Semi-Regular Functions in Markov Chains*

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I. Introduction

Let $\{X_n, n = 0,1,2,\ldots\}$ be a Markov chain in discrete time whose (minimal) state space I consists of the nonnegative integers. Each random variable X_n is defined on a probability space (Ω, F, P); ω represents an element of Ω. The transition probabilities are assumed to be stationary, i.e.,

$$P\{X_{n+1} = j \mid X_n = i\} = p_{ij} \quad i,j \in I \quad n = 0,1,2,\ldots$$

where

$$p_{ij} \geq 0, \quad \sum_{j=0}^{\infty} p_{ij} = 1 \quad i,j \in I.$$

The matrix of transition probabilities will be denoted by (p_{ij}). The n-step transition probabilities are denoted by $p_{ij}^{(n)}$, where $p_{ij}^{(0)} = \delta_{ij}$ and $p_{ij}^{(1)} = p_{ij}$. The following quantities will be used frequently:

$$p_{ij}^{*} = \sum_{n=1}^{\infty} p_{ij}^{(n)} \quad i,j \in I$$

$$f_{ij}^{(n)} = P\{X_n = j,\ X_\nu \neq j \text{ for } \nu = 1,\ldots,n-1 \mid X_0 = i\} \quad i,j \in I \quad n = 1,2,\ldots$$

$$f_{ij}^{*} = \sum_{n=1}^{\infty} f_{ij}^{(n)} \quad i,j \in I.$$

Note that $f_{ij}^{(1)} = p_{ij}$.

Following Chung [5] we define a class of states to be a set of two or more mutually communicating states or a single state which does not communicate with any state. Recall that two states i and j belong to the same recurrent class if and only if $f_{ij}^{*} = f_{ji}^{*} = 1$, and that a class C is nonrecurrent if and only if the series

$$\sum_{n=0}^{\infty} p_{ij}^{(n)} = p_{ij}^{*} + \delta_{ij}$$

converges for some pair of (not necessarily distinct) states i and j in C.

In section II we consider super and sub regular functions and measures. These functions are important in the study of Markov chains for many reasons. In particular, it has been known for some time that the existence of certain types of functions defined on the state space I can give information about the character of irreducible Markov chains. We have, for example, the well known result of Foster (theorem 7 of this paper) and the following two theorems which can be found in Foster [15]:

<u>Theorem</u>: In an irreducible Markov chain a sufficient condition for recurrence is that there exist a sequence $\{y_i\}$ such that

$y_i \geq \sum_{j=0}^{\infty} p_{ij} y_j$, $i \neq 0$ with $y_i \longrightarrow \infty$ as $i \longrightarrow \infty$.

<u>Theorem</u>: In an irreducible Markov chain, if there exists a solution $\{y_i\}$ to the system of equations $y_j = \sum_{i=0}^{\infty} y_i \, p_{ij}$ such that $y_i \neq 0$

and $\sum_{i=0}^{\infty} |y_i| < \infty$, then the chain is positive recurrent.

We will discuss some general properties of super regular functions and super regular measures and present some theorems of the type just mentioned. It is common in the literature on Markov chains to classify states and classes of states as recurrent, nonrecurrent, equitable, etc. One purpose of the present discussion is to see how these types of chains can be characterized by means of super regular functions and measures. Many of the results are applications of two fundamental inequalities which are stated as theorems 3 and 4.

In section III super regular functions will be used to study a particular type of Markov chain (non-dissipative) which has the property that almost every path function eventually enters the set of positive recurrent states. The principal tool here is the relation between super regular functions and martingales established in section II.

Section IV is concerned with certain sets of states, called sojourn sets and almost closed sets, which are of interest in the study of the asymptotic behavior of the path functions. We will present some results about these sets and show their relation to super regular functions.

II. Regular and Super Regular Functions on Measures

A real (finite) valued function u on I is called a super regular function if $u(i) \geq \sum_j p_{ij}\, u(j)$ for every i. u is a regular function if $u(i) = \sum_j p_{ij}\, u(j)$ for every i; and u is a sub regular function if its negative is super regular. (An unrestricted sum will be assumed to be over the entire state space.)

A real (finite) valued function μ on I is called a super regular measure if $\mu(j) \geq \sum_i \mu(i)\, p_{ij}$ for every j; μ is a regular measure if $\mu(j) = \sum_i \mu(i)\, p_{ij}$ for every j; and μ is a sub regular measure if its negative is super regular.

Suppose u is a nonnegative super regular function. Then

$$u(i) \geq \sum_k p_{ik}\, u(k) \geq \sum_k p_{ik} \sum_j p_{kj}\, u(j) = \sum_j \sum_k p_{ik} p_{kj}\, u(j) = \sum_j p_{ij}^{(2)}\, u(j)$$

for every i by the Chapman Kolmogorov equation. Repeating this we conclude that

$$u(i) \geq \sum_j p_{ij}\, u(j) \geq \sum_j p_{ij}^{(2)}\, u(j) \geq \cdots \geq \sum_j p_{ij}^{(n)}\, u(j) \geq \cdots \geq 0$$

for every i. This relation will be useful in many of the theorems to follow.

If i and k belong to the same communicating class then there exists an integer m such that $p_{ik}^{(m)} > 0$. Therefore $u(i) \geq \sum_j p_{ij}^{(m)}\, u(j) \geq p_{ik}^{(m)}\, u(k)$ and we see that $u(k) = 0$ if $u(i) = 0$.

It follows that a nonnegative super regular function u defined on a communicating class C is either identically zero or always positive on C. Similar statements can be made about super regular measures.

Let u be a nonnegative super regular function and let $w(i) = u(i) - \sum_j p_{ij} u(j)$. The letter w will always represent this difference in this paper. If we imagine $u(i)$ to place an evaluation on the state i, then $w(i)$ gives us the difference between the evaluation on our present position (known to be state i) and the conditional expected evaluation after one more step in the Markov chain.

Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a sequence of random variables defined on the probability space (Ω, F, P) with $E(|Z_n|) < \infty$, $n = 0,1,2,\ldots$, and suppose that to each $n = 0,1,2,\ldots$ there corresponds a σ-field F_n (a sub σ-field of F) with the following properties:

(i) $F_n \subset F_{n+1}$ $n = 0,1,2,\ldots$

(ii) Z_n is an F_n measurable function $n = 0,1,2,\ldots$

(iii) $E(Z_{n+1}|F_n) \leq Z_n$ with probability one $n = 0,1,2,\ldots$

Then the sequence $\{Z_n, F_n, n = 0,1,2,\ldots\}$ is called a super martingale. A special case of a celebrated convergence theorem of Doob [10] is the Super Martingale Convergence Theorem: Suppose $\{Z_n, F_n, n = 0,1,2,\ldots\}$ is a nonnegative super martingale. Then there exists a nonnegative random variable Z such that $\lim_{n \to \infty} Z_n = Z$ with probability one, and $E(Z) < \infty$.

For our purposes observe that if u is a nonnegative super regular function such that $E[u(X_0)] < \infty$, then

(i) $E[u(X_n)] = E\{E[u(X_n)|X_0]\} = \sum_i E[u(X_n)|X_0 = i]\ P(X_0 = i)$

$$= \sum_i \sum_j p_{ij}^{(n)}\ u(j)\ P(X_0 = i) \leq \sum_i u(i)\ P(X_0 = i) = E[u(X_0)] < \infty,$$

and

(ii) $E[u(X_{n+1})|X_0,\ldots,X_n] = E[u(X_{n+1})|X_n] = \sum_j p_{X_n j}\ u(j) \leq u(X_n)$

where the first equality in (ii) follows from the Markov property and the inequality in (ii) from super regularity. Let F_n be the smallest σ-field with respect to which the random variables $X_0,\ldots,X_n$ are measurable. We see that $\{u(X_n), F_n, n = 0,1,2,\ldots\}$ is a nonnegative super martingale and the above stated convergence theorem applies. All these considerations will be expressed in the sequel by the statement: The functional process $\{u(X_n), n = 0,1,2,\ldots\}$ is a super martingale.

If the Markov chain is irreducible, it is possible to translate results about super regular functions to corresponding results about super regular measures by a well known device (called duality) which is described in detail in Kemeny and Snell [16]. This method uses the following theorem which is due to Kendall [17]:

Theorem: An irreducible Markov chain admits at least one positive super regular measure.

Let $\alpha(\cdot)$ be a positive super regular measure. If u is a super regular function, then $\mu(i) = u(i)/\alpha(i)$ is a super regular measure with respect to a new Markov chain with transition probabilities $\hat{p}_{ij}$ given by

$$\hat{p}_{ij} = \frac{\alpha(i)}{\alpha(j)} p_{ji} ,$$

since

$$\sum_i \mu(i)\ \hat{p}_{ij} = \frac{1}{\alpha(j)} \sum_i p_{ji}\ u(i) \leq \frac{u(j)}{\alpha(j)} = \mu(j).$$

A mapping of super regular measures into super regular functions can be defined in a similar fashion. The new chain with transition probabilities $\hat{p}_{ij}$ has the property that it is nonrecurrent if and only if the original chain is nonrecurrent.

The following theorem provides a description of the behavior of the iterates $\sum_j p_{ij}^{(n)}\ u(j)$ which were discussed above. The theorem is due to Feller [12].

<u>Theorem</u>: If u is a nonnegative super regular function, then

$$a(i) = \lim_{n \to \infty} \sum_j p_{ij}^{(n)}\ u(j)$$

exists for every i. Moreover, a is a regular function. It is the maximal regular function which is dominated by u, i.e., if b is a regular function such that $b(i) \leq u(i)$ for every i, then $b(i) \leq a(i)$ for every i.

Similar results are true for sub regular functions, and for super and sub regular measures. For u a finite nonnegative sub regular function we find that $\lim_{n \to \infty} \sum_j p_{ij}^{(n)}\ u(j)$ is the smallest regular function that is greater than or equal to $u(i)$ at every state i. For μ a finite nonnegative super regular measure it turns out that $\lim_{n \to \infty} \sum_i \mu(i)\, p_{ij}^{(n)}$ is the maximal regular measure which is dominated by μ.

<u>Theorem 1</u>: Suppose u is a nonnegative super regular function such that $\sum_i u(i) < \infty$, and let $a(i) = \lim_{n \to \infty} \sum_j p_{ij}^{(n)}\, u(j)$. Then $a(i) = \sum_j \pi_{ij}\, u(j)$ where $\pi_{ij} = \lim_{n \to \infty} \frac{1}{n} \sum_{\nu=1}^{n} p_{ij}^{(\nu)}$ is the Cesaro limit of the $p_{ij}^{(n)}$'s. (We know that $0 \le a \le u$ and that a is regular. Now we see that $a \equiv 0$ if the chain has no positive recurrent states.)

Proof:

$$a(i) = \lim_{n \to \infty} \sum_j p_{ij}^{(n)}\, u(j) = \lim_{n \to \infty} \frac{1}{n} \sum_{\nu=1}^{n} \sum_j p_{ij}^{(\nu)}\, u(j)$$

$$= \lim_{n \to \infty} \sum_j \frac{1}{n} \sum_{\nu=1}^{n} p_{ij}^{(\nu)}\, u(j) = \sum_j \lim_{n \to \infty} \frac{1}{n} \sum_{\nu=1}^{n} p_{ij}^{(\nu)}\, u(j) = \sum_j \pi_{ij} u(j)$$

by the dominated convergence theorem. QED

If the chain is irreducible, then in order for the convergence condition of the theorem to hold the chain must be nonrecurrent unless u is identically zero. (This follows from Corollary 1 to Theorem 3).

<u>Theorem 2</u>: Suppose μ is a nonnegative super regular measure such that $\sum_i \mu(i) < \infty$, and let $b(j) = \lim_{n \to \infty} \sum_i \mu(i)\, p_{ij}^{(n)}$. Then $b(j) = \sum_i \mu(i)\, \pi_{ij}$.

(We know that $0 \leq b \leq \mu$ and that μ is regular. Now we see that $b(j) \equiv 0$ if the chain has no positive recurrent states.)

Theorem 3: In any Markov chain, if u is a super regular function which is bounded below, then $u(i) \geq f^*_{ik}\, u(k)$ for every i and k.

Proof: We may assume, without loss of generality, that u is non-negative, since adding a constant, say $-\inf u(i)$ if $\inf u(i)$ is negative, to a super regular function yields a super regular function. Pick any state k and fix it. Now

$$u(i) \geq \sum_j p_{ij}\, u(j) \geq p_{ik}\, u(k) = f^{(1)}_{ik}\, u(k).$$

Assume $u(i) \geq \sum_{\nu=1}^{n} f^{(\nu)}_{ik}\, u(k)$ for every i. Then

$$u(i) \geq \sum_j p_{ij}\, u(j) = p_{ik}\, u(k) + \sum_{j \neq k} p_{ij}\, u(j) \geq$$

$$f^{(1)}_{ik}\, u(k) + \sum_{j \neq k} p_{ij} \sum_{\nu=1}^{n} f^{(\nu)}_{jk}\, u(k) = f^{(1)}_{ik}\, u(k) +$$

$$\sum_{\nu=1}^{n} f^{(\nu+1)}_{ik}\, u(k) = \sum_{\nu=1}^{n+1} f^{(\nu)}_{ik}\, u(k)$$

for every i, since $\sum_{j \neq k} p_{ij}\, f^{(\nu)}_{jk} = f^{(\nu+1)}_{ik}$. Hence, by induction, $u(i) \geq \sum_{\nu=1}^{N} f^{(\nu)}_{ik}\, u(k)$ for every N, and this implies that, for every i,

$u(i) \geq f^*_{ik}\, u(k)$. QED

Corollary 1: In a recurrent class the only super regular function which is bounded below is a constant.

Proof: By the theorem we have that $u(i) \geq f^*_{ik}\, u(k)$ for every i. By recurrence $f^*_{ik} = 1$ for every i and k. Therefore, $u(i) \geq u(k)$ for every i. But k was chosen arbitrarily. We conclude that u is a constant function. QED

We remark that the property of recurrent chains established in this theorem does not characterize recurrent chains. It will be shown in section IV that for a certain type of nonrecurrent Markov chain the only bounded regular functions are constants.

Corollary 2: Let u be a nonnegative super regular function and let $w(i) = u(i) - \sum_j p_{ij}\, u(j)$. Then $A = \{i : w(i) > 0\}$ is a set of nonrecurrent states.

Proof: Let i be an element of A and suppose i is recurrent. Then i is an element of some recurrent class C. But u is constant and w therefore identically zero on C. This contradicts $i \in A$. QED.

We remark that it is not true that $w(i) = 0$ implies that i is a recurrent state. This can be seen by considering Foster's theorem (Theorem 7 of this paper) which shows that in a nonrecurrent chain there exists a function u with the property that $w(i) = 0$ for all $i \neq 0$.

We define

$$e^{(n)}_{ij} = P\{X_n = j,\ X_\nu \neq i \quad \text{for} \quad \nu = 1, \ldots, n-1 \mid X_0 = i\}$$

$$i, j \in I \qquad n = 1, 2, \ldots,$$

$$e^{(0)}_{ij} = \delta_{ij}, \qquad e^{(1)}_{ij} = p_{ij}$$

$$e^*_{ij} = \sum_{n=1}^{\infty} e^{(n)}_{ij} \qquad i,j\epsilon I.$$

The quantity e^*_{ij} is the expected number of times the Markov chain, starting at i, visits j before returning to i. It has been studied in Chung [5] under the notation e_{ij} and in Chung [4] under the present notation. It can be shown that $e^*_{ij}\, e^*_{ji} > 0$ if and only if i and j communicate; and $0 < e^*_{ij} < \infty$ for every i and j in the same class. Derman [8] has proved that in a recurrent class the only nonnegative regular measure μ is given by $\mu(i) = c\, e^*_{hi}$ where h is any fixed state and c is a nonnegative constant independent of i.

<u>Theorem 4</u>: In any Markov chain, if μ is a nonnegative super regular measure, then $\mu(j) \geq \mu(k)\, e^*_{k_j}$ for every j and k.

Proof: Choose and fix a state k. Now

$$\mu(j) \geq \sum_i \mu(i)\, p_{ij} \geq \mu(k)\, p_{kj} = \mu(k)\, e^{(1)}_{kj}$$

for every j. Assume $\mu(j) \geq \mu(k) \sum_{\nu=1}^{n} e^{(\nu)}_{kj}$ for every j. Then

$$\mu(j) \geq \sum_i \mu(i)\, p_{ij} = \mu(k)\, p_{kj} + \sum_{i \neq k} \mu(i)\, p_{ij} \geq$$

$$\mu(k)\, p_{kj} + \sum_{i \neq k} \sum_{\nu=1}^{n} e^{(\nu)}_{ki} \mu(k) p_{ij} = \mu(k) e^{(1)}_{kj} + \sum_{\nu=1}^{n} e^{(\nu+1)}_{kj} \mu(k) = \mu(k) \sum_{\nu=1}^{n+1} e^{(\nu)}_{kj}$$

for every j, since $\sum_{i \neq k} e_{ki}^{(\nu)} p_{ij} = e_{kj}^{(\nu+1)}$. Hence, by induction,

$\mu(j) \geq \mu(k) \sum_{\nu=1}^{N} e_{kj}^{(\nu)}$ for every N, and this implies that, for every j,

$\mu(j) \geq \mu(k)\ e_{kj}^{*}$ = QED

Corollary 1: In a recurrent class the only nonnegative super regular measure is given by $\mu(j) = \mu(k)\ e_{kj}^{*}$ where k is an arbitrary but fixed state.

Proof: By the theorem we have that $\mu(j) \geq \mu(k)\ \epsilon_{kj}^{*}$ for every j. Let $y(j) = \mu(j) - \mu(k)\ e_{kj}^{*}$. For every j we have

$$\sum_{i} y(i)\ p_{ij} = \sum_{i} \mu(i) p_{ij} - \mu(k) \sum_{i} e_{ki}^{*}\ p_{ij} \leq \mu(j) - \mu(k) e_{kj}^{*} = y(j)$$

by the super regularity of μ and Derman's theorem. We see that y is also a nonnegative super regular measure. But $y(k) = 0$, since $e_{kk}^{*} = 1$ by recurrence. Therefore, $y(j) = 0$ for every j, and $\mu(j) = \mu(k) e_{kj}^{*}$ for every j. QED

Corollary 2: If C is a recurrent class and μ is a nonnegative super regular measure on C, then $\sum_{j} \mu(j)$ converges or diverges according as C is positive or null.

Proof: This follows immediately from a corollary to theorem 6 page 49 in Chung [5] which states that $\sum_{j} e_{kj}^{*}$ converges in a positive recurrent class and diverges in a null recurrent class. QED

We next consider the concept of an equitable class and show that such a class can be characterized by the nature of the super regular functions and measures defined on it. We also state a proposition which demonstrates the equivalence of several properties of a recurrent class of states.

A class C of states is called <u>equitable</u> if $e^*_{ij} = 1$ for every i and j in C. Chung [4] has shown that an equitable class is recurrent, and that a recurrent class is equitable if and only if the transition matrix (p_{ij}) restricted to C is doubly stochastic.

<u>Theorem 5</u>: In an equitable class all nonnegative super regular functions and measures are constants.

<u>Theorem 6</u>: A recurrent class C is equitable if and only if the only nonnegative super regular measures are constants.

Proof: Suppose C is equitable. Then $\mu(i) = \mu(k)\, e^*_{ki} = \mu(k)$ for some fixed k and every i in C, and we see that μ is a constant. Now suppose that all nonnegative super regular measures are constants. Then $M = \mu(i) = \mu(k)\, e^*_{ki} = M\, e^*_{ki}$ for every i. Therefore, $e^*_{ki} = 1$ for every k and i in C. Hence C is equitable. QED

Derman [8] makes the following definition: the states of a recurrent class C are equally likely if $e^*_{0k} = 1$ for every state k in C, where 0 is a fixed state in C. This is equivalent to C being equitable. Hence we have the

<u>Proposition</u>: In a recurrent class C, the following 4 statements are equivalent:

1. All states in C are equally likely,
2. C is equitable,
3. (p_{ij}) restricted to C is doubly stochastic,
4. All nonnegative super regular measures are constant.

We conclude the discussion of equitable classes with an example.

Let Y be an integer valued random variable and let $p_i = P(Y=i)$, $-\infty < i \infty$. We will assume that 1 is the greatest common divisor of all the values of i for which $p_i > 0$, and that $E(Y) = 0$. Let $\{Y_1, Y_2, Y_3, \ldots\}$ be a sequence of independent random variables identically distributed as Y. Define

$$S_0 = 0, \qquad S_n = \sum_{k=1}^{n} Y_k, \qquad n = 1,2,\ldots .$$

A result of Chung and Fuchs [7] shows that the stochastic process $\{S_n, n = 0,1,2,\ldots\}$ is a recurrent homogeneous Markov chain with state space $I = \{\ldots,-2,-1,0,1,2,\ldots\}$. Moreover, the transition probabilities are given by

$$p_{ij} = P(S_{n+1} = j | S_n = i) = P(Y_{n+1} = j-i) = p_{j-i}$$

so that $\{S_n, n=0,1,2,\ldots\}$ has stationary independent increments. We observe that

$$\sum_i p_{ij} = \sum_i p_{j-i} = 1$$

for every j, i.e., the transition matrix is doubly stochastic, and it follows that this chain is equitable. It is clear in general, from this consideration, that a recurrent Markov chain with stationary independent increments is equitable, and that this property is not affected by whether the chain is positive or null.

Any irreducible finite Markov chain is positive recurrent, but certainly may be non-equitable. For example, the chain with $I = \{1,2\}$ and transition matrix given by

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

is positive recurrent, but

$$e^*_{12} = \sum_{n=1}^{\infty} p_{12}(p_{22})^{n-1} = \frac{1}{2}\,\frac{4}{3} = \frac{2}{3}, \quad \text{and} \quad e^*_{21} = \sum_{n=1}^{\infty} p_{21}(p_{11})^{n-1} = \frac{3}{2}$$

so that it is not equitable, ($e^*_{11} = e^*_{22} = 1$ by recurrence). It is not the case, however, that a recurrent equitable Markov chain necessarily has stationary independent increments. Consider the finite chain with $I = \{1,2,3\}$ and transition matrix given by

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix} .$$

This chain is positive recurrent and equitable (because the transition matrix is doubly stochastic), but it does not have stationary independent increments.

We remark that an infinite Markov chain with a doubly stochastic transition matrix cannot be positive recurrent (see Feller [10]).

The following theorem was first proved by Foster [15]. His method was to make the zero state absorbing and to work with the resulting modified chain.

Theorem 7: An irreducible Markov chain is nonrecurrent if and only if there exists a bounded nonconstant solution to the system of equations

$$* \qquad u(i) = \sum_j p_{ij}\, u(j), \quad i \neq 0$$

Proof: Suppose there are no bounded nonconstant solutions to $*$, that is, every bounded solution to $*$ is of the form $u(i) \equiv c$. Define $u(i) = f^*_{i0}$ for $i \neq 0$, and $u(0) = 1$. This function satisfies $*$. Hence $f^*_{i0} \equiv c = 1$ and the chain is recurrent. Since there is always at least one bounded super regular function, namely $u(i) \equiv 1$, it follows that I nonrecurrent implies that there exists a bounded nonconstant solution to $*$.

Now suppose that there exists a bounded nonconstant solution to $*$. It must occur that either (i) $u(0) \geq \sum_j p_{0j}\, u(j)$ or (ii) $u(0) < \sum_j p_{0j}\, u(j)$.

If (i) holds, then u is a bounded nonconstant super regular function and the chain must be nonrecurrent by corollary 1 to theorem 3. If (ii) holds, let $v(i) = -u(i)$ for every i. Then v is a bounded nonconstant super regular function and the chain is nonrecurrent for the same reason. QED

Theorem 8: Let μ be a nonnegative super regular measure and let $v(j) = \mu(j) - \sum_i \mu(i)\, p_{ij}$. If $\mu(j) < \infty$, and if there exists a state i

which leads to j such that $v(i) > 0$, then j is nonrecurrent.

Proof: Let $b(j) = \lim_{n \to \infty} \sum_i \mu(i)\, p_{ij}^{(n)}$.

$$\infty > \mu(j) = \sum_{n=0}^{\infty} \left[\sum_i \mu(i)\, p_{ij}^{(n)} - \sum_i \mu(i)\, p_{ij}^{(n+1)} \right] + b(j) =$$

$$\sum_{n=0}^{\infty} \sum_i \left[\mu(i) - \sum_k \mu(k)\, p_{ki} \right] p_{ij}^{(n)} + b(j) = \sum_{n=0}^{\infty} \sum_i v(i)\, p_{ij}^{(n)} + b(j).$$

Therefore,

$$\infty > \sum_{n=0}^{\infty} \sum_i v(i)\, p_{ij}^{(n)} = \sum_i v(i) \sum_j p_{ij}^{(n)}.$$

Hence, if there exists a state i which leads to j (so that $p_{ij}^{(n)}$ is not equal to zero for every n) with $v(i) > 0$, then $\sum_{n=0}^{\infty} p_{ij}^{(n)} < \infty$ and j is a nonrecurrent state. QED

<u>Corollary</u>: An irreducible Markov chain is nonrecurrent if and only if there exists a nonnegative super regular measure μ such that $\mu(k) > \sum_i \mu(i)\, p_{ik}$ for some state k.

Proof: Assume such a measure μ exists. Then k leads to j for every j and $v(k) > 0$. Hence the chain is nonrecurrent by the theorem. If the chain is nonrecurrent, then $\mu(i) = p_{0i}^{*}$ satisfies the conditions. QED

<u>Theorem 9</u>: Suppose u is a nonnegative super regular function such that $\lim_{i \to \infty} u(i) = 0$. Then $\lim_{i \to \infty} \pi_{ik} = 0$ for every k for which $u(k) > 0$.

Proof: Fix any state k. By theorem 4

$$u(i) \geq f^*_{ik}\, u(k) \geq p^{(\nu)}_{ik}\, u(k)$$

for every ν and every i. Hence

$$u(i) \geq \frac{1}{n} \sum_{\nu=1}^{n} p^{(\nu)}_{ik}\, u(k)$$

for every n and every i, and we see that

$$u(i) \geq \lim_{n \to \infty} \frac{1}{n} \sum_{\nu=1}^{n} p^{(\nu)}_{ik}\, u(k) = \pi_{ik}\, u(k)$$

for every i. But $0 = \lim_{i \to \infty} u(i) \geq \lim_{i \to \infty} \pi_{ik}\, u(k)$. Therefore, if $u(k) > 0$ we have $\lim_{i \to \infty} \pi_{ik} = 0$. QED

We remark that $\lim_{k \to \infty} \pi_{ik} = 0$ for every i since $\sum_k \pi_{ik} \leq 1$ for every i. Now we see that if there exists a strictly positive super regular function u such that $\lim_{i \to \infty} u(i) = 0$, then $\lim_{k \to \infty} \pi_{ik} = 0$ for every k also.

III. Non-Dissipative Markov Chains

A Markov chain is called <u>non-dissipative</u> if $\sum_j \pi_{ij} = 1$ for every i. Equivalently, the chain is non-dissipative if, for every i,

$$1 = f^*(i,D) = P\{X_n \in D \text{ for some } n = 1,2,\ldots | X_0 = i\},$$

where D is the set of all positive recurrent states in the chain. See Chung [5], page 35, for proof of this equivalence. We note immediately that if a Markov chain is non-dissipative, then D is non-empty and there are no null recurrent states at all in the chain.

A nonnegative super regular function u will be called <u>properly divergent</u> if $u(i) \longrightarrow \infty$ as $i \longrightarrow \infty$.

The following theorem is a generalization of one due to Kendall [18] who proved that if a properly divergent super regular function exists, then the chain is non-dissipative. Foster [13] had previously proved the same theorem for the special case of $u(i) = i$. The terms non-dissipative and properly divergent are due to Foster.

<u>Theorem 10</u>: In any Markov chain, if a properly divergent super regular function u exists, then

(a) there are no null recurrent states in the chain,

(b) D is finite (if non-empty), and

(c) the chain is non-dissipative.

Proof: Suppose the set E of all null recurrent states is non-empty Then there is a null recurrent class $C \subset E$ and C is infinite. Therefore, $u \longrightarrow \infty$ on C. But u must be constant on C by corollary 1 to theorem 3. This is a contradiction. Hence E is empty. Similarly, if D is infinite, then $u \longrightarrow \infty$ on D which provides another contradiction.

Consider the Markov chain $\{X_{kn}, n = 0,1,2,\ldots\}$ which is the original chain $\{X_n, n = 0,1,2,\ldots\}$ restricted to start at state k, i.e., $X_{k0} = k$ with probability one (k is arbitrary but fixed.) The functional process $\{u(X_{kn}), n = 0,1,2,\ldots\}$ is a nonnegative super martingale. It follows from the super martingale convergence theorem that there exists an ω function v such that $u(X_{kn}) \longrightarrow v$ with probability one as $n \longrightarrow \infty$, and $0 \leq E(v) < \infty$. Let $T = I-D$ be the set of all nonrecurrent states in the chain. We know that there are no null recurrent states at all and that D if non-empty, is finite. We want to show that $f^*(k,D) = 1$. If $k \in D$ we are done. Assume k is not an element of D. Let $B = \{\omega: X_{kn}(\omega) \in T \text{ for every } n\}$. The subset of B for which $\{X_{kn}(\omega)\}$ is a finite subset of T has probability zero. Hence for $\omega \in B$ $v(\omega) = \lim_{n \longrightarrow \infty} u(X_{kn}(\omega))$ equals $+\infty$ by the proper divergence of u. Therefore, B is a null set, since otherwise $E(v) = \infty$. This shows that D is non-empty (and hence finite) and that $f^*(i,D) = 1$. But k was arbitrary. The result follows. QED

Remark on how fast a function can diverge and still be super regular. Let u be a super regular function. Then

$$(q_{ij}) = \left(\frac{u(j)}{u(i)} p_{ij}\right)$$

is a (sub stochastic) transition matrix. $\sum_j q_{ij} \leq 1$ implies that $q_{ij} \longrightarrow 0$ as $j \longrightarrow \infty$ for every i. Hence $u(j)\, p_{ij} \longrightarrow 0$ as $j \longrightarrow \infty$ for each i, i.e., $u(j) = o\left(\frac{1}{p_{ij}}\right)$ for every i. We see that u cannot go to infinity faster than the slowest row probabilities go to zero.

The state space of the Markov chains under consideration is denoted by I. Let A be an arbitrary subset of I. Let X denote the Markov chain. Define a new Markov chain X(A) by making each state in I-A into an absorbing state, i.e., X(A) = X stopped when, if ever, a path leaves A, so that after a path reaches a point in I-A it remains at that point throughout all later steps. The primary result in the remainder of this section is due to Mauldin [19] and is stated below. The proof uses a series of lemmas which study the properties of the differences $w(i) = u(i) - \sum_j p_{ij}u(j)$ and the Markov chain X(A) when I-A is a finite set.

<u>Theorem</u>: In any Markov chain, if there exists a function u such that $\liminf_{i \to \infty} w(i) > 0$, then the chain is non-dissipative.

We precede the lemmas and the proof of this theorem by some examples of non-dissipative Markov chains.

<u>Example 1</u>: A one dimensional random walk on the nonnegative integers with an absorbing state at zero. $p_{00} = 1$. If $i \neq 0$, then $p_{i,i+1} = p > 0$, $p_{i,i} = r$, $p_{i,i-1} = q > 0$, with $p + r + q = 1$. In this case $D = \{0\}$ and every other state is nonrecurrent. Let $u(i) = i$. Then

$$\sum_j p_{0j}\, u(j) = u(0) = 0,$$

and for $i \neq 0$

$$\sum_j p_{ij}\, u(j) = q(i-1) + ri + p(i+1) = i+(p-q) = u(i)+(p-q).$$

Therefore, if $p \leq q$, u is super regular. Clearly u is properly divergent. Hence, by theorem 10, we have the well known result that if $p \leq q$ then eventual absorbtion into the zero state is certain, regardless of the initial state.

Example 2: A random walk on the nonnegative integers with an absorbing state at zero. Here $p_{00} = 1$. If $i \neq 0$, then $p_{i,i+1} = p_i > 0$, $p_{i,i} = r_i \geq 0$, $p_{i,i-1} = q_i > 0$, $p_{ij} = 0$ if $|i-j| > 1$. Here again $D = \{0\}$ and every other state is nonrecurrent. Let $u(i) = i$. Then $\sum_j p_{0j}\, u(j) = 0$ and $\sum_j p_{ij}\, u(j) = u(i) + (p_i - q_i)$ for $i \neq 0$. Hence, if $p_i \leq q_i$ for every i, then we have that $f^*(i,D) = f^*_{i0} = 1$ for every i by theorem 10. In addition

$$w(i) = u(i) - \sum_j p_{ij}\, u(j) = p_i - q_i.$$

Therefore, if $\lim_{i \to \infty} \inf(q_i - p_i) > 0$, then $f^*_{i0} = 1$ for every i by the theorem to be proved. It is well known (see Feller [11]) that absorbtion into the origin is certain if and only if the series

$$\sum_{n=1}^{\infty} \frac{q_1 q_2 \cdots q_n}{p_1 p_2 \cdots p_n}$$

diverges. Foster [14] has shown that this condition is necessary and sufficient for the existence of a properly divergent super regular function in this case. (Actually, Foster considers only the case where $r_i = 0$ for every i, but his proof goes through in the more general situation).

Example 3: Now consider a two dimensional random walk, i.e., a Markov chain on the lattice points (x,y) in the plane with integer coordinates. If $(x,y) \neq (0,0)$ we have probability $\frac{1}{4}$ of going to any of the four adjacent states $(x,y-1)$, $(x,y+1)$, $(x-1,y)$ and $(x+1,y)$. The state $(0,0)$ is absorbing. Here $D = \{(0,0)\}$ and every other state is nonrecurrent. We enumerate the states as follows: $(0,0)$ is state 0. Now number all those states (x,y) with $|x| + |y| = 1$, then those with $|x| + |y| = 2$, etc. Define a function u on these states by

$$
\begin{aligned}
u(0,0) &= 0 \\
u(x,0) &= |x| + 1 \qquad \text{if } x \neq 0 \\
u(0,y) &= |y| + 1 \qquad \text{if } y \neq 0 \\
u(x,y) &= |x| + |y| \qquad \text{if } x \neq 0 \text{ and } y \neq 0.
\end{aligned}
$$

This u is regular and properly divergent. Hence eventual absorbtion into the origin is inevitable. (This is not a new result. It follows directly from a theorem of Polya. (See Feller [11])).

Finally, consider a symmetric random walk in three dimensions. Since there are no recurrent states in this case (also from Polya's theorem), it follows that there cannot be any properly divergent super regular functions. In particular, the functions $u(x,y,z)=|x|+|y|+|z|$ and $u(x,y,z) = (x^2+ y^2+ z^2)^{1/2}$ are not super regular.

Lemma 11: If u is a nonnegative super regular function, then $\inf_i w(i) = 0$.

Proof: Suppose $\inf w(i) > 0$. Then $w(i)$ is bounded away from zero, say, $w(i) = u(i) - \sum_j p_{ij}u(j) \geq \epsilon > 0$ for every i. Hence

$$u(X_n) - E[u(X_{n+1})|X_n] \geq \epsilon, \text{ i.e.,}$$

$$E[u(X_{n+1})|X_n] \leq u(X_n) - \epsilon, \text{ and}$$

$$E[u(X_{n+1})|X_0] = E\{E[u(X_{n+1})|X_n]|X_0\} \leq E[u(X_n)|X_0] - \epsilon.$$

Similarly,

$$E[u(X_{n+2})|X_0] = E[u(X_{n+1})|X_0] - \epsilon \leq E[u(X_n)|X_0] - 2\epsilon .$$

Continuing in this way we see that, for any positive integer k,

$$E[u(X_{n+k})|X_0] \leq E[u(X_n)|X_0] - k\epsilon.$$

Now let i_0 be an arbitrary state. We have

$$E[u(X_{n+k})|X_0 = i_0] \leq E[u(X_n)|X_0 = i_0] - k\epsilon = \sum_j p_{i_0 j}^{(n)} u(j) - k\epsilon \leq u(i_0) - k\epsilon$$

which is negative for a sufficiently large k. This contradicts the assumed nonnegativeness of u. Hence w cannot be bounded away from zero. QED

In a similar fashion it can be shown that if u is any bounded non-negative sub regular function, then $\sum_j p_{ij} u(j) - u(i)$ cannot be bounded away from zero.

Lemma 12: Let $\{X_n, n=0,1,2,\ldots\}$ be a Markov chain with state space $I = \{0,1,2,\ldots\}$ and let u be a nonnegative, super regular function such that $E[u(X_0)] < \infty$. Then, with probability one, $\lim_{n\to\infty} \inf w(X_n)=0$.

Proof: $\sum_j p_{ij}u(j) = E[u(X_{n+1})|X_n = i]$ and this does not depend on n. Hence $w(X_n) = u(X_n) - E[u(X_{n+1})|X_n]$. For every n we have that $E[u(X_n)] = \sum_i E[u(X_n)|X_0 = i]\, P(X_0 = i) = \sum_i \sum_j p_{ij}^{(n)} u(j)P(X_0=i) \leq \sum_i u(i)\, p(X_0 = i) = E[u(X_0)] < \infty$. Therefore, $E[w(X_n)]$ exists and $E[w(X_n)] = E[u(X_n)] - E[u(X_{n+1})]$. Furthermore, $\lim_{n\to\infty} E[w(X_n)]$ exists and equals 0. Hence, using Fatou's lemma,

$$0 = \lim_{n\to\infty} E[w(X_n)] \geq E[\lim_{n\to\infty} \inf w(X_n)].$$

and this implies, in view of the fact that $w(X_n)$ is a nonnegative rancom variable for every n, that $\lim_{n\to\infty} \inf w(X_n) = 0$ with probability one. QED

Corollary: If the sequence of random variables $w(X_n)$, $n = 0,1,\ldots$ is a.s. uniformly bounded, then with probability one, $w(X_n) \to 0$ as $n \to \infty$. [We note that this will be the case if u is bounded since $0 \leq w(i) \leq u(i)$ for every i.]

Lemma 13: Let u be a super regular function. Suppose there exists a non-empty finite set B of states such that $w(i) = 0$ if $i \in B$ and $w(i) > 0$ if $i \in I-B$. Suppose also that $\lim_{i\to\infty} \inf w(i) = \epsilon > 0$. Then the chain is non-dissipative.

Proof: Consider the process $\{X_{kn}, n = 0,1,2,\ldots\}$. k is an arbitrary but fixed state. $E[u(X_{k0})] = u(k) < \infty$. By lemma 12,
$\lim_{n\to\infty} \inf w(X_{kn}) = 0$ with probability one. Let
$N = \{\omega: \lim_{n\to\infty} \inf w(X_{kn}) \neq 0\}$, and let $\omega' \in N$. If the sequence of states
$\{X_{kn}(\omega')\}_{n=0}^{\infty}$ runs through an infinite subset of I, then
$\lim_{n\to\infty} \inf w[X_{kn}(\omega')] \geq \lim_{i\to\infty} \inf w(i) = \epsilon > 0$. Thus $\{X_{kn}(\omega')\}_{n=0}^{\infty}$ takes on only a finite number of values. But $\lim_{n\to\infty} \inf w[X_n(\omega')] = 0$. Hence there exists an integer $N(\omega')$ such that $w[X_n(\omega'] = 0$ for every $N(\omega')$, i.e., $X_{kn}(\omega') \in B$ for every $n > N(\omega')$. Therefore, for every $\omega' \in N^c$ the path function $\{X_{kn}(\omega')\}_{n=0}^{\infty}$ eventually enters B and remains there throughout all the remaining steps. It follows that $D \neq \phi$ and $f^*(k,D) = 1$. But k was an arbitrary state. The result follows. QED

Let A be an arbitrary subset of I. Define $h^*(i,A) = P(X_n \in A$ for some $n = 0,1,2,\ldots | X_0 = i)$. This function is closely related to the function $f^*(i,A)$ previously defined. In particular $h^*(i,A) \geq f^*(i,A)$ for every i and every set of states A. Hence a Markov chain is non-dissipative if and only if $h^*(i,D) = 1$ for every i (since D is a closed set of states). $h^*(i,A)$ is a super regular function; regular if $i \in I-A$. The following lemma concerning the function h^* can be found in Doob [9].

Lemma 14: In any Markov chain, $h^*(X_n, A)$ has limit one along almost every path function of $\{X_n, n = 0,1,\ldots\}$ which hits A infinitely often, and limit zero otherwise.

Corollary: Let A be the set of all recurrent states in the Markov chain. Then $h^*(X_n, A)$ has limit zero along every path function of

$\{X_n, n = 0,1,2,\ldots\}$ which never hits A, i.e., which remains always in the nonrecurrent states. Stated another way, this says that $\lim_{n \to \infty} h^*(X_n, A) = 1$ or 0 according as X_n ever hits A or not.

Lemmas 15 and 16 are originally due to Mauldon [19]. The proofs are new.

Lemma 15: Let $X = \{X_n, n = 0,1,2,\ldots\}$ be a Markov chain with no null recurrent states. Fix any state k, and consider the Markov chain $\hat{X} = X(I - \{k\})$. Then $\hat{X}$ is non-dissipative if and only if X is non-dissipative.

Proof: That X non-dissipative implies $\hat{X}$ non-dissipative is clear. Suppose $\hat{X}$ is non-dissipative. Let D be the set of all positive recurrent states for X. Then $D \cup \{k\}$ is the set of all positive recurrent states for $\hat{X}$. The path functions of $\hat{X}$ differ in behavior from those of X only when they reach the state k. If $k \in D$, then D is the set of all positive recurrent states for $\hat{X}$ and since $\hat{X}$ is non-dissipative, almost every path function of $\hat{X}$ enters D. Therefore, almost every path function of X enters D, and we conclude that X is non-dissipative.

Now suppose that $k \notin D$. Let $B_1 = \{\omega: X_n(\omega) \in D$ for some $n = 0,1,2,\ldots\}$ and $B_2 = \{\omega: X_n(\omega) = k$ for some $n = 0,1,2,\ldots\}$. Now $f^*(i,D) = P(B_1|X_0 = i) = 1 - P(B_1^c \cap B_2^c|X_0 = i) - P(B_1^c \cap B_2|X_0 = i) = P(B_1 \cup B_2|X_0 = i) - P(B_1^c \cap B_2|X_0 = i) = P(B_1 \cup B_2|X_0 = i) - P(B_2|X_0 = i) \cdot P(B_1^c|X_0 = k) = f^*(i, D \cup \{k\} - f^*_{ik}[1 - f^*(k,D)]$. This is proved by the method of first entrance (into k). But $\hat{X}$ is non-dissipative. Therefore, $\hat{f}^*(i, D \cup \{k\}) = 1$ for every i, and, putting $i = k$, we get $f^*(k,D) = 1 - f^*_{kk}[1 - f^*(k,D)]$. However, since $k \notin D$, k is a nonrecurrent state for X, and $f^*_{kk} < 1$. Therefore, solving for $f^*(k,D)$, we conclude that $f^*(k,D) = 1$. Substituting this in the relation $f^*(i,D) = 1-f^*_{ik}[1-f^*(k,D)]$, it follows that $f^*(i,D) = 1$ for every i. QED

Corollary: Let X be a Markov chain with no null recurrent states and consider X(A), where I-A is finite. Then X(A) is non-dissipative if and only if X is non-dissipative

Lemma 16: Let X be an arbitrary Markov chain and consider X(A), where I-A is finite and no state in I-A is null recurrent. Then X and X(A) have the same null recurrent states, i.e., if i is a null recurrent state for X, then i is a null recurrent state for X(A), and conversely.

Proof: It will suffice to demonstrate the result for $I-A = \{k\}$, a single state. The state k is not null recurrent for X or X(A). Suppose $i \neq k$ and i leads to k. Then i is nonrecurrent for X(A) and is not null recurrent for X (since if it were it could not lead to k which is not null recurrent). Suppose $i \neq k$, and i does not lead to k. Then the character of i is the same in both chains. Thus X and X(A) have the same null recurrent states. QED

The theorem of Mauldon follows.

Theorem 17: In any Markov chain, if there exists a function u such that $\liminf_{i \to \infty} w(i) > 0$, then the chain is non-dissipative.

Proof: Suppose u is super regular and that $\liminf_{i \to \infty} w(i) = \epsilon > 0$.

There exist at most a finite number of states, say a set $J = \{j_1, \ldots, j_m\} \subset I$, such that $w(j_k) < \epsilon/2$, $k = 1, \ldots, m$. Suppose $J = \phi$ or if $J \neq \phi$, that

$\min_{k=1,\ldots,m} w(j_k) = \delta > 0$. Then $w(i) \geqq \delta > 0$ for every $i \in I$, and w is bounded away from zero. But this cannot happen by lemma 11. Therefore, $j \neq \phi$ and there exists at least one state $j_k \in J$ for which $w(j_k) = 0$. Hence the set $B = \{i \in I: w(i) = 0\}$ is non-empty and finite. The result follows from lemma 13. Now assume that $\lim_{i \to \infty} \inf w(i) = \epsilon > 0$, but u is not super regular. Let $A = \{i \in I: w(i) \geq 0\}$ and define $X(A)$ as above. Properties of lim inf assure us that $I-A$ is finite. The transition probabilities for the Markov chain $X(A)$ are given by

$$\hat{p}_{ij} = p_{ij} \quad \text{if} \quad i \in A, \quad \text{and}$$

$$\hat{p}_{ij} = \delta_{ij} \quad \text{if} \quad i \in I-A.$$

Consider the function u for the Markov chain $X(A)$. For $i \in I-A$ we have

$$\sum_j \hat{p}_{ij}\, u(j) = \sum_j \delta_{ij}\, u(j) = u(i),$$

and for $i \in A$ we have

$$\sum_j \hat{p}_{ij}\, u(j) = \sum_j p_{ij}\, u(j) \leq u(i).$$

Hence u is a nonnegative super regular function for $X(A)$ and the set $B = \{i \in I: w(i) = 0 \text{ for } X(A)\} \subset I-A$ is finite and non-empty. By the proof of lemma 13 $X(A)$ and consequently X (by lemma 16), have no null recurrent states. $X(A)$ is non-dissipative by lemma 13. It follows from the corollary to lemma 15 that X is non-dissipative. QED

IV. Almost Closed and Sojourn Sets

The previous results, in particular theorems 10 and 17 are concerned with a situation in which almost every path function of the Markov chain under consideration eventually enters and never leaves a certain set of states, e.g., the set of all positive recurrent states in the chain. In recent years there has been considerable interest in the classification and characterization of sets of states that are entered infinitely often by a set of path functions with positive probability, but not necessarily probability one.

The first work along this line seems to have been done by Blackwell [1] in 1955; some of which will be described below. Further work has come from Brieman [2], [3], and Chung and Derman [6].

Feller [12] has also contributed to the analytic study of these sets. We will define these sets and study some of their properties, putting particular emphasis on their relation to super regular functions. We conclude with results which give some knowledge about the behavior of a super regular function in the evolution in time of a Markov chain.

Let A be an arbitrary set of states. We define (Chung [5])

$$\overline{L}(A) = \limsup_{n \to \infty}\{\omega : X_n(\omega) \in A\} = \{\omega : X_n(\omega) \in A \text{ for infinitely many } n\},$$

and

$$\underline{L}(A) = \liminf_{n \to \infty}\{\omega : X_n(\omega) \in A\} = \{\omega : X_n(\omega) \in A \text{ for all but a finite number of } n\}.$$

The set A will be called a sojourn set (Chung [5]) if $P\{\underline{L}(A)\} > 0$, and will be called almost closed (Blackwell [1]) if $0 < P\{\underline{L}(A)\} = P\{\overline{L}(A)\}$. Clearly every almost closed set is also a sojourn set. We note that in an irreducible recurrent Markov chain the only sojourn set is the entire state space which is also almost closed. In general, every closed set of states is also almost closed.

A set A of states is called transient if $P\{\overline{L}(A)\} = 0$. In a recurrent Markov chain the only transient set is the empty set. In a nonrecurrent chain every finite set is a transient set. In a non-dissipative chain the set $I-D$ is transient, and D is (almost) closed.

Example of a sojourn set which is not almost closed.

Consider a Markov chain with state space $I = \{0, \pm 1, \pm 2, \ldots\}$ which consists of three communicating classes: $I_1 = \{\ldots, -3, -2, -1\}$ which is nonrecurrent. $I_2 = \{0\}$ in which 0 is an absorbing state, and $I_3 = \{1, 2, 3, \ldots\}$ which is recurrent. We assume an initial distribution $\{p_i, -\infty < i < \infty\}$ such that $\sum_{i \in I_1} p_i > 0$, $p_0 > 0$, and $\sum_{i \in I_3} p_i > 0$.

Let $A = \{-1, 0, 1\}$. Now $P\{\underline{L}(A)\} = p_0 > 0$, and $P\{\overline{L}(A)\} > P\{\underline{L}(A)\}$ since the state 1 is recurrent. Hence A is a sojourn set but it is not almost closed.

Example of a sojourn set which is not almost closed in an irreducible Markov chain. Let I be the set of nonnegative integers, $p_{01} = p_{02} = \frac{1}{2}$, $p_{2i-1,2i+1} = p_{2i,2i+2} = 1 - \frac{1}{(i+1)^2}$, and $p_{2i+1,0} = p_{2i,0} = \frac{1}{(i+1)^2}$, $i \geq 1$.

Then I is a nonrecurrent class; the set of even integers and the set of odd integers are two disjoint almost closed sets. Let A be the set consisting of the even integers and ''half'' the odd integers. A is a sojourn set which is not almost closed.

We observe that the sets $\underline{L}(A)$ and $\overline{L}(A)$ have an invariance property which is related to the fact that they do not depend on the behavior of the path functions for the first finite number of steps. This can be expressed more carefully in the following way: Consider a fixed $\omega \in \Omega$. This ω yields the path function $\{X_0(\omega), X_1(\omega), \ldots\}$. $\omega \in \underline{L}(A)$ if and only if there exists an integer $N(\omega)$ such that $X_m(\omega) \in A$ for every $m > N(\omega)$, and this remains true even if we ignore the first K steps (for any K), i.e., $\omega \in \underline{L}(A)$ if and only if there exists a number $N(\omega)$ such that $X_m(\omega) \in A$ for every $m > \max(N(\omega), K)$. Similarly, $\omega \in \overline{L}(A)$ if and only if for every $N > K$ there exists $m > N$ such that $X_m(\omega) \in A$, and this is true for every K. It follows that the probabilities $P\{\underline{L}(A)|X_n=i\}$ and $P\{\overline{L}(A)|X_n=i\}$ do not depend on n.

Theorem 18: A set A of states is transient if and only if $P\{\overline{L}(A)|X_0=i\} = 0$ for every $i \in A$.

Proof: Suppose A is transient. Then $0 = P\{\overline{L}(A)\} = \sum_i P\{\overline{L}(A)|X_0= i\} P\{X_0= i\}$. Hence $P\{\overline{L}(A)|X_0= i\} = 0$ for all states i such that $P\{X_0= i\} > 0$. Similarly, $0 = P\{\overline{L}(A)\} = \sum_i P\{\overline{L}(A)|X_1= i\} P\{X_1=i\}$. Hence $0 = P\{\overline{L}(A)|X_1= i\} = P\{\overline{L}(A)|X_0=i\}$ (by invariance) for all states i such that $P\{X_1= i\} > 0$, and so on. All states must eventually be accounted for in this way by definition of minimal state space.

Now suppose that $P\{\overline{L}(A)|X_0= i\} = 0$ for every $i \in A$. Then the probability of entering A infinitely often, starting from a state in A, is zero. If there exists a state $j \in I-A$ such that $P\{\overline{L}(A)|X_0= j\} > 0$, then $P\{L(A)|X_0= k\} > 0$ where k is the state in A first hit by a path from j, and this is impossible by assumption. Hence $P\{\overline{L}(A)|X_0= i\} = 0$ for every i and $P\{\overline{L}(A)\} = 0$. QED

An almost closed set will be called atomic if it does not contain two disjoint almost closed subsets, and completely nonatomic if it does not contain any atomic almost closed subset. If A is atomic and B is transient, then $A \cup B$ is atomic. A Markov chain whose state space consists of a single atomic almost closed set is called <u>atomic</u>.

The most important theorem concerning almost closed sets is the following decomposition theorem which is due to Blackwell [1].

<u>Theorem (Decomposition Theorem)</u>: We have the following decomposition of the state space:

$$I = A_0 \cup A_1 \cup A_2 \cup \dots$$

where the A's are a finite or countable number of disjoint almost closed sets, at most one of which is completely nonatomic and the others are atomic; and

$$\sum_{n=0}^{\infty} P\{\underline{L}(A)\} = 1.$$

The decomposition is unique modulo transient sets. Each existing recurrent class may be taken as one of the atomic A_n's; each of the remaining A_n's including the completely nonatomic one, if present, contains only nonrecurrent states.

Blackwell [1] has also shown that the decomposition consists of a single atomic almost closed set if and only if the only bounded regular function is a constant. This result is known as Blackwell's theorem.

The concepts of transient and almost closed sets can be used to provide still another proof of Foster's theorem (Theorem 7 of this paper) in the following manner:

Let C be an essential class of states. Choose an arbitrary state 0 in C. Make 0 absorbing by setting $\hat{p}_{ij} = p_{ij}$ if $i \neq 0$ and $\hat{p}_{0j} = \delta_{0j}$.

<u>Lemma 19</u>: C becomes atomic under this change if and only if it was originally recurrent.

Proof: Suppose C was originally recurrent. Under the change 0 becomes an absorbing state and every other state is nonrecurrent. By the original recurrence we have that $P\{\text{absorbtion into } 0 | X_0 = i\} = 1$ for every $i \in C$ so that $P\{\overline{L}(C - \{0\}) | X_0 = i\} = 0$ for every $i \in C$ which says that $C - \{0\}$ is a transient set. Clearly C (before and after changing) is almost closed. $\{0\}$ is almost closed since $\overline{L}(\{0\}) = \{\omega : X_n(\omega) = 0 \text{ infinitely often}\}$ and $\underline{L}(\{0\}) = \{\omega : X_n(\omega) = 0 \text{ for all but a finite number of } n\}$ are both sets having probability one (in the changed chain). $\{0\}$ is atomic. Therefore, $C = [C - \{0\}] \cup \{0\}$ is the disjoint union of an atomic almost closed set and a transient set, from which we conclude that (the new) C is atomic.

Now suppose that the changed C is atomic. Then C does not contain two disjoint almost closed subsets. $\{0\}$ is a positive recurrent class in the changed chain and is therefore almost closed. Hence $C - \{0\}$ is not almost closed and either (i) $P\{\overline{L}(C-\{0\})\} = P\{\underline{L}(C-\{0\})\} = 0$, or (ii) $P\{\overline{L}(C-\{0\})\} > P\{\underline{L}(C-\{0\})\}$. Suppose (i) is true. Then $C - \{0\}$ is a transient set and almost every path function eventually leaves $C - \{0\}$ and stays away forever after. Hence 0 is a recurrent state for the original chain and the original chain must have been recurrent. Now suppose (ii) is true. Let $A = \overline{L}(C - \{0\}) - \underline{L}(C - \{0\})$. Then $A = \{\omega : \text{given any } N \text{ there exist } N_1 > N \text{ and } N_2 > N \text{ such that } X_{N_1}(\omega) \neq 0 \text{ and } X_{N_2}(\omega) = 0\}$ and since 0 is absorbing in the new chain

A must be a null set. Hence (ii) cannot hold, and the original chain is recurrent. QED.

The theorem of Foster follows.

Theorem: Let 0 be an arbitrary state in an essential class C. The system of equations

$$* \qquad u(i) = \sum_{j \in C} p_{ij}\, u(j), \quad i \neq 0$$

has a bounded nonconstant solution if and only if C is nonrecurrent.

Proof: Suppose * has a bounded nonconstant solution. Make 0 absorbing as in lemma 19. In this new chain the system of equations

$$** \qquad u(i) = \sum_{j} \hat{p}_{ij}\, u(j)$$

has a bounded nonconstant solution. Hence, by Blackwell's theorem, the new chain is not atomic which implies, by lemma 19, that the original chain is nonrecurrent.

Now suppose that the original chain is nonrecurrent. Make 0 absorbing as in lemma 19. By lemma 18 the new chain is not atomic. Hence, by Blackwell's theorem, there exists a bounded nonconstant solution to **. Therefore, there exists a bounded nonconstant solution to * in the original chain. QED

There are two functions, denoted by σ_A and S_A (defined below) which are super regular and are useful in the study of sojourn sets and almost closed sets. These functions have been extensively studied by Feller [12]. In order to define them we must introduce the notion of taboo probability.

Let H be an arbitrary (possible empty) subset of the state space I. We define the taboo probabilities ${}_{H}p_{ij}^{(n)}$ as follows:

$${}_{H}p_{ij}^{(n)} = P\{X_n = j,\ X_\nu \notin H \text{ for } \nu = 1,2,\ldots,n \mid X_0 = i\} \text{ for } i \in I-H,\ j \in I-H, \text{ and } n = 1,2,\ldots$$

$${}_{H}p_{ij}^{(n)} = 0 \text{ if } i \in H \text{ or } j \in H,\ n = 0,1,2,\ldots$$

$${}_{H}p_{ij}^{(0)} = \delta_{ij} \text{ if } i \in I-H \text{ and } j \in I-H.$$

The matrix ${}_{H}P = ({}_{H}p_{ij})$ in sub stochastic. This definition of taboo probability differs slightly from the standard definition of Chung [5] in that we here take account of the first and last steps of the transition. This definition is, however, more convenient for our present purposes. It has been used by Kemeny and Snell [16] under a different notation, and is essentially the definition employed by Feller [12].

We note that ${}_{\phi}P = P$. We will say that a function u is super regular with respect to ${}_{H}P$ if

$$u(i) \geq \sum_{j} {}_{H}p_{ij}\, u(j) \quad \text{for every } i \in I.$$

Let A be an arbitrary set of states. We define (Feller [12])

$$\sigma_A(i) = \begin{cases} \lim\limits_{n \to \infty} \sum\limits_{j \in A} {}_{I-A}p_{ij}^{(n)} = P\{X_n \in A \text{ for all } n \mid X_0 = i\} & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}$$

The following theorem gives some properties of the function σ_A. Parts (i) and (ii) are due to Feller [12], as is all of theorem 21, and (2) of theorem 23.

<u>Theorem 20</u>: For any set A of states

(i) $\sigma_A(i)$ exists for every i, and the limit is attained monotonically,

(ii) σ_A is a sub regular function with respect to P,

(iii) σ_A is a regular function with respect to ${}_{I-A}P$, and

(iv) If $\sigma_A(i) > 0$, then $\sum_{j\in A} {}_{I-A}p_{ij}^{(n)} > 0$ for every n.

Proof: For $i \in A$ we have $\sum_{j\in A} {}_{I-A}p_{ij}^{(n)} = P\{X_\nu \in A,\ \nu=1,\dots,n|X_0=i\}$ $\geq P\{X_\nu \in A \text{ for } \nu = 1,\dots,n+1|X_0 = i\} = \sum_{j\in A} {}_{I-A}\bar{p}_{ij}^{(n+1)}$ from which (i) follows immediately. (ii) and (iii) are shown by direct computation. $\sigma_A(i) = P\{X_n \in A \text{ for every } n|X_0 = i\} \leq P\{X_\nu \in A\ \nu = 1,\dots,n|X_0 = i\} = \sum_{j\in A} {}_{I-A}p_{ij}^{(n)}$. Hence (iv). QED

Let A be an arbitrary set of states. We define (Feller [12]) for each $i \in I$

$$S_A(i) = \lim_{n\to\infty} \sum_j p_{ij}^{(n)}\ \sigma_A(j).$$

<u>Theorem 21</u>: For any set A of states

(i) S_A is a regular function with respect to P, and

(ii) $0 \leq \sigma_A(i) \leq S_A(i) \leq 1$ for every i.

Proof: Both parts follow from the theorem of Feller in section II of this paper. QED

It follows from this theorem that if i leads to j and $S_A(j) > 0$, then $S_A(i) > 0$. Hence, if i and j communicate, then $S_A(i)$ and $S_A(j)$ are either both positive or both zero.

Theorem 22: There exists a state $i \in A$ for which $\sigma_A(i) > 0$ if and only if there exists a state k for which $S_A(k) > 0$.

Proof: Suppose $\sigma_A(i) > 0$. Then, since $0 \leq \sigma_A(i) \leq S_A(i)$ for every i, we see that $S_A(i) > 0$. Now suppose $S_A(k) > 0$. Then $0 < S_A(k) = \lim_{n \to \infty} \sum_j p_{kj}^{(n)} \sigma_A(j)$ and there must be a state $i \in A$ for which $\sigma_A(i) > 0$. QED

Theorem 23: Let A be an arbitrary set of states. Then, for every $i \in I$, $S_A(i) = P\{\underline{L}(A) | X_0 = i\}$ and the following 3 statements are equivalent:

(1) A is a sojourn set,

(2) For some $i \in I$, $S_A(i) > 0$ (equivalently, $\sigma_A(i) > 0$ for some $i \in A$),

(3) There exists a state $i \in A$ and an $\epsilon > 0$ such that $\sum_{j \in A} {}_{I-A}p_{ij}^{(n)} > 0$ for every n.

Proof: For $i \in A$ we have $\sigma_A(i) = P\{X_n \in A \text{ for all } n | X_0 = i\}$. Therefore, $S_A(i) = \lim_{n \to \infty} \sum_{j \in A} p_{ij}^{(n)} \sigma_A(j) = \lim_{n \to \infty} P\{X_m \in A \text{ for } m = n, n+1, \ldots | X_0 = i\} = P\{\underline{L}(A) | X_0 = i\}$. That (2) implies (3) follows easily from the fact that $\sigma_A(i)$ is approached monotonically from above. It is clear that (3) implies (2). Suppose (2) is not true. Then $P\{\underline{L}(A) | X_0 = i\} = 0$ for every i and $P\{\underline{L}(A)\} = \sum_i P\{\underline{L}(A) | X_0 = i\} P(X_0 = i) = 0$. Hence, by contraposition, if A is a sojourn set, then $S_A(i) > 0$ for some i. Now suppose that $S_A(i) > 0$ for some i. Then, there exists a state i

such that $0 < P\{\underline{L}(A)|X_0 = i\} = P\{\underline{L}(A)|X_m = i\}$ and this conditional probability (whenever defined) does not depend on m. Let m be such that $P\{X_m = i\} > 0$. Such an m exists by definition of minimal state space. Now $P\{\underline{L}(A)\} = \sum_j P\{\underline{L}(A)|X_m = j\} P\{X_m = j\} \geq P\{\underline{L}(A)|X_m = i\} P\{X_m = i\} > 0$. Thus (2) implies (1) and the result follows. QED

<u>Theorem 24</u>: Let u be a nonnegative sub regular function with $0 < ||u|| = \sup_{i \in I} u(i) < \infty$. Then for any number $a \in (0, ||u||)$ the set $B(a) = \{i \in I: u(i) > ||u|| - a\}$ is a sojourn set.

Proof: The functional process $\{u(X_n), n = 0,1,2,\ldots\}$ is a nonnegative sub martingale which is bounded above (by $||u||$). Therefore, by a sub martingale convergence theorem, there exists a nonnegative random variable v such that $\lim_{n \to \infty} u(X_n) = v$ with probability one, and $E[u(X_0)] \leq E[u(X_1)] \leq \ldots \leq E(v)$. Let $A_0 = \{\omega: u[X_n(\omega)]$ does not converge$\}$. A_0 is a null set. Now suppose, for some number $a \in (0, ||u||)$, that $B(a)$ is not a sojourn set. Then, clearly, no subset of $B(a)$ is a sojourn set. Let $a_k = ||u||/k$, for $k = 1,2,\ldots$. We note that $a_1 = ||u||$, $\lim_{k \to \infty} a_k = 0$, and that the sequence $\{a_k\}$ is decreasing. It follows that $B(a_1) \supset B(a_2) \supset B(a_3) \supset \ldots$. Let a_ℓ be the largest element of $\{a_k\}$ which is less than or equal to a. Such an a_ℓ exists since $\lim_{k \to \infty} a_k = 0$. Thus $B(a_\ell) \subset B(a)$ and $P\{\underline{L}(B(a_\ell))\} = 0$. Hence $P\{\overline{L}(I - B(a_\ell))\} = 1$ and almost every path function hits $I - B(a_\ell)$ infinitely often. Let $A_\ell = \{\omega: X_n(\omega)$ does not hit $I - B(a_\ell)$ infinitely often$\}$. A_ℓ is a null set. If $\omega \in \Omega - (A_0 \; A_\ell)$, then $X_n(\omega) \in I - B(a_\ell)$ infinitely often, i.e., $u[X_n(\omega)] \leq ||u|| - a_\ell$ infinitely often and

$v(\omega) = \lim_{n \to \infty} u[X_n(\omega)] \leq ||u|| - a_\ell$. Now let $A = A_0 \cup \bigcup_{m=\ell}^{\infty} A_m$ where $A_m = \{\omega: X_n(\omega)$ does not hit $I-B(a_m)$ infinitely often$\}$.

$P(A) \leq P(A_0) + \sum_{m=\ell}^{\infty} P(A_m) = 0$. For $\omega \in I-A$ we have that $v(\omega) \leq ||u||-a_m$ $m = \ell, \ell+1, \ldots$, Therefore, $v(\omega) \leq 0$ which implies that $v(\omega) = 0$ with probability one. We conclude that $E(v) = 0$ and, consequently, that $E[u(X_n)] = 0$ for every n. We will show now that this implies that $u \equiv 0$ which will contradict $||u|| > 0$ and the result will be proved. Suppose there exists a state i for which $u(i) > 0$. By definition of minimal state space there exists an n for which $P\{X_n = i\} > 0$. Therefore, $E[u(X_n)] \geq u(i)\ P\{X_n = i\} > 0$, which is inconsistent with $E[u(X_n)] = 0$. QED

Corollary: Let u be a nonnegative super regular function and let α be any positive number. Then the set $A = \{i \in I: u(i) < \alpha\}$ is either empty or a sojourn set.

Proof: u is nonnegative and super regular. Therefore, the function S defined by

$$S(i) = -(\min[u(i), \alpha] - \alpha)$$

is nonnegative, sub regular, and $A = \{i: u(i) < \alpha\} = \{i: S(i) > 0\}$. Now either $S \equiv 0$ in which case A is empty, or $||S|| = \sup_{i \in I} S(i) > 0$. In the latter situation choose any number $a_1 \in (0, ||S||)$ and let $a = ||S|| - a_1$. Now $B(a) = \{i: S(i) > a_1\} \subset \{i: S(i) > 0\} = A$. But $B(a)$ is a sojourn set by theorem 25. Therefore A, being a superset of a sojourn set, is also a sojourn set. QED

Let $Y_1, Y_2, \ldots$ be a sequence of independent identically distributed integer valued random variables such that 1 is the absolute value of the greatest common divisor of the set of all integers i for which $0 < P(Y_n = i) = p_i$. Let $X_0, X_1, \ldots$ be a sequence of random variables defined by

$$X_0 = 0 \quad \text{with probability one}$$

$$X_n = X_0 + \sum_{k=1}^{n} Y_k, \quad n \geq 1.$$

Then $\{X_n, n=0,1,2,\ldots\}$ is an irreducible Markov chain with stationary transition probabilities and state space $I = \{0, \pm 1, \pm 2, \ldots\}$. The one step transition probabilities are given by $p_{ij} = p_{j-i}$. The following theorem shows that all bounded regular functions on such a chain are constants. Of course, if the chain is recurrent, then the result is known in a more general form (see corollary 1 to theorem 4 of this paper). Thus we may assume that the chain is nonrecurrent-although this assumption plays no part in the proof to be given.

Theorem 25: Let $\{X_n, n=0,1,2,\ldots\}$ be a Markov chain of the type described in the preceding paragraph. Then every bounded regular function on this chain is a constant.

Proof: Let u be a bounded regular function. We may assume, without loss of generality, that $u(i) > 0$ for every i. The functional process $\{u(X_n), n=0,1,2,\ldots\}$ is a bounded nonnegative martingale. Therefore, with probability one, there exists a random variable v such that $\lim_{n \to \infty} u(X_n) = v$, and $E(v) = E[u(X_n)] = E[u(X_0)] = u(0)$, a constant.

Let C be any nonnegative real number and let $\Lambda = \{\omega: v(\omega) = C\}$. $\Lambda \in$

tail σ-field of $\{u(X_n), n=0,1,2,\ldots\} \subset$ tail σ-field of $\{X_n, n=0,1,\ldots\}$ = tail σ-field of $\{Y_n, n=1,2,\ldots\}$ which consists entirely of events having probability zero or one, by the Kolmogorov zero-one law. Hence v is a constant with probability one, and, since $E(v) = u(0)$, we see that $v = u(0)$ with probability one. Now suppose that u is nonconstant. We may also assume, without loss of generality, that $0 < u(0) < ||u||$ (since if $u(i) = ||u||$ for any i, then $u \equiv ||u||$ and is a constant). Let a be chosen in such a way that $||u|| - a = u(0) + \epsilon$ for some number $\epsilon > 0$. Then, by theorem 24, the set $A = \{i: u(i) > u(0) + \epsilon\}$ is a sojourn set. Hence $0 < P\{\underline{L}(A)\} = P\{\omega: X_n(\omega) \in A$ for all but a finite number of $n\} = P\{\omega: u[X_n(\omega)] > u(0) + \epsilon$ for all but a finite number of $n\} \leq P\{\omega: v(\omega) \geq u(0) + \epsilon\}$. Therefore, $P\{\omega: v(\omega) \geq u(0) + \epsilon\} = 1$. But this contradicts $v = u(0)$ with probability one. Hence u is constant. QED

REFERENCES

[1] Blackwell, D., ''On transient Markov chains with a countable number of states and stationary transition probabilities,'' Annals Math. Stat., Vol. 26 (1955), pp. 654-658.

[2] Brieman, L., ''On transient Markov chains with application to the uniqueness problem for Markov processes'', Annals Math. Stat.,Vol. 28 (1957), pp. 499-503.

[3] Brieman, L., ''Transient atomic Markov chains with a denumerable number of states'', Annals Math. Stat., Vol. 29 (1958), pp. 212-218.

[4] Chung, K. L., ''Some remarks on taboo probabilities'', Illinois Journal of Math., Vol. 5 (1961), pp. 431-435.

[5] Chung, K. L., ''Markov chains with stationary transition probabilities'', Berlin (1960).

[6] Chung, K. L. and C. Derman, ''Non recurrent random walks'', Pacific Journal of Mathematics, Vol. 6 (1956), pp. 441-447.

[7] Chung, K. L. and W. H. J. Fuchs, ''On the distribution of values of sums of random variables'', Memoirs Amer. Math. Soc. No. 6, pp. 1-12.

[8] Derman, C., ''A solution to a set of fundamental equations in Markov chains'', Proc. Amer. Math. Soc., Vol. 5 (1954), pp. 332-334.

[9] Doob, J. L., ''Discrete potential theory and boundaries'', Journal of Math. and Mech., Vol. 8 (1959), pp. 433-458.

[10] Doob, J. L., ''Stochastic Processes'', New York, 1953.

[11] Feller, W., ''An Introduction to probability theory and its applications'', New York, 1957.

[12] Feller, W., ''Boundaries induced by non negative matrices'', <u>Transactions Amer. Math. Soc.</u>, Vol. 83 (1956), pp. 19-54.

[13] Foster, F. G., ''Markov chains with an enumerable number of states and a class of cascade processes'', <u>Proc. Camb. Phil. Soc.</u>, Vol. 47 (1951), pp. 77-85.

[14] Foster, F. G., ''On Markoff chains with an enumerable infinity of states'', <u>Proc. Camb. Phil. Soc.</u>, Vol. 48 (1952), pp. 587-591.

[15] Foster, F. G., ''On the stochastic matrices associated with certain queueing processes'', <u>Annals Math. Stat.</u>, Vol. 24 (1953), pp. 355-361.

[16] Kemeny, J. G. and J. L. Snell, ''Potentials for denumerable Markov chains'', <u>Journal of Math. Anal. and Appl.</u>, Vol. 3 (1960), pp. 196-260.

[17] Kendall, D., ''Unitary dilations of Markov transition operators, and the corresponding integral representations for transition-probability matrices'', In U. Grenander (ed.),"Probability and Statistics'', New York, 1959, pp. 139-161.

[18] Kendall, D., ''On non dissipative Markov chains with an enumerable number of states'', <u>Proc. Camb. Phil. Soc.</u>, Vol. 47 (1951), pp. 633-634.

[19] Mauldon, J. G., ''On non dissipative Markov chains'', <u>Proc. Camb. Phil. Soc.</u>, Vol. 53 (1957), pp. 825-835.

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13. ABSTRACT

In this paper some properties of a discrete parameter Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ are studied.

A function u defined on the state space I of the chain is a super regular function if $u(i) \geq \sum_j p_{ij} u(j)$ for every i. A function μ on I is a super regular measure if $\mu(j) \geq \sum_j \mu(i) p_{ij}$ for every j.

In sections I and II some general properties of super regular functions and measures are established. Theorems discussing super regular functions and measures on recurrent classes are proved, a new proof of Foster's theorem characterizing nonrecurrent classes is given, and a theorem which characterizes a nonrecurrent class in terms of the existence of a super regular measure possessing a simple property is proved. In addition, an equitable class of states is shown to be one on which all super regular functions and measures are constants.

Section III is devoted to the study of non-dissipative chains, i. e., those chains in which almost every path function eventually enters the set of positive recurrent states. A new proof of the theorem of Kendall and Foster's given which uses the super martingale convergence theorem. Mauldon's theorem is proved, making use of a series of lemmas which study in detail the path functions of the Markov chain.

Section IV is concerned with certain sets of states called sojourn sets and almost closed sets, which are of interest in the study of the asymptotic behavior of the path functions. General properties of these sets are established and their relation to super regular functions is demonstrated

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