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ON MIXTURES OF χ^2 AND F DISTRIBUTIONS WHICH YIELD DISTRIBUTIONS OF THE SAME FAMILY.

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Which Yield Distributions of the Same Family[†].

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1. Introduction. Robbins and Pitman proved in 1949 [3], that the distribution of a linear combination $\sum_{i=1}^{r} a_i \chi_i^2$, $a_i > 1$, can be expressed as a mixture of χ^2 distribution, whose number of degrees of freedom are random variables distributed like the sum of independent negative binomials. Teicher used this relationship in his 1960 paper [4] to demonstrate a mixture of gamma distributions, with a common scale parameter, which yields a gamma random variable with a different scale parameter. The relationship exhibited is:

(1.1) $$G(\lambda\rho,\beta) \sim G(\lambda,\beta+M) \ , \ 0<\lambda<\infty \ , \ 0<\rho<1 \ ,$$

where $G(\lambda,\nu)$ denotes a gamma random variable, with a scale parameter λ; and M denotes a random variable having a negative binomial distribution, with probability density

(1.2) $$P[M=m] = \binom{-\beta}{m} \rho^{\beta} (\rho-1)^{m}, \ m = 0, 1, 2, \ldots, \ 0<\rho<1.$$

The mixing identity for χ^2 random variables, which was derived by Robbins and Pitman in [3] is obtained from (1.1) and (1.2) by substituting $\beta=\nu/2$, $\lambda=1/2$, and $\rho=(1+\phi)^{-1}$, $0<\phi<\infty$. The effects of mixing gamma random variables with probabilities following a Pascal distribution was also studied by Gurland [1,2].

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In the present note we prove two theorems (in section 2), and apply the mixing identities derivable to prove certain expressions for the moments of negative binomial random variables. The first theorem gives a necessary and sufficient condition on the mixing probabilities of $G(\lambda,\beta+M)$ distribution laws, which yield a $G(\lambda\rho,\beta)$ distribution law. The second theorem proves that only the family of $G(\lambda,\beta)$ distribution laws has the property that a mixture of powers of its Laplace transform, $\psi(s;\lambda,\beta)$ say, with negative binomial probabilities yields a Laplace transform of a gamma $G(\lambda\rho,\beta)$ distribution law, i.e.,

$$\sum_{\nu=0}^{\infty} \binom{-\beta}{\nu} \rho^{\beta} (\rho-1)^{\nu} (\psi(s;\lambda,\beta))^{\nu} = \psi(\frac{s}{\rho};\lambda,\beta) \quad . \tag{1.3}$$

This is a further characterization of the mixing identities studied here. In section 3 we utilize some of the mixing identities derivable from (1.1) to obtain various moments of negative binomial random variables, and of their inverses. The derivations of expressions for these moments by the aid of such mixing identities becomes a very simple task, compared to the computation requirements for deriving those expressions directly. In section 4 we show a case of a linear model (ANOVA model II) in which a mixtrue of F-distributions yields an F-distribution, which is a mixing identity derived from (1.1).

2. The characterization of mixing probabilities. Consider an additive Markov process $X(t)$, for which:

$$E\left\{e^{-sX(t)}\right\} = \left(\frac{\lambda}{\lambda+s}\right)^{\alpha t+\gamma} , \tag{2.1}$$

where, $0<\lambda<\infty$, $-\lambda<s<\infty$, $t\geq 0$, $\alpha\geq 0$ and $\gamma\geq 0$. Let T be a random variable, independent of $\left\{X(t);\ t\geq 0\right\}$, and such that: $P[T\leq x] = F(x)$. We consider the problem of characterizing the conditions under which $X(t)$ has a gamma distribution. In other words, the first problem we consider is that of characterizing the distribution $F(t)$, which satisfies:

$$\int_0^{\infty} F(dt) \left(\frac{\lambda}{\lambda+s}\right)^{\alpha t+\gamma} = \left(\frac{\mu}{\mu+s}\right)^{\beta} , \tag{2.2}$$

for some $\mu>0$ and $\beta>0$. $F(t)$ is the mixing probability distribution of the (random) gamma distributions, $G(\lambda,\alpha T+\gamma)$, which yield a gamma distribution $G(\mu,\beta)$.

It is sufficient to solve equation (2.2) for $s\geq 0$. Setting,

$$\left(\frac{\lambda}{\lambda+s}\right)^{\alpha} = e^{-u}, \quad s\geq 0,\ u\geq 0, \tag{2.3}$$

we obtain,

$$\int_0^\infty e^{-ut} F(dt) = e^{-\frac{\beta-\gamma}{\alpha}u} \rho^{\beta}\left[1-(1-\rho)e^{-\frac{u}{\alpha}}\right]^{-\beta}, \tag{2.4}$$

for $u>0$, $\rho=\mu\lambda^{-1}$, $0<\rho\leq 1$. The R.H.S. of (2.4) is the Laplace transform of a negative binomial distribution. This implies that $\alpha T-(\beta-\gamma)$ is a negative binomial random variable on the non-negative integers, i.e.,

$$P\left[T = \alpha^{-1}(\beta-\gamma) + \alpha^{-1} m\right] = \binom{-\beta}{m} \rho^{\beta} (\rho-1)^{m}, \quad m = 0, 1, \ldots \tag{2.5}$$

for $\beta\geq\gamma$. We have thus proven that a mixture of $G(\lambda,\alpha t+\gamma)$ distribution laws, which yields a $G(\lambda\rho,\beta)$ distribution law must be a negative binomial one. Using characteristic functions, it is simple to verify that a negative binomial of $(G\lambda,\alpha t+\gamma)$ distribution laws yields a $G(\lambda\rho,\beta)$ distribution law. We have thus proved.

Theorem 2.1: *A mixture on* t *of the family of gamma distributions laws* $G(\lambda,\alpha t+\gamma)$; $t\geq 0$; *is again a gamma distribution law* $G(\lambda\rho,\beta)$ *if, and only if, the mixing distribution law is a negative binomial, with parameters* ρ *and* β, *on the lattice points*

$$\left\{\alpha^{-1}(\beta-\gamma) + k\alpha^{-1},\ k = 0, 1, 2, \ldots\right\}, \quad \beta\geq\gamma . \tag{2.6}$$

The mixing identity (1.1) is a special case, where $\gamma=\beta$, $\alpha=1$ and $T=M$. We proceed now to state and prove the second characterizing theorem.

Theorem 2.2: *Let* $\psi(s)$ *be the (bilateral) Laplace transform of a distribution function, and let* $\psi(s)$ *satisfy the equation*

$$\sum_{\nu=0}^{\infty} \binom{-\beta}{\nu} \rho^{\beta} (\rho-1)^{\nu} \psi^{\beta+\nu}(s) = \psi^{\beta}\left(\frac{s}{\rho}\right), \tag{2.7}$$

<u>in a neighborhood of</u> $s=0$, <u>with</u> $0<\rho<1$, $\beta>0$. <u>Then, the distribution function is either degenerate or negative exponential.</u>

<u>Proof</u>: From (2.7) with $\beta=1$ we obtain the equation,

$$\rho\psi(s)\left[1-(1-\rho)\ \psi\ (s)\right]^{-1} = \psi\left(\frac{s}{\rho}\right) . \tag{2.8}$$

Since $\psi(s) \neq 0$ for $s\geq 0$, set $\theta(s) = \psi^{-1}(s)$. Then, (2.8) is reduced to

$$\rho\left[\theta\left(\frac{s}{\rho}\right) - 1\right] = \theta(s) - 1. \tag{2.9}$$

Setting $\prod(s) = (\theta(s)-1)/s$, we obtain:

$$\prod(\tau) = \prod(\rho\tau) \qquad , \text{ for } \tau\geq 0. \tag{2.10}$$

Hence, by iteration,

$$\prod(\rho^k) = \prod(\rho) \qquad , \text{ for all } k = 1,2,\ldots \tag{2.11}$$

Since $\prod(s)$ is analytic at $s=0$, equation (2.11) implies that $\prod(s)$ = constant. Therefore,

$$\psi(s) = (1+Cs)^{-1} , \tag{2.12}$$

where $c\geq 0$. Hence the distribution function, of which $\psi(s)$ is the Laplace transform, is either degenerate ($c=0$) or a negative exponential ($c>0$).

Finally, if $\psi(s)$ is the Laplace transform of the negative exponential law $G(\lambda,1)$, then $\psi^{\beta}(s)$ is the Laplace transform of the gamma distribution law $G(\lambda,\beta)$. Thus, theorem 2.2 proves (1.3).

3. <u>Extensions and applications of mixing identities</u>. In the present section we derive from mixing identity (1.1) further mixing identities, which relate χ^2, F and β-distributions, and indicate few possible applications of these identities.

As is well known, a χ^2 random variable with ν degrees of freedom is related to a gamma random variable $G(\lambda,p)$ according to the relationship,

$$G(\lambda,p) \sim \frac{1}{2\lambda}\chi^2[2p].$$

Thus, identity (1.1) yields the following mixing identity of χ^2 random variables:

$$(1+\phi)\ \chi^2[\nu] \sim \chi^2\ [\nu+2M] \qquad , 0<\phi<\infty, \tag{3.1}$$

$\nu=1, 2, \ldots$, where M is a negative binomial, with density

(3.2) $$P_{\nu,\phi}[M=m] = \frac{(-1)^m}{(1+\phi)^{\nu/2}} \cdot \binom{-\nu/2}{m} \left(\frac{\phi}{1+\phi}\right)^m \qquad , m=0,1,\ldots$$

Relationship (3.1) yields immediately the expected value and variance of a negative binomial random variable M, having a probability density (3.2). Indeed from (3.1),

(3.3) $$\begin{aligned}(1+\phi)\, E\left\{\chi^2[\nu]\right\} &= E\left\{\chi^2[\nu+2M]\right\} \\ &= E\left\{\nu+2M\right\} \quad .\end{aligned}$$

Hence, since $E\left\{\chi^2[\nu]\right\} = \nu$, we obtain,

(3.4) $$E_{\nu,\phi}\left\{M\right\} = \frac{\nu}{2}\,\phi \qquad , \nu = 1, 2, \ldots, 0<\phi<\infty .$$

Similarly, from (3.1) we have

(3.5) $$\begin{aligned}(1+\phi)^2\, \mathrm{Var}\left\{\chi^2[\nu]\right\} &= \mathrm{Var}\left\{\chi^2[\nu+2M]\right\} \\ &= E_{\nu,\phi}\left\{\mathrm{Var}\left\{\chi^2[\nu+2M]\,\middle|\,M\right\}\right\} + \mathrm{Var}_{\nu,\phi}\left\{E\left\{\chi^2[\nu+2M]\,\middle|\,M\right\}\right\} .\end{aligned}$$

From (3.4), (3.5) and $\mathrm{Var}\left\{\chi^2[\nu]\right\} = 2$, we obtain

(3.6) $$\mathrm{Var}_{\nu,\phi}\left\{M\right\} = \frac{\nu}{2}\,\phi(1+\phi), \qquad \nu = 1, 2, \ldots, 0<\phi<\infty .$$

A straightforward computation of the first two moments of M, without a use of identity (3.1) is considerably more complicated.

Division of $(1+\phi)\chi^2[\nu]$ by an independent $\chi^2[s]/s$, yields according to (3.1) to following mixing identity for F-distributions,

(3.7) $$(1+\phi)\,\nu\, F[\nu,s] \sim (\nu+2M)\, F\,[\nu+2M,s],$$

for every $\nu = 1, 2, \ldots$ and $s = 1, 2, \ldots$; where M has the density (3.2). Since

$$E\left\{F^2[\nu,s]\right\} = \frac{(\nu+2)\, s^2}{\nu(s-2)(s-4)} \qquad , s>4 ,$$

identity (3.7) implies,

(3.8) $$E_{\nu,\phi}\left\{(\nu+2M)(\nu+2M+2)\right\} = (1+\phi)^2\, \nu(\nu+2) ,$$

for all $\nu = 1, 2, \ldots$ and $0<\phi<\infty$. This is another application of a mixing identity, which can yield the second moment of M. However, expression (3.8) is more useful for the derivation of the expectation of $(\nu+2M-2)^{-1}(\nu+2M-4)^{-1}$, when $\nu>4$. We start, however, with the derivation of a double mixture identity, which yields the expectation of $(\nu+2M-2)^{-1}$, for all $\nu>2$. Dividing $(1+\phi)\chi^2[\nu]$ by an independent $(1+\phi)\chi^2[\nu]$, we obtain,

$$F[\nu,\nu] \sim \frac{\nu+2M_1}{\nu+2M_2} F[\nu+2M_1, \nu+2M_2], \tag{3.9}$$

where M_1 and M_2 are independent negative binomial random variables, identically distributed like (3.2). Hence, for every $\nu>2$, the expectation of (3.9) yields,

$$\frac{\nu}{\nu-2} = E_{\nu,\phi}\left\{\frac{\nu+2M_1}{\nu+2M_2-2}\right\} \qquad , \nu>2,\ 0<\phi<\infty \tag{3.10}$$

Since M_1 and M_2 are independent, we obtain form (3.4) and (3.10),

$$E_{\nu,\phi}\left\{(\nu-2+2M)^{-1}\right\} = \frac{1}{(1+\phi)(\nu-2)} \qquad , \nu>2,\ 0<\phi<\infty \tag{3.11}$$

This result proves that for all $\nu>2$, $\frac{\nu-2}{\nu-2+2M}$ is an unbiased estimator (when ν is known) of $\rho = (1+\phi)^{-1}$, where M is a negative binomial with density,

$$P[M=m] = \binom{-\nu/2}{m}\rho^{\nu/2}(\rho-1)^m \qquad , m = 0, 1, \ldots, 0<\rho<1.$$

To obtain the expectation of $(\nu-2+2M)^{-1}(\nu-4+2M)^{-1}$, for all $\nu>4$, we consider the second moment of $F[\nu,\nu]$. From (3.9) we obtain,

$$E_{\nu,\phi}\left\{\frac{(\nu+2M_1)(\nu+2M_1+2)}{(\nu-2-2M_2)(\nu-4-2M_2)}\right\} = \frac{\nu(\nu+2)}{(\nu-2)(\ -4)} \ . \tag{3.12}$$

The independence of M_1 and M_2, and formula (3.8) imply, for all $\nu>4$, $0<\phi<\infty$,

$$E_{\nu,\phi}\left\{(\nu-2+2M)^{-1}(\nu-4+2M)^{-1}\right\} = \frac{1}{(1+\phi)^2(\nu-2)(\nu-4)} \ . \tag{3.13}$$

Mixing identity (3.9) cannot be applied for the derivation of $E\left\{(1+2M)^{-1}\right\}$ when $\nu=1$. It is interesting to note that straightforward but somewhat novel derivations yield for $\nu=1$,

$$E_\phi\left\{(1+2M)^{-1}\right\} = \frac{1}{\sqrt{\phi}}\sin^{-1}\left(\sqrt{\frac{\phi}{1+\phi}}\right) \qquad , 0<\phi<\infty \ . \tag{3.14}$$

For $\nu=3$ the result is simpler, and is given by (3.11); namely:

$E_{3,\phi}\left\{(1+2M)^{-1}\right\} = (1+\phi)^{-1}$.

We conclude the present section with another example of a useful application of a mixing identity derivable from (1.1). It is well known that if $G_1(\lambda,p)$ and $G_2(\lambda,q)$ are two independent gamma variables, having the same scale parameter $\lambda, 0<\lambda<\infty$, then

$$\frac{G_1(\lambda,p)}{G_1(\lambda,p)+G_2(\lambda,q)} \sim \beta(p,q) \quad ,$$

where $\beta(p,q)$ desingates a beta random variable, with expectation $p/(p+q)$. We thus obtain, from relationship (1.1) the following mixing identity:

(3.15) $$\frac{1}{1+\frac{\nu_2}{\nu_1}\rho\ F[\nu_2,\nu_1]} \sim \beta\left(\frac{1}{2}\nu_1+M,\frac{1}{2}\nu_2\right),\ 0<\rho<1,$$

where M has the negative binomial distribution (1.2). In particular, for $\nu_1=\nu_2$ one obtain,

(3.16) $$\frac{1}{1+\rho\ F[\nu,\nu]} \sim \beta\left(\frac{1}{2}\nu+M,\ \frac{1}{2}\nu\right) \qquad ,\ 0<\rho<1.$$

This relationship (3.16) can be utilized in the following problem; Consider two random samples $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ from normal distributions $\mathcal{N}(\mu,\sigma_1^2)$ and $\mathcal{N}(\mu,\sigma_2^2)$, respectively having a common mean μ, $-\infty<\mu<\infty$, and an unknown variance ratio, $\rho=\sigma_2^2/\sigma_1^2$. A common estimator of μ (see Zacks [5]), is:

(3.17) $$\hat{\mu} = \bar{X}_n \frac{S_2/S_1}{1+S_2/S_1} + \bar{Y}_n \frac{1}{1+S_2/S_1} \quad ,$$

where $\bar{X}_n$ and $\bar{Y}_n$ are the respective sample means, and $S_1 = \sum_{i=1}^{n} (X_i-\bar{X}_n)^2$, $S_2 = \sum_{i=1}^{n} (Y_i-\bar{Y}_n)^2$. The efficiency of this unbiased estimator, relative to that of the best unbiased estimator when ρ is known, is given by:

(3.18) $$\text{eff.}(\hat{\mu}|\rho) = \frac{1}{(1+\rho)\ E\left\{\frac{1+\rho F^2[n-1,n-1]}{(1+\rho F[n-1,n-1])^2}\right\}} \ .$$

The computation of the expectation in the denominator of (3.18) is a very tedious task whenever $n>3$. For $n=3$ one can show that, (3.19) $$E\left\{\frac{1+\rho F^2[2,2]}{(1+\rho F[2,2])^2}\right\} = \frac{2(\rho^2-1) - 4\rho \ln \rho}{(\rho-1)^3} .$$

Very tedious computations yield, for n=5,

(3.20) $$E\left\{\frac{1+\rho F^2[4,4]}{(1+\rho F[4,4])^2}\right\} =$$

$$= \frac{1}{(\rho-1)^5}\left[1-26\rho+26\rho^3-\rho^4-6\rho \ln \rho-36\rho^2 \ln \rho-6\rho^3 \ln \rho\right] .$$

To obtain a closed formula for the expectation in the denominator of (3.18) when $n>5$ is a very difficult task, which becomes impractical as n grows. Numerical integration is always a possible solution. We utilize here the mixing identity (3.16), for $\nu=n-1$, to give this expectation in a power-series form. From (3.16) we obtain, for every $0<\rho<1$,

(3.21) $$E_\rho\left\{(1+\rho F[\nu,\nu])^{-2}\right\} = E_{\nu,\rho}\left\{\beta^2\left(\frac{\nu}{2}+M,\frac{\nu}{2}\right)\right\}.$$

It is easy to verify that,

(3.22) $$E\left\{\beta^2(p,q)\right\} = \frac{(p+1)p}{(p+1+q)(p+q)}$$

Hence, from (3.21) and (3.22), for every $0<\rho<1$,

(3.23) $$E_\rho\left\{\left(1+\rho F[\nu,\nu]\right)^{-2}\right\} = \frac{1}{4}\sum_{m=0}^{\infty}\frac{(\nu+2m)(\nu+2+2m)}{(\nu+m)\ (\nu+1+m)}\binom{-\nu/2}{m}\rho^{\nu/2}(\rho-1)^m.$$

Furthermore, for every $0<\rho<1$,

(3.24) $$E_\rho\left\{\frac{\rho F^2[\nu,\nu]}{(1+\rho F[\nu,\nu])^2}\right\} = \frac{1}{\rho}\ E_{\nu,\rho}\left\{\left(1-\beta\left(\frac{\nu}{2}+M,\ \frac{\nu}{2}\right)\right)^2\right\}.$$

Indeed, $\dfrac{\rho^2F^2[\nu,\nu]}{(1+\rho F[\nu,\nu])^2} \sim \left(1-\dfrac{1}{1+\rho F[\nu,\nu]}\right)^2 \sim \left(1-\beta\left(\dfrac{\nu}{2}+M,\ \dfrac{\nu}{2}\right)\right)^2$.

Thus,

(3.25) $$E_\rho\left\{\frac{\rho F^2[\nu,\nu]}{(1+\rho F[\nu,\nu])^2}\right\} = \frac{1}{\rho}\left[1-E_{\nu,\rho}\ \frac{\nu+2M}{\nu+M} + \frac{1}{4}E_{\nu,\rho}\left\{\frac{(\nu+2M)(\nu+2+2M)}{(\nu+M)(\nu+1+M)}\right\}\right] .$$

Combining (3.23) and 3.25), we obtain, for all $0<\rho<1$,

$$(3.26) \qquad E_\rho\left\{\frac{1+\rho F^2[\nu,\nu]}{(1+\rho F[\nu,\nu])^2}\right\} = \frac{1}{\rho}\left[1 + \frac{\rho+1}{4}\sum_{m=0}^{\infty}\frac{\nu+2m}{\nu+m}\cdot\right.$$

$$\left.\cdot\left(\frac{\nu+2+2m}{\nu+1+m} - \frac{4}{\rho+1}\right)\binom{-\nu/2}{m}\rho^{\nu/2}(\rho-1)^m\right]$$

4. An example from linear models. Consider a Model II in ANOVA, namely,

$$(4.1) \qquad X_i = a + e_i \quad (i = 1, \ldots, n)$$

where $e_i \sim N(0,\sigma^2)$, for all $i = 1, \ldots, n$, independently of a; and $a \sim N(\alpha,B^2)$.

For testing the hypotheses;

$$(4.2) \qquad H_0 : \alpha = 0$$

versus

$$H_1 : \alpha \neq 0 ,$$

consider the test statistic:

$$(4.3) \qquad F = \frac{n\bar{X}^2}{\hat{\sigma}^2} ,$$

where $\bar{X}$ is the sample mean, $\frac{1}{n}\sum_{i=1}^{m} X_i$, and $\hat{\sigma}^2$ is the sample variance, $\frac{1}{n-1}\sum_{i=1}^{m}(X_i - \bar{X})^2$. To exhibit identity (3.2) with $\nu = 1$ and $s = n-1$ consider the distribution of F, under H_0 . The conditional distribution law of $\hat{\sigma}^2$, given a, is

$$\mathcal{L}(\hat{\sigma}^2|a) = \mathcal{L}\left(\frac{\sigma^2}{n-1}\chi^2[n-1]\right) \text{ independently of } a.$$

Hence, $\bar{X}$ and $\hat{\sigma}^2$ are independent. Moreover, $\mathcal{L}(\bar{X}|a) = \mathcal{N}(a,\frac{\sigma^2}{n})$. Hence, $\mathcal{L}(n\bar{X}^2|a) = \mathcal{L}(\sigma^2\chi^2[1;\frac{a^2n}{2\sigma^2}])$. The conditional distribution of F, given a, is therefore like that of the non-central $F[1,n-1;\lambda]$; where $\lambda = \frac{a^2n}{2\sigma^2}$. According to the assumption of the model, $\lambda \sim \frac{\phi}{2}\chi^2[1]$, where $\phi = \frac{n\beta^2}{\sigma^2}$. Thus, since $\chi^2[1,\lambda] \sim \chi^2[1+2M]$, where M is a Poisson random variable with parameter λ, one obtains:

(4.4)
$$P[M = m] = \frac{1}{m!}\int_0^\infty e^{-\lambda}\lambda^m dP\left[\frac{\phi}{2}\chi^2[1] \le \lambda\right]$$
$$= \frac{1}{\sqrt{1+\phi}} \cdot \frac{\binom{2m}{m}}{4^m} \cdot \left(\frac{\phi}{1+\phi}\right) .$$
$$= \frac{(-1)^m}{(1+\phi)^{1/2}} \binom{-1/2}{m} \left(\frac{\phi}{1+\phi}\right)^m .$$

It follows that, under H_0,

(4.5)
$$P[F \le x] = \frac{1}{\sqrt{1+\phi}} \sum_{m=0}^{\infty} \frac{\binom{2m}{m}}{4^m} \cdot \left(\frac{\phi}{1+\phi}\right)^m F\left(\frac{x}{1+2m} \mid 1+2m,\ n-1\right) .$$

On the other hand, since $Var\left\{\bar{X}\right\} = B^2 + \sigma^2/n$, the distribution of $n\bar{X}^2$, under H_0, is like that of $(mB^2 + \sigma^2)\ \chi^2[1] = \sigma^2(1+\phi)\ \chi^2[1]$. Hence,

$F = \frac{n\bar{X}^2}{\hat{\sigma}^2} \sim (1+\phi)\ F[1,\ n-1]$. Therefore,

(4.6)
$$P[F \le x] = F\left(\frac{x}{1+\phi} \mid 1,\ n-1\right) .$$

The comparison of (4.5) with (4.6) yields the mixing identity (3.7) with the special values of $\nu = 1$ and $s = n-1$.

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