

On the distribution of the largest characteristic root of a matrix  
and percentage points\*

by

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1. Introduction. The joint distribution of  $s$  non-null characteristic roots of a matrix in multivariate analysis given by Fisher (1939), Girshick (1939), Hsu (1939) and Roy (1939) can be expressed in the form

$$(1.1) \quad f(\theta_1, \dots, \theta_s) = C(s, m, n) \prod_{i=1}^s \theta_i^m (1-\theta_i)^n \prod_{i>j} (\theta_i - \theta_j) \quad (0 < \theta_1 \leq \dots \leq \theta_s < 1),$$

where

$$(1.2) \quad C(s, m, n) = \frac{\prod_{i=1}^s \Gamma(\frac{1}{2}(2m+2n+s+i+2))}{\prod_{i=1}^s \Gamma(\frac{1}{2}(2m+i+1)) \Gamma(\frac{1}{2}(2n+i+1)) \Gamma(\frac{1}{2}i)}$$

and the parameters  $m$  and  $n$  are defined differently for various situations as described in detail by Pillai (1955, 1957). Now, the c.d.f. of the largest root,  $\theta_s$ , can be presented in the following determinantal form:

$$(1.3) \quad \Pr(\theta_s \leq x) = C(s, m, n) \begin{vmatrix} \int_0^x \theta_s^{m+s-1} (1-\theta_s)^n d\theta_s & \dots & \int_0^x \theta_s^m (1-\theta_s)^n d\theta_s \\ \dots & \dots & \dots \\ \int_0^{\theta_2} \theta_1^{m+s-1} (1-\theta_1)^n d\theta_1 & \dots & \int_0^{\theta_2} \theta_1^m (1-\theta_1)^n d\theta_1 \end{vmatrix}$$

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In order to overcome the difficulty of integrating each of the  $s!$  multiple integrals in the expansion of the determinant in (1.3), Pillai (1954, 1956b) suggested a reduction formula and gave exact expressions for the c.d.f. of the largest root in terms of incomplete beta functions or functions of incomplete beta functions for values of  $s$  from 2 to 10. In addition, Pillai (1954) suggested a method of approximating the cdf of the largest root for computing the upper percentage points which was used to obtain approximations to the cdf and upper percentage points for  $s = 2$  to 7 (Pillai, 1954, 1956a, 1957, 1960, 1964, Pillai and Bantegui, 1959). Further, more recently, a general expression approximating the cdf at the upper end was obtained by Pillai (1965a) for odd values of  $s$  and another for even values of  $s$ , from which the approximation to the c.d.f. worked out earlier for each value of  $s$  from 2 to 7 could be deduced by the substitution of the appropriate value of  $s$ . In addition, these general expressions were used to compute upper 5 and 1 per cent points of the largest root for  $s = 8, 9$  and 10 (1965a) and further for  $s = 11$  to 20 (Pillai 1965b, 1966).

In the present paper, upper 5 and 1 per cent points of the largest root are presented for values of  $m = 0(1)5, 7, 10, 15$ , and  $n = 5(5)30, 40, 60, 80, 100, 130, 160, 200, 300, 500$  and 1000, and  $s = 2$  to 10. Five significant digits are provided in all the tables except those for upper 5% points for  $s = 7$  to 10 which give only four significant digits. The tabulations made earlier for  $s = 2$  to 6 give the values mostly up to four significant digits or less and for  $s = 4$  to 6, values were computed only for  $m = 0(1)4$  (Pillai, 1960). For  $s = 7$  percentage points were not computed earlier for  $m = 15$  and for  $m = 5, 7$  and 10, only four significant digits were provided (Pillai, 1964). The present attempt is to bring the tables to a standard form. Tables for  $s = 8, 9$  and 10 were already available in this

standard form (Pillai, 1965a), but some values which had been extrapolated in the bottom right corner of those tables were now computed exactly. All the entries are believed to be accurate within a unit in the last decimal quoted.

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Table 1. Upper 5% points of the largest root for  $s = 2$ 

$n \backslash m$	0	1	2	3	4	5	7	10	15
5	.56464	.65072	.70630	.74591	.77578	.79921	.83373	.86764	.90110
10	.37371	.45503	.51429	.56055	.59808	.62932	.67864	.73156	.78891
15	.27805	.34773	.40150	.44547	.48256	.51449	.56703	.62661	.69547
20	.22114	.28095	.32866	.36877	.40344	.43394	.48551	.54625	.61976
25	.18350	.23555	.27798	.31433	.34627	.37478	.42394	.48347	.55804
30	.15678	.20273	.24076	.27378	.30314	.32964	.37600	.43331	.50707
40	.12140	.15849	.18982	.21751	.24254	.26546	.30638	.35848	.42823
60	.08362	.11030	.13331	.15405	.17312	.19089	.22332	.26606	.32600
80	.06377	.084565	.10271	.11922	.13456	.14897	.17561	.21139	.26299
100	.051534	.068565	.083526	.097228	.11003	.12213	.14467	.17532	.22031
130	.040016	.053405	.065244	.076151	.086397	.096133	.11441	.13956	.17714
160	.032705	.043735	.053526	.062581	.071119	.079259	.094616	.11591	.14809
200	.026299	.035228	.043184	.050566	.057549	.064225	.076878	.094549	.12152
300	.017654	.023703	.029118	.034166	.038961	.043565	.052341	.064715	.083871
500	.010651	.014327	.017632	.020723	.023669	.026508	.031946	.039674	.051780
1000	.0053478	.0072039	.0088770	.010447	.011947	.013396	.016182	.020165	.026463

Table 2. Upper 1% points of the largest root for  $s = 2$ 

$n \backslash m$	0	1	2	3	4	5	7	10	15
5	.67697	.74461	.78725	.81714	.83943	.85676	.88204	.90661	.93059
10	.47011	.54427	.59713	.63775	.67030	.69715	.73907	.78344	.83088
15	.35727	.42470	.47570	.51680	.55108	.58032	.62792	.68117	.74180
20	.28755	.34730	.39410	.43292	.46612	.49506	.54351	.59981	.66696
25	.24043	.29347	.33599	.37195	.40323	.43093	.47821	.53473	.60452
30	.20652	.25398	.29265	.32582	.35504	.38120	.42652	.48188	.55211
40	.16101	.20001	.23247	.26084	.28626	.30937	.35026	.40173	.46971
60	.11170	.14026	.16458	.18627	.20606	.22438	.25756	.30082	.36075
80	.085502	.10797	.12733	.14479	.16088	.17592	.20350	.24020	.29251
100	.069252	.087754	.10382	.11840	.13193	.14464	.16816	.19985	.24585
130	.053887	.068509	.081293	.092973	.10387	.11417	.13338	.15956	.19832
160	.044101	.056185	.066797	.076531	.085650	.094297	.11051	.13279	.16615
200	.035504	.045315	.053965	.061927	.069411	.076531	.089939	.10851	.13660
300	.023870	.030542	.036454	.041924	.047088	.052022	.061372	.074456	.094533
500	.014420	.018487	.022107	.025468	.028654	.031709	.037528	.045737	.058492
1000	.0072468	.0093051	.011143	.012854	.014481	.016045	.019036	.023283	.029945

Table 3. Upper 5% points of the largest root for  $s = 3$ 

$n \backslash m$	0	1	2	3	4	5	7	10	15
5	.66893	.72918	.76981	.79942	.82208	.84004	.86677	.89335	.91988
10	.47182	.53729	.58624	.62489	.65643	.68279	.72457	.76956	.81849
15	.36199	.42187	.46902	.50786	.54073	.56907	.61575	.66869	.72985
20	.29313	.34649	.38983	.42648	.45822	.48615	.53336	.58889	.65592
25	.24612	.29371	.33316	.36714	.39703	.42371	.46966	.52516	.59444
30	.21204	.25478	.29074	.32210	.35002	.37521	.41920	.47343	.54292
40	.16599	.20129	.23157	.25845	.28276	.30501	.34466	.39498	.46199
60	.11568	.14167	.16445	.18505	.20402	.22168	.25385	.29610	.35504
80	.088753	.10927	.12745	.14407	.15951	.17401	.20078	.23661	.28799
100	.071992	.088924	.10403	.11793	.13092	.14319	.16602	.19696	.24213
130	.056097	.069505	.081546	.092689	.10317	.11312	.13177	.15736	.19539
160	.045951	.057046	.067051	.076346	.085118	.093478	.10923	.13101	.16374
200	.037022	.046041	.054203	.061811	.069015	.075902	.088935	.10709	.13466
300	.024917	.031060	.036646	.041877	.046852	.051629	.060723	.073510	.093220
500	.015065	.018815	.022238	.025456	.028528	.031487	.037149	.045175	.057699
1000	.0075763	.0094754	.011215	.012854	.014424	.015940	.018812	.023005	.029547

Table 4. Upper 1% points of the largest root for  $s = 3$ 

$n \backslash m$	0	1	2	3	4	5	7	10	15
5	.75816	.80398	.83442	.85638	.87305	.88617	.90558	.92473	.94368
10	.55857	.61641	.65903	.69231	.71924	.74159	.77671	.81414	.85442
15	.43751	.49373	.53742	.57304	.60293	.62852	.67033	.71724	.77079
20	.35857	.41036	.45193	.48674	.51665	.54279	.58663	.63764	.69850
25	.30343	.35061	.38928	.42228	.45110	.47666	.52035	.57257	.63699
30	.26286	.30585	.34164	.37258	.39994	.42446	.46699	.51889	.58462
40	.20728	.24344	.27418	.30125	.32557	.34772	.38689	.43614	.50101
60	.14556	.17272	.19632	.21752	.23692	.25490	.28744	.32981	.38834
80	.11213	.13378	.15282	.17011	.18609	.20103	.22844	.26485	.31657
100	.091176	.10915	.12508	.13963	.15318	.16591	.18947	.22117	.26703
130	.071210	.085527	.098295	.11004	.12104	.13144	.15083	.17725	.21618
160	.058416	.070306	.080956	.090794	.10004	.10881	.12526	.14786	.18155
200	.047125	.056820	.065537	.073620	.081239	.088495	.10216	.12107	.14958
300	.031772	.038402	.044395	.049979	.055268	.060327	.069915	.083318	.10383
500	.019236	.023297	.026984	.030432	.033710	.036857	.042855	.051308	.064412
1000	.0096841	.011746	.013624	.015386	.017066	.018684	.021778	.026169	.033042

Table 5. Upper 5% points of the largest root for  $s = 4$ 

n \ m	0	1	2	3	4	5	7	10	15
5	.73875	.78253	.81315	.83596	.85368	.86787	.88924	.91078	.93256
10	.54719	.60044	.64116	.67369	.70041	.72293	.75877	.79761	.84013
15	.43074	.48224	.52350	.55777	.58692	.61213	.65379	.70118	.75608
20	.35430	.40174	.44086	.47417	.50311	.52864	.57187	.62277	.68426
25	.30062	.34388	.38024	.41173	.43953	.46437	.50721	.55896	.62351
30	.26095	.30041	.33406	.36356	.38989	.41367	.45523	.50644	.57197
40	.20635	.23963	.26854	.29432	.31770	.33911	.37726	.42563	.48990
60	.14538	.17044	.19267	.21287	.23150	.24885	.28046	.32191	.37959
80	.11219	.13220	.15015	.16663	.18198	.19640	.22300	.25856	.30942
100	.091326	.10796	.12299	.13687	.14988	.16217	.18502	.21596	.26100
130	.071404	.084672	.096728	.10794	.11850	.12853	.14734	.17311	.21130
160	.058615	.069642	.079704	.089096	.097977	.10645	.12240	.14443	.17747
200	.047315	.056313	.064553	.072271	.079593	.086596	.099852	.11829	.14623
300	.031926	.038086	.043756	.049090	.054174	.059059	.068359	.081421	.10151
500	.019343	.023119	.026608	.029904	.033057	.036096	.041913	.050152	.062981
1000	.0097428	.011662	.013441	.015125	.016741	.018303	.021305	.025582	.032310

Table 6. Upper 1% points of the largest root for  $s = 4$ 

$n \backslash m$	0	1	2	3	4	5	7	10	15
5	.81099	.84364	.86625	.88297	.89588	.90618	.92160	.93705	.95257
10	.62472	.67082	.70571	.73336	.75596	.77484	.80474	.83688	.87174
15	.50160	.54903	.58666	.61768	.64389	.66644	.70344	.74517	.79303
20	.41751	.46272	.49966	.53089	.55787	.58153	.62134	.66779	.72334
25	.35705	.39918	.43430	.46451	.49101	.51459	.55496	.60333	.66306
30	.31168	.35070	.38370	.41246	.43797	.46091	.50074	.54942	.61110
40	.24831	.28186	.31080	.33647	.35961	.38071	.41810	.46513	.52703
60	.17634	.20213	.22486	.24541	.26428	.28179	.31351	.35481	.41180
80	.13666	.15747	.17603	.19299	.20872	.22345	.25048	.28638	.33731
100	.11154	.12895	.14459	.15898	.17241	.18505	.20846	.23996	.28546
130	.087419	.10139	.11402	.12572	.13670	.14710	.16650	.19294	.23184
160	.071874	.083530	.094116	.10396	.11323	.12204	.13858	.16130	.19513
200	.058097	.067640	.076340	.084457	.092131	.099451	.11325	.13234	.16109
300	.039274	.045836	.051851	.057489	.062845	.067977	.077713	.091324	.11213
500	.023831	.027868	.031584	.035081	.038416	.041624	.047742	.056367	.069724
1000	.012017	.014074	.015973	.017767	.019482	.021136	.024304	.028800	.035833

Table 7. Upper 5% points of the largest root for  $s = 5$ 

n \ m	0	1	2	3	4	5	7	10	15
5	.78821	.82102	.84469	.86266	.87682	.88829	.90576	.92359	.94186
10	.60690	.65074	.68489	.71250	.73540	.75475	.78579	.81968	.85705
15	.48830	.53281	.56899	.59930	.62522	.64772	.68505	.72770	.77731
20	.40723	.44948	.48475	.51498	.54137	.56470	.60432	.65109	.70772
25	.34883	.38813	.42154	.45066	.47644	.49953	.53942	.58770	.64798
30	.30490	.34127	.37261	.40026	.42499	.44738	.48655	.53487	.59673
40	.24337	.27465	.30209	.32670	.34905	.36956	.40614	.45252	.51412
60	.17323	.19729	.21883	.23850	.25669	.27364	.30453	.34504	.40133
80	.13442	.15386	.17145	.18767	.20282	.21706	.24334	.27846	.32861
100	.10980	.12608	.14090	.15467	.16759	.17981	.20255	.23331	.27803
130	.086130	.099194	.11117	.12236	.13292	.14297	.16180	.18760	.22577
160	.070851	.081756	.091797	.10121	.11013	.11864	.13469	.15684	.19001
200	.057298	.066228	.074483	.082246	.089627	.096696	.11008	.12869	.15685
300	.038758	.044905	.050613	.056006	.061157	.066112	.075552	.088810	.10918
500	.023531	.027314	.030841	.034188	.037395	.040492	.046422	.054819	.067882
1000	.011871	.013799	.015603	.017319	.018968	.020565	.023635	.028011	.034886

Table 8. Upper 1% points of the largest root for  $s = 5$ 

$n \backslash m$	0	1	2	3	4	5	7	10	15
5	.84775	.87192	.88922	.90229	.91254	.92081	.93336	.94612	.95911
10	.67620	.71362	.74255	.76580	.78497	.80111	.82685	.85477	.88532
15	.55436	.59479	.62742	.65459	.67769	.69767	.73061	.76798	.81109
20	.46770	.50743	.54037	.56845	.59283	.61430	.65053	.69300	.74398
25	.40376	.44155	.47348	.50114	.52552	.54727	.58463	.62951	.68507
30	.35490	.39042	.42085	.44754	.47132	.49275	.53006	.57574	.63373
40	.28543	.31660	.34381	.36807	.39003	.41010	.44571	.49057	.54966
60	.20484	.22933	.25116	.27100	.28928	.30626	.33707	.37723	.43261
80	.15964	.17965	.19768	.21424	.22965	.24409	.27064	.30592	.35594
100	.13076	.14763	.16293	.17708	.19032	.20282	.22597	.25713	.30213
130	.10284	.11646	.12891	.14049	.15139	.16173	.18106	.20740	.24613
160	.084734	.096150	.10663	.11641	.12566	.13447	.15101	.17374	.20758
200	.068623	.078007	.086650	.094754	.10244	.10978	.12363	.14282	.17168
300	.046511	.052998	.059005	.064665	.070056	.075231	.085060	.098809	.11982
500	.028282	.032290	.036017	.039544	.042916	.046164	.052368	.061120	.074670
1000	.014285	.016334	.018245	.020059	.021799	.023480	.026704	.031282	.038443

Table 9. Upper 5% points of the largest root for  $s = 6$ 

n \ m	0	1	2	3	4	5	7	10	15
5	.82466	.84990	.86857	.88301	.89453	.90395	.91846	.93347	.94906
10	.65522	.69172	.72062	.74424	.76395	.78076	.80786	.83768	.87083
15	.53719	.57589	.60773	.63462	.65774	.67789	.71147	.75003	.79511
20	.45355	.49128	.52312	.55059	.57466	.59601	.63237	.67545	.72779
25	.39186	.42762	.45832	.48521	.50911	.53057	.56773	.61280	.66920
30	.34469	.37825	.40741	.43326	.45647	.47751	.51440	.55999	.61842
40	.27755	.30694	.33293	.35634	.37766	.39726	.43225	.47666	.53567
60	.19949	.22256	.24338	.26245	.28012	.29662	.32672	.36620	.42104
80	.15562	.17447	.19164	.20755	.22242	.23642	.26229	.29687	.34621
100	.12755	.14343	.15801	.17159	.18436	.19646	.21899	.24947	.29375
130	.10037	.11320	.12505	.13616	.14667	.15668	.17546	.20119	.23923
160	.082735	.093491	.10347	.11285	.12176	.13028	.14635	.16853	.20172
200	.067029	.075871	.084102	.091871	.099273	.10637	.11982	.13852	.16681
300	.045453	.051568	.057288	.062713	.067906	.072906	.082440	.095835	.11640
500	.027650	.031430	.034979	.038359	.041606	.044743	.050758	.059280	.072529
1000	.013970	.015903	.017723	.019462	.021137	.022760	.025884	.030340	.037336

Table 10. Upper 1% points of the largest root for  $s = 6$ 

$n \backslash m$	0	1	2	3	4	5	7	10	15
5	.87453	.89295	.90650	.91694	.92524	.93201	.94240	.95310	.96417
10	.71732	.74816	.77243	.79217	.80860	.82251	.84488	.86936	.89639
15	.59862	.63342	.66188	.68580	.70627	.72405	.75354	.78719	.82624
20	.51110	.54623	.57571	.60103	.62312	.64264	.67573	.71468	.76164
25	.44498	.47904	.50813	.53349	.55593	.57602	.61063	.65236	.70417
30	.39362	.42608	.45416	.47894	.50110	.52112	.55607	.59899	.65357
40	.31936	.34838	.37395	.39687	.41768	.43674	.47063	.51341	.56982
60	.23147	.25476	.27569	.29480	.31246	.32889	.35876	.39773	.45150
80	.18138	.20062	.21810	.23423	.24927	.26340	.28940	.32399	.37304
100	.14907	.16541	.18035	.19423	.20725	.21955	.24238	.27313	.31754
130	.11762	.13091	.14314	.15457	.16536	.17561	.19479	.22096	.25942
160	.097122	.10830	.11864	.12834	.13754	.14630	.16279	.18547	.21921
200	.078803	.088028	.096591	.10466	.11232	.11966	.13352	.15272	.18162
300	.053547	.059958	.065940	.071601	.077008	.082205	.092090	.10593	.12708
500	.032628	.036606	.040334	.043876	.047272	.050549	.056815	.065664	.079364
1000	.016506	.018546	.020465	.022293	.024051	.025752	.029019	.033663	.040929

Table 11. Upper 5% points of the largest root for  $s = 7$ 

n \ m	0	1	2	3	4	5	7	10	15
5	.8523	.8722	.8872	.8990	.9085	.9163	.9286	.9413	.9548
10	.6950	.7257	.7503	.7707	.7878	.8025	.8263	.8527	.8823
15	.5792	.6130	.6412	.6651	.6858	.7039	.7342	.7692	.8104
20	.4944	.5282	.5570	.5820	.6040	.6236	.6570	.6968	.7453
25	.4305	.4631	.4914	.5162	.5384	.5583	.5930	.6351	.6879
30	.3809	.4119	.4390	.4632	.4850	.5048	.5395	.5825	.6378
40	.3093	.3369	.3615	.3837	.4040	.4227	.4561	.4986	.5552
60	.2244	.2465	.2665	.2850	.3021	.3181	.3474	.3858	.4392
80	.1759	.1942	.2109	.2264	.2410	.2547	.2801	.3141	.3626
100	.1447	.1601	.1744	.1878	.2004	.2123	.2346	.2647	.3085
130	.1142	.1268	.1385	.1495	.1599	.1698	.1885	.2141	.2519
160	.09431	.1049	.1148	.1241	.1330	.1415	.1575	.1797	.2128
200	.07654	.08528	.09347	.1012	.1086	.1157	.1292	.1479	.1763
300	.05203	.05810	.06382	.06926	.07447	.07950	.08910	.1026	.1233
500	.03171	.03548	.03904	.04245	.04572	.04889	.05497	.06359	.07699
1000	.01605	.01798	.01981	.02157	.02326	.02491	.02807	.03260	.03969

Table 12. Upper 1% points of the largest root for  $s = 7$ 

$n \backslash m$	0	1	2	3	4	5	7	10	15
5	.89470	.90908	.91991	.92839	.93521	.94083	.94956	.95867	.96821
10	.75082	.77656	.79714	.81405	.82824	.84033	.85992	.88154	.90563
15	.63628	.66646	.69144	.71260	.73083	.74673	.77323	.80367	.83923
20	.54905	.58029	.60677	.62966	.64973	.66753	.69782	.73365	.77704
25	.48171	.51253	.53909	.56238	.58307	.60165	.63377	.67262	.72105
30	.42859	.45835	.48430	.50732	.52798	.54670	.57946	.61982	.67128
40	.35059	.37767	.40170	.42335	.44306	.46116	.49341	.53419	.58806
60	.25649	.27866	.29872	.31712	.33415	.35003	.37895	.41673	.46890
80	.20204	.22055	.23748	.25317	.26783	.28162	.30705	.34091	.38894
100	.16660	.18243	.19700	.21058	.22336	.23544	.25791	.28820	.33196
130	.13187	.14483	.15684	.16810	.17875	.18889	.20788	.23381	.27195
160	.10911	.12006	.13025	.13985	.14897	.15767	.17406	.19663	.23023
200	.088695	.097764	.10624	.11424	.12187	.12918	.14301	.16220	.19107
300	.060419	.066754	.072702	.078352	.083759	.088964	.098880	.11277	.13402
500	.036892	.040839	.044561	.048111	.051521	.054817	.061129	.070051	.083871
1000	.018692	.020724	.022645	.024483	.026254	.027971	.031272	.035971	.043324

Table 13. Upper 5% points of the largest root for  $s = 8$ 

n \ m	0	1	2	3	4	5	7	10	15
5	0.8739	0.8898	0.9020	0.9118	0.9197	0.9263	0.9367	0.9478	0.9595
10	.7281	.7542	.7754	.7931	.8080	.8209	.8419	.8655	.8921
15	.6156	.6453	.6703	.6917	.7103	.7266	.7541	.7859	.8236
20	.5307	.5611	.5872	.6100	.6302	.6481	.6789	.7157	.7607
25	.4655	.4953	.5213	.5443	.5648	.5834	.6157	.6551	.7047
30	0.4141	0.4428	0.4680	0.4906	0.5110	0.5296	0.5623	0.6029	0.6552
40	.3388	.3648	.3880	.4091	.4284	.4463	.4782	.5188	.5730
60	.2480	.2692	.2885	.3063	.3229	.3384	.3668	.4041	.4560
80	.1954	.2131	.2293	.2445	.2587	.2722	.2971	.3304	.3779
100	.1612	.1763	.1902	.2034	.2157	.2275	.2494	.2792	.3224
130	0.1276	0.1399	0.1515	0.1623	0.1727	0.1825	0.2010	0.2264	0.2639
160	.1056	.1160	.1258	.1351	.1439	.1523	.1683	.1904	.2234
200	.08586	.09449	.1026	.1103	.1177	.1248	.1383	.1570	.1853
300	.05850	.06453	.07023	.07567	.08090	.08594	.09557	.1091	.1299
500	.03573	.03949	.04305	.04647	.04976	.05295	.05908	.06777	.08130
1000	.01811	.02004	.02188	.02365	.02536	.02702	.03022	.03480	.04198

Table 14. Upper 1% points of the largest root for  $s = 8$ 

$n \backslash m$	0	1	2	3	4	5	7	10	15
5	0.91031	0.92176	0.93055	0.93754	0.94323	0.94796	0.95537	0.96320	0.97152
10	.77855	.80027	.81787	.83248	.84482	.85541	.87267	.89189	.91349
15	.66867	.69502	.71708	.73589	.75218	.76647	.79039	.81803	.85053
20	.58250	.61043	.63429	.65505	.67334	.68961	.71741	.75045	.79066
25	.51467	.54266	.56696	.58839	.60751	.62472	.65456	.69080	.73614
30	0.46038	0.48773	0.51176	0.53317	0.55245	0.56997	0.60071	0.63869	0.68727
40	.37948	.40480	.42741	.44786	.46653	.48371	.51438	.55327	.60475
60	.28011	.30123	.32046	.33814	.35456	.36990	.39787	.43447	.48507
80	.22175	.23958	.25597	.27120	.28547	.29892	.32375	.35686	.40386
100	.18344	.19878	.21298	.22626	.23878	.25064	.27271	.30251	.34559
130	0.14565	0.15829	0.17007	0.18115	0.19165	0.20166	0.22043	0.24609	0.28386
160	.12076	.13149	.14152	.15100	.16002	.16865	.18492	.20734	.24074
200	.098336	.10725	.11562	.12356	.13114	.13841	.15219	.17132	.20013
300	.067151	.073412	.079320	.084948	.090347	.095551	.10548	.11940	.14070
500	.041086	.045003	.048715	.052266	.055685	.058994	.065337	.074316	.088230
1000	.020850	.022872	.024794	.026638	.028420	.030148	.033477	.038221	.045650

Table 15. Upper 5% points of the largest root for  $s = 9$ 

n \ m	0	1	2	3	4	5	7	10	15
5	0.8910	0.9039	0.9141	0.9222	0.9289	0.9346	0.9435	0.9531	0.9635
10	.7560	.7784	.7968	.8122	.8253	.8367	.8554	.8765	.9006
15	.6473	.6736	.6959	.7151	.7318	.7466	.7716	.8007	.8353
20	.5631	.5906	.6143	.6351	.6536	.6701	.6985	.7326	.7745
25	.4972	.5245	.5485	.5697	.5888	.6061	.6363	.6732	.7198
30	0.4446	0.4712	0.4947	0.5158	0.5350	0.5524	0.5832	0.6216	0.6711
40	.3664	.3908	.4128	.4329	.4512	.4682	.4987	.5376	.5894
60	.2704	.2907	.3093	.3265	.3426	.3576	.3852	.4214	.4718
80	.2141	.2312	.2470	.2618	.2757	.2888	.3132	.3458	.3924
100	.1772	.1918	.2055	.2184	.2306	.2421	.2637	.2930	.3356
130	0.1407	0.1528	0.1641	0.1748	0.1850	0.1947	0.2131	0.2383	0.2755
160	.1166	.1269	.1366	.1457	.1545	.1629	.1787	.2007	.2335
200	.09500	.1035	.1116	.1192	.1266	.1336	.1471	.1658	.1941
300	.06488	.07086	.07654	.08197	.08719	.09224	.1019	.1155	.1364
500	.03970	.04345	.04701	.05043	.05374	.05694	.06311	.07187	.08548
1000	.02015	.02208	.02393	.02571	.02743	.02910	.03233	.03696	.04422

Table 16. Upper 1% points of the largest root for  $s = 9$ 

$n \backslash m$	0	1	2	3	4	5	7	10	15
5	0.92264	0.93192	0.93917	0.94499	0.94979	0.95381	0.96017	0.96696	0.97430
10	.80179	.82030	.83548	.84818	.85899	.86832	.88362	.90079	.92029
15	.69676	.71994	.73951	.75631	.77093	.78381	.80548	.83067	.86048
20	.61221	.63727	.65886	.67774	.69444	.70936	.73493	.76547	.80282
25	.54440	.56992	.59221	.61197	.62965	.64561	.67339	.70724	.74977
30	0.48940	0.51462	0.53691	0.55685	0.57486	0.59126	0.62013	0.65591	0.70182
40	.40632	.43003	.45131	.47064	.48833	.50464	.53383	.57092	.62013
60	.30247	.32262	.34104	.35804	.37386	.38866	.41570	.45114	.50020
80	.24060	.25778	.27365	.28844	.30232	.31542	.33965	.37199	.41796
100	.19966	.21454	.22837	.24134	.25359	.26521	.28688	.31617	.35853
130	0.15901	0.17134	0.18288	0.19377	0.20411	0.21398	0.23252	0.25789	0.29524
160	.13209	.14260	.15248	.16183	.17075	.17929	.19541	.21766	.25082
200	.10775	.11652	.12479	.13265	.14017	.14739	.16110	.18015	.20885
300	.073760	.079948	.085812	.091413	.096795	.10199	.11191	.12584	.14716
500	.045220	.049108	.052807	.056355	.059777	.063093	.069460	.078480	.092470
1000	.022983	.024997	.026918	.028767	.030555	.032293	.035644	.040426	.047919

Table 17. Upper 5% points of the largest root for  $s = 10$ 

$n \backslash m$	0	1	2	3	4	5	7	10	15
5	0.9049	0.9155	0.9240	0.9309	0.9366	0.9414	0.9492	0.9576	0.9672
10	.7798	.7991	.8151	.8287	.8403	.8504	.8671	.8861	.9079
15	.6752	.6986	.7185	.7358	.7510	.7644	.7871	.8138	.8457
20	.5922	.6171	.6387	.6578	.6747	.6900	.7162	.7479	.7870
25	.5261	.5512	.5733	.5930	.6108	.6269	.6551	.6897	.7336
30	0.4726	0.4973	0.5193	0.5391	0.5570	0.5734	0.6025	0.6388	0.6857
40	.3922	.4152	.4360	.4550	.4725	.4887	.5178	.5550	.6047
60	.2918	.3113	.3292	.3458	.3613	.3758	.4025	.4377	.4867
80	.2321	.2487	.2640	.2784	.2920	.3048	.3286	.3605	.4062
100	.1926	.2069	.2203	.2329	.2448	.2562	.2774	.3063	.3482
130	0.1534	0.1653	0.1764	0.1870	0.1970	0.2066	0.2248	0.2497	0.2865
160	.1274	.1375	.1471	.1561	.1648	.1731	.1888	.2107	.2433
200	.1040	.1124	.1204	.1280	.1352	.1423	.1556	.1743	.2025
300	.07117	.07711	.08275	.08816	.09338	.09842	.1081	.1217	.1426
500	.04364	.04736	.05092	.05435	.05766	.06088	.06707	.07588	.08958
1000	.02218	.02411	.02596	.02774	.02947	.03116	.03441	.03908	.04640

Table 18. Upper 1% points of the largest root for  $s = 10$ 

$n \backslash m$	0	1	2	3	4	5	7	10	15
5	0.93258	0.94020	0.94624	0.95116	0.95524	0.95869	0.96419	0.97016	0.97680
10	.82149	.83740	.85058	.86170	.87122	.87948	.89311	.90854	.92620
15	.72134	.74183	.75927	.77433	.78751	.79916	.81886	.84189	.86931
20	.63873	.66133	.68093	.69814	.71344	.72714	.75072	.77900	.81378
25	.57137	.59468	.61519	.63344	.64982	.66465	.69054	.72222	.76217
30	0.51602	0.53932	0.56002	0.57862	0.59547	0.61085	0.63798	0.67172	0.71517
40	.43131	.45356	.47362	.49189	.50867	.52416	.55195	.58733	.63440
60	.32368	.34292	.36058	.37692	.39216	.40644	.43257	.46688	.51445
80	.25867	.27524	.29059	.30495	.31844	.33120	.35482	.38640	.43132
100	.21530	.22974	.24321	.25588	.26786	.27924	.30049	.32925	.37088
130	.17197	.18401	.19532	.20602	.21619	.22592	.24420	.26925	.30617
160	0.14313	0.15344	0.16315	0.17238	0.18119	0.18963	0.20560	0.22765	0.26054
200	.11696	.12559	.13375	.14153	.14898	.15615	.16977	.18871	.21728
300	.080258	.086375	.092191	.097760	.10312	.10830	.11820	.13213	.15345
500	.049301	.053160	.056844	.060386	.063808	.067128	.073509	.082560	.096608
1000	.025096	.027100	.029020	.030871	.032666	.034411	.037780	.042593	.050141

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