On Non-Dissipative Markov Chains

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I. Introduction

A Markov chain is non-dissipative if almost every path function eventually enters (and remains in) the set of positive recurrent states. In section II we give a short discussion of discrete parameter non-dissipative chains including a new proof of the celebrated theorem of Foster and Kendall. Section III contains generalizations to the continuous parameter case.

II. Let $X = \{X_n, n = 0,1,2,\ldots\}$ be a Markov chain in discrete time whose (minimal) state space I consists of the non-negative integers. Each random variable X_n is defined on the probability space (Ω, F, P); ω represents an element of Ω. The transition probabilities are assumed to be stationary, i.e.,

$$P(X_{n+1} = j \mid X_n = i) = p_{ij} \qquad i, j \in I \quad n = 0,1,2,\ldots$$

where

$$p_{ij} \geq 0 \, , \quad \sum_{j=0}^{\infty} p_{ij} = 1 \qquad i, j \in I \, .$$

The n-step transition probabilities are denoted by $p_{ij}^{(n)}$ where $p_{ij}^{(0)} = \delta_{ij}$ and $p_{ij}^{(1)} = p_{ij}$. As usual, the Cesàro limit of $\{p_{ij}^{(n)}\}_{n=0}^{\infty}$ is denoted by π_{ij}, i.e.,

$$\pi_{ij} = \lim_{n \to \infty} \frac{1}{n} \sum_{\nu=0}^{n} p_{ij}^{(\nu)} \qquad i, j \in I \, .$$

We recall that $\pi_{ij} > 0$ if and only if j is accessible from i and j is a positive recurrent state; and that $\sum_{j=0}^{\infty} \pi_{ij} \leq 1$ for every i (see Chung [2], p. 32).

Concerning the limits π_{ij} there are three possibilities, namely:

(a) $\pi_{ij} = 0$ for all i, and j,

(b) $\pi_{ij} > 0$ for some pair i,j, but $\sum_{j=0}^{\infty} \pi_{kj} < 1$ for some k,

(c) $\sum_{j=0}^{\infty} \pi_{ij} = 1$ for all i.

Foster [5] has dubbed these three types of chains dissipative, semi-dissipative, and non-dissipative, respectively. It is clear that a Markov chain is dissipative if and only if all its states are either null recurrent or non recurrent. Every irreducible positive recurrent Markov chain is non-dissipative. In view of the standard theorem on the determining system of a Markov chain (see Chung [2], p. 33) and the equally well known fact that for every i and j $\pi_{ij} = \sum_{k=0}^{\infty} \pi_{ik}\pi_{kj}$; (see Bharucha-Reid [1], p. 32), it follows immediately that a Markov chain is dissipative if and only if the only non-negative convergent solution $\{x_0, x_1, \ldots\}$ to the system of equations

$$x_j = \sum_{i=0}^{\infty} x_i p_{ij}$$

is the zero solution $x = (0,0,\ldots)$. (This is theorem 2 of Foster [5]).

Let D denote the set of positive recurrent states in the chain, and let $f^*(i,D) = P(X_n \in D \text{ for some } n = 0,1,2,\ldots | X_0 = i)$, i.e., $f^*(i,D)$ is the probability when starting at state i of eventual absorption into some positive recurrent class. Chung [2], pp. 35 and 37, has shown that

$f^*(i,D) = \sum_{j=0}^{\infty} \pi_{ij}$. Hence a Markov chain is non-dissipative if and only if $f^*(i,D) = 1$ for ever i , i.e., if and only if almost every path function eventually enters D . We note that if a Markov chain is non-dissipative, then D is non-empty and there are no null recurrent states at all in the chain.

Let $u = \{u(i)\}_{i=0}^{\infty}$ be a (finite valued) funtion on I. u is called a <u>super regular function</u> if

$$u(i) \geq \sum_{j=0}^{\infty} p_{ij}\, u(j) \tag{1}$$

for every i . A non-negative super regular function u is <u>properly divergent</u> if $u(i) \to \infty$ as $i \to \infty$.

<u>Theorem 1</u>: In any Markov chain, if a properly divergent super regular function exists, then

(a) there are no null recurrent states in the chain,

(b) every positive recurrent class is finite, and

(c) the chain is non-dissipative.

<u>Proof</u>: Suppose the set E of all null recurrent states in the chain is non-empty. Then there is a null recurrent class $C \subset E$ and C is infinite. Therefore, $u \to \infty$ on C . But it is well known that a non-negative super regular function on a recurrent class must be identically equal to a constant (see e.g. Karlin [8], p. 142 or Holmes [7], p. 10). This is a contradiction.

Hence E is empty. Similarly, if a positive recurrent class F is infinite, then $u \to \infty$ on F which provides another contradiction.

Let $\{X_{kn}, n=0,1,2,\ldots\}$ be the original chain $\{X_n, n = 0,1,2,\ldots\}$ restricted to start at state k , i.e., $X_{k0} = k$ with probability one (k is arbitrary but fixed). Consider the functional process $\{u(X_{kn}), n=0,1,2,\ldots\}$. We note that

$$\text{(i)} \qquad E[u(X_{kn})] = \sum_{j=0}^{\infty} p_{kj}^{(n)} u(j) \leq u(k) < \infty$$

and

$$\text{(ii)} \qquad E[u(X_{k,n+1}) \mid X_{k0},\ldots,X_{kn}] = E[u(X_{k,n+1}) \mid X_{k\,n}] =$$

$$\sum_{j=0}^{\infty} p_{X_{k\,n}j} \quad u(j) \leq u(X_{k\,n})$$

(The inequality in (i) comes from iterating the defining relation of super regularity; the first equality in (ii) follows from the Markov property and the inequality in (ii) from super regularity). Hence $\{u(X_{k\,n}), n = 0,1,\ldots\}$ is a non-negative super martingale. A convergence theorem of Doob [3], p. 324, tells us that there exists a non-negative random variable v such that $u(X_{k\,n}) \to v$ with probability one as $n \to \infty$, and $0 \leq E(v) < \infty$. We know that there are no null recurrent states in the chain. Let $T = I - D$ be the set of all non-recurrent states. Assume $T \neq \phi$. (If not, then $D = I$ and there is nothing to prove.) We want to prove that $f^*(k,D) = 1$. If $k \in D$ we are done. Assume $k \notin D$ and let $B = \{\omega: X_{kn}(\omega) \in T \text{ for every } n\}$. The subset of B for which $\{X_{kn}(\omega)\}$ is a finite subset of T has probability zero. Hence for almost every $\omega \in B$ $v(\omega) = \lim_{n \to \infty} u[X_{kn}(\omega)]$ equals $+\infty$ by the proper divergence of u . Therefore, B is a null set, since otherwise $E(v) = \infty$. This shows that D is non-empty and that $f^*(k,D) = 1$. But K was arbitrary. The result follows. QED

Theorem 1 was originally proved by Foster [5], p. 81, for the special case $u(i) = i$, and subsequently generalized to its present state by Kendall [9]. The proof herein presented is new. A partial converse to this theorem has been given by Foster [6], p. 588.

It is clear that if a properly divergent super regular function exists, then there is a finite set C of states such that all transitions out of C are of probability zero — we need only let C be the set of all states at which u achieves its minimum, or let C be any positive recurrent class. It does not follow, however, that D is finite. As an example consider the Markov chain in which every state is absorbing and let $u(i) = i$.

<u>Remark</u> on how fast a function can diverge and still be super regular. Let u be a positive super regular function and define

$$q_{ij} = \frac{u(j)}{u(i)} p_{ij} .$$

The matrix of the q_{ij}'s is sub-stochastic. $\sum_{j=0}^{\infty} q_{ij} \leq 1$ implies that $q_{ij} \to 0$ as $j \to \infty$ for every i. Hence $u(i)p_{ij} \to 0$ as $j \to \infty$ for every i, i.e.,

$$u(i) = o\left(\frac{1}{p_{ij}}\right)$$

as $j \to \infty$ for every i. We see that u cannot go to infinity faster than the slowest row probabilities go to zero.

We state another interesting theorem (due to Mauldin [10]) which gives a sufficient condition for a Markov chain to be non-dissipative. Proofs can be found in Mauldon [10] and Holmes [7].

Theorem 2: In any Markov chain, if there exists a super regular function u such that $\liminf_{i \to \infty}[u(i) - \sum_{j=0}^{\infty} p_{ij}\, u(j)] > 0$, then the chain is non-dissipative.

In another vein, Chung [2], p. 37, has shown that a Markov chain is non-dissipative if and only if the series in j

$$\sum_{j}^{\infty} \left(\frac{1}{n} \sum_{\nu=1}^{n} p_{ij}^{(\nu)}\right)$$

converges uniformly with respect to n.

Example 1: A one dimensional random walk on the nonnegative integers with an absorbing state at zero. $p_{00} = 1$. If $i \neq 0$, then $p_{i,i+1} = p > 0$, $p_{i,i} = r, p_{i,i-1} = q > 0$, with $p + r + q = 1$. In this case $D = \{0\}$ and every other state is nonrecurrent. Let $u(i) = i$. Then

$$\sum_{j=0}^{\infty} p_{0j}\, u(j) = u(0) = 0 ,$$

and for $i \neq 0$

$$\sum_{j=0} p_{ij}\, u(j) = q(i-1) + ri + p(i+1) = i+(p-q) = u(i) + (p-q) .$$

Therefore, if $p \leq q$, u is super regular. Clearly u is properly divergent. Hence, by the theorem of Foster and Kendall, we have the well known result that if $p \leq q$ then eventual absorbtion into the zero state is certain, regardless of the initial state.

Example 2: A random walk on the nonnegative integers with an absorbing state at zero. Here $p_{00} = 1$. If $i \neq 0$, then $p_{i,i+1} = p_i > 0$, $p_{i,i} = r_i > 0$, $p_{i,i-1} = q_i > 0$, $p_{ij} = 0$ if $|i-j| > 1$. Here again $D = \{0\}$ and every other state is nonrecurrent. Let $u(i) = i$. Then

$\sum_{j=0}^{\infty} p_{0j}\, u(j) = 0$ and $\sum_{j=0}^{\infty} p_{ij}\, u(j) = u(i) + (p_i - q_i)$ for $i \neq 0$. Hence, if $p_i \leq q_i$ for every i, then we have that $f^*(i,D) = f^*_{i0} = 1$ for every i. In addition

$$w(i) = u(i) - \sum_{j=0}^{\infty} p_{ij}\, u(j) = p_i - q_i .$$

Therefore, if $\lim_{i \to \infty} \inf (q_i - p_i) > 0$, then $f^*_{i0} = 1$ for every i

It is well known (see Feller [4]) that absorption into the origin is certain if and only if the series

$$\sum_{n=1}^{\infty} \frac{q_1 q_2 \cdots q_n}{p_1 p_2 \cdots p_n}$$

diverges. Foster [6] has shown that this condition is necessary and sufficient for the existence of a properly divergent super regular function in this case. (Actually, Foster considers only the case where $r_i = 0$ for every i, but his proof goes through in the more general situation).

Example 3: Now consider a two dimensional random walk, i.e., a Markov chain on the lattice points (x,y) in the plane with integer coordinates. If $(x,y) \neq (0,0)$ we have probability $\frac{1}{4}$ of going to any of the four adjacent states $(x,y-1)$, $(x,y+1)$, $(x-1,y)$ and $(x+1,y)$. The state $(0,0)$ is absorbing. Here $D = \{(0,0)\}$ and every other state is nonrecurrent.

We enumerate the states as follows: $(0,0)$ is state 0. Now number all those states (x,y) with $|x| + |y| = 1$, then those with $|x| + |y| = 2$, etc. Define a function u on these states by

$$u(0,0) = 0$$
$$u(x,0) = |x| + 1 \quad \text{if} \quad x \neq 0$$
$$u(0,y) = |y| + 1 \quad \text{if} \quad y \neq 0$$
$$u(x,y) = |x| + |y| \quad \text{if} \quad x \neq 0$$

This u is regular and properly divergent. Hence eventual absorbtion into the origin is inevitable. (This is not a new result. It follows directly from a theorem of Polya. (See Feller [4])).

Finally, consider a symmetric random walk in three dimensions. Since there are no recurrent states in this case (also from Polya's theorem), it follows that there cannot be any properly divergent super regular functions. In particular, the functions $u(x,y,z)=|x|+|y|+|z|$ and $u(x,y,z) = (x^2+ y^2+ z^2)^{1/2}$ are not super regular.

III. In this section we extend some of the results of section II to the continuous parameter case.

Let $X = \{X_t \;\; t \geq 0\}$ be a Markov chain in continuous time with state space $I = \{0,1,2,\ldots\}$. The transition probability matrix $P(t) = (p_{ij}(t))$ is assumed to be standard and satisfy

$$p_{ij}(t) \geq 0 , \quad \sum_{j=0}^{\infty} p_{ij}(t) = 1 \qquad i,j \in I$$

$$P(t+s) = P(t)\,P(s), \quad P(0+) = I \quad ,$$

for all $s, t \geq 0$. In addition, the states are all assumed to be stable so that

$$0 > p'_{ii}\,(0) = \lim_{t \downarrow 0} \frac{p_{ii}(t)-1}{t} = q_{ii} = -\,q_i > -\,\infty \qquad i \in I$$

$$0 \leq p'_{ij}\,(0) = \lim_{t \downarrow 0} \frac{p_{ij}(t)}{t} = q_{ij} < \infty \qquad i \neq j \in I \quad .$$

We further assume that matrix of the quantities q_{ij} is conservative , $\left(\sum_{j=0}^{\infty} q_{ij} = 0,\; i \in I\right)$, and that the quantities q_{ij} determine the process uniquely. This excludes from consideration those processes which can explode to $+\infty$ in finite time. Various necessary and sufficient conditions for these assumptions to hold have been obtained and can be found, for example, in Chung [2] and Reuter [12].

X will be called <u>non-dissipative</u> if almost every path function eventually enters D, the set of positive recurrent states.

We will prove a continuous time version of the Foster-Kendall theorem. The equations analogous to the defining relation of super regularity (1) in continuous time are

(2) $$\sum_{j=0}^{\infty} q_{ij}\, u(j) \leq 0 \qquad i \in I \quad .$$

It will be shown that if there exists a non-negative solution $u = \{u(0), u(1), \ldots\}$ to (2) such that $u(i) \to \infty$ as $i \to \infty$, then X is non-dissipative.

Every continuous time Markov chain X has an imbedded discrete time Markov chain $Y = \{Y_n, n = 0,1,2,\ldots\}$ whose transitions consist of the successive state changes in the continuous chain whenever they occur. The elements of the transition matrix R for the imbedded chain are

$$r_{ij} = \begin{cases} \dfrac{q_{ij}}{q_i} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases} .$$

If a state is recurrent in the continuous time chain it is also recurrent in the imbedded chain, and vice versa. However, this does not extend to the positiveness of the recurrence. (For examples, see Miller [ ]).

<u>Lemma</u>: If $Q\,u \leq 0$, then u is a super regular function for R .

Proof: If $0 \geq Q\,u$, then

$$0 \geq \frac{1}{q_i} \sum_{j=0}^{\infty} q_{ij}\, u_j = \sum_{\substack{j=0 \\ j\neq i}}^{\infty} \frac{q_{ij}}{q_i}\, u_j + \frac{q_{ii}}{q_i}\, u_i = \sum_{j=0}^{\infty} r_{ij}\, u_j - u_i .$$

Therefore,

$$u_i \geq \sum_{j=0}^{\infty} r_{ij}\, u_j. \quad \text{Q E D} .$$

If u is a non-negative solution to (2) and $u(i) \to \infty$ as $i \to \infty$, it follows from the lemma that u is a properly divergent super regular function for the discrete parameter Markov chain Y, and consequently, that the Foster-Kendall theorem applies to Y.

Let N be the set of null recurrent states for X, and let D be the set of positive recurrent states for X. Further, let η be the set of null recurrent states for Y, and let $\mathcal{D}$ be the set of positive recurrent states for Y.

We now have the following information at our disposal:

$N \cup D = \eta \cup \mathcal{D}$, $\eta = \phi$, each class in $\mathcal{D}$ is finite, and almost every path function of Y eventually enters (and necessarily remains in) $\mathcal{D}$. It follows that almost every path function of X eventually enters (and necessarily remains in) $N \cup D$. If we can demonstrate that $N = \phi$ the proof will be complete.

Assume N is non-empty. Then there exists a communicating class $N' \subset N$ and N' is infinite. If the X process starts at a state $k \in N'$, then (with probability one) as time progresses the X process must pass through every state in N', and indeed must hit each state in N' infinitely often. But k is also an element of $\mathcal{D}$ and is consequently in some closed communicating class $\mathcal{D}' \subset \mathcal{D}$, and $\mathcal{D}'$ is finite. This provides a contradiction, Hence N' is empty. We summarize this discussion as a theorem.

Theorem 3: If $u = \{u_0, u_1, \ldots\}$ is a finite valued non-negative solution to the system of inequalities $\sum_{j=0}^{\infty} q_{ij}\, u_j \leq 0$, $i = 0,1,2,\ldots$, and $\lim_{i\to\infty} u(i) = \infty$, then X is non-dissipative.

Remarks: 1. In this section we have insisted that $q_i > 0$ for every i. This means that there are no absorbing states in the chain (see Chung [3], p. 181). If a state i happens to be absorbing we can modify the Y process by making i absorbing for Y also and the above given analysis will hold. 2. It is possible to show, using some results of Reuter [13], that if u is a non-negative solution to $Q\,u \leq 0$, then $u \geq \Pi\, u$, where Π is the sub-stochastic matrix of the quantities $\pi_{ij} = \lim_{t \to \infty} p_{ij}(t)$. It is clear, however, that the mere existence of a divergent solution to $u \geq \pi\, u$ is not enough to guarantee non-dissipativeness. As an example consider

a Birth and Death process on the non-negative integers with an absorbing state at zero. For such a process the Q matrix is

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \cdots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

and the transition matrix of the imbedded Markov chain Y is

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ q_1 & 0 & p_1 & 0 & 0 & \cdots \\ 0 & p_2 & 0 & Y_2 & 0 & \cdots \\ 0 & 0 & q_3 & 0 & p_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

where $p_i = \dfrac{\lambda_1}{\lambda_1 + \mu_1} = 1 - q_i \quad (i \geq 1)$. [See Karlin [9], p. 189 and p. 202].

In the case of a linear growth Birth and Death process where $\mu_n = n\,\mu$, $\lambda_n = n\,\lambda$, and $\mu < \lambda$, the probability of absorption into state 0 when the initial state is m given by $(\frac{\mu}{\lambda})^m$ which is less than one for every $m \geq 1$ (see Karlin [9], p. 203). For this process the quantities π_{ij} are given by:

$\pi_{oo} = 1, \pi_{oi} = 0$ for $i \geq 1$

$\pi_{io} = P(\text{absorption into state } 0(X_o=i) = (\frac{\mu}{\lambda})^i, i \neq 0$

$\pi_{ij} = 0 \qquad i \neq 0, j \neq 0.$

and there are numerous non-negative divergent solutions to $u \geq \cdot u$, e.g., just take $u_o = 1$ and $u_n = n$ for $n \geq 1$. 3. We conclude with a remark about the relationship between the non-dissipative character of X and that of its discrete skeletons.

For $h > 0$ let $X_h = \{X_{nh}, n = 0,1,2,\ldots\}$ be the discrete skeleton of X at the scale h. X_h is a discrete parameter, aperiodic Markov chain and has one-step transition probabilities $p_{ij}(h)$, and n-step transition probabilities $p_{ij}(nh)$. The classification of states in each X_h is the same as in X.

<u>Theorem 4</u>: If, for some $h > 0$, X_h is non-dissipative, then X is non-dissipative. Conversely, if X is non-dissipative, then so is each $X_h, h > 0$.

<u>Proof:</u> Assume X_h is non-dissipative for some $h > 0$. Let $\Lambda_h = \{\omega\colon X_{nh}(\omega) \in D$ for some $n = 0,1,2,\ldots\}$. $P(\Lambda_h) = 1$. Suppose ω is such that $X_t(\omega) \notin D$ for every $t \geq 0$. Then $X_{nh}(\omega) \notin D$ for $n = 0,1,2,\ldots$, i.e., $\omega \notin \Lambda_h$. Hence $P\{\omega\colon X_t(\omega) \notin$ for every $t \geq 0\} = 0$, and X is non-dissipative. The converse is obvious. QED

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