# A Special Team Decision Problem and its Solution through Stochastic Programming

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Suppose a team of n members has the following problem. The team observes an n dimensional Bernoulli random variable $X = (X_1, X_2, \ldots, X_n)$. For example, X could indicate the availability of an item in each market location, where outcome $x_i = 1$ implies the item is available at market i and $x_i = 0$ implies the item is not available. The team is to define a real, non negative decision vector $a = (a_1, a_2, \ldots, a_n)$. This could be the quantity of the item to be purchased at each market. This decision vector is to be chosen prior to observing the market.

From the observed value of the random event and the predetermined decision vector we shall assume that the team experiences a team outcome defined by the inner product

$$(1) \qquad r = \rho(x,a) = \sum_{i=1}^{n} a_i x_i = (a,x).$$

Various values of r have different effects on the team, some of which are preferred over others. A numerical function that describes the preference ordering of the possible values of r is called a utility function. Given a utility function $v$ and an outcome function $\rho$, the team experiences a loss as defined by

$$(2) \qquad w(x,a) = v(r) = v\,[\rho(x,a)]$$

Let us assume $v$ is a piecewise linear, convex function in $r$, with a minimum at $r = r_o$, and the objective is to minimize $v$. In the context of the marketing example given above this would imply that the most desirable total quantity to be purchased by the team is $r_o$, and furthermore, as the total quantity purchased varies from $r_o$, the team suffers increasingly severe losses. The objective of the team, of course, is to determine the decision vector, a, such that this loss is minimized.

This objective is impossible to obtain as long as the value of $X$ is unknown. However, if the joint probability function, $\varphi_X(x)$, is known, then the team can determine a decision rule, a, which minimizes the expected loss

$$(3) \qquad W(a) = E\,[w(X,a)] = \sum_x v\,[\rho(x,a)]\,\varphi_X(x)$$

In this case the decision rule is a Bayes decision rule. The problem is to find values of the decision variable, a, that minimize $W$, the expected value of v. It is not evident that a minimum exists, and if so, how this can be found.

It is evident that $W$ is a piecewise linear, convex function of $r$, since, $W$, is a linear combination (with all coefficients non negative) of convex piecewise linear functions.

In addition, the set of all possible decision vectors is a convex set bounded from below. In fact, it is the non negative orthant. Thus the minimum of W exists, and furthermore, any local minimum is a global minimum.

If the vector $\hat{a}$ is the optimal decision rule, then $\hat{a}_i \leq r_o$. This is easily seen through a contradiction argument; that is, assume $\hat{a}_i > r_o$. Then define another (non-optimal decision rule $a \leq r_o$ such that

$$a_i = \begin{cases} \hat{a}_i & \text{if } \hat{a}_i \leq r_o \\ r_o & \text{if } \hat{a}_i > r_o \end{cases}$$

From this we see that

$$(\hat{a},x) > (a\ x)$$

and as a result, $v(\hat{a},x) \geq v(a,x)$, which in turn implies that $W(a) \leq W(\hat{a})$. Thus, if $\hat{a}$ is minimal, then so is $a$, and we can conclude that $a_i \leq r_o$.

Suppose that $P(X_i = 1) = \varphi_1$ for $i = 1, 2, \ldots, n$, where $\varphi_i$ is the appropriate marginal value of the $n$ dimensional probability function $\varphi_X$. We can say that if $\varphi_i = \varphi_j$, then $a_i = a_j$; also, if $\varphi_i < \varphi_j$, then $a_i \leq a_j$.

Since W is a convex, piecewise linear function, it is continuous, and has a derivative at all but a finite number of points. Let us denote each point where the derivative does not exist as a corner point. Then, we can say that Min W(a) occurs at a corner point of W, although such a corner point is not necessarily a unique minimum. The fact that the minimum occurs at a corner point implies that the problem is a finite one.

Let us now turn our attention to the specific problem of finding a decision rule, a, which minimizes W. The summation in (3) is over all the $2^n$ possible outcome locations of X in the n dimensional space. These $2^n$ points can be denoted as $(0,0,\ldots,0)$, $(1,0,0,\ldots,0)$, $(0,1,0,\ldots,0)\ldots$, $(0,0,\ldots,0,1)$ $(1,1,0,\ldots,0),\ldots$, $(1,1,\ldots,1)$, and the corresponding probabilities can be denoted $p_{00\ldots0}$, $p_{10\ldots0}$, $p_{010\ldots0},\ldots$, $p_{00\ldots1}$, $p_{110\ldots0},\ldots,_0$ $p_{11\ldots1}$. It is evident that the inner products are then, respectively, as follows: $0,a_1,a_2,\ldots,a_n,a_1+a_2,\ldots,a_1+a_2+\ldots+a_n$. If we let $2^n = N$, it is convenient to denote the inner products by, respectively, $0,a_1,a_2,\ldots,a_m,a_{n+1},\ldots,a_{N-1}$ where, of course, $a_{n+1} = a_1+a_2,\ldots,a_{N-1}=a_1+a_2+\ldots a_n$. Furthermore, we can denote the corresponding respective probabilities as $p_0,p_1,p_2,\ldots,p_n,p_{n+1},\ldots,p_{N-1}$. Now the problem can be stated as follows: Find values of the decision variables, $a_1$ $a_2,\ldots,$ $a_{N-1}$ which minimize

$$(4)\qquad W(a) = v(0)\, p_0 + \sum_{i=1}^{N-1} v(a_i)\, p_i$$

subject to the N-n-1 conditions

$$(5)\qquad \begin{aligned} &a_1+a_2-a_{n+1} = 0 \\ &\quad\vdots \\ &a_1+a_2+\ldots+a_n-a_N = 0 \end{aligned}$$

Thus the problem has been transformed to a problem of stochastic programming.

If $v(r)$ has a derivative everywhere, then the minimum of $W(a)$ can be found by classical methods. However, a more interesting problem occurs when $v(a)$ is a piecewise linear, convex function. They $W$ is also a piecewise linear, convex function, and thus the above results still hold.

Suppose $v(r)$ is a piecewise linear function. Then this problem can be easily solved by a rather well known procedure. First of all, it can be noted that each of the components of the sum in (4) is determined by different values of the same piecewise linear function. Suppose for each $i$ that $v(a_i)$ has $m-1$ corner points. Let the corner points occur at $a_{i1}, a_{i2}, \ldots, a_{i,m-1}$. Also let us denote by $\alpha_i$ an upper bound to the set of possible values of $a_i$. Of course, zero is a lower bound for this set of $a_i$. Now, any $a_i$ in the interval $[a_{ik}, a_{i,k+1}]$ can be written as

(6) $$a_i = y_{ik}a_{ik} + y_{i,k+1}\, a_{i,k+1} \qquad k = 0, 1, \ldots, m-1$$

where $y_{ik}+y_{i,k+1} = 1$, $y_{ik}, y_{i,k+1} \geq 0$, $a_{i0} = 0$ and $a_{im} = \alpha_i$ .

Likewise the corresponding value of $v$ is

(7) $$v(a_i) = y_{ik}v(a_{ik}) + y_{i,k+1}\, v(a_{i,k+1}), \qquad k = 0, 1, \ldots, m-1 ,$$

where again $y_{ik} + y_{i,k+1} = 1$, $y_{ik}, y_{i,k+1} \geq 0$, $a_{i0} = 0$ and $a_{im} = \alpha_i$ .

Indeed, for any $a_i \in [0,\alpha_i]$, $i = 1, 2, \ldots, n$ , we can write

$$a_i = \sum_{k=0}^{m} y_{ik} a_{ik} \quad , \quad a_{i0} = 0, a_{im} = \alpha_i$$

(8)
$$v(a_i) = \sum_{k=0}^{m} y_{ik} v(a_{ik})$$

$$\sum_{k=0}^{m} y_{ik} = 1$$

$$y_{ik} \geq 0, \quad k = 0, 1, \ldots, m$$

provided that at most two $y_{ik} \geq 0$ for each $i$; furthermore, if for any $i$ two $y_{ik}$ are positive and if $y_{ik} > 0$, then either $y_{i,k-1}$ or $y_{i,k+1}$ is also positive. That is, positive values must be adjacent.

Substituting from (8) the expressions for $a_i$ and $v(a_i)$ into (4) and (5) we have the following problem: Find the values of the $m(N-1)$ variables

$y_{11}, y_{12}, \ldots, y_{1m}, y_{21}, \ldots, y_{2m}, \ldots, y_{N-1,1}, \ldots, y_{N-1,m}$.

(9) $$W(y) = \sum_{i=1}^{N-1} \sum_{k=0}^{m} v(a_{ik}) \; y_{ik} p_i + v(0) \; p_0$$

Subject to the $N-n-1$ conditions

$$\sum_{k=0}^{m} a_{1k} y_{1k} + \sum_{k=0}^{m} a_{2k} y_{2k} - \sum_{k=0}^{m} a_{m+1,k} \, y_{m+1,k} = 0$$

(10)
$$\sum_{k=0}^{n} a_{1k}y_{1k} + \sum_{k=0}^{m} a_{2k}y_{2k} + \dots + \sum_{k=0}^{m} a_{nk}y_{nk} - \sum_{k=1}^{m} a_{Nk}y_{Nk} = 0$$

and also the N-1 conditions

(11)
$$\sum_{k=0}^{m} y_{ik} = 1 \qquad i = 1, 2, \dots, N-1$$

Thus the stochastic programming problem is now one of linear programming with the non linear restriction that for each i, at most two $y_{ik}$ are positive and then only if they adjacent. Such a restriction to adjacent positive values can be made readily in most simplex computer codes. In fact, many such codes are already available. Actually, in almost all cases in a series of trial problems the usual simplex method yielded a minimum which satisfied this condition of adjacent positive values.

After the $y_{ik}$ are determined, from (10) and (11) then $a_i$ can be found by substituting the $y_{ik}$ values into (8).

To illustrate, let us consider the following three memberteam example: that is, $n = 3$. Suppose that the $x_i$ are independent with parameters $\varphi_1 = .4$, $\varphi_2 = .5$ and $\varphi_3 = .6$, and suppose that $v(r) = |r - 1000|$; i.e., $r_o = 1{,}000$. Since $a_i \leq r_o$ for all i and $n = 3$, let us take $c_i = 3{,}000$. We first define the following decision variables:

$a_1, a_2, a_3, a_4 = a_1 + a_2,\ a_5 = a_1 + a_3,\ a_6 = a_2 + a_3,\ a_7 = a_1 + a_2 + a_3$ .

Furthermore, it is easy to see that because of independence, $p_0 = .120$, $p_1 = .080$, $p_2 = .120$, $p_3 = .180$, $p_4 = .080$, $p_5 = .120$, $p_6 = .180$, and $p_7 = .120$. The problem then is to find the minimum of

$$W(a) = .12v(0) + .08v(a_1) + .12v(a_2) + .18v(a_3) + .08v(a_4) + .12v(a_5) + .18v(a_6) + .12v(a_7)$$

where $v(r) = |r - 1{,}000|$, subject to the conditions

$$a_1 + a_2 - a_4 = 0$$

$$a_1 + a_3 - a_5 = 0$$

$$a_2 + a_3 - a_6 = 0$$

$$a_1 + a_2 + a_3 + a_7 = 0$$

Using the suggested transformation to linearity we can take $a_{i0} = 0$, $a_{i1} = 1{,}000$ and $a_{i2} = \alpha_i = 3{,}000$. There is only one corner point and this occurs when $a_i = 1{,}000$ for each $i$. The corresponding values of $v$ are $v(a_{i0}) = v(0) = 1{,}000$, $v(a_{i1}) = v(1000) = 0$ and $v(a_{i2}) = v(3{,}000 = 2{,}000$. Thus we can now restate the problem as follows: Find $y_{ik} \geq 0$, $i = 1, 2, \ldots, 7$ $k = 0, 1, 2$ which minimizes

$$W(y) = 120 + 80y_{10} + 120y_{20} + 180y_{30} + 80(y_{40} + 2y_{42}) + 120(y_{50} + 2y_{52}) + 180(y_{60} + 2y_{62}) + 240y_{72}$$

subject to the conditions

$$y_{11} + y_{21} - y_{41} - 3y_{42} = 0$$

$$y_{11} + y_{31} - y_{51} - 3y_{51} = 0$$

$$y_{21} + y_{31} - y_{61} - 3y_{61} = 0$$

$$y_{11} + y_{21} + y_{31} - y_{71} - 3y_{71} = 0$$

and the conditions

$$y_{10} + y_{11} = 1$$

$$y_{20} + y_{21} = 1$$

$$y_{30} + y_{31} = 1$$

$$y_{40} + y_{41} + y_{42} = 1$$

$$y_{50} + y_{51} + y_{52} = 1$$

$$y_{60} + y_{61} + y_{62} = 1$$

$$y_{71} + y_{72} = 1$$

and such that for each i, any pair of solutions must be adjacent. Note that it is assumed that $y_{i3} = 0$ since $a_i \leq 1{,}000$ for $i = 1, 2, 3,$ and also that $y_{70} = 0$ since $a_7 = a_1 + a_2 + a_3$. The solution to this problem as found by computer* calculations is as follows:

$y_{10} = .5$, $y_{11} = .5$, $y_{20} = .5$, $y_{21} = .5$, $y_{30} = .5$, $y_{31} = .5$, $y_{41} = 1$, $y_{51} = 1$, $y_{61} = 1$, $y_{71} = .75$, $y_{72} = .25$, and the rest

*Siemens 2002, Astronomisches Recheninstitut Heidelberg.

of the variables are zero. This gives a value of $W(y) = 370$. Substituting these values for $y_{ik}$ into expressions for $a_i$ we have: $a_1 = 500$, $a_2 = 500$, $a_3 = 500$ as the optimal decision rule. That is, each team member is to purchase an amount equal to 500 and the expected loss is 370 when using this rule.

A modification of this problem occurs when each $a_i$ is restricted to the range $[b_i, c_i]$ where $0 \leq b_i \leq c_i \leq 1$ for all i. In the marketing example, the lower limit could be interpreted as requiring the team member i to always purchase at least the amount $b_i$ if the product is available at his market. The upper limit, $c_i$, can be interpreted as the known amount of the product that would be available at the market if, perchance, any is available. Such restrictions as these do not create any additional difficulties outside of adding a few additional restraint inequalities.

In addition, the relative advantages that result from the various forms of communication are immediately available. To illustrate, suppose in numerical example there exists a one way communication, such as a telegram, from team member 2 to each of the other two team members. Let us suppose further that the message of this communication is his observation. In this case, the optimal rule is as follows: if $x_2 = 1$, then $a_1 = 0$, $a_2 = 1{,}000$, and $a_3 = 0$; on the other hand, if $x_2 = 0$, then $a_1 = 0$ $a_3 = 1{,}000$ and $a_2$ can be anything. The expected loss is now $W(a) = 200$, which is a reduction of 170 from the no communication case.

Thus, if telegrams cost less than 170, then it is to the advantage of the team to use this method of communication. In addition, the cost can be reduced to as low as 120 by complete two way communication among the team members.