A General Class of Bulk Queues with Poisson Input

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Abstract

We assume that customers arrive at a counter according to a homogeneous Poisson process and are served in groups, according to the following policy: If there are less than L customers waiting at the time of a departure, the server must wait until there are L customers present, whereupon he serves them together. If there are L or more, but less than K ($K \geq L$) customers waiting, all are served together. If there are K or more customers waiting, a group of K customers is served and the others must wait. The service times of successive groups are assumed to be conditionally independent given the bulk sizes, but may depend on the group sizes. We obtain 1. a description of the output process, 2. the queue length in discrete time, 3. the distribution of the busy period, 4. the queue length in continuous time and 5. some limit theorems for the number of customers served over a long period of time.

The order of service is irrelevant in this paper. The method used throughout is that of the imbedded semi-Markov process.

1. Introduction

This paper is devoted to the study of the following queueing model. We assume that customers arrive at a counter, according to a homogeneous Poisson process of rate λ. They are served in groups, according to the following rule. If immediately after a departure there are less than L customers

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present, the server must wait until there are L customers, whereupon all L enter service. If there are L or more, but less than K customers waiting, then all customers present are served together ($L \leq K$). If there are more than K customers waiting, a group of K enters service and the others must wait. In this paper the order of service is immaterial. We assume that the successive service times are conditionally independent, given the batch sizes, but their distributions may depend on the batch size.

It is of interest to describe some actual situations, which may be described by this model. The operation of an unscheduled car ferry or a single ground floor station of an elevator may be approximated by the above description. The author is indebted to Professor G. Newell for supplying the following application to traffic flow. We consider a main road and a minor road merging into it. A traffic light on the main road interrupts its traffic flow after a certain length of time if at least L cars have activated a tripplate on the minor road. Otherwise the light stays green until L cars have arrived. The red cycle is timed so that at most K cars can merge during it. We count as the successive service times the time required for the plattoon to merge, plus the fixed length of the green cycle on the main road. The model then studies the queue forming on the minor road, under the assumptions of Poisson arrivals and the rule that cars arriving during the time that vehicles ahead are merging must wait for the next cycle. This assumption is not too unrealistic in very light or very heavy traffic, or if K is not too large.

There are some generalizations of the model, which may occur in practice. We may want to serve a group of less than L customers if its waiting time exceeds a given value. This generalization is easy to work out, along the same lines as the discussion below.

Another generalization is obtained when L and K are random variables on the lattice points (a,b) with $a \leq b$. This may occur if the service times of individual customers are independent, identically distributed random variables and the server accepts only as many customers as to satisfy the condition that their total service time lies between given lower and upper bounds. If K is a bounded random variable, this model may be analyzed by the same reasoning as given below, but the calculations become exceedingly involved.

We now denote the distribution of the service time for a batch of j customers by $H_j(\cdot)$, $j=L,\ldots,K$. Let ξ_n be the number of customers in the system after the nth departure and let X_n be the length of time between the $(n-1)$st and the nth departure. It follows immediately from the assumptions of Poisson input and the conditional independence of the service times, that the bivariate sequence $\{(\xi_n,X_n),\ n \geq 0\}$ is a semi-Markov sequence as defined by Pyke [5]. We set ξ_o equal to the queue length at time $t = o+$ and $X_o = o$ a.s. Without loss of generality and with a substantial gain in computational simplicity, we assume that the point $t = o$ is a departure point, so that the sequence $\{(\xi_n,X_n)\}$ is an ordinary semi-Markov sequence.

The semi-Markov process is completely characterized by the transition probability distributions:

$$(1) \qquad Q_{ij}(x) = P\{\xi_n = j,\ X_n \leq x \mid \xi_{n-1} = i\}, \quad i,j=o,1,\ldots,\ n \geq 1.$$

If we write $E_\nu(x)$ for the distribution function of an Erlang variable of order ν with parameter λ, then the probabilities $Q_{ij}(x)$ are given by:

(2) For $o \leq i \leq L$, $j \geq o$:

$$Q_{ij}(x) = \int_o^x E_{L-i}(x-y)e^{-\lambda y} \frac{(\lambda y)^j}{j!} \, d\, H_L(x),$$

for $L \leq i \leq K$, $j \geq o$:

$$Q_{ij}(x) = \int_o^x e^{-\lambda y} \frac{(\lambda y)^j}{j!} \, d\, H_i(y),$$

for $i \geq K$, $j < i-K$:

$$Q_{ij}(x) = o,$$

for $i \geq K$, $j \geq i-K$:

$$Q_{ij}(x) = \int_o^x e^{-\lambda y} \frac{(\lambda y)^{j-i+K}}{(j-i+K)!} \, d\, H_K(y),$$

The Laplace-Stieltjes transforms of the $Q_{ij}(x)$ are denoted by $q_{ij}(s)$ and are given by:

(3) For $o \leq i \leq L$, $j \geq o$:

$$q_{ij}(s) = \left(\frac{\lambda}{\lambda+s}\right)^{L-i} \int_o^\infty e^{-(\lambda+s)y} \frac{(\lambda y)^j}{j!} \, d\, H_L(y),$$

for $L \leq i \leq K$, $j \geq o$:

$$q_{ij}(s) = \int_o^\infty e^{-(\lambda+s)y} \frac{(\lambda y)^j}{j!} \, d\, H_i(y),$$

for $i \geq K$, $j < i-K$:

$$q_{ij}(s) = o,$$

for $i \geq K$, $j \geq i-K$:

$$q_{ij}(s) = \int_o^\infty e^{-(\lambda+s)y} \frac{(\lambda y)^{j-i+K}}{(j-i+K)!} \, d\, H_K(y),$$

Particular cases:

For particular choices of L and K, several queueing models are obtained, which have been studied earlier. For $L = K$, we obtain the bulk queue with fixed batch size, which has been studied by Takács [11].

For $L = 1$, we obtain the case in which the server is operating as soon as one customer is present. This model has been investigated by Bloemena [1], Le Gall [2], Runnenburg [9] and Neuts [3,4]. The M|G 1 queue is of course obtained for $L = K = 1$.

2. The Output Process and the Imbedded Markov Chain.

The nth departure from the queue occurs at a random time $\tau_n' = X_1 + \ldots + X_n$. We propose to calculate the n-step transition probabilities:

$$Q_{ij}^{(n)}(x) = P\{\tau_n' \leq x, \ \xi_n = j \,|\, \xi_o = i\}$$

for the discrete Markov process $\{(\xi_n, \tau_n'), \ n \geq o\}$. We introduce the following generating functions:

(4) a. $$U_i^{(n)}(z,s) = \sum_{\sigma=0}^{\infty} q_{i,K+\sigma}^{(n)}(s)\, z^{\sigma}, \qquad \text{Re } s \geq 0,\ |z| \leq 1$$

b. $$W_{ij}(s,w) = \sum_{n=0}^{\infty} q_{ij}^{(n)}(s)\, w^{n}, \qquad \text{Re } s \geq 0,\ |w| \leq 1,\ j=0,1,\ldots,K-1$$

c. $$V_i(z,s,w) = \sum_{n=0}^{\infty} U_i^{(n)}(z,s) w^{n}, \qquad \text{Re } s \geq 0,\ |w| \leq 1,\ |z| \leq 1.$$

and we denote the Laplace-Stieltjes transforms of the distributions $H_\nu(x)$ by $h_\nu(s)$ for $\nu = L,\ldots,K$ and $\text{Re } s \geq 0$.

We then have the following theorem:

Theorem 1

The generating function $V_i(z,s,w)$ is given by:

(5) $$V_i(z,s,w) =$$

$$\left[z^K - w h_K(s+\lambda-\lambda z)\right]^{-1} \left\{ z^i + \sum_{\nu=0}^{L-1} \left[w\left(\frac{\lambda}{\lambda+s}\right)^{L-\nu} h_L(s+\lambda-\lambda z) - z^\nu\right] W_{i\nu}(s,w) \right.$$

$$\left. + \sum_{\nu=L}^{K-1} \left[w h_\nu(s+\lambda-\lambda z) - z^\nu\right] W_{i\nu}(s,w) \right\}$$

in which the functions $W_{i\nu}(s,w)$ are the solutions to the system of linear equations:

(6) $\gamma_\rho^i(w,s) =$

$$\sum_{\nu=0}^{L-1} \left\{\gamma_\rho^\nu(w,s) - w\left(\frac{\lambda}{\lambda+s}\right)^{L-\nu} h_L\left[s+\lambda-\lambda\gamma_\rho(w,s)\right]\right\} W_{i\nu}(s,w) +$$

$$\sum_{\nu=L}^{K-1} \left\{\gamma_\rho^\nu(w,s) - w h_\nu\left[s+\lambda-\lambda\gamma_\rho(w,s)\right]\right\} W_{i\nu}(s,w),$$

$\rho=1,\ldots,K.$

The functions $\gamma_\rho(w,s)$, $\rho=1,\ldots,K$ are the K roots of the equation:

$$z^K = w\, h_K(s+\lambda-\lambda z), \tag{7}$$

which lie in the unit disk $|z| \leq 1$ and defined analytically for $\operatorname{Re} s \geq 0$, $|w| < 1$.

Proof:

The transforms $q_{ij}^{(n)}(s)$ satisfy the recurrence relations:

$$q_{ij}^{(0)}(s) = \delta_{ij}, \tag{8}$$

$$q_{ij}^{(n+1)}(s) = \sum_{\nu=0}^{L-1} q_{i\nu}^{(n)}(s)\left(\frac{\lambda}{\lambda+s}\right)^{L-\nu} \int_0^\infty e^{-(\lambda+s)y} \frac{(\lambda y)^j}{j!}\, d\, H_L(y)$$

$$+ \sum_{\nu=L}^{K-1} q_{i\nu}^{(n)}(s) \int_0^\infty e^{-(\lambda+s)y} \frac{(\lambda y)^j}{j!}\, d\, H_\nu(y)$$

$$+ \sum_{\nu=K}^{j+K} q_{i\nu}^{(n)}(s) \int_0^\infty e^{-(\lambda+s)y} \frac{(\lambda y)^{j-\nu+K}}{(j-\nu+K)!}\, d\, H_K(y),$$

for $n \geq o$.

We obtain:

$$(9)\qquad \sum_{j=o}^{K-1} q_{ij}^{(n+1)}(s)\, z^j + z^K\, U_i^{(n+1)}(z,s) =$$

$$\sum_{\nu=o}^{L-1} q_{i\nu}^{(n)}(s)\left(\frac{\lambda}{\lambda+s}\right)^{L-\nu} h_L(s+\lambda-\lambda z) + \sum_{\nu=L}^{K-1} q_{i\nu}^{(n)}(s)\, h_\nu(s+\lambda-\lambda z)$$

$$+ U_i^{(n)}(z,s)\, h_K(s+\lambda-\lambda z),$$

and upon multiplication by w^{n+1} and summation we obtain (5). The function $V_i(z,s,w)$ is analytic in its region of definition. The denominator has exactly K roots in the unit disk $|z| \leq 1$ - Takács [11] p. 82 - We denote them by $\gamma_\rho(w,s)$, $\rho=1,\ldots,K$ and define them so as to be analytic in the region of interest. Expressing that the zeros of numerator and denominator must coincide, we obtain the system (6).

Theorem 2

The n-step transition probabilities $P_{ij}^{(n)}$ for the imbedded Markov chain $\{\xi_n,\ n \geq o\}$ are given by:

$$(10)\qquad \sum_{n=o}^{\infty}\sum_{j=o}^{\infty} z^j\, w^n\, P_{ij}^{(n)} =$$

$$\left[z^K - wh_K(\lambda-\lambda z)\right]^{-1}\left\{z^{K+i} + \sum_{\nu=o}^{L-1} wW_{i\nu}(o,w)\left[z^K h_L(\lambda-\lambda_z) - z^\nu h_K(\lambda-\lambda z)\right]\right.$$

$$\left. + \sum_{\nu=L}^{K-1} w\, W_{i\nu}(o,w)\left[z^K h_\nu(\lambda-\lambda z) - z^\nu\, h_K(\lambda-\lambda z)\right]\right\},$$

The limiting probabilities $\pi_\nu = \lim_{n\to\infty} P_{i\nu}^{(n)}$ are obtained by multiplying (10) on both sides by $1-w$ and letting w tend to one. We have:

$$(11) \qquad \pi_\nu = \lim_{w\to 1} (1-w)\, W_{i\nu}(o,w), \qquad \text{for } \nu = o,1,\ldots,K-1,$$

and:

$$(12) \qquad \sum_{j=o}^{\infty} \pi_j z^j = \left[z^K - h_K(\lambda-\lambda z)\right]^{-1} \left\{ \sum_{\nu=o}^{L-1} \pi_\nu \left[z^K h_L(\lambda-\lambda z) - z^\nu h_K(\lambda-\lambda z)\right] \right.$$

$$\left. + \sum_{\nu=L}^{K-1} \pi_\nu \left[z^K h_\nu(\lambda-\lambda z) - z^\nu h_K(\lambda-\lambda z)\right] \right\}$$

Let α_K denote the first moment of $H_K(\cdot)$, then the Markov chain $\{\xi_n, n \geq o\}$ is positive recurrent if and only if $K - \lambda\alpha_K > o$, null-recurrent if and only if $K - \lambda\alpha_K = o$ and transient if and only if $K - \lambda\alpha_K < o$.

Proof:

Formula (10) follows from (5) and (6) by setting $s = o+$ in

$$\sum_{n=o}^{\infty} \sum_{j=o}^{\infty} z^j w^n q_{ij}^{(n)}(s) = \sum_{j=o}^{K-1} W_{ij}(w,s) z^j + z^K V_i(z,s,w).$$

The proof of (12) is well-known.

From the system (6) we obtain $W_{i\nu}(o,w)$ as the ratio of two determinants, namely the determinant

$$\left|\left| 1-wh_L(\lambda-\lambda\gamma_\rho), \ldots, \gamma_\rho^{L-1} - wh_L(\lambda-\lambda\gamma_\rho),\ \gamma_\rho^{L} - wh(\lambda-\lambda\gamma_\rho), \ldots, \gamma_\rho^{K-1} - wh_{K-1}(\lambda-\lambda\gamma_\rho) \right|\right|$$

in the denominator and a determinant, obtained by replacing the νth column

of the above by the column with entries $\gamma_\rho^i(w,o)$, $\rho=1,\ldots,K$. We form the product $(1-w)\, W_{i\nu}(o,w)$ by dividing the first row of the denominator by $1-w$. The equation:

$$z^K = h_K(\lambda-\lambda z) \tag{13}$$

has a real, positive root of smallest value in the interval $[0,1]$. Let us define $\gamma_1(w,s)$ to be the root of the equation (7) which corresponds to this positive root of (13) for $w=1$, $s=o+$. By a well-known argument, we have:

$$\gamma_1(1,o+) = 1 \quad \text{if and only if} \quad K - \lambda\, \alpha_K \geq o \tag{14}$$

$$\gamma_1(1,o+) < 1 \quad \text{if and only if} \quad K - \lambda\, \alpha_K < o$$

Consider the case $\gamma_1(1,o+) = 1$. The numerator in the expression for $(1-w)W_{i\nu}(o,w)$ tends to the cofactor of the element $\gamma_1^i(w,o)$ in the first row and the νth column, evaluated at $w=1$. This quantity does not depend on the initial state i. The denominator converges to a determinant in which the first row is given by the constants:

$$(A_o,\ldots,A_{L-1},A_L,\ldots,A_{K-1})$$

where:

$$A_\nu = \lim_{w\to 1} (1-w)^{-1}\left[\gamma_1^\nu(w,o)-wh_L(\lambda-\lambda\gamma_1(w,o))\right], \quad \text{for} \quad \nu=o,1,\ldots,L-1 \tag{15}$$

$$A_\nu = \lim_{w\to 1} (1-w)^{-1}\left[\gamma_1^\nu(w,o)-wh_\nu(\lambda-\lambda\gamma_1(w,o))\right], \quad \text{for} \quad \nu=L,\ldots,K-1\ .$$

These limits may be evaluated explicitely using de l'Hopital's rule and the fact that:

$$(16) \qquad \frac{\partial}{\partial w}\ \gamma_1(w,s)\Big|_{\substack{w=1\\ s=0}} = (K - \lambda\ \alpha_K)^{-1},$$

We find:

$$(17) \qquad A_\nu = (K-\nu+\lambda\ \alpha_L - \lambda\ \alpha_K)(K-\lambda\ \alpha_K)^{-1}, \qquad \text{for} \quad \nu=0,1,\ldots,L-1.$$

$$A_\nu = (K-\nu+\lambda\ \alpha_\nu - \lambda\ \alpha_K)(K-\lambda\ \alpha_K)^{-1}, \qquad \text{for} \quad \nu=L,\ldots,K-1,$$

in which α_ν, $\nu = L,\ldots,K$ is the expected duration of the service time for a group of ν customers.

The other rows of the determinant in the denominator are given by:

$$1-h_L\left[\lambda-\lambda\ \gamma_\rho(1,0)\right],\ldots,\gamma_\rho^{L-1}(1,0)-h_L\left[\lambda-\lambda\ \gamma_\rho(1,0)\right],\ldots,$$

$$\gamma_\rho^{K-1}(1,0)-h_{K-1}\left[\lambda-\lambda\gamma_\rho(1,0)\right]$$

for $\rho = 2,\ldots,K$.

If $K-\lambda\ \alpha_K > 0$, this procedure yields the generating function for the stationary probability distribution of the queue length immediately after departures. The imbedded Markov chain is then positive recurrent.

If $K - \lambda\ \alpha_K = 0$, all π_ν, $\nu = 0,1,\ldots$ are equal to zero and the chain is null recurrent.

If $K - \lambda\ \alpha_K < 0$, we need an additional argument to show that the imbedded chain is transient. We need a slight extension of Foster's theorem

given in the proposition below:

Proposition

An irreducible Markov chain is transient if and only if the system of equations:

$$(*) \qquad \sum_{j=o}^{\infty} P_{ij}\, y_j = y_i, \qquad i \notin C$$

where C is any finite subset of the state space, has a bounded nonconstant solution.

The proof of this proposition is essentially the same as for Foster's theorem.

Applying this property, it is easy to see that if we remove the equations, corresponding to the first K rows the resulting system has a bounded nonconstant solution if $K-\lambda\, \alpha_K < o$.

3. The Successive Busy Periods.

The server will become free as soon as there are less than L customers left in the queue immediately after a departure. We define a busy period as the length of time between the beginning of service for the first batch, which contains L customers and the time when the number of customers in the system drops below L for the first time thereafter. The initial busy period will depend on the initial conditions in the queue and we will not derive its distribution here. It may be obtained by an analogous reasoning to the one given here. We denote the lengths of the busy periods by Y_n, $n=1,2,\ldots$ and the number of customers left in the queue upon termination of the nth busy period by I_n. It is obvious that the sequence

$$\{(I_n, Y_n),\ n \geq o,\ Y_o = o,\ I_o \text{ arbitrary}\}$$

forms a semi-Markov sequence with L states $\{o,\ldots,L-1\}$. We will now obtain transforms for the transition probability distributions

$$G_{ij}(x) = P\{I_n = j,\ Y_n \leq x \mid I_{n-1} = i\} \tag{12}$$

for $i,j=o,1,\ldots,L-1$ and $n > 1$.

Theorem 3

The transition probability distributions $G_{ij}(x)$ do not depend on i. Their Laplace-Stieltjes transforms $E_j(1,s)$ $(j=o,.,\ldots,L-1)$ are the solutions to the system of linear equations

$$\sum_{r=o}^{L-1} E_r(1,s)\ \gamma_\rho^r(1,s) + \sum_{r=L}^{K-1} \left[\gamma_\rho^r(1,s) - h_r[s+\lambda-\lambda\gamma_\rho(1,s)]\right] E_r(1,s) \tag{19}$$

$$= h_L[s+\lambda-\lambda\gamma_\rho(1,s)], \quad \rho = 1,\ldots,K.$$

Proof: The proof is an extension of an argument by Takács [10] for the $M|G|1$ queue.

Let $G(k,n,x)$ be the probability that a busy period consists of at least n services, which last a length of time of at most x and such that at the end of the nth service there are h customers waiting.

It is clear that the $G(k,n,x)$ do not depend on the number of customers in the queue at the end of the previous busy period.

The probabilities $G(k,n,x)$ satisfy the recurrence relations:

(20) $$G(k,1,x) = \int_0^x e^{-\lambda y} \frac{(\lambda y)^k}{k!} \, d\, H_L(y),$$

and for $n > 1$:

$$G(k,n,x) = \sum_{r=L}^{K-1} \int_0^x G(r,n-1,x-y) e^{-\lambda y} \frac{(\lambda y)^k}{k!} \, d\, H_r(y) +$$

$$\sum_{\nu=K}^{K+k} \int_0^x G(\nu,n-1,x-y) e^{-\lambda y} \frac{(\lambda y)^{k+K-\nu}}{(k+K-\nu)!} \, d\, H_K(y),$$

We denote the Laplace-Stieltjes transforms of the $G(k,n,x)$ by $\Gamma(k,n,s)$ and obtain the transformed version of equations (20). We also introduce the generating functions:

(21) a. $$C_K(z,n,s) = \sum_{\nu=0}^{\infty} z^\nu \, \Gamma(\nu+K,n,s), \qquad |z| \leq 1, \ \mathrm{Re}\ s \geq 0.$$

b. $$D_K(z,w,s) = \sum_{n=1}^{\infty} C_K(z,n,s) w^n, \qquad |z| \leq 1, \ |w| \leq 1$$

c. $$E_r(w,s) = \sum_{n=1}^{\infty} \Gamma(r,n,s) w^n, \qquad |w| \leq 1, \ \mathrm{Re}\ s \geq 0$$
$$r = 0,1,\ldots,K-1.$$

We obtain successively:

(22) $$\sum_{k=0}^{\infty} \Gamma(k,1,s) z^k = h_L(s+\lambda-\lambda z)$$

and for $n > 1$:

$$\sum_{k=o}^{K-1} \Gamma(k,n,s)\, z^k + z^K\, C_K(z,n,s) =$$

$$\sum_{r=L}^{K-1} \Gamma(r,n-1,s)\, h_r(s+\lambda-\lambda z) + h_K(s+\lambda-\lambda z)\, C_K(z,n-1,s),$$

$$(23)\quad D_K(z,w,s) = \frac{wh_L(s+\lambda-\lambda z) - \sum_{r=o}^{L-1} E_r(w,s) z^r + \sum_{r=L}^{K-1} [wh_r(s+\lambda-\lambda z) - z^r]\, E_r(w,s)}{z^K - wh_K(s+\lambda-\lambda z)}$$

Applying the standard argument, we obtain the unknown functions $E_r(w,s)$ as the solutions to the system of linear equations:

$$(24)\quad \sum_{r=o}^{L-1} E_r(w,s)\gamma_\rho^r(w,s) + \sum_{r=L}^{K-1} \left[\gamma_\rho^r(w,s) - wh_r\left[s+\lambda-\lambda\gamma_\rho(w,s)\right]\right] E_r(w,s)\Big]$$

$$= wh_L\left[s+\lambda-\lambda\gamma_\rho(w,s)\right], \quad \rho = 1,\ldots,K.$$

in which the $\gamma_\rho(w,s)$ are the roots of equation (7) in $|z| \leq 1$. For $w=1$ and $j = o,\ldots,L-1$ the $E_j(1,s)$ are the transforms of $G_{ij}(x)$.

Since the $G_{ij}(x)$ do not depend on i, the successive busy periods form a semi-Markov process of zero order as defined by Pyke [7].

If $K-\lambda\,\alpha_K \geq o$, we have $\gamma_1(1,o+) = 1$. If we set $w=1$, $s=o+$ and $\rho = 1$ in equation (24), we obtain

$$\sum_{r=o}^{L-1} E_r(1,o+) = 1,$$

This quantity is less than one if $K - \lambda \alpha_K < o$, because the imbedded semi-Markov process is transient. $E_r(1,o+)$, r=o,...,L-1, is the probability that at the end of a busy period there will be r customers left behind.

The sequence of successive idle periods is now also easy to describe. They form a semi-Markov process of zero order in which the transition probabilities are $E_r(1,o+)$ and the corresponding idle periods have an Erlang distribution with parameter λ and order L-r, r=o,1,...,L-1. The moments of the busy periods may be calculated from the determinant representation of $E_r(1,s)$, r=o,...,L-1. One verifies directly that the first moment is infinite if $K = \lambda \alpha_K$.

4. The Queue length in Continuous Time.

The distribution of the queue length in continuous time is readily obtained from the renewal functions for the imbedded semi-Markov process $\{(\xi_n, X_n),\ n \geq o\}$. For simplicity we again assume that the point $t = o$ is a departure point. We define $M_{ij}(t)$, i,j = o,1,... as the expected number of visits to state j in the closed interval [o,t], starting in state i, in the imbedded semi-Markov process. The Laplace-Stieltjes transforms of the $M_{ij}(t)$ are denoted by $\mu_{ij}(s)$. We denote by μ_j^* the mean recurrence time of the state j in the imbedded semi-Markov process.

Let $\xi(t)$ be the queue length at time t and let:

$$P_{ij}(t) = P\{\xi(t) = j \mid \xi_o = i\}, \tag{25}$$

We denote the Laplace transform of $P_{ij}(t)$ by $p_{ij}(s)$ and we introduce the generating function:

$$\Pi(s,z) = \sum_{j=0}^{\infty} p_{ij}(s)\, z^j, \qquad |z| \leq 1,\ \mathrm{Re}\ s \geq 0 \tag{26}$$

We now prove the following theorem:

Theorem 4:

The generating function $\Pi(s,z)$ is given by:

$$\begin{aligned}\Pi(s,z) = {} & \frac{1}{s+\lambda-\lambda z} \sum_{\nu=0}^{L-1} W_{i\nu}(1,s) z^{\nu} \left[1-\left(\frac{\lambda z}{\lambda+s}\right)^{L-\nu}\right] \\ & + \frac{z^L}{s+\lambda-\lambda z}\left[1-h_L(s+\lambda-\lambda z)\right] \sum_{\nu=0}^{L-1} W_{i\nu}(1,s)\left(\frac{\lambda}{\lambda+s}\right)^{L-\nu} \\ & + \frac{1}{s+\lambda-\lambda z} \sum_{\nu=L}^{K-1} W_{i\nu}(1,s)\left[1-h_{\nu}(s+\lambda-\lambda z)\right] z^{\nu} \\ & + \frac{z^K}{s+\lambda-\lambda z}\, V_i(z,s,1)\left[1-h_K(s+\lambda-\lambda z)\right],\end{aligned} \tag{27}$$

in which the functions $W_{i\nu}(1,s)$, $\nu=0,1,\ldots,K-1$ and $V_i(z,s,1)$ are given by formulae (5) and (6) for $w=1$.

The limits $P_j^* = \lim_{t \to \infty} P_{ij}(t)$ exist and are given below for the positive recurrent case. In the null-recurrent and transient cases these limits are equal to zero.

Proof:

We have

$$M_{ij}(t) = \sum_{\nu=0}^{\infty} Q_{ij}^{(\nu)}(t), \tag{28}$$

and hence:

$$\sum_{j=0}^{\infty} \mu_{ij}(s) z^j = \sum_{j=0}^{K-1} W_{ij}(1,s) + z^K V_i(z,s,1), \tag{29}$$

by formulae (4), (5) and (6).

By enumeration of cases and the law of total probability, we obtain:

(30) For $0 \leq j \leq L-1$:

$$P_{ij}(t) = \sum_{\nu=0}^{j} \int_0^t e^{-\lambda(t-\tau)} \frac{[\lambda(t-\tau)]^{j-\nu}}{(j-\nu)!} \, d\, M_{i\nu}(t),$$

for $L \leq j \leq K-1$:

$$P_{ij}(t) = \sum_{\nu=0}^{L-1} \int_0^t dM_{i\nu}(t) e^{-\lambda(t-\tau)} \int_0^{t-\tau} \frac{(\lambda u)^{L-\nu-1}}{(L-\nu-1)!} \frac{[\lambda(t-\tau-u)]^{j-L}}{(j-L)!}$$

$$[1-H_L(t-\tau-u)] \, \lambda \, du.$$

$$+ \sum_{\nu=L}^{j} \int_0^t dM_{i\nu}(t) e^{-\lambda(t-\tau)} \frac{[\lambda(t-\tau)]^{j-\nu}}{(j-\nu)!} [1-H_\nu(t-\tau)],$$

for $j \geq K$:

$$P_{ij}(t) =$$

$$\sum_{\nu=0}^{L-1} \int_0^t dM_{i\nu}(\tau) \int_0^{t-\tau} e^{-\lambda(t-\tau)} \frac{(\lambda u)^{L-\nu-1}}{(L-\nu-1)!} \frac{[\lambda(t-\tau-u)]^{j-L}}{(j-L)!} [1-H_L(t-\tau-u)]\, \lambda\, du$$

$$+ \sum_{\nu=1}^{K-1} \int_0^t dM_{i\nu}(\tau) e^{-\lambda(t-\tau)} \frac{[\lambda(t-\tau)]^{j-\nu}}{(j-\nu)!} [1-H_\nu(t-\tau)] +$$

$$\sum_{\nu=K}^{j} \int_0^t dM_{i\nu}(t) e^{-\lambda(t-\tau)} \frac{[\lambda(t-\tau)]^{j-\nu}}{(j-\nu)!} [1-H_K(t-\tau)],$$

Taking Laplace transforms, we obtain:

(31) For $o \leq j \leq L-1$:

$$p_{ij}(s) = \sum_{\nu=o}^{j} \mu_{i\nu}(s) \frac{1}{\lambda} \left(\frac{\lambda}{\lambda+s}\right)^{j-\nu+1},$$

for $L \leq j \leq K-1$:

$$p_{ij}(s) = \sum_{\nu=o}^{L-1} \mu_{i\nu}(s) \left(\frac{\lambda}{\lambda+s}\right)^{L-\nu} \int_o^\infty e^{-(\lambda+s)v} \frac{(\lambda v)^{j-L}}{(j-L)!} [1-H_L(v)]\, dv$$

$$+ \sum_{\nu=L}^{j} \mu_{i\nu}(s) \int_o^\infty e^{-(\lambda+s)v} \frac{(\lambda v)^{j-\nu}}{(j-\nu)!} [1-H_\nu(v)] dv,$$

and for $j \geq K$:

$$p_{ij}(s) = \sum_{\nu=0}^{L-1} \mu_{i\nu}(s)\left(\frac{\lambda}{\lambda+s}\right)^{L-\nu} \int_0^\infty e^{-(\lambda+s)v} \frac{(\lambda v)^{j-L}}{(j-L)!} [1-H_L(v)]dv$$

$$+ \sum_{\nu=L}^{K-1} \mu_{i\nu}(s) \int_0^\infty e^{-(\lambda+s)v} \frac{(\lambda v)^{j-\nu}}{(j-\nu)!} [1-H_\nu(v)]dv$$

$$+ \sum_{\nu=K}^{j} \mu_{i\nu}(s) \int_0^\infty e^{-(\lambda+s)v} \frac{(\lambda v)^{j-\nu}}{(j-\nu)!} [1-H_K(v)]dv,$$

Formula (27) follows by taking the generating function.

The mean recurrence times μ_j^* in the imbedded semi-Markov process are found as the limits:

(32) $$\lim_{t \to \infty} \frac{M_{jj}(t)}{t} = \frac{1}{\mu_j^*},$$

and are independent of i. If $K\text{-}\lambda\ \alpha_K \leq 0$, these limits are zero. If $K\text{-}\lambda\ \alpha_K > 0$, the μ_j^* are finite. Applying the key renewal theorem to the integrals in formula (30) we obtain:

(33) For $0 \leq j \leq L\text{-}1$

$$P_j^* = \frac{1}{\lambda} \sum_{\nu=0}^{j} \frac{1}{\mu_\nu^*},$$

for $L \leq j \leq K\text{-}1$:

$$P_j^* = \sum_{\nu=0}^{L-1} \frac{1}{\mu_\nu^*} \int_0^\infty e^{-\lambda v} \frac{(\lambda v)^{j-L}}{(j-L)!} [1-H_L(v)]dv$$

$$+ \sum_{\nu=L}^{j} \frac{1}{\mu_\nu^*} \int_0^\infty e^{-\lambda v} \frac{(\lambda v)^{j-\nu}}{(j-\nu)!} [1-H_\nu(v)]dv,$$

and for $j \geq K$:

$$P_j^* = \sum_{\nu=0}^{L-1} \frac{1}{\mu_\nu^*} \int_0^\infty e^{-\lambda v} \frac{(\lambda v)^{j-L}}{(j-L)!} [1-H_L(v)]dv$$

$$+ \sum_{\nu=L}^{K-1} \frac{1}{\mu_\nu^*} \int_0^\infty e^{-\lambda v} \frac{(\lambda v)^{j-\nu}}{(j-\nu)!} [1-H_\nu(v)]dv$$

$$+ \sum_{\nu=K}^{j} \frac{1}{\mu_\nu^*} \int_0^\infty e^{-\lambda v} \frac{(\lambda v)^{j-\nu}}{(j-\nu)!} [1-H_K(v)]dv,$$

There seems to be no simple relation in general, between these limiting probabilities and those for the queue length in discrete time.

The formulae (33) may be obtained directly by considering the stationary version of the semi-Markov process.

5. Limit Theorems

In this section we apply theorems due to Pyke and Schaufele [8] to obtain the strong law of large numbers and the central limit theorem for the output of the queueing model in the positive recurrent case.

Let ζ_n be the size of the batch, leaving the queue at the nth departure. We then have:

$$\zeta_n = f(\xi_{n-1}), \quad n \geq 1 \tag{34}$$

in which:

$$f(\nu) = \begin{cases} L & 0 \leq \nu \leq L \\ \nu & L \leq \nu \leq K \\ K & \nu \geq K \end{cases} \tag{35}$$

Let $N(t)$ denote the number of customers leaving the queue in the interval $(0,t]$, then it follows from theorems of Pyke and Schaufele that the following theorems hold:

Theorem 5

As $t \longrightarrow \infty$, we have

$$\frac{N(t)}{t} \xrightarrow{a.s.} A = \frac{L\sum_{\nu=0}^{L-1}\pi_\nu + \sum_{\nu=L}^{K-1}\nu\pi_\nu + K\left[1-\sum_{\nu=0}^{K-1}\pi_\nu\right]}{\sum_{\nu=0}^{L-1}\pi_\nu\left[\alpha_L+(L-\nu)\lambda^{-L}\right] + \sum_{\nu=L}^{K-1}\pi_\nu\alpha_\nu + \alpha_K\left[1-\sum_{\nu=0}^{K-1}\pi_\nu\right]}. \tag{36}$$

The quantity A may be interpreted as the asymptotic average number of customers leaving the queue per unit of time. The numerator is the asymptotic average number of customers per batch and the denominator is a measure of the average time between departures.

Theorem 6

Let A be the constant found in theorem 5, then of the service time distributions have finite second moments and if the queue is ergodic, the random variable

$$t^{-1/2}\,[N(t) - tA]$$

converges in law to a normal random variable with mean zero and variance σ^2, given by:

$$(37)\quad \sigma^2 = \Big(\sum_{i=o}^{\infty} \eta_i \pi_i\Big)^{-1} \Big\{ \sum_{i=o}^{\infty} \pi_i \zeta_i^{(2)} + 2 \sum_{i=o}^{\infty} \sum_{k \neq j} \sum_{r \neq j} \xi_{ik} \zeta_r^{(1)} \pi_i \, {}_jM_{kr}(\infty) \Big\}$$

in which:

$$(38)\quad \sum_{i=o}^{\infty} \eta_i \pi_i = \sum_{i=o}^{L-1} \pi_i \big[\alpha_L + (L-i)\lambda^{-1}\big] + \sum_{i=L}^{K-1} \pi_i \alpha_i + \alpha_K \Big[1 - \sum_{\nu=o}^{K-1} \pi_\nu\Big],$$

and:

$$\zeta_i^{(2)} = f^2(i) - A \sum_{k=o}^{\infty} \int_o^{\infty} x^2 \, dQ_{ik}(x),$$

$$\xi_{ik} = f(i) Q_{ik}(\infty) - A \int_o^{\infty} x \, D \, Q_{ik}(x)$$

$$\zeta_r^{(1)} = f(r) - A \sum_{k=o}^{\infty} \int_o^{\infty} x \, d \, Q_{rk}(x)$$

$${}_jM_{kr}(\infty) = \frac{\mu_{kj} + \mu_{jr} - \mu_{kr}}{\mu_{rr}}$$

in which the quantities μ_{kj} are the first moments of the first passage times from state k to state j in the semi-Markov process. This central limit theorem is difficult to apply because of the involved calculations for the μ_{kj}.

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13. ABSTRACT

We assume that customers arrive at a counter according to a homogeneous Poisson process and are served in groups, according to the following policy. If there are less than L customers waiting at the time of a departure, the server must wait until there are L customers present, whereupon he serves them together. If there are L or more, but less than K ($K \geq L$) customers waiting, all are served together. If there are K or more customers waiting, a group of K customers is served and the others must wait. The service times of successive groups are assumed to be conditionally independent given the bulk sizes, but may depend on the group sizes. We obtain 1. a description of the output process, 2. the queue length in discrete time, 3. the distribution of the busy period, 4. the queue length in continuous time and 5. some limit theorems for the number of customers served over a long period of time.

The order of service is irrelevant in this paper. The method used throughout is that of the imbedded semi-Markov process.

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A GENERAL CLASS OF BULK QUEUES WITH POISSON INPUT[1]

BY MARCEL F. NEUTS

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We assume that customers arrive at a counter according to a homogeneous Poisson process and are served in groups, according to the following policy: If there are less than L customers waiting at the time of a departure, the server must wait until there are L customers present, whereupon he serves them together. If there are L or more, but less than $K(K \geqq L)$ customers waiting, all are served together. If there are K or more customers waiting, a group of K customers are served and the others must wait. The service times of successive groups are assumed to be conditionally independent given the bulk sizes, but may depend on their magnitude. We obtain 1. a description of the output process, 2. the queue length in discrete time, 3. the distribution of the busy period, 4. the queue length in continuous time and 5. some limit theorems for the number of customers served over a long period of time.

The order of service is irrelevant in this paper. The method used throughout is that of the imbedded semi-Markov process.

1. Introduction. This paper is devoted to the study of the following queueing model. We assume that customers arrive at a counter, according to a homogeneous Poisson process of rate λ. They are served in groups, according to the following rule. If immediately after a departure there are less than L customers present, the server must wait until there are L customers, whereupon all L enter service. If there are L or more, but at most K customers waiting, then all customers present are served together ($L \leqq K$). If there are more than K customers waiting, a group of K enters service and the others must wait. In this paper the order of service is immaterial. We assume that the successive service times are conditionally independent, given the batch sizes, but their distributions may depend on the batch size.

It is of interest to describe some actual situations, which may be described by this model. The operation of an unscheduled car ferry or a single ground floor station of an elevator may be approximated by the above description. The author is indebted to Professor G. Newell for supplying the following application to traffic flow. We consider a main road and a minor road merging into it. A trafic light on the main road interrupts its traffic flow after a certain length of time if at least L cars have activated a tripplate on the minor road. Otherwise the light stays green until L cars have arrived. The red cycle is timed so that at most K cars can merge during it. We count as the successive service times the

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time required for the platoon to merge, plus the fixed length of the green cycle on the main road. The model then studies the queue forming on the minor road, under the assumptions of Poisson arrivals and the rule that cars arriving during the time that vehicles ahead are merging must wait for the next cycle. This assumption is not too unrealistic in very light or very heavy traffic, or if K is not too large.

There are some generalizations of the model, which may occur in practice. We may want to serve a group of less than L customers if its waiting time exceeds a given value. This generalization is easy to work out, along the same lines as the discussion below.

Another generalization is obtained when L and K are random variables on the lattice points (a, b) with $a \leqq b$. This may occur if the service times of individual customers are independent, identically distributed random variables and the server accepts only as many customers as to satisfy the condition that their total service time lies between given lower and upper bounds. If K is a bounded random variable, this model may be analyzed by the same reasoning as given below, but the calculations become exceedingly involved.

We now denote the distribution of the service time for a batch of j customers by $H_j(\cdot), j = L, \cdots, K$. Let ξ_n be the number of customers in the system after the nth departure and let X_n be the length of time between the $(n-1)$st and the nth departure. It follows immediately from the assumptions of Poisson input and the conditional independence of the service times, that the bivariate sequence $\{(\xi_n, X_n), n \geqq 0\}$ is a semi-Markov sequence as defined by Pyke [5]. We set ξ_0 equal to the queue length at time $t = 0+$ and $X_0 = 0$ a.s. Without loss of generality and with a substantial gain in computational simplicity, we assume that the point $t = 0$ is a departure point, so that the sequence $\{(\xi_n, X_n)\}$ is an ordinary semi-Markov sequence.

The semi-Markov process is completely characterized by the transition probability distributions

$$(1)\quad Q_{ij}(x) = P\{\xi_n = j,\ X_n \leqq x \mid \xi_{n-1} = i\}, \qquad i, j = 0, 1, \cdots,\ n \geqq 1.$$

If we write $E_\nu(x)$ for the distribution function of an Erlang variable of order ν with parameter λ, then the probabilities $Q_{ij}(x)$ are given by:

(2) For $0 \leqq i \leqq L, j \geqq 0$

$$Q_{ij}(x) = \int_0^x E_{L-i}(x-y)e^{-\lambda y}(\lambda y)^j/j!\, dH_L(y),$$

for $L \leqq i \leqq K, j \geqq 0$

$$Q_{ij}(x) = \int_0^x e^{-\lambda y}(\lambda y)^j/j!\, dH_i(y),$$

for $i \geqq K, j < i - K$

$$Q_{ij}(x) = 0,$$

for $i \geqq K, j \geqq i - K$

$$Q_{ij}(x) = \int_0^x e^{-\lambda y}(\lambda y)^{j-i+K}/(j-i+K)!\, dH_K(y),$$

The Laplace-Stieltjes transforms of the $Q_{ij}(x)$ are denoted by $q_{ij}(s)$ and are given by:

(3) For $0 \leqq i \leqq L, j \geqq 0$

$$q_{ij}(s) = (\lambda/(\lambda + s))^{L-i} \int_0^\infty e^{-(\lambda+s)y}(\lambda y)^j/j!\, dH_L(y),$$

for $L \leqq i \leqq K, j \geqq 0$

$$q_{ij}(s) = \int_0^\infty e^{-(\lambda+s)y}(\lambda y)^j/j!\, dH_i(y),$$

for $i \geqq K, j < i - K$

$$q_{ij}(s) = 0,$$

for $i \geqq K, j \geqq i - K$

$$q_{ij}(s) = \int_0^\infty e^{-(\lambda+s)y}(\lambda y)^{j-i+K}/(j - i + K)!\, dH_K(y),$$

Particular cases. For particular choices of L and K, several queueing models are obtained, which have been studied earlier. For $L = K$, we obtain the bulk queue with fixed batch size, which has been studied by Takács [11].

For $L = 1$, we obtain the case in which the server is operating as soon as one customer is present. This model has been investigated by Bloemena [1], Le Gall [2], Runnenburg [9] and Neuts [3, 4]. The $M/G/1$ queue is obtained for $L = K = 1$.

2. The output process and the imbedded Markov chain. The nth departure from the queue occurs at a random time $\tau_n' = X_1 + \cdots + X_n$. We propose to calculate the n-step transition probabilities

$$Q_{ij}^{(n)}(x) = P\{\xi_n = j, \tau_n' \leqq x \mid \xi_0 = i\}$$

for the discrete Markov process $\{(\xi_n, \tau_n')\}, n \geqq 0\}$. We introduce the following generating functions for the L-S-transforms of the $Q_{ij}^{(n)}(x)$:

(4) (a)
$$U_i^{(n)}(z, s) = \sum_{\sigma=0}^\infty q_{i,K+\sigma}^{(n)}(s)z^\sigma, \quad \text{Re } s \geqq 0, |z| \leqq 1,$$

(b)
$$W_{ij}(s, w) = \sum_{n=0}^\infty q_{ij}^{(n)}(s)w^n, \quad \text{Re } s \geqq 0, |w| < 1, j = 0, 1, \cdots, K - 1,$$

(c)
$$V_i(z, s, w) = \sum_{n=0}^\infty U_i^{(n)}(z, s)w^n, \quad \text{Re } s \geqq 0, |w| < 1, |z| \leqq 1;$$

and we denote the Laplace-Stieltjes transforms of the distributions $H_\nu(x)$ by $h_\nu(s)$ for $\nu = L, \cdots, K$ and Re $s \geqq 0$.

We then have the following theorem:

THEOREM 1. *The generating function $V_i(z, s, w)$ is given by:*

$$V_i(z, s, w) = [z^K - wh_K(s + \lambda - \lambda z)]^{-1}$$

(5) $$\cdot\{z^i + \sum_{\nu=0}^{L-1} [w(\lambda/(\lambda + s))^{L-\nu}h_L(s + \lambda - \lambda z) - z^\nu]W_{i\nu}(s, w) + \sum_{\nu=L}^{K-1} [wh_\nu(s + \lambda - \lambda z) - z^\nu]W_{i\nu}(s, w)\}$$

in which the functions $W_{i\nu}(s, w)$ are the solutions to the system of linear equations:

$$\begin{aligned}(6)\quad &\gamma_\rho{}^i(w, s)\\ &= \sum_{\nu=0}^{L-1}\{\gamma_\rho{}^\nu(w, s) - w(\lambda/(\lambda+s))^{L-\nu}h_L[s+\lambda-\lambda\gamma_\rho(w, s)]\}W_{i\nu}(s, w)\\ &+ \sum_{\nu=L}^{K-1}\{\gamma_\rho{}^\nu(w, s) - wh_\nu[s+\lambda-\lambda\gamma_\rho(w, s)]\}W_{i\nu}(s, w), \quad \rho = 1, \cdots, K.\end{aligned}$$

The functions $\gamma_\rho(w, s)$, $\rho = 1, \cdots, K$ are the K roots of the equation

$$(7)\qquad z^K = wh_K(s+\lambda-\lambda z),$$

which lie in the unit disk $|z| \leqq 1$ and defined analytically for $\operatorname{Re} s \geqq 0$, $|w| < 1$.

PROOF. The transforms $q_{ij}^{(n)}(s)$ satisfy the recurrence relations:

$$\begin{aligned}&q_{ij}^{(o)}(s) = \delta_{ij},\\ (8)\quad &q_{ij}^{(n+1)}(s) = \sum_{\nu=0}^{L-1} q_{i\nu}^{(n)}(s)(\lambda/(\lambda+s))^{L-\nu}\int_0^\infty e^{-(\lambda+s)y}[(\lambda y)^j/j!]\,dH_L(y)\\ &\qquad + \sum_{\nu=L}^{K-1} q_{i\nu}^{(n)}(s)\int_0^\infty e^{-(\lambda+s)y}[(\lambda y)^j/j!]\,dH_\nu(y)\\ &\qquad + \sum_{\nu=K}^{j+K} q_{i\nu}^{(n)}(s)\int_0^\infty e^{-(\lambda+s)y}[(\lambda y)^{j-\nu+K}/(j-\nu+K)!]\,dH_K(y),\\ &\qquad\qquad \text{for} \quad n \geqq 0.\end{aligned}$$

We obtain

$$\begin{aligned}&\sum_{j=0}^{K-1} q_{ij}^{(n+1)}(s)z^j + z^K U_i^{(n+1)}(z, s)\\ (9)\quad &= \sum_{\nu=0}^{L-1} q_{i\nu}^{(n)}(s)(\lambda/(\lambda+s))^{L-\nu}h_L(s+\lambda-\lambda z)\\ &\quad + \sum_{\nu=L}^{K-1} q_{i\nu}^{(n)}(s)h_\nu(s+\lambda-\lambda z)\\ &\quad + U_i{}^{(n)}(z, s)h_K(s+\lambda-\lambda z),\end{aligned}$$

and upon multiplication by w^{n+1} and summation we obtain (5). The function $V_i(z, s, w)$ is analytic in its region of definition. The denominator has exactly K roots in the unit disk $|z| \leqq 1$ (Takács [11] p. 82). We denote them by $\gamma_\rho(w, s)$, $\rho = 1, \cdots, K$ and define them so as to be analytic in the region of interest. Expressing that the zeros of numerator and denominator must coincide, we obtain the system (6).

THEOREM 2. *The n-step transition probabilities $P_{ij}^{(n)}$ for the imbedded Markov chain $\{\xi_n, n \geqq 0\}$ are given by:*

$$\begin{aligned}&\sum_{n=0}^\infty \sum_{j=0}^\infty z^j w^n P_{ij}^{(n)} = [z^K - wh_K(\lambda-\lambda z)]^{-1}\\ (10)\quad &\cdot\{z^{K+i} + \sum_{\nu=0}^{L-1} wW_{i\nu}(0, w)[z^K h_L(\lambda-\lambda z) - z^\nu h_K(\lambda-\lambda z)]\\ &\quad + \sum_{\nu=L}^{K-1} wW_{i\nu}(0, w)[z^K h_\nu(\lambda-\lambda z) - z^\nu h_K(\lambda-\lambda z)]\},\end{aligned}$$

The limiting probabilities $\pi_\nu = \lim_{n\to\infty} P_{i\nu}^{(n)}$ are obtained by multiplying (10) on both sides by $1-w$ and letting w tend to one. We have:

$$(11)\qquad \pi_\nu = \lim_{w\to 1}(1-w)W_{i\nu}(0, w), \quad \textit{for} \quad \nu = 0, 1, \cdots, K-1,$$

and

$$(12)\quad \sum_{j=0}^{\infty}\pi_j z^j = [z^K - h_K(\lambda-\lambda z)]^{-1}\{\sum_{\nu=0}^{L-1}\pi_\nu[z^K h_L(\lambda-\lambda z) - z^\nu h_K(\lambda-\lambda z)] + \sum_{\nu=L}^{K-1}\pi_\nu[z^K h_\nu(\lambda-\lambda z) - z^\nu h_K(\lambda-\lambda z)]\}$$

Let α_K denote the first moment of $H_K(\cdot)$, then the Markov chain $\{\xi_n, n \geqq 0\}$ is positive recurrent if and only if $K - \lambda\alpha_K > 0$, null-recurrent if and only if $K - \lambda\alpha_K = 0$ and transient if and only if $K - \lambda\alpha_K < 0$.

PROOF. Formula (10) follows from (5) and (6) by setting $s = 0+$ in

$$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} z^j w^n q_{ij}^{(n)}(s) = \sum_{j=0}^{K-1} W_{ij}(w, s)z^j + z^K V_i(z, s, w).$$

The proof of (12) is well-known.

From the system (6) we obtain $W_{i\nu}(0, w)$ as the ratio of two determinants, namely the determinant

$$\|1 - wh_L(\lambda-\lambda\gamma_\rho), \cdots, \gamma_\rho^{L-1} - wh_L(\lambda-\lambda\gamma_\rho), \gamma_\rho^{L} - wh_L(\lambda-\lambda\gamma_\rho), \cdots, \gamma_\rho^{K-1} - wh_{K-1}(\lambda-\lambda\gamma_\rho)\|$$

in the denominator and a determinant, obtained by replacing the νth column of the above by the column with entries $\gamma_\rho^{\,i}(w, 0)$, $\rho = 1, \cdots, K$. We form the product $(1 - w)W_{i\nu}(0, w)$ by dividing the first row of the denominator by $1 - w$. The equation

$$(13)\qquad z^K = h_K(\lambda - \lambda z)$$

has a real, positive root of smallest value in the interval $[0, 1]$. Let us define $\gamma_1(w, s)$ to be the root of the equation (7) which corresponds to this positive root of (13) for $w = 1$, $s = 0+$. By a well-known argument, we have

$$(14)\qquad \gamma_1(1, 0+) = 1 \quad \text{if and only if} \quad K - \lambda\alpha_K \geqq 0$$
$$\gamma_1(1, 0+) < 1 \quad \text{if and only if} \quad K - \lambda\alpha_K < 0$$

Consider the case $\gamma_1(1, 0+) = 1$. The numerator in the expression for $(1 - w)W_{i\nu}(0, w)$ tends for $w \uparrow 1$ to the cofactor of the element $\gamma_1^{\,i}(w, 0)$ in the first row and the νth column, evaluated at $w = 1$. This quantity does not depend on the initial state i. The denominator converges to a determinant in which the first row is given by the constants

$$(A_0, \cdots, A_{L-1}, A_L, \cdots, A_{K-1})$$

where

$$(15)\qquad A_\nu = \lim_{w\to 1}(1 - w)^{-1}[\gamma_1^{\,\nu}(w, 0) - wh_L(\lambda - \lambda\gamma_1(w, 0))], \quad \text{for } \nu = 0, 1, \cdots, L-1$$
$$A_\nu = \lim_{w\to 1}(1 - w)^{-1}[\gamma_1^{\,\nu}(w, 0) - wh_\nu(\lambda - \lambda\gamma_1(w, 0))], \quad \text{for } \nu = L, \cdots, K-1.$$

These limits may be evaluated explicitly using de l'Hopital's rule and the fact that

$$(\partial/\partial w)\gamma_1(w, s)|_{w=1,s=0} = (K - \lambda\alpha_K)^{-1}. \tag{16}$$

We find

$$A_\nu = (K - \nu + \lambda\alpha_L - \lambda\alpha_K)(K - \lambda\alpha_K)^{-1} \quad \text{for} \quad \nu = 0, 1, \cdots, L-1, \tag{17}$$

$$A_\nu = (K - \nu + \lambda\alpha_\nu - \lambda\alpha_K)(K - \lambda\alpha_K)^{-1}, \quad \text{for} \quad \nu = L, \cdots, K-1,$$

in which α_ν, $\nu = L, \cdots, K$ is the expected duration of the service time for a group of ν customers.

The other rows of the determinant in the denominator are given by:

$$1 - h_L[\lambda - \lambda\gamma_\rho(1, 0)], \cdots, \gamma_\rho{}^{L-1}(1, 0) - h_L[\lambda - \lambda\gamma_\rho(1, 0)],$$

$$\gamma_\rho{}^{L}(1, 0) - h_L[\lambda - \lambda\gamma_\rho(1, 0)], \cdots, \gamma_\rho{}^{K-1}(1, 0) - h_{K-1}[\lambda - \lambda\gamma_\rho(1, 0)]$$

for $\rho = 2, \cdots, K$.

If $K - \lambda\alpha_K > 0$, this procedure yields the generating function for the stationary probability distribution of the queue length immediately after departures. The imbedded Markov chain is then positive recurrent, since an absolutely convergent solution to the stationarity equations is exhibited.

If $K - \lambda\alpha_K = 0$, all π_ν, $\nu = 0, 1, \cdots$ are equal to zero and the chain is null recurrent.

If $K - \lambda\alpha_K < 0$, we need an additional argument to show that the imbedded chain is transient. Slightly extending a theorem due to Foster, we have:

PROPOSITION. *An irreducible Markov chain is transient if and only if the system of equations:*

$$(*) \qquad \sum_{j=0}^{\infty} P_{ij}y_j = y_i, \qquad i \notin C$$

where C is any finite subset of the state space, has a bounded nonconstant solution.

The proof of this proposition is essentially the same as for Foster's theorem.

Applying this property, it is easy to see that if we remove the equations, corresponding to the first K rows the resulting system has a bounded nonconstant solution if $K - \lambda\alpha_K < 0$.

3. The successive busy periods. The server will become free as soon as there are less than L customers left in the queue immediately after a departure. We define a busy period as the length of time between the beginning of service for the first batch, which contains L customers and the time when the number of customers in the system drops below L for the first time thereafter. The initial busy period will depend on the initial conditions in the queue and we will not derive its distribution here. It may be obtained by an analogous reasoning to the one given here. We denote the lengths of the busy periods by Y_n, $n = 1, 2, \cdots$ and the number of customers left in the queue upon termination of the nth busy period by I_n. It is obvious that the sequence

$$\{(I_n, Y_n), n \geqq 0, Y_0 = 0, I_0 \quad \text{arbitrary}\}$$

forms a semi-Markov sequence with L states $\{0, \cdots, L-1\}$. We will now obtain transforms for the transition probability distributions

$$G_{ij}(x) = P\{I_n = j,\ Y_n \leqq x \mid I_{n-1} = i\} \tag{18}$$

for $i, j = 0, 1, \cdots, L-1$ and $n > 1$.

THEOREM 3. *The transition probability distributions $G_{ij}(x)$ do not depend on i. Their Laplace-Stieltjes transforms $E_j(1, s)(j = 0, \cdot, \cdots, L-1)$ are the solutions to the system of linear equations*

$$\begin{aligned} &\textstyle\sum_{r=0}^{L-1} E_r(1, s)\gamma_\rho{}^r(1, s) \\ &+ \textstyle\sum_{r=L}^{K-1} [\gamma_\rho{}^r(1, s) - h_r[s + \lambda - \lambda\gamma_\rho(1, s)]]E_r(1, s) \\ &\qquad = h_L[s + \lambda - \lambda\gamma_\rho(1, s)], \quad \rho = 1, \cdots, K. \end{aligned} \tag{19}$$

PROOF. The proof is an extension of an argument by Takács [10] for the $M\ |G|\ 1$ queue.

Let $G(k, n, x)$ be the probability that a busy period consists of at least n services, which last a length of time of at most x and such that at the end of the nth service there are h customers waiting.

It is clear that the $G(k, n, x)$ do not depend on the number of customers in the queue at the end of the previous busy period.

The probabilities $G(k, n, x)$ satisfy the recurrence relations

$$G(k, 1, x) = \textstyle\int_0^x e^{-\lambda y}[(\lambda y)^k/k!]\, dH_L(y), \tag{20}$$

and for $n > 1$:

$$\begin{aligned} G(k, n, x) = &\textstyle\sum_{r=L}^{K-1} \int_0^x G(r, n-1, x-y)e^{-\lambda y}[(\lambda y)^k/k!]\, dH_r(y) \\ &+ \textstyle\sum_{\nu=K}^{K+k} \int_0^x G(\nu, n-1, x-y)e^{-\lambda y}[(\lambda y)^{k+K-\nu}/(k+K-\nu)!]\, dH_K(y). \end{aligned}$$

We denote the Laplace-Stieltjes transforms of the $G(k, n, x)$ by $\Gamma(k, n, s)$ and obtain the transformed version of equations (20). We also introduce the generating functions:

$$\begin{aligned} &\text{(a)} \quad C_K(z, n, s) = \textstyle\sum_{\nu=0}^{\infty} z^\nu \Gamma(\nu + K, n, s), && |z| \leqq 1, \operatorname{Re} s \geqq 0; \\ &\text{(b)} \quad D_K(z, w, s) = \textstyle\sum_{n=1}^{\infty} C_K(z, n, s)w^n, && |z| \leqq 1, |w| \neq 1, \operatorname{Re} s \geqq 0; \\ &\text{(c)} \quad E_r(w, s) = \textstyle\sum_{n=1}^{\infty} \Gamma(r, n, s)w^n, && |w| \leqq 1, \operatorname{Re} s \geqq 0, r = 0, \\ &&& \qquad 1, \cdots, K-1. \end{aligned} \tag{21}$$

We obtain successively

$$\textstyle\sum_{k=0}^{\infty} \Gamma(k, 1, s)z^k = h_L(s + \lambda - \lambda z) \tag{22}$$

and for $n > 1$

$$\begin{aligned} &\textstyle\sum_{k=0}^{K-1} \Gamma(k, n, s)z^k + z^K C_K(z, n, s) \\ &\quad = \textstyle\sum_{r=L}^{K-1} \Gamma(r, n-1, s)h_r(s + \lambda - \lambda z) + h_K(s + \lambda - \lambda z)C_K(z, n-1, s), \end{aligned}$$

$$(23)\quad D_K(z, w, s) = [wh_L(s + \lambda - \lambda z) - \sum_{r=0}^{L-1} E_r(w, s)z^r + \sum_{r=L}^{K-1} [wh_r(s + \lambda - \lambda z) - z^r]E_r(w, s)][z^K - wh_K(s + \lambda - \lambda z)]^{-1}.$$

Applying the standard argument, we obtain the unknown functions $E_r(w, s)$ as the solutions to the system of linear equations

$$(24)\quad \sum_{r=0}^{L-1} E_r(w, s)\gamma_\rho{}^r(w, s) + \sum_{r=L}^{K-1} [\gamma_\rho{}^r(w, s) - wh_r[s + \lambda - \lambda\gamma_\rho(w, s)]]E_r(w, s) = wh_L[s + \lambda - \lambda\gamma_\rho(w, s)], \quad \rho = 1, \cdots, K$$

in which the $\gamma_\rho(w, s)$ are the roots of equation (7) in $|z| \leqq 1$. For $w = 1$ and $j = 0, \cdots, L - 1$ the $E_j(1, s)$ are the transforms of $G_{ij}(x)$.

Since the $G_{ij}(x)$ do not depend on i, the successive busy periods form a semi-Markov process of zero order as defined by Pyke [7].

If $K - \lambda\alpha_K \geqq 0$, we have $\gamma_1(1, 0+) = 1$. If we set $w = 1$, $s = 0+$ and $\rho = 1$ in equation (24), we obtain

$$\sum_{r=0}^{L-1} E_r(1, 0+) = 1.$$

This quantity is less than one if $K - \lambda\alpha_K < 0$, because the imbedded semi-Markov process is transient. $E_r(1, 0+)$, $r = 0, \cdots, L - 1$, is the probability that at the end of a busy period there will be r customers left behind.

The sequence of successive idle periods is now also easy to describe. They form a semi-Markov process of zero order in which the transition probabilities are $E_r(1, 0+)$ and the corresponding idle periods have an Erlang distribution with parameter λ and order $L - r$, $r = 0, 1, \cdots, L - 1$. The moments of the busy periods may be calculated from the determinant representation of $E_r(1, s)$, $r = 0, \cdots, L - 1$. One verifies directly that the first moment is infinite if $K = \lambda\alpha_K$.

4. The queue length in continuous time. The distribution of the queue length in continuous time is readily obtained from the renewal functions for the imbedded semi-Markov process $\{(\xi_n, X_n), n \geqq 0\}$. For simplicity we again assume that the point $t = 0$ is a departure point. We define $M_{ij}(t)$, $i, j = 0, 1, \cdots$ as the expected number of visits to state j in the *closed* interval $[0, t]$, starting in state i, in the imbedded semi-Markov process. The Laplace-Stieltjes transforms of the $M_{ij}(t)$ are denoted by $\mu_{ij}(s)$. We denote by $\mu_j{}^*$ the mean recurrence time of the state j in the imbedded semi-Markov process.

Let $\xi(t)$ be the queue length at time t and let:

$$(25)\quad P_{ij}(t) = P\{\xi(t) = j \mid \xi_0 = i\},$$

We denote the Laplace transform of $P_{ij}(t)$ by $p_{ij}(s)$ and we introduce the generating function:

$$(26)\quad \Pi(s, z) = \sum_{j=0}^{\infty} p_{ij}(s)z^j, \qquad |z| < 1, \operatorname{Re} s \geqq 0$$

We now prove the following theorem:

THEOREM 4. *The generating function* $\Pi(s, z)$ *is given by:*

$$
\begin{aligned}
\Pi(s, z) &= [1/(s + \lambda - \lambda z)] \textstyle\sum_{\nu=0}^{L-1} W_{i\nu}(1, s) z^{\nu} [1 - (\lambda z/(\lambda + s))^{L-\nu}] \\
&+ [z^L/(s + \lambda - \lambda z)][1 - h_L(s + \lambda - \lambda z)] \textstyle\sum_{\nu=0}^{L-1} W_{i\nu}(1, s)(\lambda/(\lambda + s))^{L-\nu} \\
&+ [1/(s + \lambda - \lambda z)] \textstyle\sum_{\nu=L}^{K-1} W_{i\nu}(1, s)[1 - h_\nu(s + \lambda - \lambda z)] z^{\nu} \\
&+ [z^K/(s + \lambda - \lambda z)] V_i(z, s, 1)[1 - h_K(s + \lambda - \lambda z)],
\end{aligned} \tag{27}
$$

in which the functions $W_{i\nu}(1, s)$, $\nu = 0, 1, \cdots, K - 1$ *and* $V_i(z, s, 1)$ *are given by formulae* (5) *and* (6) *for* $w = 1$.

The limits $P_j^* = \lim_{t\to\infty} P_{ij}(t)$ exist and are given below for the positive recurrent case. In the null-recurrent and transient cases these limits are equal to zero.

PROOF. We have

$$
M_{ij}(t) = \textstyle\sum_{\nu=0}^{\infty} Q_{ij}^{(\nu)}(t), \tag{28}
$$

and hence

$$
\textstyle\sum_{j=0}^{\infty} \mu_{ij}(s) z^j = \sum_{j=0}^{K-1} W_{ij}(1, s) z^j + z^K V_i(z, s, 1), \tag{29}
$$

by formulae (4), (5) and (6).

By enumeration of cases and the law of total probability, we obtain

(30) For $0 \leqq j \leqq L - 1$

$$
P_{ij}(t) = \textstyle\sum_{\nu=0}^{j} \int_0^t e^{-\lambda(t-\tau)} [[\lambda(t - \tau)]^{j-\nu}/(j - \nu)!]\, dM_{i\nu}(\tau),
$$

for $L \leqq j \leqq K - 1$

$$
\begin{aligned}
P_{ij}(t) = {} & \textstyle\sum_{\nu=0}^{L-1} \int_0^t dM_{i\nu}(\tau) e^{-\lambda(t-\tau)} \\
& \cdot \textstyle\int_0^{t-\tau} [(\lambda u)^{L-\nu-1}/(L - \nu - 1)!][[\lambda(t - \tau - u)]^{j-L}/(j - L)!] \\
& \cdot [1 - H_L(t - \tau - u)]\lambda\, du + \textstyle\sum_{\nu=L}^{j} \int_0^t dM_{i\nu}(\tau) e^{-\lambda(t-\tau)} \\
& \cdot [[\lambda(t - \tau)]^{j-\nu}/(j - \nu)!][1 - H_\nu(t - \tau)],
\end{aligned}
$$

for $j \geqq K$

$$
\begin{aligned}
P_{ij}(t) = {} & \textstyle\sum_{\nu=0}^{L-1} \int_0^t dM_{i\nu}(\tau) \int_0^{t-\tau} e^{-\lambda(t-\tau)} [(\lambda u)^{L-\nu-1}/(L - \nu - 1)!] \\
& \cdot [[\lambda(t - \tau - u)]^{j-L}/(j - L)!][1 - H_L(t - \tau - u)]\lambda\, du \\
& + \textstyle\sum_{\nu=L}^{K-1} \int_0^t dM_{i\nu}(\tau) e^{-\lambda(t-\tau)} [[\lambda(t - \tau)]^{j-\nu}/(j - \nu)!][1 - H_\nu(t - \tau)] \\
& + \textstyle\sum_{\nu=K}^{j} \int_0^t dM_{i\nu}(\tau) e^{-\lambda(t-\tau)} [[\lambda(t - \tau)]^{j-\nu}/(j - \nu)!][1 - H_K(t - \tau)],
\end{aligned}
$$

Taking Laplace transforms, we obtain

(31) For $0 \leqq j \leqq L - 1$

$$
p_{ij}(s) = \textstyle\sum_{\nu=0}^{j} \mu_{i\nu}(s)(1/\lambda)(\lambda/(\lambda + s))^{j-\nu+1},
$$

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