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ON THE MOMENTS OF SOME ONE-SIDED STOPPING RULES

by

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1. Introduction. The moments of stopping rules (or stopping times) have been discussed in [1,3,4], and the following results have been proved. Let x_n be independent random variables with $Ex_1 = o$, $Ex_n^2 = 1$, and $S_n = x_1 + \ldots + x_n$. For $c > o$ and $m = 1, 2, \ldots$, define t_m to be the first $n \geq m$ such that $|S_n| > c\, n^{1/2}$. If $c \geq 1$, then $Et_1 = \infty$. If $P[|x_n| \leq K] = 1$ for some $K < \infty$ and $n = 1, 2, \ldots$, then $Et_m < \infty$ for every m if $c < 1$, $Et_m^2 < \infty$ for every m if $c < 3-\sqrt{6}$, and $Et_m^2 = \infty$ for some large m if $c > 3-\sqrt{6}$

In this note, we are interested in the following one-sided stopping rules, instead of the above stated two-sided stopping rules. For $c > o$ and $1 > p \geq o$, define

$$s = \text{first } n \geq 1 \text{ such that } S_n \geq c\, n^p \quad .$$

One of the results states that, if x_n are independent, $Ex_n = \mu > o$, and $Ex_n^2 - \mu^2 = \sigma^2 < \infty$, then $Es^2 < \infty$ and

$$(1) \qquad \lim_{c\to\infty} \mu^2\, Es^2/(c^2\, Es^{2p}) = \lim \mu Es^2/(c\, Es^{1+p}) = 1 \quad .$$

When $p = 0$, $Es^2 < \infty$ implies that $P[S_1 < c, \ldots, S_n < c] = P[s > n] = o(n^{-2})$ as $n \to \infty$, which completes a result of Morimura [8]. Also (1) extends the elementary renewal theorem from first moments to second moments and generalizes some results due to Chow and Robbins [2] and Hatori [6].

2. The first moment.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and x_n be a sequence of integrable random variables. Let $\mathcal{F}_n$ be the Borel field generated by $x_1, \ldots, x_n$ and $\mathcal{F}_0 = \{\phi, \Omega\}$. Put $S_n = x_1 + \ldots + x_n$, $S_0 = 0$, $m_n = E(x_n | \mathcal{F}_{n-1})$ and $T_n = \sum_1^n m_j$. Assume that for some constant $\infty > \mu > 0$ and for some null set N,

$$\lim_{n \to \infty} T_n / n = \mu, \text{ uniformly on } \Omega - N. \tag{2}$$

For $c > 0$ and $1 > p \geq 0$, define

$$s = \text{first } n \geq 1 \text{ such that } S_n \geq c\, n^p .$$

<u>Theorem 1</u>. (i) If for some $0 < \delta < \mu/3$, $P[x_n \leq m_n + n\delta] = 1$ for all large n, then $Es < \infty$. (ii) If $E([(x_n - m_n)^+]^\alpha | \mathcal{F}_{n-1}) \leq K < \infty$ for some $\alpha > 1$, then $Es < \infty$ and

$$\lim_{c \to \infty} \mu\, Es/(c\, Es^p) = 1 = \lim_{c \to \infty} ES_s/(c\, Es^p). \tag{3}$$

Proof. (i) Set $t = \min(s,k)$ for $k = 1,2,\ldots$. T
Then by the Wald identity for martingales [see 5, p. 302; or 3],

$$ET_t = ES_t = E(S_{t-1} + x_t) = \leq c\ Et^p + E(m_t + \delta t).$$

Let $0 < \epsilon < \delta$. As $k \to \infty$, by (2)

$$ET_t \geq (\mu - \epsilon)\ Et + O(1), \qquad Em_t = O(1) + o(Et).$$

Hence

$$(\mu-\epsilon)\ Et \leq c\ Et^p + \delta\ Et + O(1) + o(Et),$$

$$\int_{[s\leq k]} s\, dP + k\, P[s > k] = Et = O(1)\ ,$$

as $k \to \infty$. Therefore $P[s < \infty] = 1$ and $Es < \infty$.

(ii) For any $0 < \delta < \mu/3$, define $x_n' = \min(x_n, m_n + n\delta)$, $m_n' = E(x_n' | \mathcal{F}_{n-1})$, and $T_n' = m_1' + \ldots + m_n'$. Let $I(A)$ be the indicator function of the set A. Then

$$
\begin{aligned}
0 \leq m_n - m_n' &= E((x_n - m_n - n\delta) I[x_n > m_n + n\delta] | \mathcal{F}_{n-1}) \\
(4) \qquad &\leq E((x_n - m_n)\ I[x_n > m_n + n\delta] | \mathcal{F}_{n-1}) \\
&\leq E^{1/\alpha}([(x_n - m_n)^+]^\alpha | \mathcal{F}_{n-1})\ P^{1/\alpha'}(x_n - m_n > n\delta | \mathcal{F}_{n-1}) \qquad (\alpha + \alpha' = \alpha\alpha') \\
&\leq K(n\delta)^{-\alpha/\alpha'}
\end{aligned}
$$

Therefore $\lim_{n\to\infty} T_n'/n = \mu$ uniformly on $\Omega - N$. Define

$$t = \text{first } n \geq 1 \text{ such that } x_1' + \ldots + x_n' \geq c\, n^p .$$

Then $s \leq t$. By (i), $Et < \infty$. Therefore $Es < \infty$ and it follows by the Wald identity again [5, p. 302; or 3] that

$$(5) \qquad E(c\, s^p + x_s) \geq ES_s = ET_s \geq c\, Es^p .$$

Let $Z_n = \sum_1^n [(x_j - m_j)^+]^\alpha$. Then by Lemma 6 of [3],

$$(6) \qquad E^\alpha (x_s - m_s)^+ \leq EZ_s = E \sum_1^s E([(x_j - m_j)^+]^\alpha | \mathfrak{F}_{j-1}) \leq K\, Es .$$

Since (2) implies that as $c \to \infty$, $Em_s = O(1) + o(Es)$ and $ET_s = O(1) + (\mu + o(1))\, Es$, we have

$$(7) \qquad Ex_s = O(E^{1/\alpha} s) + o\,(Es) + O(1)$$

from (6), and

$$\lim_{c\to\infty} \mu\, Es/(c\, Es^p) = \lim ET_s/(c\, Es^p) = \lim ES_s/(c\, Es^p) = 1$$

from (5) and (7), since $\lim_{c\to\infty} Es = \infty$. The proof is completed.

When $p = 0$, part (ii) of Theorem 1 reduces to an elementary renewal theorem, which was proved in [2], in a slightly restricted form by requiring that $m_n = E(x_n)$ for each n.

3. The second moment.

Assume that $EX_n^2 < \infty$ for each n, let $V_n = \sum_1^n E((x_j - m_j)^2 | \mathcal{F}_{j-1})$ for $n = 1, 2, \ldots$, and define s as before. For a random variable y, put $||y|| = (Ey^2)^{1/2}$.

Theorem 2. If (2) holds and $E((x_n - m_n)^2 | \mathcal{F}_{n-1}) \leq K < \infty$, then $Es^2 < \infty$, $E\, S_s^2 < \infty$, and as $c \to \infty$

(8) $$ES_s^2 + ET_s^2 = EV_s + 2\, ES_s\, T_s ,$$

(9) $$\lim ES_s^2 / ET_s^2 = 1 ,$$

(10) $$\lim \mu^2\, Es^2 / (c^2\, Es^{2p}) = 1,$$

(11) $$\lim ES_s^2 / (c^2\, Es^{2p}) = 1,$$

(12) $$\lim \mu\, Es^2 / (c\, Es^{1+p}) = 1.$$

Proof. (i) First, assume that for some $o < \delta < \mu/8$ and $o < M < \infty$, $P[x_n \leq m_n + n\delta + M] = 1$. Set $t = \min(t, k)$ for $k = 1, 2, \ldots$. Then by Theorem 1 and Lemma 6 of [3],

$$E(S_t - T_t)^2 = EV_t \leq K\, Et .$$

Hence by Schwarz inequality

(13) $$ES_t^2 + ET_t^2 \leq K\, Et + 2||T_t|| \cdot ||S_t|| .$$

Assume, on the contrary, that $Es^2 = \infty$. Then $\lim_{k\to\infty} Et^2 = \infty$ and (2) implies that

$$Em_t^{\ 2} = O(1) + o(Et^2) = o(Et^2) ,$$

as $k \to \infty$. Hence

$$(14) \qquad ||S_t|| \le ||\ ct^p + m_t + \delta t + M|| \le c||t^p|| + \delta||t|| + o(||t||)$$
$$= (\delta + o(1)||t|| ;$$

and from (2),

$$(15) \qquad ET_t^{\ 2} = O(1) + (\mu^2 + o(1))\ Et^2 = (\mu^2 + o(1))\ Et^2 \quad .$$

By (13), (14) and (15), we have

$$1 + ES_t^{\ 2}/ET_t^{\ 2} \le O(||t||^{-1}) + 2||S_t||/||T_t||$$
$$\le O(||t||^{-1}) + (2\delta + o(1)/\mu = 2\delta/\mu + o(1) \quad .$$

Since $\delta < \mu/8$, we have a contradiction when k is large. Therefore $Es^2 < \infty$. From (14), (13) and Fatou's lemma $ES_s^{\ 2} < \infty$ and $ET_s^{\ 2} < \infty$.

(ii) For the general case, let $x_n' = \min\ (x_n, m_n + n\delta + M)$ for an arbitrary constant $\infty > M > o$ and $o < \delta < \mu/8$. Define m_n', T_n' and t as in the proof of part (ii) of Theorem 1. Then by (4) (for $\alpha=2$), $0 \le m_n - m_n' \le K(n\delta)^{-1}$ Hence

$$\lim T_n'/n = \mu \qquad \text{uniformly on } \Omega\text{-}N \ .$$

It is not too difficult to see that

$$E((x_n-m_n)^2|\mathcal{F}_{n-1})-E((x_n'-m_n')^2|\mathcal{F}_{n-1})\geq E([(x_n-m_n-n\delta-M)^+]^2|\mathcal{F}_{n-1})-$$
$$-E^2((x_n-m_n-n\delta-M)^+|\mathcal{F}_{n-1})\geq 0.$$

Therefore $E((x_n'-m_n')^2|\mathcal{F}_{n-1})\leq K$. Since $t \geq s$ and from part (i) $Et^2 < \infty$, we have that $Es^2 < \infty$. By Theorem 1 and Lemma 6 of [3] again,

$$(16) \qquad E(S_s-T_s)^2 = EV_s \leq K\ Es \ .$$

For $\epsilon > 0$, (2) implies that there exists a constant $\infty > L>0$ such that

$$ET_s^2 \leq L + (\mu^2 + \epsilon)\ Es^2 \ .$$

Hence $ET_s{}^2 < \infty$ and from (16), $ES_s{}^2 < \infty$. Thus (8) follows.

Now by (16),

$$|ES_s{}^2-ET_s{}^2|\leq E|S_s{}^2-T_s{}^2|\leq||S_s-T_s||\cdot||S_s+T_s||\leq(K\ Es)^{1/2}\ ||S_s+T_s|| \ .$$

Since $ES_s{}^2 \geq c^2\ Es^{2p}$, from (3)

$$|1-ET_s{}^2/ES_s{}^2|\leq(K\ Es/ES_s{}^2)^{1/2}(1+||T_s||/||S_s||) = o(1) + o(||T_s||/||S_s||)$$

as $c\to\infty$. Hence (9) follows.

Since (2) implies that $ET_s^{\,2} = O(1) + (\mu^2 + o(1))\, Es^2$ as $c\to\infty$, from (9)

(17) $$\lim_{c\to\infty} \mu^2\, Es^2/ET_s^{\,2} = 1 = \lim \mu^2\, Es^2/\, ES_s^{\,2} \;.$$

Let $Z_n = \sum_1^n (x_j - m_j)^2$. Applying Lemma 6 of [3], we have

$$E(x_s - m_s)^2 \le EZ_s = E \sum_1^s E((x_j - m_j)^2 | \mathcal{F}_{j-1}) \le KEs \quad .$$

From (2), $Em_s^{\,2} = O(1) + o\,(Es^2) = o\,(Es^2)$ as $c \to \infty$. Hence

(18) $$Ex_s^{\,2} = E(x_s - m_s + m_s)^2 = o\,(Es^2),\ ||x_s|| = o(||s||) \;.$$

Now from (18), as $c \to \infty$

(19) $$c||s^p|| \le ||S_s|| \le ||\ cs^p + x_s|| \le c||s^p|| + ||x_s|| = c||s^p|| + o\,(||s||).$$

Therefore (10) follows from (17) and (19), and (11) follows from (17) and (10).

Now $ET_s\, S_s = O(ES_s) + (\mu + o(1))\, Es\, S_s$ as $c\to\infty$. By the definition of s and (18), as $c\to\infty$

$$(20) \quad C\, Es^{1+p} \le Es\, S_s \le c\, Es^{1+p} + Es\, x_s \le c\, Es^{1+p} + ||s||\cdot||x_s||$$
$$\le c\, Es^{1+p} + o(Es^2)\ .$$

Since $EV_s \le KEs$, from (8), (9), (10), and (11), $\lim ES_s\, T_s/(\mu^2 Es^2) = 1$. Hence

$$\lim Es\, S_s/(\mu\, Es^2) = 1$$

and then (20) implies (12).

4. Corollaries and comments.

In this section we assume that x_n is a sequence of random variables and p=o. Define S_n, m_n, T_n, $\mathcal{F}_n$ and s as in Section 2.

Corollary 1. If (2) holds and if

$$(21) \qquad E((x_n - m_n)^2 | \mathcal{F}_{n-1}) \le K < \infty\ ,$$

then $Es^2 < \infty$ and

$$(22) \qquad \lim_{c\to\infty} Es^{\alpha}/c^{\alpha} = \mu^{-\alpha} \qquad \text{for } o \le \alpha \le 2\ .$$

Proof. Since (21) implies $E(x_n - m_n)^2 \le K$, from (2) and (21) it follows [7] that $\lim S_n/n = \mu$ a.e. Hence

$$1 \leq \liminf_{c\to\infty} S_s/c \leq \limsup \mu\, s/c = \limsup \mu(s-1)/c$$
$$= \limsup S_{s-1}/c \leq 1 \quad .$$

Therefore $\lim s/c = \mu^{-1}$ a.e. Theorem 2 implies that $E(s/c)^2 \leq M < \infty$ for all $c > o$. Hence [see 5, p. 629] for every $o \leq \alpha < 2$, $(s/c)^\alpha$ is uniformly integrable and

$$(23) \qquad \lim_{c\to\infty} E|\mu^{-1}-s/c|^\alpha = o, \quad \lim Es^\alpha/c^\alpha = \mu^{-\alpha} \quad .$$

Thus (22) follows from (23) and (10).

<u>Corollary 2</u>. Let x_n be a sequence of independent, identically distributed random variables such that $E\,x_1 > o$ and $E(x_1-Ex_1)^2 > \infty$. Then for every $c > o$, as $n \to \infty$,

$$(24) \qquad P[S_1 < c, \ldots, S_n < c] = o\,(n^{-2}) \quad .$$

<u>Proof</u>. Since $[s > n] = [S_1 < c, \ldots, S_n < c]$, $Es^2 < \infty$ implies (24) and thus Corollary 2 follows from Corollary 1.

(22) has been proved by Hatori [6] for every $\alpha > o$, by requiring, in addition to the assumptions of Corollary 1, that x_n be independent, $P[x_n \geq o] = 1$ and $m_n \geq L > o$ for each n.

Under the conditions of Corollary 2, Morimura [8] proves that $P[S_1 < c, \ldots, S_n < c] = O(n^{-\delta})$ for $o \leq \delta < (1 + \sqrt{5})/2$ and that there exists an example such that for some $D > o$ and for each $\epsilon > o$, $P[S_1 < c, \ldots, S_n < c] \geq D\, n^{-2-\epsilon}$ when n is large enough. Thus (24) is the best possible. Clearly, Corollary 2 completes Morimura's work.

The counter example in [8] satisfies the condition $Es^{2+\epsilon} = \infty$ for every $\epsilon > o$, since $P[s > n] \neq o\,(n^{-2-\epsilon})$. Therefore (22) can not be extended to the cases where $\alpha > 2$, without some conditions as $P[x_n \geq o] = 1$ imposed in [6].

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