}-\tfrac{1}{4}r_0^{(3)}]+\delta(\nu+4)[\tfrac{1}{8}r_0^{(3)}] \ ,$$

and

$$(5.20) \qquad E(V^{(p)}-1)^4 = E(V_0^{(p)}-1)^4+\delta(\nu)[4r_3^{(1)}-3r_2^{(2)}+\tfrac{1}{2}r_1^{(3)}-\frac{1}{48}r_0^{(4)}]$$

$$+\delta(\nu+2)[9r_2^{(2)}-r_1^{(3)}+\frac{1}{16}\ r_0^{(4)}] \cdot$$

$$+\delta(\nu+4)\left[\tfrac{1}{2}r_1^{(3)}-\frac{1}{16}r_0^{(4)}\right]+\delta(\nu+6)\left[\frac{1}{48}r_0^{(4)}\right].$$

Now, using the lemma 1 of [15] and results of V type determinants in [18] the moments of $V^{(p)}$ can be obtained [6].

Hence, we get

$$E(V^{(p)}) = 1+\frac{(p-1)(f_2-1)}{\nu-1}+f_1\left\{\frac{p-1}{\nu-1}-1\right\}a_1 \tag{5.21}$$

and

$$E(V^{(p)})^2 = 1+\frac{(p-1)(f_2-1)}{\nu-1}\left\{2+\frac{f_2+1}{\nu+1}+\frac{(p-2)(f_2-2)}{\nu+2}\right. \tag{5.22}$$

$$+\frac{f_1(p-2)}{(\nu+1)(\nu-2)}-2\left[f_1\left\{1-\frac{p-1}{\nu-1}+\frac{(p-1)(f_2-1)}{\nu-1}\right\}\right.$$

$$+\frac{(p-1)(f_2-1)}{\nu-1}\left\{1-p+\frac{f_2+1}{\nu+1}+\frac{(f_2-2)(p-2)}{\nu-2}\right.$$

$$\left.\left.+\frac{f_1(p-2)}{(\nu+1)(\nu+2)}\right\}\right]a_1+f_1(f_1+2)\left[1-\frac{2(p-1)}{\nu-1}+\frac{3(p-1)}{(\nu-1)(\nu+1)}\right.+$$

$$\left.+\frac{(p-1)(p-2)}{(\nu-1)(\nu-2)(f_1+2)}\left\{f_1-1+\frac{3(f_2-1)}{\nu+1}\right\}\right]a_2,$$

where

$$(5.23) \qquad a_1 = \left\{ \sum_{i=0}^{\infty} (\lambda^2)^i / [i!(\nu+2i)] \right\} \exp(-\lambda^2) ,$$

and

$$(5.24) \qquad a_2 = \left\{ \sum_{i=0}^{\infty} (\lambda^2)^i / [i!(\nu+2i)(\nu+2i+2)] \right\} \exp(-\lambda^2) .$$

$E(V^{(p)})^3$ and $E(V^{(p)})^4$ are given in [7].

6. Moments of $U^{(p)}$ in the linear case. It may be noted that

$$(6.1) \qquad U^{(p)} = \mathrm{tr}(\underset{\sim}{I}_p - \underset{\sim}{L})^{-1} - p .$$

Since, as pointed out in section 1, $\lambda_i = \theta_i/(1-\theta_i)(i=1,\ldots,p)$

Now consider the following lemma.

Lemma 3. If $\underset{\sim}{L}$: p x p is a symmetric and positive definite matrix and $U^{(p)} = \mathrm{tr}(\underset{\sim}{I}_p - \underset{\sim}{L})^{-1} - p$, then

$$(6.2) \qquad 1 + U^{(p)} = \{(1-\ell_{11})(1-\underset{\sim}{u}'\underset{\sim}{u})\}^{-1} + (1-\underset{\sim}{u}'\underset{\sim}{u})^{-1}(\underset{\sim}{u}'\underset{\sim}{M}\underset{\sim}{u}) + \mathrm{tr}\,\underset{\sim}{M}$$

where

$$L = \begin{pmatrix} \ell_{11} & \underset{\sim}{\ell}' \\ \\ \underset{\sim}{\ell} & \underset{\sim}{L}_{11} \end{pmatrix}, \quad \underset{\sim}{\ell}:(p-1)\mathrm{x}1 = \{\ell_{11}(1-\ell_{11})\}^{\frac{1}{2}} (\underset{\sim}{I}_{p-1}-\underset{\sim}{L}_{22})^{\frac{1}{2}} \underset{\sim}{u},$$

$$\underset{\sim}{L}_{22}:\ (p-1)\mathrm{x}(p-1) = \underset{\sim}{L}_{11} - \underset{\sim}{\ell}\underset{\sim}{\ell}'/\ell_{11} \quad \text{and} \quad \underset{\sim}{M}:(p-1)\mathrm{x}(p-1) = (\underset{\sim}{I}_{p-1}-\underset{\sim}{L}_{22})^{-1} - \underset{\sim}{I}_{p-1} .$$

<u>Proof:</u> We may note that

$$(\mathbf{I}_p - \mathbf{L})^{-1} = \begin{pmatrix} (1-\ell_{11})^{\frac{1}{2}} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I}_{p-1}-\mathbf{L}_{22})^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{\ell_{11}}\,\mathbf{u}' \\ -\sqrt{\ell_{11}}\,\mathbf{u} & \mathbf{I}_{p-1}-(1-\ell_{11})\mathbf{u}\mathbf{u}' \end{pmatrix}^{-1}$$

$$\begin{pmatrix} (1-\ell_{11})^{-\frac{1}{2}} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I}_{p-1}-\mathbf{L}_{22})^{-\frac{1}{2}} \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -\sqrt{\ell_{11}}\,\mathbf{u}' \\ -\sqrt{\ell_{11}}\,\mathbf{u} & \mathbf{I}_{p-1}-(1-\ell_{11})\mathbf{u}\,\mathbf{u}' \end{pmatrix}^{-1} = \begin{pmatrix} 1+\ell_{11}\,\mathbf{u}'\,\mathbf{u}/(1-\mathbf{u}'\mathbf{u}) & \sqrt{\ell_{11}}\,\mathbf{u}'/(1-\mathbf{u}'\mathbf{u}) \\ \sqrt{\ell_{11}}\,\mathbf{u}/(1-\mathbf{u}'\mathbf{u}) & \mathbf{I}_{p-1}+\mathbf{u}\mathbf{u}'/(1-\mathbf{u}'\mathbf{u}) \end{pmatrix}.$$

Hence

$$\text{(6.3)}\quad \operatorname{tr}(\mathbf{I}_p-\mathbf{L})^{-1} = 1-(1-\mathbf{u}'\mathbf{u})^{-1} + \{(1-\ell_{11})(1-\mathbf{u}'\mathbf{u})\}^{-1} + \operatorname{tr}(\mathbf{I}_{p-1}-\mathbf{L}_{22})^{-1}$$

$$+\mathbf{u}'(\mathbf{I}_{p-1}-\mathbf{L}_{22})^{-1}\mathbf{u}/(1-\mathbf{u}'\mathbf{u}).$$

From this, the lemma follows.

Now the distribution of M is given by

$$(6.4)\quad \prod_{i=1}^{p-1}\left[\Gamma\left(\frac{f_1+f_2-i}{2}\right)/\left\{\Gamma\left(\frac{f_1-i+1}{2}\right)\Gamma\left(\frac{f_2-i}{2}\right)\right\}\right]\pi^{\frac{-(p-1)(p-2)}{4}}$$

$$\frac{|M|^{\frac{1}{2}\{f_2-1-(p-1)-1\}}}{|I_{p-1}+M|^{\frac{1}{2}(f_1+f_2-1)}} .$$

Further

$$(6.5)\quad E(1-\ell_{11})^{-1}-E(1-\ell_{11,0})^{-1}=2\lambda^2/(f_1-2), E(1-\ell_{11})^{-2}-E(1-\ell_{11,0})^{-2}$$

$$=1/[(f_1-2)(f_1-4)][(2\lambda^2)^2+2(\nu-2)(2\lambda^2)] ,$$

$$E(1-\ell_{11})^{-3}-E(1-\ell_{11,0})^{-3}$$

$$=\left[\frac{(2\lambda^2)^3+3(\nu-2)(2\lambda^2)^2+3(\nu-2)(\nu-4)(2\lambda^2)}{(f_1-2)(f_1-4)(f_1-6)}\right]$$

and

$$E(1-\ell_{11})^{-4}-E(1-\ell_{11,0})^{-4}$$

$$=\left[\frac{(2\lambda^2)^4+4(\nu-2)(2\lambda^2)^3+6(\nu-2)(\nu-4)(2\lambda^2)^2+4(\nu-2)(\nu-4)(\nu-6)(2\lambda^2)}{(f_1-2)(f_1-4)(f_1-6)(f_1-8)}\right] .$$

Let $\beta_2 = 1/(1-u'u)$ and $\alpha_1 = \mathrm{tr}\underset{\sim}{M}+(u'\underset{\sim}{M}u)/(1-u'u)$;

(6.6) $E(\beta_2^i) = [(f_1-2)(f_1-4) \ldots (f_1-2i)]/[(f_1-p-1)(f_1-p-3) \ldots$

$(f_1-p-2i+1)]$ for $i = 1,2,3,4$;

$= \eta_i^{(0)}(f_1-2)(f_1-4) \ldots (f_1-2i)$.

(6.7) $$E(\alpha_1\beta_2^i) = \frac{(f_1-2)(f_1-4) \ldots (f_1-2i)(f_1-p-2i)}{(f_1-p-1)(f_1-p-3) \ldots (f_1-p-2i-1)} \quad E(\mathrm{tr}\underset{\sim}{M})$$

$=(f_1-2)(f_1-4) \ldots (f_1-2i)\ \eta_i^{(1)}$ for $i = 1,2,3.$

(6.8) $E(\alpha_1^2\beta_2^i)$

$$= \frac{(f_1-2)(f_1-4)\ldots(f_1-2i)}{(f_1-p-1)(f_1-p-3)\ldots(f_1-p-2i+1} E[\frac{(f_1-p-2i)(f_1-p-2i-2)}{(f_1-p-2i-1)(f_1-p-2i-3)}(\mathrm{tr}\underset{\sim}{M})^2$$

$$- \frac{4(\mathrm{tr}_2\underset{\sim}{M})}{(f_1-p-2i-1)(f_1-p-2i-3)}]$$

$= (f_1-2)(f_1-4)\ldots(f_1-2i)\ \eta_i^{(2)}$ for $i = 1,2$.

(6.9) $E(\alpha_1^3\beta_2)$

$$= \frac{(f_1-2)}{(f_1-p-1)(f_1-p-3)(f_1-p-5)(f_1-p-7)} E[(f_1-p-2)(f_1-p-4)(f_1-p-6)(\mathrm{tr}\underset{\sim}{M})^3$$

$$-12(f-p-4)(\mathrm{tr}\underset{\sim}{M})(\mathrm{tr}_2\underset{\sim}{M})+24(\mathrm{tr}_3\underset{\sim}{M})]$$

$$= (f_1-2)\,\eta_1^{(3)}.$$

Hence, we get

(6.10) $$E(1+U^{(p)}) = E(1+U_0^{(p)}) + (2\lambda^2)\ \eta_1^{(0)}$$

(6.11) $$E(1+U^{(p)})^2 = E(1+U_0^{(p)})^2+(2\lambda^2)^2\eta_2^{(0)}+2(2\lambda^2)[(\nu-2)\eta_2^{(0)}+\eta_1^{(1)}]$$

(6.12) $$E(1+U^{(p)})^3=E(1+U_0^{(p)})^3+(2\lambda^2)^3\eta_3^{(0)}+3(2\lambda^2)^2[(\nu-2)\eta_3^{(0)}+\eta_2^{(1)}]$$

$$+3(2\lambda^2)[(\nu-2)(\nu-4)\eta_3^{(0)}+2(\nu-2)\eta_2^{(1)}+\eta_1^{(2)}]$$

(6.13) $$E(1+U^{(p)})^4=E(1+U_0^{(p)})^4+(2\lambda^2)^4\eta_4^{(0)}+4(2\lambda^2)^3[(\nu-2)\eta_4^{(0)}+\eta_3^{(1)}]$$

$$+6(2\lambda^2)^2[(\nu-2)(\nu-4)\eta_4^{(0)}+2(\nu-2)\eta_3^{(1)}+\eta_2^{(2)}]$$

$$+4(2\lambda^2)[(\nu-2)(\nu-4)(\nu-6)\eta_4^{(0)}+3(\nu-2)(\nu-4)\eta_3^{(1)}+3(\nu-2)\eta_2^{(2)}+\eta_1^{(3)}] .$$

Now by using lemmas 1 and 2 and the values of the determinants in [18] the moments of $U^{(p)}$ are obtained as follows:

(6.14) $$E(U^{(p)}) = (pf_2+2\lambda^2)/(f_1-p-1),$$

and for $f_1 > p + 3$,

(6.15) $$Var(U^{(p)}) = \frac{2[4\lambda^4(f_1-p)+(4\lambda^2+pf_2)(f_1-1)(f_1+f_2-p-1)]}{(f_1-p)(f_1-p-1)^2(f_1-p-3)} .$$

The third and fourth moments are given in [7] .

Some comparative power function studies of Wilks' criterion, $V^{(s)}$ and $U^{(s)}$ in the linear case have been carried out and are presented in [17]. Approximations to the distributions of $V^{(s)}$ and $U^{(s)}$ also have been attempted [17] .

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