On Some Selection and Ranking Procedures for Multivariate Normal Populations Using Distance Functions*

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Introduction and Summary

An important class of problems is concerned with the selection and ranking of k populations. For most univariate problems the selection and ranking has been defined in terms of location or scale parameters (see, [6], [8], [9], [10], [11], [12], [13], [18], [20], [21] and [22]). In problems dealing with multivariate populations, one is usually interested in the ranking and selection problems in terms of suitably defined functions of the several parameters. These functions are usually some scalar quantities. For example, for k multivariate normal populations with mean vectors $\mu_i'(i=1,2,\ldots,k)$ each of which has p components, a function that naturally arises and is of interest, is the Mahalanobis distance function $\lambda_i = \mu_i' \Sigma^{-1} \mu_i$ where Σ is the common covariance matrix of the k populations. Thus the ranking of multivariate populations in terms of λ_i reduces to the ranking with respect to parameters of non-centrality of non-central chi-square populations each with p degrees of freedom.

In this paper, we are primarily concerned with the selection problem for the k $(k \geq 2)$ non-central chi-square populations. We are interested in selecting populations with large values of the parameters λ_i, selecting as far as possible the best ones. The procedure to be defined is such that

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the probability is at least equal to a given number P^* $(1/k < P^* < 1)$ that the population with the largest value of the parameter is included in the selected subset. The size of the subset is an integer-valued random variable which takes values $1,2,\ldots,k$. We are clearly interested in procedures which select a non-empty subset which is small and yet large enough to guarantee the basic probability requirement and which have some desirable properties. In the above formulation, the loss connected with the selection and non-selection of the population with the largest value of the parameter i.e. the best population is 0 or 1 and the risk is bounded above: We guarantee that the least upper bound of this risk be $\leq 1-P^*$.

It should be pointed out that this ''selecting a subset'' formulation is different from the ''indifference zone'' approach where the selection is in terms of a subset of fixed size, which is usually one. In the latter formulation, the width (ratio) of an indifference zone in the parameter space is specified, the number of observations needed is tabulated and the final decision is the selection of a single population which is asserted to be the best one. Contributions to selection and ranking problems using this approach are presented in [3], [4], [5] and [7]. The present formulation is more flexible in that a decision can be made on any given number of observations with the assertion that a certain subset contains the best population, the size of the subset depending upon the observed results.

In Section 2, a formal statement of the problem is given. Section 3 deals with a result relevant to ranking and selection in terms of any general parameters (not necessarily scale or location). The procedure for the non-central χ^2 populations is given in Section 4 and its application to the ranking of multivariate populations is presented. An approximation to the probability of a correct selection and its infimum is given which enables us to compute constants (approximate) to carry out the procedure.

Sections 5 and 7 deal with the distribution function, and the moments of the ratio of the maximum of several correlated statistics each of which has a non-central χ^2, in the numerator and a non-central χ^2 in the denominator. An exact evaluation of the probability of the correct selection and its infimum is accomplished in Section 6 and it is shown that for this case the infimum of this probability is attained at $\lambda_1 = \lambda_2 = 0$.

2. Statement of the Problem

Suppose each of the k populations $\pi_1, \pi_2, \ldots, \pi_k$ has an observable random variable $Y_i (i=1,2,\ldots,k)$ whose density function is a non-central χ^2 given by (2.1)

$$(2.1) \qquad f_{\lambda_i}(x) = \frac{e^{-x/2} e^{-\lambda_i/2}}{2^{p/2}} \sum_{j=0}^{\infty} \frac{\lambda_i^j x^{\frac{p}{2}+j-1}}{j! 2^{2j} \Gamma(\frac{p}{2}+j)}, \quad x \geq 0,$$

where λ_i (≥ 0) is the non-centrality parameter and p is the degrees of freedom. Let the ranked λ's be denoted by

$$\lambda_{[1]} \leq \lambda_{[2]} \leq \cdots \leq \lambda_{[k]},$$

and it is assumed that there is no a priori information available about the correct pairing of the ordered $\lambda_{[i]}$ and the k given populations.

Any population associated with $\lambda_{[k]}$ will be called the best population. A correct selection is defined as the selection of any subset of the k populations which includes the best population. Our problem is to define a selection procedure which selects a, small, non-empty subset of the k populations and guarantees that the best population has been included with probability at least P^*, P^* being a specified number between $1/k$ and 1. Mathematically, if CS stands for a correct selection then our goal is to define a decision rule R such that

$$(2.2) \qquad \inf_{\Omega} P\{CS|R\} = P^*$$

where $\Omega = \{\underline{\lambda} = (\lambda_1,\ldots,\lambda_k): \lambda_i \geqq 0\}$. In the limiting case where all λ's are equal (we need to consider such cases for evaluating the infimum of the probability of a correct selection) the definition of a correct selection is modified to mean the selection of a ''tagged'' population.

The above formulation uses a zero-one loss function i.e.

$$(2.3) \qquad L(S_j, \underline{\lambda}) = 0 \quad \text{if } \pi_{[k]} \text{ with } \lambda_{[k]} \in S_j, \text{ the selected subset,}$$
$$= 1 \quad \text{if } \pi_{[k]} \notin S_j .$$

Thus

$$(2.4) \qquad \text{Risk} = E_{\underline{\lambda}}\, L(S_j,\underline{\lambda}) = 1-P\{CS|R\}.$$

Hence, the above formulation requires that we find a procedure whose risk $\leqq 1\text{-}P^*$.

However, the problem could be considered within the framework of more general loss functions. Multiple decision problems for the subset selection and with a more general loss function of the type

$$(2.5) \qquad L(S_j,\underline{\lambda}) = \sum_{q \in S_j} \alpha_{jq}\,(\lambda_{[k]} - \lambda_q),\ \alpha_{jq} \geqq 0$$

have been considered recently by Deely (1965).

After giving the procedure in Section 4, we also show that the procedure possesses a desirable monotonicity property, namely, that the probability of selecting or population with a larger value of λ is at least as large as the probability of selecting a population with a smaller value of λ. An approximation to the procedure is given which also can be used to provide simultaneous upper (lower) confidence bounds on certain ratios of the linear functions of the parameters. An important application of this problem is to the ranking and selections of multivariate populations in terms of the Mahalanobis distance from the origin.

3. A Class of Selection and Ranking Procedures

Let π_j be the population with density $f_{\lambda_j}(x)$, $j = 1,2,\ldots,k$. Let

$$\lambda_{[1]} \leqq \lambda_{[2]} \leqq \cdots \leqq \lambda_{[k]} \tag{3.1}$$

be the ordered λ_i's. It is not known what the correct pairing of the ordered and unordered λ's is. Let $\underline{x}' = (x_1,x_2,\ldots,x_k)$ be an observation of $\underline{X}' = (X_1,X_2,\ldots,X_k)$ whose components are independent random variables; $f_\lambda(x_j)$ being the density of X_j. Based on the observation vector $\underline{x}'$ we are interested in selecting a subset such that the probability is at least P^* that the best i.e. population with largest $\lambda_{[k]}$, is included in the subset. Let $h_b(x)$, $b \in [0,\infty)$ be a class of functions such that for every x,

(i) $h_b(x) > x$,

(ii) $h_o(x) = x$,

(iii) $\lim_{b \to \infty} h_b(x) = \infty$,

(iv) $h_b(x)$ is continuous and mononotone increasing in b.

Then the class $\mathcal{C}$ of procedures R_{h_b} is defined as follows:

R_{h_b}: ''Select π_i iff

$$(3.2) \qquad h_b(\hat{\mathcal{T}}_i) \geq \hat{\mathcal{T}}_{max}.''$$

The above procedure selects a non-empty set of random size in view of (1). The justification for conditions (ii), (iii) and (iv) will be provided later. It is clear that the probability of correct selection (correct selection $\Longleftrightarrow$ selection of any subset which includes the best) for this procedure is

$$(3.3) \qquad P\{CS|R_{h_b}\} = P\{h_b(\hat{\mathcal{T}}_{(k)}) \geq \hat{\mathcal{T}}_{(j)},\ j=1,2,\ldots,k-1\}$$

$$= \int \left[\prod_{j=1}^{k-1} F_{\lambda_{[j]}}(h_b(\hat{\mathcal{T}}))\right] f_{\lambda_{[k]}}(\hat{\mathcal{T}})\, d\hat{\mathcal{T}}.$$

Assume now that $F_\lambda(x)$ is stochastically increasing in λ, then,

$$(3.4) \qquad \inf_{\Omega} P\{CS|R_{h_b}\} = \inf_{\lambda} \int F_\lambda^{k-1}(h_b(\hat{\mathcal{T}}))\, f_\lambda(\hat{\mathcal{T}})\, d\hat{\mathcal{T}}$$

where Ω is the space of $\underline{\lambda}' = (\lambda_1,\ldots,\lambda_k)$.

Now we discuss the infimum over λ of

$$\int F_\lambda^{k-1}(h_b(\hat{\mathcal{T}}))\, f_\lambda(\hat{\mathcal{T}})\, d\hat{\mathcal{T}}.$$

First we prove a lemma.

Lemma: X is a random variable with density $f_\lambda(x)$ and the cumulative distribution function $F_\lambda(x)$ ~~which is stochastically increasing in λ.~~ Let $h_b(x)$, $b \in [0,\infty)$ be a class of functions and suppose there exists a density $f(x)$ with c.d.f. $F(x)$ such that

$$(3.5) \qquad h_b(g_\lambda(x)) \geq g_\lambda(h_b(x)), \quad \text{for all } \lambda \text{ and all } x,$$

where $g_\lambda(x)$ is defined by

$$(3.6) \qquad F_\lambda(g_\lambda(x)) = F(x), \qquad \text{for all } x.$$

Then for any $t > 0$,

$$\int \left[F_\lambda(h_b(x))\right]^t f_\lambda(x)dx \geq \int \left[F(h_b(x))\right]^t f(x)dx,$$

where the integral extends over the whole range of values of x.

<u>Proof</u>: Let

$$(3.7) \qquad \psi(\lambda, b) = \int F_\lambda^t(h_b(x))\ f_\lambda(x)dx$$

$$= \int F_\lambda^t\ (h_b(g_\lambda(z))\ f_\lambda(g_\lambda(z))g_\lambda'(z)dz\ .$$

From (3.6), it follows that

$$(3.7) \qquad f_\lambda(g_\lambda(x))\ g_\lambda'(x) = f(x).$$

Thus

$$(3.8) \qquad \psi(\lambda, b) = \int F_\lambda^t(h_b(g_\lambda(z))\ f(z)dz\ .$$

Now using (3.5) and (3.6),

$$(3.9) \qquad \psi(\lambda, b) \geq \int F^t(h_b(x)\ f(x)dx.$$

Application of the lemma in (3.4) allows us to write

$$\inf_{\Omega} P\{CS|R_{h_b}\} \geq \int F^{k-1}(h_b(y))\, f(y)dy. \tag{3.10}$$

From (3.10) we see that in order for the procedure R_{h_b} to guarantee the probability of a correct selection to be at least equal to P^* for all points in Ω, $h_b(y)$ may be chosen so as to make

$$\int F^{k-1}(h_b(y))\, f(y)dy = P^* . \tag{3.11}$$

It should be noted that conditions (ii) and (iii) and (iv) ensure that the probability of a correct selection will always be $\geq \frac{1}{k}$ and that this probability $\longrightarrow 1$ as $b \to \infty$.

<u>Remark</u>: (3.5) is a sufficient condition and gives an actual infimum if equality holds.

<u>Examples of $h_b(x)$ satisfying the lemma</u>

A. If λ is a location (translation) or a scale parameter in $f_\lambda(x)$ then

(i) $$\int_{-\infty}^{g_\lambda(x)} f_\lambda(\xi)d\xi = \int_{-\infty}^{g_\lambda(x)} f(\xi-\lambda)d\xi = \int_{-\infty}^{g_\lambda(x-\lambda)} f(\xi)d\xi = \int_{-\infty}^{x} f(\xi)d\xi$$

$$\text{or,} \quad g_\lambda(x-\lambda) = x$$

(ii) $$\frac{1}{\lambda}\int_{0}^{g_\lambda(x)} f(\tfrac{\xi}{\lambda})d\xi = \int_{0}^{g_\lambda(\frac{x}{\lambda})} f(t)dt = \int_{0}^{x} f(\xi)d\xi$$

$$\text{or,} \quad g_\lambda(\tfrac{x}{\lambda}) = x.$$

Hence

(i) if $h_b(x) = x+b$, then $h_b(g_\lambda(x)) = h_b(x+\lambda) = x+\lambda+b = g_\lambda(h_b(x))$ for all λ and all x.

(ii) if $h_b(x) = xb$, $h_b(g_\lambda(x)) = h_b(x\lambda) = x\lambda b = g_\lambda(h_b(x))$, for all λ and all x.

Thus, from the lemma, it follows that,

$$\inf_{\Omega} P\{CS|R_{h_b}\} = \int F^{k-1}(h_b(x))\, f(x)dx. \tag{3.12}$$

Note here $f(x)$ is the density corresponding to $\lambda = 0$ (central) for the location parameter case and it is the density corresponding to $\lambda = 1$ for the scale parameter case.

B. For the problem of selection and ranking of the non-central χ^2 parameters, the function $h_c(x) = \frac{x}{c}$, $0 < c$. Let $\chi_\alpha'^2$ and χ_α^2

$$\int_0^{\chi_\alpha'^2} f_\lambda(x)dx = 1-\alpha = \int_0^{\chi_\alpha^2} f(x)dx \tag{3.13}$$

where $f_\lambda(x)$ and $f(x)$ are, respectively, the densities of the non-central and central chi-square r.v. with p degrees of freedom, λ being the non-central parameter. It is known that for large p

$$g_\lambda(x) = \chi_\alpha'^2 \approx \frac{p+2\lambda}{p+\lambda}\, \chi_\alpha^2 = \frac{p+2\lambda}{p+\lambda}\, x, \quad \text{all } \lambda \text{ and all } x \tag{3.14}$$

$$h_c(g_\lambda(x)) \approx \frac{(p+2\lambda)x}{(p+\lambda)c} = g_\lambda(h_c(x)), \quad \text{all } \lambda \text{ and all } x. \tag{3.15}$$

Thus the lemma may be applied to show that if the approximation is sufficiently good then the infimum of the probability of a correct selection in using the above rule occurs when $\lambda = 0$ in (3.4).

Monotonicity Property of the Procedure R_{h_b}

We will now show that the procedure R_{h_b} satisfies the following property. If $\lambda_i \geqq \lambda_j$, then

(3.16) $$P\left\{\text{Selecting the population with parameter } \lambda_i\right\} \geqq P\left\{\text{Selecting the population with parameter } \lambda_j\right\},$$

provided the function $h_b(x)$ is a non-decreasing function of x.

Without loss of generality we may assume $\lambda_1 \geqq \lambda_2$. Then,

(3.17) $$P\{\text{Select } \Pi_1\} = \int\left[\prod_{j=3}^{k} F_{\lambda_j}(h_b(y))\right] F_{\lambda_2}(h_b(y))\, f_{\lambda_1}(y)dy$$

$$\geqq \int\left\{\left[\prod_{j=3}^{k} F_{\lambda_j}(h_b(y))\right] F_{\lambda_2'}(h_b(y))\right\} f_{\lambda_1}(y)dy,$$

for any $\lambda_2' > \lambda_2$, since F is a stochastically increasing.

Since the function within the braces on the right hand side of the inequality is a non-decreasing function of $h_b(x)$ and hence a non-decreasing function of x by our assumption, it follows from a result in Lehmann (1959) [Lemma 2, p.74] that the integral is non-decreasing in λ_1. Hence

(3.18) $$P\{\text{Select } \Pi_1\} \geqq \int\left\{\left[\prod_{j=3}^{k} F_{\lambda_j}(h_b(y))\right] F_{\lambda_2'}(h_b(y))\right\} f_{\lambda_2}(y)dy$$

$$\geqq P\{\text{Select } \Pi_2\}, \text{ by choosing } \lambda_2' = \lambda_1 .$$

4. Procedure R and the Probability of Correct Selection.

Let Y_i be a non-central χ^2 random variable with p degrees of freedom and non-centrality parameter λ_i.

Let $\tilde{n}$ independent observations $y_1, y_2, \ldots, y_n$ on Y_i be given and let $\bar{y}_i$ be the arithmetic mean of the y_i's. Then the selection procedure for parameters λ_i is as follows.

Procedure R: Select the population π_i if

$$(4.1) \qquad \bar{y}_i \geq c\, \bar{y}_{max}$$

where $\bar{y}_{max} = \max\{\bar{y}_j, j = 1,2,\ldots,k\}$ and where $c = c(k,p,n,P^*)$ is a predetermined number between 0 and 1 which is such that the procedure R satisfies the basic P^* probability requirement.

Such a number c clearly exists since by taking $c = 0$, we select all the populations and guarantee the probability P^* condition to be unity.

Expression for the Probability of a Correct Selection.

Let $\bar{y}_{(j)}$ (unknown) be the observed value of $\bar{y}_i$ which is associated with $\lambda_{[j]}$ $(j = 1,2,\ldots,k)$ where $\lambda_{[1]} \leq \lambda_{[2]} \leq \cdots \leq \lambda_{[k]}$ are the ordered values of the vector $\underline{\lambda}' = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Then

$$(4.2) \qquad P\{CS|R\} = P\{\bar{y}_{(k)} \geq c\, \bar{y}_{max}\}$$

$$= P\{\bar{y}_{(k)} \geq c\, \bar{y}_{(j)}, \quad j = 1,2,\ldots,k-1\}$$

$$= \int_0^\infty [\prod_{j=1}^{k-1} F_{\lambda'_{[j]}}(\tfrac{x}{c})]\, f_{\lambda'_{[k]}}(x)\, dx$$

where $F_{\lambda'}(\cdot)$ and $f_{\lambda'}(\cdot)$ refer to the cumulative distribution function and the density of the non-central χ^2 distribution with parameter $\lambda'=n\lambda$ and degrees of freedom $p'= np$.

Infimum of $P\{CS|R\}$

The number c in our procedure R is defined such that

$$\inf_{\Omega} P\{CS|R\} = P^*$$

where Ω is the space of all possible configurations of all λ,

(4.2a) $$\inf_{\Omega} P\{CS|R\} = \inf_{\Omega} \int_0^{\infty} [\prod_{j=1}^{k-1} F_{\lambda'_{[j]}} (\tfrac{x}{c})] f_{\lambda'_{[k]}} (x)\, dx.$$

Now it is known that distribution of a non-central χ^2 random variable has the property TP_2, i.e. total positivity of order 2 which is equivalent to the property of monotone likelihood ratio. From the TP_2 or MLR property, it follows that the distribution of Y is stochastically increasing, i.e. $F_{\lambda'}(y)$, the cdf, is a nonincreasing function of λ' for all y. It follows that

(4.3) $$\int_0^{\infty} [\prod_{j=1}^{k-1} F_{\lambda'_{[j]}} (\tfrac{x}{c})] f_{\lambda'_{[k]}} (x)dx \geq \int_0^{\infty} F^{k-1}_{\lambda'_{[k]}} (\tfrac{x}{c}) f_{\lambda'_{[k]}} (x)dy,$$

and thus

(4.4) $$\inf_{\Omega} P\{CS|R\} = \inf_{\lambda \geq 0} \int_0^{\infty} F^{k-1}_{\lambda'} (\tfrac{x}{c}) f_{\lambda'}(x)\, dx.$$

It will be shown later that for the case $k=2$, the infimum takes place at $\lambda' = 0$. In this case the exact values of c are obtained by

$$(4.5) \qquad \int_0^\infty G_{m'}\left(\frac{x}{c}\right) g_{m'}(x)\, dx = P^*$$

where $G_{m'}(\cdot)$ and $g_{m'}(\cdot)$ refer to the c.d.f. and the density of a standard gamma chance variable with $g_{m'}(x) = e^{-x} x^{m'-1}/\Gamma(m')$ and $m' = p'/2$. It has been shown elsewhere (Gupta (1963)) that for the problem of ranking and selection of the scale parameters of gamma populations (4.5) provides the solution in c. The values of the constants c for this case have been tabulated in the above paper and also in a paper by Armitage and Krishnaiah (1964).

Normal Approximation.

In the sequel we will write p for p' and λ for λ'. Define $a = p+\lambda$, $b = \lambda/(p+\lambda)$, then it is known that $(\frac{\bar{Y}}{a})^{1/3}$ is approximately normally distributed with mean $d = 1- \frac{2(1+b)}{9a}$ and variance $\frac{2(1+b)}{9a}$.

Using this fact (4.4) can be approximated by

$$(4.6) \qquad \inf_{\lambda \geq 0} P\{CS|R\} = \inf_{\lambda \geq 0} \int_{-\infty}^{\infty} \Phi^{k-1}\left(c^{-1/3} x - \frac{d(1-c^{-1/3})}{\sqrt{1-d}}\right) \varphi(x) dx$$

where $\Phi(\cdot)$ and $\varphi(\cdot)$ are the cumulative distribution function and the density of a standard normal variable.

Since $0 < d < 1$, $1-c^{-1/3} < 0$, the integral on the right hand side of (4.6) is a mononote increasing function of d. Since the derivative of d with respect to λ is positive, it follows that the integral on the right hand side has its minimum value at $\lambda = 0$. Hence the approximate c is determined by

$$(4.7) \qquad \int_{-\infty}^{\infty} \Phi^{k-1}\left(c^{-1/3}\, x + \frac{(c^{-1/3}-1)(9p-2)}{3\sqrt{2p}}\right) \varphi(x)dx = P^*.$$

The above integral equation has been solved for c for selected values of p, k and P^* and these approximate values have been tabulated at the end of this paper. A comparison with the exact values for $k = 2$, indicates good accuracy for this approximation.

The approximation based on the Wilson-Hilferty cube root transformation used above has been discussed along with other approximations by Abdel-Aty (1954). In this paper the tabulated values indicate that the approximation is good. Other approximations for the distribution of non-central χ^2 have been discussed by Patnaik (1949), Johnson (1959) and Roy and Mohamad (1964).

<u>Procedure for Selecting the Subset to Contain the Multivariate Population with the Largest Value of the Mahalanobis Distance from the Origin.</u>

Let π_i: $N(\underline{\mu}_i, \Sigma)$, $i = 1,2,\ldots,k$, be p-variate normal populations with mean vectors $\underline{\mu}_i$, respectively, and with a common known positive definite matrix Σ. Let $\lambda_i = \underline{\mu}_i' \Sigma^{-1} \underline{\mu}_i$ denote the Mahalanobis distance function for the population π_i from the origin.

We take a sample of n independent observations from each of the k populations. Let $\underline{x}_{ij}$ denote the jth observation of the p-dimensional random vector on the ith population; then for each $j = 1,2,\ldots,n$, we compute

$$(4.8) \qquad y_{ij} = \underline{x}_{ij}' \Sigma^{-1} \underline{x}_{ij}, \quad i=1,2,\ldots,k;\ j=1,2,\ldots,n.$$

Since $y_{ij}(j=1,2,\ldots,n)$ correspond to the n independent observations on a non-central χ^2 for each i and since $Y_i = \sum_{j=1}^{n} Y_{ij}$ is distributed as a non-central χ^2 with non-centrality parameter $\lambda_i' = n\lambda_i = n\,\underline{\mu}_i'\,\Sigma^{-1}\,\underline{\mu}_i$, and degrees of freedom $p' = np$, it follows that the selection rule for the population with the largest value of the Mahalanobis distance function is:

Rule R: Select the population π_i if

$$\sum_{j=1}^{n} y_{ij} \geq c \max_i \left\{ \sum_{j=1}^{n} y_{ij} : i = 1,2,\ldots,k \right\} \tag{4.9}$$

where values of c are tabulated at the end of the paper. It should be pointed out that the appropriate value of c for the above procedure are obtained by using $p' = np$ as the number for the d.f. p in the tables. It will be noticed that for a fixed p as n increases, the tabulated values of c (for fixed k and P^*) increase which would be expected.

Another procedure for this problem is as follows: Compute

$$z_i = \bar{\underline{x}}_i'\,\Sigma^{-1}\,\bar{\underline{x}}_i \tag{4.10}$$

where $\bar{\underline{x}}_i' = (\bar{x}_{i1}, \bar{x}_{i2}, \ldots, \bar{x}_{ip})$ is the sample mean vector based on n observations. Then the procedure is: Select π_i iff

$$z_i \geq z_{max} - d \tag{4.11}$$

where $d = d(k,p,n,P^*)$ is given by

$$\inf_{\lambda' \geq 0} \int_0^\infty F_{\lambda'}^{k-1}(x+d)\, f_{\lambda'}(x)\, dx = P^* \tag{4.12}$$

where F and f refer to the non-central χ^2 with p d.f. and non-central parameter $\lambda' = n\lambda$. Properties and relative performance of this procedure are being developed at this time and will be published later.

Simultaneous (approximate) Confidence Bounds on the (k-1) Ratios $(p+\lambda_j)/(p+\lambda_i)$, $j \neq i$ for a fixed i.

We have shown earlier in this Section that for any $i = 1,2,\ldots,k$, we have

$$(4.13) \qquad P^* = \int_{-\infty}^{\infty} \Phi^{k-1}\left(c^{-1/3}\, x + \frac{(c^{-1/3}-1)(9p-2)}{3\sqrt{2p}}\right) \varphi(x)dx$$

$$\approx \inf_{\lambda \geq 0} P\left\{\max_j \left(\frac{y_j}{p+\lambda}\right)^{1/3} \leq c\left(\frac{y_i}{p+\lambda}\right)^{1/3}\right\}$$

$$= \inf_{\Omega} P\left\{\max_j \left(\frac{y_j}{p+\lambda_j}\right)^{1/3} \leq c\left(\frac{y_i}{p+\lambda_i}\right)^{1/3}\right\}$$

$$= \inf_{\Omega} P\left\{\frac{p+\lambda_j}{p+\lambda_i} \geq \frac{1}{c^3}\frac{y_j}{y_i}\ ,\ j=1,2,\ldots,k,\ j \neq i\right\}$$

Thus,

$$(4.14) \qquad P\left\{\frac{p+\lambda_j}{p+\lambda_i} \geq \frac{1}{c^3}\frac{y_j}{y_i}\ ,\quad j = 1,2,\ldots,k, j \neq i\right\} \approx P^*$$

provides simultaneous (approximate) lower 100 P* confidence bounds on the desired ratios.

Selection of the Subset with respect to the Minimum Value of the Parameter.

If we are interested in selecting a subset to contain the population with the smallest value of λ_i, the procedure R' is as follows:

R': Select π_i iff

(4.15) $$\bar{y}_i \leq b\, \bar{y}_{min}$$

where $b = b(k,p,n,P^*)$ is a number greater than one which is such that R' satisfies the basic probability requirement. As before it can be shown that the values of the constants b are obtained from

(4.16) $$\inf_{\lambda \geq 0} \int_0^\infty \left[1-F_\lambda\left(\frac{x}{b}\right)\right]^{k-1} f_\lambda(x)dx = P^* .$$

In the special case k=2, it follows from the derivations (in particular (6.10)) of Section 6, that the function

(4.17) $$I_1(\lambda) = \int_0^\infty \left(1-F_\lambda\left(\frac{x}{b}\right)\right) f_\lambda(x)dx, \quad b > 1$$

is a monotone increasing function of λ ($\lambda \geq 0$), its minimum being at zero. Again from arguments similar to those given earlier it follows that the approximate value of b in (4.16) are obtained by setting $\lambda = 0$, and solving

(4.18) $$\int_{-\infty}^\infty \left[1-\Phi\left(xb^{-1/3}+ \frac{(b^{-1/3}-1)(9p-2)}{3\sqrt{2p}}\right)\right]^{k-1} \varphi(x)dx = P^*.$$

5. Distribution of an Associated Statistic

Consider the statistic

(5.1) $$Z = \max_{j=1,\dots,t} (Y_j/Y) = Y_{max}/Y$$

where $Y_1, Y_2, \dots, Y_t$ and Y are independent non-central χ^2 random variables each with p degrees of freedom and non-centrality parameter λ.

Now,

(5.2) $$P(Z \leq b) = \int_0^\infty F_\lambda^t(by)\, f_\lambda(y)dy$$

$$= I_\lambda(b,t)$$

where F_λ and f_λ refer to the cumulative distribution function and the density function of the non-central χ^2 random variable with p degrees of freedom and non-centrality parameter $= \lambda (\lambda \geq 0)$.

Bounds on $I_\lambda(b,t)$

If we keep λ fixed, then

(5.3) $$\frac{1}{t+1} \leq I_\lambda(b,t) \leq 1$$

(5.2) $$I_\lambda(b,t-1) \geq I_\lambda(b,t)$$

(5.3) $$I_\lambda(b,t) \geq I_\lambda^t(b,1)$$

[(5.3) follows from Jensen's inequality since $F_\lambda^t(b\,x)$ is convex for $t > 1$.]

$$(5.4) \qquad I_\lambda(b,t) \geq I_\lambda(b,1) + 1-t$$

$$(5.5) \qquad e^{-\lambda} \sum_0^\infty a_\ell \lambda^\ell \leq I_\lambda(b,1) \leq e^{-\lambda} \sum_0^\infty a_\ell \lambda^\ell + R(n)$$

where n is any positive integer and p = even integer = $2m$, say and $R(n)$ and a_ℓ are defined as follows,

$$R(n) = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j\,!} \left[1- \sum_{\alpha=0}^{m+n-1} \frac{b^\alpha}{\alpha!\,(1+b)^{\alpha+m+j}} \frac{\Gamma(\alpha+m+j)}{\Gamma(m+j)}\right]$$

$$(5.6) \qquad a_\ell = \begin{cases} \dfrac{1}{2^\ell(\ell!)} \displaystyle\sum_{r=0}^{\ell} \binom{\ell}{r}\left[1- \sum_{\alpha=0}^{m+\ell-r-1} \frac{b^\alpha\, \Gamma(m+\alpha+r)}{r!\,(1+b)^{\alpha+m+r}\, \Gamma(m+r)}\right], & \ell \leq n-1 \\[2ex] \dfrac{1}{2^\ell(\ell!)} \displaystyle\sum_{r=0}^{\ell} \binom{\ell}{r}\left[1- \sum_{\alpha=0}^{m+r-1} \frac{b^\alpha\, \Gamma(\alpha+m+r)}{\alpha!\,(1+b)^{\alpha+m+\ell-r}\, \Gamma(m+r)}\right], & \ell > n-1 \,. \end{cases}$$

It should be pointed out that for p = a positive even integer = $2m$, say,

$$(5.7) \qquad I_\lambda(b,1) = e^{-\lambda} \sum_0^\infty a_\ell \lambda^\ell$$

where a_ℓ is given by the first (top) part of (5.6) for all ℓ.

6. The infimum of the probability of correct selection for the case k=2.

We have already seen that the probability of correct selection is minimized on the hyperplane $\lambda_1 = \lambda_2 = \ldots = \lambda_k = \lambda$. Now we would like to find out for what value of $\lambda \geq 0$, the infimum of the probability of correct selection takes place. Let us define, for any positive numbers n_1 and n_2 and any c $(0 < c < 1)$,

$$(6.1) \qquad I(n_1,n_2,c) = \int_0^\infty G_{n_1}(\tfrac{t}{c})\ g_{n_2}(t)\ dt$$

where $g_n(x) = \frac{d}{dx} G_n(x) = e^{-x}\ x^{n-1}/\Gamma(n)$. First we give two lemmas.

Lemma 1.

$$(6.2) \qquad I(n_1,n_2,c) = I(n_1-r-1,n_2,c) - \frac{c^{n_2}}{\Gamma(n_2)(1+c)^{n_1+n_2-1}} \sum_{j=0}^{r} \frac{\Gamma(n_1+n_2-j-1)}{\Gamma(n_1-j)} (1+c)^j .$$

The proof of the lemma follows by observing that

$$(6.3) \qquad G_{n_1}(t) = G_{n_1-r-1}(t) - \sum_{j=0}^{r} g_{n_1-j}(t), \quad r < n_1$$

and that

$$(6.4) \qquad \int_0^\infty g_{n_2}(t)\ g_{n_1-j}(\tfrac{t}{c})dt = \frac{c^{n_2}}{(1+c)^{n_1+n_2-j-1}} \frac{\Gamma(n_1+n_2-j-1)}{\Gamma(n_2)\ \Gamma(n_1-j)} .$$

Lemma 2.

$$(6.5) \quad I(n_1,n_2,c) = I(n_1,n_2-r-1,c) + \frac{c^{n_2-1}}{\Gamma(n_1)(1+c)^{n_1+n_2-1}} \sum_{j=0}^{r} \frac{\Gamma(n_1+n_2+j-1)(1+c)^j}{\Gamma(n_2-j)\ c^j} .$$

Now for any $p = 2m$ (m is not necessarily an integer), and any $x > 0$, the c.d.f. and the density of the random variable X which is non-central χ^2 with parameter λ and degrees of freedom p,

$$(6.5a) \qquad F_\lambda(\tfrac{x}{c}) = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!2^j} G_{m+j}\,(\tfrac{x}{2c}), \quad c > 0$$

$$(6.6) \qquad f_\lambda(x) = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!2^{j+1}} g_{m+j}\,(\tfrac{x}{2}) .$$

From (6.5) and (6.6), we obtain, after term by term integration

$$(6.7) \qquad \int_0^\infty F_\lambda(\tfrac{x}{c})\, f_\lambda(x)dx = e^{-\lambda} \sum_{\alpha=0}^{\infty} \lambda^\alpha \frac{1}{\alpha!2^\alpha} \sum_{r=0}^{\alpha} \binom{\alpha}{r} \int_0^\infty G_{m+\alpha-r}(\tfrac{t}{c}) g_{m+r}(t)dt$$

$$= e^{-\lambda} \sum_{0}^{\infty} a_\alpha\, \lambda^\alpha$$

where a_α is defined by

$$(6.8) \qquad \alpha!\; 2^\alpha\, a_\alpha = \sum_{r=0}^{\alpha} \binom{\alpha}{r} I(m+\alpha-r,\; m+r, c).$$

Using Lemma 1 and the fact that $\binom{\alpha+1}{r} = \binom{\alpha}{r} + \binom{\alpha}{r-1}$, we obtain, after some simplification,

$$(6.9) \qquad (\alpha+1)!\; 2^{\alpha+1}\, a_{\alpha+1} = \alpha!\; 2^\alpha\, a_\alpha - \frac{c^m}{(1+c)^{2m+\alpha}} \sum_{r=0}^{\alpha} \binom{\alpha}{r} \frac{c^r}{m+\alpha-r} \frac{\Gamma(2m+\alpha)}{\Gamma(m+\alpha-r)\;\Gamma(m+r)}$$

$$+ \sum_{r=0}^{\alpha} \binom{\alpha}{r} I(m+\alpha-r,\; m+r+1, c) .$$

Using lemma 2 on the last term in (6.9), we obtain, after some algebraic simplification,

$$(6.10)\quad (\alpha+1)\, a_{\alpha+1} - a_\alpha = \frac{c^m\, \Gamma(2m+\alpha)}{\alpha!\, 2^{\alpha+1}(1+c)^{2m+\alpha}} \sum_{r=0}^{[\frac{\alpha-1}{2}]} \binom{\alpha}{r} \frac{(\alpha-2r)\, c^r}{\Gamma(m+r+1)\, \Gamma(m+\alpha-r+1)} (1-c^{\alpha-2r})$$

$$> 0 \quad \text{since} \quad \alpha-2r > 0,\ 0 < c < 1.$$

Thus, $a_1 = a_o$, $2a_1 - a_o > 0$. Now,

$$(6.11)\qquad \frac{dI_\lambda}{d\lambda} = e^{-\lambda} \sum_{o}^{\infty} [(\alpha+1)\, a_{\alpha+1} - a_\alpha]\, \alpha$$

From (6.10) and (6.11), it follows that the derivative is positive for all values of λ except at $\lambda = 0$, where it equals zero. Thus we have the theorem.

Theorem. For $k = 2$, the infimum of the probability of a correct selection occurs at $\lambda = 0$ and

$$\inf_{\Omega} P\{CS|R\} = \inf_{\lambda \geq 0} \int_o^\infty F_\lambda\left(\frac{x}{c}\right) f_\lambda(x)dx = \int_o^\infty G_m\left(\frac{x}{c}\right) g_m(x)dx\ .$$

Remark 1: In the derivation of the above theorem it is seen that for any c, $0 < c < 1$, the function $I_\lambda\left(\frac{1}{c}, t\right)$ is a monotone increasing function of λ, its minimum being at $\lambda = 0$.

7.1. Moments of the statistic Z

Since the random variables Y_{max} and Y are independent,

$$(7.1) \qquad E(Z^r) = E(Y_{max}^r / Y^r)$$

$$= E(Y^{-r})\, E(Y_{max}^r), \qquad \text{provided } EY^{-r} \text{ exists.}$$

Thus the evaluation of the moments of Z depends upon the negative moments of Y as in (7.2) and the moments of the largest of t independent non-central χ^2.

$$(7.2) \qquad E(Y^{-r}) = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \frac{1}{2^{j+r}} \frac{\Gamma(m+j-r)}{\Gamma(m+j)}, \qquad \text{provided } r < m = \frac{p}{2}.$$

$$(7.3) \qquad E(Y_{max}^r) = t \int_0^{\infty} y^r F_{\lambda}^{t-1}(y)\, f_{\lambda}(y)\, dy$$

Special Cases. Now we discuss the evaluation of (5.10) for some cases. For $t = 1$, (7.3) reduces to the moment μ_r' of the non-central χ^2, which can be obtained in terms of κ_r, the cumulants of the non-central χ^2, which are

$$(7.4) \qquad \kappa_r = 2^{r-1}(r-1)!\,(p + r\lambda).$$

Case $t = 2$. Following the methods used in Section 5, we obtain

$$(7.5) \qquad E(Y_{max}^r) = 2\, e^{-\lambda} \sum_{\alpha=0}^{\infty} \lambda^{\alpha} A_{\alpha}$$

where

$$(7.6)\qquad A_\alpha = \sum_{j=0}^{\alpha} \binom{\alpha}{j} \frac{2^{r-\alpha}}{\alpha!} \frac{\Gamma(m+j+r)}{\Gamma(m+j)} \int_0^\infty G_{m+\alpha-j}(t)\; g_{m+j+r}(t)dt.$$

Now using the notation of Section 6, we can write

$$(7.7)\qquad A_\alpha = \sum_{j=0}^{\alpha} \binom{\alpha}{j} \frac{2^{r-\alpha}}{\alpha!} \frac{\Gamma(m+j+r)}{\Gamma(m+j)} I(m+\alpha-j\ ,\ m+j+r,1)$$

in which the result of lemma 1 in Section 6 can be used to express $I(m+\alpha-j,\ m+j+r,1)$ explicitly as

$$(7.8)\qquad I(m+\alpha-j,\ m+j+r,1) = 1- \frac{1}{2^{m+j+r}} - \frac{1}{2^{2m+\alpha+r-1}\ \Gamma(m+j+r)} \cdot \left[\sum_{h=0}^{m+\alpha-j-2} \frac{2^h\ \Gamma(2m+\alpha+r-1-h)}{\Gamma(m+h-j-h)}\right],$$

if m is a positive integer.

For $t > 2$, the computation of the moments of Y_{max} will follow from an extension of the lemma 1 of Section 6. However, the computations become very tedious.

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Probability that the ratio of two independent non-central χ^2, each with p degrees of freedom and non-centrality parameter λ, does not exceed $1/c$.

$p = 2$

λ \ c	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50	.55	.60	.65	.70	.75	.80	.85	.90	.95	1.00
0	.952	.909	.870	.833	.800	.769	.741	.714	.690	.667	.645	.625	.606	.588	.571	.556	.541	.526	.513	.500
.2	.953	.909	.870	.834	.800	.770	.741	.715	.690	.667	.645	.625	.606	.588	.572	.556	.541	.526	.513	.500
.4	.953	.910	.871	.835	.802	.771	.742	.716	.691	.668	.646	.626	.607	.589	.572	.556	.541	.527	.513	.500
.6	.954	.912	.873	.837	.804	.773	.744	.718	.693	.669	.648	.627	.608	.590	.573	.557	.541	.527	.513	.500
.8	.955	.913	.875	.839	.806	.775	.746	.720	.695	.671	.649	.629	.609	.591	.574	.557	.542	.527	.513	.500
1.0	.956	.915	.877	.842	.809	.778	.749	.722	.697	.673	.651	.630	.611	.592	.575	.558	.542	.528	.513	.500
2.0	.963	.927	.892	.859	.826	.795	.766	.738	.712	.687	.664	.641	.620	.600	.581	.563	.546	.530	.515	.500
5.0	.983	.962	.937	.909	.881	.851	.821	.791	.762	.733	.705	.678	.652	.627	.604	.581	.559	.538	.519	.500

Probability that the ratio of two independent non-central χ^2, each with p degrees of freedom and non-centrality parameter λ, does not exceed $1/c$.

$p = 4$

λ \ c	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50	.55	.60	.65	.70	.75	.80	.85	.90	.95	1.00
0	.993	.977	.953	.926	.896	.865	.833	.802	.771	.741	.712	.684	.657	.631	.606	.583	.561	.539	.519	.500
.2	.993	.977	.953	.926	.896	.865	.833	.802	.771	.741	.712	.684	.657	.631	.606	.583	.561	.539	.519	.500
.4	.993	.977	.954	.926	.897	.865	.834	.802	.771	.741	.712	.684	.657	.631	.607	.583	.561	.540	.519	.500
.6	.994	.977	.954	.927	.897	.866	.835	.803	.772	.742	.713	.685	.658	.632	.607	.584	.561	.540	.519	.500
.8	.994	.977	.955	.928	.898	.867	.836	.804	.773	.743	.714	.685	.658	.632	.608	.584	.561	.540	.519	.500
1.0	.994	.978	.955	.928	.899	.868	.837	.805	.774	.744	.715	.686	.659	.633	.608	.584	.562	.540	.520	.500
2.0	.995	.980	.959	.934	.906	.876	.844	.813	.782	.751	.721	.692	.664	.638	.612	.587	.564	.542	.520	.500
5.0	.997	.988	.973	.954	.930	.903	.874	.843	.811	.780	.748	.716	.686	.656	.627	.599	.573	.547	.523	.500

Probability that the ratio of two independent non-central χ^2, each with p degrees of freedom and non-centrality parameter λ, does not exceed $1/c$.

$p = 6$

λ \ c	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50	.55	.60	.65	.70	.75	.80	.85	.90	.95	1.00
0	.999	.993	.982	.965	.942	.916	.886	.855	.823	.790	.757	.725	.693	.662	.632	.603	.576	.549	.524	.500
.2	.999	.993	.982	.965	.942	.916	.887	.855	.823	.790	.757	.725	.693	.662	.632	.603	.576	.549	.524	.500
.4	.999	.994	.982	.965	.942	.916	.887	.856	.823	.790	.758	.725	.693	.662	.632	.603	.576	.549	.524	.500
.6	.999	.994	.982	.965	.943	.916	.887	.856	.824	.791	.758	.725	.694	.663	.633	.604	.576	.549	.524	.500
.8	.999	.994	.982	.965	.943	.917	.888	.857	.824	.791	.758	.726	.694	.663	.633	.604	.576	.550	.524	.500
1.0	.999	.994	.982	.965	.943	.917	.888	.857	.825	.792	.759	.726	.694	.663	.633	.604	.576	.550	.524	.500
2.0	.999	.994	.984	.968	.946	.921	.892	.862	.829	.796	.763	.730	.698	.666	.636	.606	.578	.551	.525	.500
5.0	.999	.996	.989	.976	.958	.936	.910	.880	.849	.816	.782	.747	.713	.680	.647	.615	.585	.555	.527	.500

Table of values of c based on an approximation

$P^* = .75$

p \ k	2	3	4	5	6	7	8	9	10
5	.529 (.529)	.397	.345	.316	.296	.282	.270	.262	.254
6	.562 (.561)	.433	.381	.350	.330	.316	.304	.295	.287
7	.589 (.588)	.462	.410	.380	.359	.344	.332	.323	.315
8	.610 (.610)	.486	.435	.405	.384	.369	.357	.347	.339
9	.629 (.629)	.508	.457	.427	.406	.391	.379	.369	.361
10	.645 (.645)	.526	.476	.446	.425	.410	.398	.388	.380
15	.701 (.699)	.594	.546	.517	.497	.482	.471	.461	.453
20	.737 (.736)	.637	.592	.565	.546	.531	.520	.510	.502
25	.761	.668	.626	.600	.581	.567	.556	.547	.539
30	.780 (.780)	.692	.652	.627	.609	.596	.585	.576	.569
35	.795	.711	.673	.649	.632	.619	.608	.600	.592
40	.807 (.807)	.727	.690	.667	.650	.638	.628	.619	.612
45	.817	.741	.705	.682	.666	.654	.644	.636	.629
50	.825 (.825)	.752	.717	.696	.680	.668	.659	.651	.644

1. For given k, p and P^*, the above table gives values of c which satisfy the equation

$$\int_{-\infty}^{\infty} \Phi^{k-1}\left(c^{-1/3}x + \frac{(c^{-1/3}-1)(9p-2)}{3\sqrt{2p}}\right)\varphi(x)\,dx = P^*,$$

where $\Phi(\cdot)$ and $\varphi(\cdot)$ refer to the c.d.f. and the density function of the standard normal distribution, respectively.

2. The values inside the parentheses in the column for $k=2$ are the exact values.

Table of values of c based on an approximation

$P^* = .90$

p \ k	2	3	4	5	6	7	8	9	10
5	.289 (.290)	.224	.198	.182	.172	.168	.158	.153	.149
6	.327 (.327)	.260	.232	.215	.203	.195	.188	.183	.178
7	.359 (.360)	.290	.261	.243	.231	.222	.215	.209	.204
8	.386 (.386)	.317	.286	.268	.255	.246	.239	.233	.228
9	.410 (.410)	.340	.309	.290	.277	.268	.260	.254	.248
10	.430 (.430)	.361	.330	.310	.297	.287	.280	.273	.268
15	.507 (.508)	.439	.407	.388	.374	.364	.355	.348	.343
20	.558 (.558)	.492	.461	.441	.428	.417	.409	.402	.396
25	.594	.531	.501	.482	.468	.458	.449	.443	.437
30	.623 (.622)	.561	.532	.514	.500	.490	.482	.475	.469
35	.645	.586	.558	.540	.527	.517	.509	.502	.496
40	.664 (.664)	.607	.579	.562	.549	.539	.531	.525	.519
45	.680	.625	.598	.581	.568	.559	.551	.544	.539
50	.694 (.694)	.640	.614	.597	.585	.576	.568	.562	.556

1. For given k, p and P^*, the above table gives values of c which satisfy the equation

$$\int_{-\infty}^{\infty} \Phi^{k-1}\left(c^{-1/3}\, x + \frac{(c^{-1/3}-1)(9p-2)}{3\sqrt{2p}}\right) \varphi(x)dx = P^*,$$

where $\Phi(\cdot)$ and $\varphi(\cdot)$ refer to the c.d.f. and the density function of the standard normal distribution, respectively.

2. The values inside the parentheses in the column for $k=2$ are the exact values.

Table of values of c based on an approximation

$P^* = .95$

p \ k	2	3	4	5	6	7	8	9	10
5	.195 (.198)	.154	.136	.125	.118	.113	.109	.105	.102
6	.232 (.233)	.186	.167	.155	.147	.141	.136	.132	.129
7	.263 (.264)	.215	.194	.182	.173	.166	.161	.157	.153
8	.290 (.291)	.241	.219	.205	.196	.189	.183	.179	.175
9	.314 (.314)	.263	.241	.227	.217	.210	.204	.199	.195
10	.335 (.336)	.284	.261	.246	.236	.229	.223	.218	.213
15	.416 (.417)	.363	.339	.323	.312	.304	.297	.292	.287
20	.471 (.471)	.419	.394	.378	.367	.358	.352	.346	.341
25	.511	.461	.436	.420	.409	.401	.394	.388	.383
30	.543 (.543)	.494	.470	.454	.443	.434	.427	.422	.417
35	.569	.521	.497	.482	.471	.462	.456	.450	.445
40	.591 (.591)	.544	.520	.506	.495	.486	.480	.474	.469
45	.609	.563	.540	.526	.515	.507	.500	.495	.490
50	.625 (.625)	.580	.558	.544	.533	.525	.518	.513	.508

1. For given k, p and P*, the above table gives values of c which satisfy the equation

$$\int_{-\infty}^{\infty} \Phi^{k-1}\left(c^{-1/3}\, x + \frac{(c^{-1/3}-1)(9p-2)}{3\sqrt{2p}}\right) \varphi(x)dx = P^*,$$

where $\Phi(\cdot)$ and $\varphi(\cdot)$ refer to the c.d.f. and the density function of the standard normal distribution, respectively.

2. The values inside the parentheses in the column for k=2 are the exact values.

Table of values of c based on an approximation

P* = .99

k / p	2	3	4	5	6	7	8	9	10
5	.085 (.091)	.067	.059	.054	.051	.048	.047	.045	.044
6	.113 (.118)	.092	.082	.077	.072	.069	.067	.065	.063
7	.139 (.143)	.115	.104	.098	.093	.089	.087	.084	.082
8	.162 (.166)	.137	.125	.118	.112	.108	.105	.103	.101
9	.184 (.187)	.157	.144	.136	.130	.126	.123	.120	.117
10	.204 (.206)	.175	.162	.153	.147	.143	.139	.136	.133
15	.282 (.284)	.251	.235	.225	.218	.213	.208	.204	.202
20	.340 (.340)	.306	.290	.279	.272	.266	.261	.257	.254
25	.383	.350	.333	.322	.314	.308	.303	.299	.295
30	.419 (.419)	.385	.368	.357	.349	.343	.338	.334	.330
35	.448	.415	.398	.387	.379	.372	.367	.363	.359
40	.473 (.473)	.440	.423	.412	.404	.398	.393	.388	.385
45	.494	.462	.445	.434	.426	.420	.415	.410	.407
50	.513 (.513)	.481	.464	.454	.446	.440	.434	.430	.426

1. For given k, p and P*, the above table gives values of c which satisfy the equation

$$\int_{-\infty}^{\infty} \Phi^{k-1}\left(c^{-1/3} x + \frac{(c^{-1/3}-1)(9p-2)}{3\sqrt{2p}}\right) \varphi(x)dx = P^*,$$

where $\Phi(\cdot)$ and $\varphi(\cdot)$ refer to the c.d.f. and the density function of the standard normal distribution, respectively.

2. The values inside the parentheses in the column for k=2 are the exact values.

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13. ABSTRACT

This paper deals with a multiple decision (selection and ranking) problem for $k(k \geq 2)$ multivariate normal populations. The selection and ranking problem is defined in terms of the distance function $\mu_i' \Sigma^{-1} \mu_i$ where μ_i' is the row vector whose p components are the means of the ith population and where Σ is the common covariance matrix of the k multivariate populations. The specific problems formulated in Section 2 is to define a procedure which selects a non-empty set, which is small, and for which the risk associated with not including the best population in the selected subset of random size, is bounded above by a given number $1-P^*$, $(\frac{1}{k} < P^* < 1)$. The procedure is identical with a procedure for ranking the non-centrality parameters of the non-central χ^2 distributions and is given in Section 4. Section 3 deals with a result relevant to ranking and selection in terms of any general parameters (not necessarily scale or location). An approximation to the probability of a correct selection and its infimum is obtained. This approximation is utilized to obtain constants to carry out the procedure. An exact evaluation of the probability of a correct selection is obtained in Section 6 and it is shown that for $k = 2$, the infimum of this probability is attained at $\lambda_1 = \lambda_2 = 0$. Sections 5 and 7 of the paper deal with the distribution function and the moments of the maximum of several correlated random variables each of which is the ratio of two non-central independent χ^2 with a common denominator. Some tables of the above distribution are appended.

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