ON A GENERAL SYSTEM OF DISTRIBUTIONS AND THE DISTRIBUTION OF THE SAMPLE MEDIAN*+

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This paper presents the region of coverage of third and fourth standardized moments (α_3, α_4) of a certain general distribution function, Burr [3]. It is shown to cover most of the regions of the main Pearson Types IV and VI, and an important part of that of main Type I. The density function for medians from this general distribution is available in closed form, and all moments which exist are expressible in terms of gamma functions. Important characteristics of the distribution of the median for samples of $n = 3,5,7,9,11$ are given for α_3 values of 0, .25, ..., 1.50, with a variety of values of α_4 for each. Also populations for which the median is more efficient than the mean are given.

1. INTRODUCTION

The sample median is used directly for some statistical tests and in estimation, in part because of its ease of calculation and because it is not unduly influenced

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by extreme values. Thus the latter property enables one to estimate the central position, λ, for the Cauchy distribution $f(x) = \Pi^{-1}[1 + (x-\lambda)^2]^{-1}$ $(-\infty,\infty)$, with increasing reliability as n increases, whereas the mean has infinite standard error for all n's. Various tests and estimation procedures are given in Brown and Mood [2], David and Johnson [12], McCarthy [20], Sarhan [25], Sarndal [26], Sibuya [27], Walsh [31] and Woodruff [32]. Others are available by making use of knowledge on the distribution of the median for samples from various populations.

There is considerable literature on the distribution of medians, much of it being in the form of the moments. The explicit probability density function of the median is available, however, whenever the distribution function $F(x)$ is given in closed form. Thus letting the sample median be $\tilde{x}$, and using the odd sample size $n = 2m + 1$, we have from, for example, Cramer [11],

$$g(\tilde{x}) = \frac{(2m+1)!}{(m!)^2} [F(\tilde{x})]^m [1-F(\tilde{x})]^m f(\tilde{x}) \quad . \tag{1}$$

Approximation methods, asymptotic results and some exact results are given in Cadwell [5], Chu and Hotelling [8], Clark and Williams [9], Eisenhart, Deming and Martin [13], Hojo [18], Pearson and Adyanthaya [21], Rider [22] and [23], Sibuya [27], Siddiqui [28], Tarter [29] and Teichroew [30]. Bounds for moments of the median are given in Ali and Chan [1], Chu [7], Ludwig [19] and Rosenblatt-Roth [24] and upper and lower bounds for the distribution function of the median are given in Chu [6]. Probability points for some distributions of the median are presented in Eisenhart, Deming and Martin [14], Gupta [15] and Gupta and Shah [16]. The most general distribution so far considered, Gupta [15], is the gamma or Pearson Type III distribution. In it are given moments and covariances for order statistics as well as probability points for several values of the shape parameter.

Much of the literature considers rather specialized populations, such as, the Cauchy, logistic, exponential, rectangular, Laplace (double exponential), and parabolic. Or else restrictions such as symmetrical $f(x)$ are made. Letting

$$E[(x-\mu)^3] / \sigma^3 = \alpha_3 \qquad \alpha_3^2 = \beta_1 \tag{2}$$

$$E[(x-\mu)^4] / \sigma^4 = \alpha_4 = \beta_2 , \tag{3}$$

it is seen that the foregoing distributions, along with the normal curve occupy but single points on the (α_3,α_4)—plane. The gamma distribution is more general since its possible (α_3,α_4) points are along a curve whose equation is

$$\alpha_4 = 3 + 1.5\,\alpha_3^2 \quad . \tag{4}$$

The present paper provides moments for medians for a population which occupies an important <u>region</u> in the (α_3,α_4) plane. The moments of all orders, whenever they are finite, are linear combinations of beta functions. Sample sizes of $n = 3,5,7,9$ and 11 were used. The values of mean, standard deviation, α_3,α_4 and efficiency relative to $\bar{x}$, were found for the distribution of the median for a family of curves each with $\alpha_3 = -.50$ and various α_4's, another family for $\alpha_3 = -.25$ etc. through $\alpha_3 = 1.50$.

2. THE GENERAL DISTRIBUTION

The population here considered was first developed by Burr [3]. Its distribution function is

$$\begin{aligned} F(x) &= 1 - (1+x^c)^{-k} \qquad x \geq 0 \qquad c,k > 0 \\ &= 0 \qquad\qquad\qquad\quad\; x < 0 \;. \end{aligned} \tag{5}$$

Moments around the origin, from $f(x) = F'(x)$ are

$$\mu_i' = E(x^i) = \int_0^\infty x^i\, ckx^{c-1}\,(1+x^c)^{-k-1}\,dx \quad . \tag{6}$$

Using the transformation $v = 1/(1+x^c)$ one obtains

$$\mu_i' = k \int_0^1 v^{k-i/c-1} (1-v)^{\frac{i}{c}} \, dv \tag{7}$$

$$= k \, B(k-i/c, \, i/c + 1)$$

from which the central moments μ_2, μ_3, μ_4 and α_3, α_4 are obtained. For α_4 to exist it is sufficient that $ck > 4$.

Recent calculations with an IBM 7094 and a Honeywell H-200 show that the approximate region covered by (5) lies between the upper and lower bounds shown in Fig. 1. In line with the description of the Pearson system of frequency curves given by Craig [10], the axes are taken as $\alpha_3^2 = \beta_1$ and

$$\delta = (2\alpha_4 - 3\alpha_3^2 - 6) \,/\, (\alpha_4 + 3) \,, \tag{8}$$

which proves more convenient than use of α_3 and α_4. The regions covered by the three main Pearson types I, IV and VI are shown in Fig. 1, along with the various transitional types such as III, that is, the gamma distribution, which lie along curves. Further, some distributions like the normal, rectangular, logistic and exponential are shown as points. The subscript B refers to a bell shaped function, and J to one which J shaped. The Weibull occupies the dotted curve which becomes the lower bound curve, since as k becomes infinite in (5), $F(x)$ approaches the Weibull as a limit.

It is seen that the distribution (5) covers a very important and large portion of (α_3^2, δ) space. It even goes beyond $\delta = +.4$. This was the region which K. Pearson named "heterotypic" because all Pearson system curves beyond this point have infinite eighth moments and hence infinite σ_{α_4}. Many cases of (5) lying beyond $\delta = +.4$ have $ck > 8$ and hence finite σ_{α_4} .

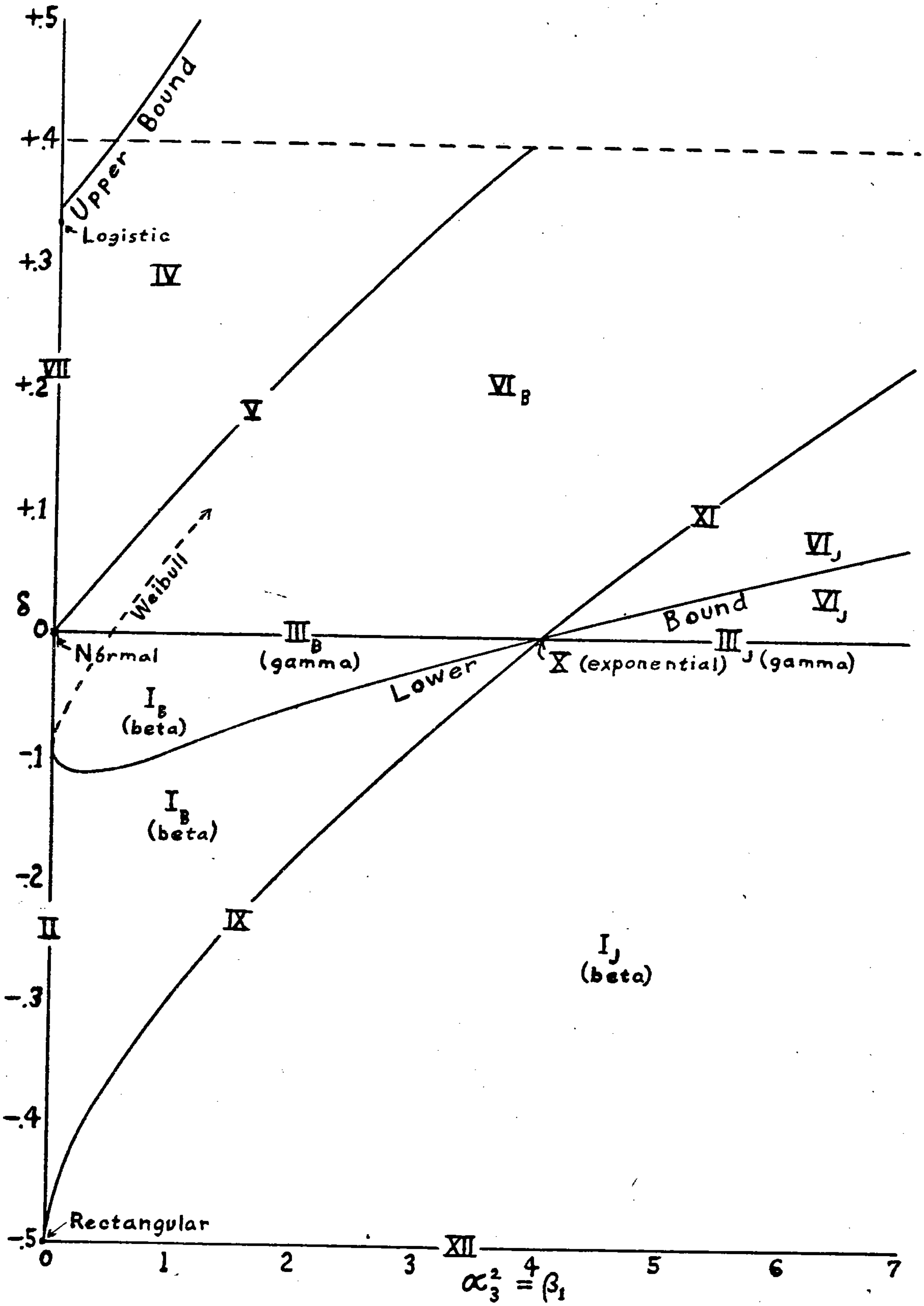

Fig. 1. Upper and lower bounds of coverage in α_3^2, δ space for general system of distributions (5), together with regions, curves and points for Pearson system and other distributions.

The c.d.f. (5) appears to have but two parameters c and k, but there are in reality four parameters. Given values of c and k in (5), these determine μ, σ, α_3, α_4. Then to obtain a distribution having these values of α_3 and α_4 but with mean and standard deviation ν and τ, one uses

$$\frac{x-\mu}{\sigma} = \frac{y-\nu}{\tau} \quad . \tag{9}$$

This may be solved for x in terms of y, and substituted into (5) to provide a distribution function having moments ν, τ, α_3 and α_4.

Finally it may be worth emphasizing that probabilities are obtainable from (5) without integrating, by

$$\begin{aligned} P(a < x < b) &= F(b) - F(a) \\ &= (1+a^c)^{-k} - (1+b^c)^{-k} \quad . \end{aligned} \tag{10}$$

3. THE SAMPLING DISTRIBUTION OF THE MEDIAN

Given the odd sample size $n = 2m+1$ we may substitute (5) into (1), obtaining the density function of the median explicitly:

$$g(\tilde{x}) = \frac{(2m+1)!}{(m!)^2} \left[1 - (1+\tilde{x}^c)^{-k}\right]^m (1+\tilde{x}^c)^{-mk} \frac{ck\tilde{x}^{c-1}}{(1+\tilde{x}^c)^{k+1}} \quad . \tag{11}$$

For moments about the origin, by definition

$$\mu_i' = \int_0^\infty \tilde{x}^i \, g(\tilde{x}) \, d\,\tilde{x} \quad . \tag{12}$$

Substituting (11) into (12), and then letting $u = (1+x^c)^{-1}$, one obtains

$$\mu_i' = \frac{(2m+1)!}{(m!)^2} \; k \int_0^1 (1-u^k)^m \, u^{km+k-i/c-1} \, (1-u)^{i/c} \, du \quad .$$

Since $(1-u^k)^m = \sum_{j=0}^{m} \binom{m}{j} (-u^k)^j$

$$\mu_i' = \frac{(2m+1)!}{(m!)^2} \, k \sum_{j=0}^{m} \binom{m}{j} \int_0^1 (-1)^j \, u^{km+k+kj-i/c-1} \, (1-u)^{i/c} \, du \tag{13}$$

$$= \frac{(2m+1)!k}{(m!)^2} \sum_{j=0}^{m} \binom{m}{j} (-1)^j \, B[k(m+j+1) - i/c, \; i/c+1] \tag{14}$$

since the beta function, $B(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1} \, du$. One can then use the well-known relation for $B(p,q) = \Gamma(p)\,\Gamma(q) \,/\, \Gamma(p+q)$ and have a sum of terms involving gamma functions. Thus we have, in this convenient form, moments about the origin, of all orders which exist.

In order to exist, it is necessary and sufficient that

$$k(m+j+1) - i/c > 0 \quad \text{for} \quad j = 0, \ldots, m$$

$$\text{or} \qquad k\,c(m+1) > i \quad .$$

With any positive k and c for (5), moments of all orders exist for sufficiently large sample size $2m+1$.

For central moments one uses

$$\begin{aligned}
\mu_2 &= \mu_2' - (\mu_1')^2 \\
\mu_3 &= \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 \\
\mu_4 &= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4
\end{aligned}$$

and then (2) and (3) for the shape parameters.

4. CHARACTERISTICS OF THE DISTRIBUTION OF THE SAMPLE MEAN

The values of the mean, standard deviation, α_3 and α_4, and the efficiency of the median as compared to that of the sample mean, are of general interest. In particular the effect of non-normality of the population on these characteristics is of importance.

A program was written for a computer (Honeywell H-200) which will take a given k value in (5) and a given value of α_3 and find c to give this value of α_3. Then it also calculates μ, σ and α_4 for the population, and finally from (12), the desired characteristics of the median for various sample sizes n, including efficiency defined by

$$\text{Eff. (median vs. mean)} = \sigma^2_{\bar{x}} \; / \; \sigma^2_{\tilde{x}} \quad . \tag{15}$$

The last is only relative efficiency, since it is not known what estimate of μ is most efficient in various cases of distribution (5). However, for such populations with skewness up to 1.5, the mean is always more efficient than the median.

4.1 EFFICIENCY, $E(\tilde{x})$ AND BIAS

Further discussion of (15) is, however, required. $\sigma_{\bar{x}}$ is, by definition, $\sigma/\sqrt{n}$ (for any population with finite σ). It is the root-mean-square deviation of $\bar{x}$'s around their mean μ. Likewise, $\sigma_{\tilde{x}}$ is the RMS deviation of $\tilde{x}$'s around their mean. This mean of $\tilde{x}$'s is, however, neither the population mean μ, nor is it the population median ξ. It can readily be shown that the median of the population of medians is exactly the population median ξ, although this fact appears to be not widely quoted.* In all the special cases of (5) which were studied the mean of the sample medians lies between μ and ξ and rapidly approaches ξ as n increases. For example, for a population (5) with $k = 3$ and $1/c = .4383$, for which $\alpha_3 = 1.500$, $\alpha_4 = 8.534$, $\sigma = .3552$, $\mu = .6154$, $\xi = .5540$, the means of the sample medians are .5795, .5699, .5655, .5630 and .5614, respectively, for $n = 3,5,7,9,11$. This approach is of course in line with the well known fact that the sample medians are asymptotically normal with mean of ξ, Cramer [11].

* A simple proof of this statement follows: The population median ξ is such that $\int_{-\infty}^{\xi} f(x)\,dx = F(\xi) = .5$. Then for the distribution of sample medians we need to show $\int_{-\infty}^{\xi} g(\tilde{x})\,d\tilde{x} = .5$. Since $\int_{-\infty}^{\infty} g(\tilde{x})\,d\tilde{x} = 1$, it will be sufficient to show that $\int_{-\infty}^{\xi} g(\tilde{x})\,d\tilde{x}$ and $\int_{\xi}^{\infty} g(\tilde{x})\,d\tilde{x}$ are equal. Substituting $y = F(\tilde{x})$ into (1), these integrals become $K\int_{0}^{.5} y^m(1-y)^m dy$ and $K\int_{.5}^{1} y^m(1-y)^m dy$. But the latter becomes identical to the former if one lets $y = 1-y'$.

Thus, in summary, $\sigma_{\bar{x}}$ measures the relative closeness of $\bar{x}$'s to what they estimate, that is, μ. Likewise the medians $\tilde{x}$ are unbiased estimates of $E(\tilde{x})$, which is not a very interesting characteristic of the population. On the other hand $\tilde{x}$'s are slightly biased estimates of population median ξ. Thus it would be perhaps fairer to take $E(\tilde{x}-\xi)^2$ instead of $\sigma_{\tilde{x}}^2$ in (15) for efficiency. Now $E(\tilde{x}-\xi)^2 = \sigma_{\tilde{x}}^2 + (\nu-\xi)^2$ where ν is the mean of the sample medians. However, in none of the populations considered, was $(\nu-\xi)^2$ more than $.0259\,\sigma_{\tilde{x}}^2$, and in most cases it was far less. Hence either definition of efficiency may well be used and thus we use the simpler (15).

4.2 EFFICIENCY OF MEDIAN

For the general distribution (5), characteristics for the median were calculated for families of distributions as shown in Table 1.

TABLE 1. FAMILIES OF DISTRIBUTIONS OF

$F(x) = 1 - (1+x^c)^{-k}$, USED FOR MEDIAN

Line	Skewness α_3	Range of k	Range of 1/c	Range of α_4
1	- .50	4 - 10	.0648 - .1042	3.692 - 3.423
2	- .25	2 - 10	.0508 - .1645	3.753 - 3.038
3	.00	2 - 10	.0925 - .2316	3.637 - 2.886
4	+ .25	2 - 10	.1353 - .3042	3.800 - 2.957
5	.50	1 - 10	.0562 - .3804	4.828 - 3.246
6	.75	1 - 10	.0821 - .4582	5.648 - 3.756
7	1.00	1 - 10	.1058 - .5359	6.858 - 4.493
8	1.25	1 - 10	.1270 - .6115	8.536 - 5.466
9	+1.50	1 - 10	.1459 - .6841	10.789 - 6.689

As seen in Table 1, there are combinations of c and k for (5) which give the same size of α_3 but opposite signs, and quite similar α_4's. The family in line 1, with skewness -.50 covers only a small range of α_4's, while that in line 2 with α_3 of -.25 are quite similar to those in line 4, but a smaller range of α_4. Consequently we shall give attention only to the families with skewness of 0 to 1.5.

The following may be stated from calculated results on the distribution of the median (n = 3 to 11) for the latter families of distributions and others not here tabulated:

(a). The relative efficiency of the median for given population α_3 and α_4 <u>decreases</u> as the sample size n <u>increases</u>, for the lower population skewnesses. But for $\alpha_3 = 1.50$ and above and high α_4, the relative efficiency <u>increases</u> as n <u>increases</u>.

(b). For fixed population α_3, the relative efficiency of the median increases as α_4 increases. Moreover in the given families of populations the efficiencies for $n = 3$ to 11 tend to approach about the same limit for the largest available α_4.

(c). Within these families of populations α_4 seems to have greater influence on efficiency than α_3, perhaps because of the strong relation of α_4 to $\sigma_{\bar{x}}$. In fact for fixed α_4, efficiencies even tend to increase for lowered α_3.

Table 2 shows some numerical results on the relative efficiency of the median.

TABLE 2. EFFICIENCIES OF MEDIAN RELATIVE TO MEAN FOR SOME FAMILIES FROM TABLE 1

Sample	Population $\alpha_3 = .00$		Population $\alpha_3 = .50$	
n	$\alpha_4 = 2.89$	$\alpha_4 = 3.64$	$\alpha_4 = 3.25$	$\alpha_4 = 4.83$
3	.732	.802	.733	.863
5	.684	.772	.684	.849
7	.665	.760	.665	.845
9	.655	.755	.654	.843
11	.648	.751	.647	.842
	Population $\alpha_3 = 1.00$		Population $\alpha_3 = 1.50$	
	$\alpha_4 = 4.49$	$\alpha_4 = 6.86$	$\alpha_4 = 6.69$	$\alpha_4 = 10.79$
3	.771	.899	.837	.947
5	.729	.891	.813	.948
7	.713	.889	.805	.951
9	.704	.889	.800	.953
11	.698	.889	.797	.954

It is also of some interest to know at about what values of α_3 and α_4 the median begins to be more efficient than the mean. Table 3 shows some populations for which the median is marginally more efficient than the mean. The transition seems to take place at α_3 about 2.4 and α_4 about 11. Although these moments are large, all of the populations in Table 3 are bell-shaped since $c > 1$, thus giving a mode with zero slope for $f(x)$, and with $f(x) = 0$ at $x = 0$. For higher α_3 and α_4 the median becomes progressively more efficient. It can of course become infinitely more efficient as for example for the Cauchy distribution.

TABLE 3. POPULATIONS (5) FOR WHICH MEDIAN IS MARGINALLY MORE EFFICIENT THAN MEAN

Population				Relative Efficiency of Mean	
k	$1/c$	α_3	α_4	$n = 3$	$n = 11$
1	.20	2.49	10.69	1.048	1.094
4	.70	2.65	20.87	1.030	1.083
6	.80	2.44	15.19	1.005	1.061
9	.90	2.44	13.99	1.011	1.088

4.3 DISTRIBUTION OF THE MEDIAN

The four general characteristics of the distribution of the median will now be discussed. As medians $E(\tilde{x})$ is different from the population median ξ. Letting the bias be defined by

$$E(\tilde{x}) = \xi + B(n) \tag{16}$$

it appears from the available data that to a first approximation $B(n)$ varies inversely as n. Meanwhile, as might be expected, the standard deviation $\sigma_{\tilde{x}}$, to a first approximation, varies inversely as $\sqrt{n}$, much as does $\sigma_{\bar{x}}$.

As is to be expected from the asymptotic normality of medians about ξ, α_3 and α_4 approach the respective normal limits of 0 and 3, rather rapidly in fact. We summarize in Table 4 these characteristics of the median, for the populations used in Table 2. These are only 8 of the 252 populations for which the characteristics of the median were evaluated for $n = 3$ to 11.

As seen in Table 4, the bias $B(n)$ rapidly approaches zero as n increases, while $\sigma_{\tilde{x}}$ less rapidly decreases. It is also to be noted that for the same α_3, $\sigma_{\tilde{x}} / \sigma_x$ is somewhat lower for higher α_4. This is because the median is less influenced by extreme deviations than is σ_x of the population.

Skewnesses α_3 of the median are in general much less than those of the population and steadily approach 0 as n increases. Likewise α_4 of the median is in general closer to 3 than is that of the population and approaches 3.

A closer examination of Table 4 reveals a behavior for curve shape characteristics of the median which is not analogous to that of the mean for which

$$\alpha_{3:\bar{x}} = \alpha_{3:x} / \sqrt{n} \qquad \alpha_{4:\bar{x}} = 3 + (\alpha_{4:x} - 3) / n \ .$$

For example take the two cases for each of which $\alpha_3 = 1.00$. One might expect that for the population with higher α_4, α_3 for the medians would be higher. It is in fact lower. Thus the distribution of medians tends more toward normality for the less normal population. Meanwhile the α_4 values go up only slightly for the population with the higher α_4 . For the two populations with $\alpha_3 = 1.50$, both α_3 and α_4 for the medians are more toward normality for the population with $\alpha_4 = 10.79$ than for that with $\alpha_4 = 6.69$.

TABLE 4. CHARACTERISTICS OF THE DISTRIBUTION OF THE MEDIAN FOR POPULATIONS (5), WITH GIVEN α_3 VALUES AND CONTRASTING α_4 VALUES

Population					Distribution of Median				
k	1/c	α_3	α_4	ξ	n	$E(\tilde{x})$	$\sigma_{\tilde{x}}/\sigma_x$	α_3	α_4
10	.2316	.00	2.89	.5433	3	.5424	.675	-.039	2.99
					5	.5426	.541	-.042	3.01
	μ = .5420				7	.5428	.464	-.040	3.01
					11	.5430	.374	-.036	3.01
2	.0925	.00	3.64	.9217	3	.9209	.645	-.050	3.34
					5	.9211	.509	-.050	3.23
	μ = .9204				7	.9213	.433	-.047	3.17
					11	.9214	.348	-.040	3.11
10	.3804	.50	3.25	.3671	3	.3726	.674	.315	3.10
					5	.3706	.541	.244	3.06
	μ = .3802				7	.3696	.464	.205	3.05
					11	.3687	.375	.162	3.03
1	.0562	.50	4.83	1.0000	3	1.0020	.621	.250	3.73
					5	1.0012	.485	.179	3.45
	μ = 1.0052				7	1.0009	.411	.145	3.32
					11	1.0006	.329	.111	3.20
10	.5359	1.00	4.49	.2438	3	.2553	.658	.672	3.66
					5	.2511	.524	.534	3.42
	μ = .2698				7	.2491	.448	.456	3.30
					11	.2472	.361	.366	3.19
1	.1058	1.00	6.86	1.0000	3	1.0072	.609	.477	4.10
					5	1.0044	.474	.339	3.62
	μ = 1.0186				7	1.0032	.401	.274	3.43
					11	1.0020	.320	.209	3.26
10	.6841	1.50	6.69	.1650	3	.1802	.631	1.015	4.64
					5	.1747	.496	.812	4.04
	μ = .1991				7	.1721	.421	.695	3.76
					11	.1696	.338	.560	3.49
1	.1459	1.50	10.79	1.0000	3	1.0138	.593	.670	4.62
					5	1.0084	.459	.472	3.85
	μ = 1.0359				7	1.0061	.388	.380	3.57
					11	1.0039	.309	.289	3.34

The explanation for this behavior is not hard to find. The median, being equal as it is to the midmost sample value, is governed more by the center of the population distribution than by the tails. But it is the latter which largely determine the population α_3 and α_4. In one instance the authors drew two graphs of standardized $f(x)$'s with the same α_3 but different α_4's. The distribution of (5) with greater α_4 had a more symmetrical central portion than that with lesser α_4, hence giving less skewness to the medians. It would be interesting to know how often such a condition exists with other general populations.

5. SUMMARY

A certain general system of distributions, Burr [3], and its coverage of α_3, α_4 space was found to contain much of the region covered by the Pearson system of frequency curves.

It is possible to find for (5) the density function of the median explicitly and all existing moments as linear combinations of beta functions. The behavior of medians from this general system of populations was studied. Exact results show the effect of departures from normality as to efficiency relative to the mean, bias in estimating population median, and curve shape of the distribution of medians.

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