On the Expected Value of a Stopped Submartingale

by

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Mimeograph Series #80

July, 1966

* Research supported by National Science Foundation under Contract No. GP-4590

(4) $$E|x_t| = \sum_{k=1}^{\infty} \int_{[t=n_k]} |x_{n_k}| = \sum_{k=1}^{\infty} \int_{D_k} |x_{n_k}| = \infty .$$

The proof of (a) is completed.

(b) The "only if" part if well known. For the "only if" part, note that the condition $\sup E|x_n| < \infty$ implies that $\lim x_n = x_\infty$ a.e., $E|x_\infty| \leq \sup E|x_n| < \infty$ and $E|x_t| < \infty$ for every stepping rule t. Put $y_n = E(x_\infty|\mathcal{F}_n)$. Then $(y_n, \mathcal{F}_n, \infty \geq n \geq 1)$ is a martingale, where $y_\infty = x_\infty$. For $\epsilon > 0$ and $m = 1, 2, \ldots,$ let

(5) $$t = \text{imf } \{n|x_n \leq y_n + \epsilon, n \geq m\} .$$

Obviously, t is a stopping time and $P[t \geq m] = 1$. Since x_∞ is finite a.e. and $\lim y_n = \lim x_\infty$ a.e. , $P[t > \infty] = 1$. Hence (1) holds and since $(y_n, \mathcal{F}_n \; n \geq 1)$ is a closed martingale with the last element t_∞,

$$Ex_n \leq Ex_t \leq Ey_t + \epsilon = Ex_\infty + \epsilon .$$

Therefore $Ex_\infty \geq \sup Ex_n$. Similarly, $Ex_\infty \leq \sup Ex_n$. Hence

(6) $$Ex_\infty = \sup Ex_n .$$

Now we prove that $(x_n, \mathcal{F}_n, \infty \geq n > 1)$ is a submartingale. Put $A_n = [y_n < x_n]$. If $PA_n > 0$, then

$$\int_{A_n} x_\infty = \int_{A_n} y_n < \int_{A_n} x_n - \epsilon$$

for some $\epsilon > 0$. Let $t = n$ on A_n, and off A_n, define

(7) $$t = \inf \{m|y_m < x_m + \epsilon, m > n\} .$$

As before, we can prove that t is a stopping rule and $P[t \geq n] = 1$. From (1), we have

$$Ex_\infty = Ey_t = \sum_{k=n}^{\infty} \int_{[t=k]} y_k < \int_{A_n} x_n - \epsilon + \sum_{k=n+1}^{\infty} \int_{[t=k]} x_k + \epsilon$$

$$= Ex_t \leq \sup Ex_n.$$

which is contradictory to (6). Therefore $PA_n = 0$ and

(8) $$x_n \leq y_n = E(x_\infty | \mathcal{F}_n) \quad \text{a.e.}$$

By a theorem of Doob ([1], p. 325), (6) and (8) imply that x_n's are uniformly integrable. Hence the proof is completed.

The proof of (a) is simpler than that of Dubins and Freedman, and the proof (6) is an adoption of D. Siegmund's approach for martingales.

References

[1] Doob, J. L. (1953). Stochastic Processes. New York, Wiley.

[2] Dubins, L. E. and Freedman, D. A. (1966). On the Expected Value of a Stopped Martingale. Ann. Math. Stat. 37, 000.