ON THE DISTRIBUTION OF RANGE FOR A GENERAL SYSTEM OF DISTRIBUTIONS*

by

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This paper gives exact results on the distribution of the sample range for $n = 2$ to 10, for a general system of distributions. The system has α_3, α_4 shape-parameter combinations covering most of the regions for main Pearson Types IV and VI and an important part of that for the other main Type I. Exact results for the distribution of the range were obtained for the following characteristics: standardized mean and standard deviation, curve shape parameters α_3, α_4, and coefficient of variation, for 81 populations. Thus the effect of non-normality on the range distribution for small samples is quite thoroughly covered.

1. INTRODUCTION

Since the early paper by Tippett [45], interest in the distribution of sample ranges and associated applications to hypothesis testing and estimation has steadily grown, perhaps reaching a climax around 1955. It is now possible to substitute the range in small samples or the average range in larger samples for the sample standard deviation and perform most of the standard tests and interval estimates. The loss of efficiency has been figured in most cases, being surprisingly small in some.

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The distribution of the range and mean range for samples from a normal population has been well covered, for example, Harter and Clemm [22], Pearson [34], Pearson and Hartley [36,37], Resnikoff [40] and Tippett [45]. The distribution of the range for correlated normal variables is given by Gupta, Pillai and Steck [20]. Distributions for approximating the distribution of the range from a normal population appear in Cadwell [7], Cox [10], Patnaik [33], Pearson [35] and Thompson [44].

The calculation of the distribution of the range for the normal population is quite formidable. Faced with such difficulty which is often greater for other populations, research workers have (a) gone to very small samples, for example, Bland, Gilbert, Kapadia and Owen [4] and Shone [42]; or (b) sought asymptotic results: Cadwell [8], Cox [9], Elving [15], Gumbel [18,19] often making restrictive assumptions such as symmetry of $f(x)$ or all moments finite; or (c) taken rather specialized populations: Belz and Hooke [3], Cox [11], David [12], Hyrenius [25]. Range distributions for discrete populations were considered by Burr [6] and Leti [28]. An extensive experimental approach was used by Niemann [32] in which 4000 samples of four each were drawn from a gamma distribution of skewness 1.15. Some inequalities for moments or probabilities making few restrictions on $f(x)$ have also been developed: Barnard [2], Hartley and David [23], Masuyama [29], Moriguti [31], Plackett [38] and Winsten [46].

Applications to point estimates of σ or σ^2 are given in David [13], Davies and Pearson [14], Godwin [16], Grubbs and Weaver [17] and King [26]. Applications to interval estimates of parameters and to tolerance limits appear in, for example, Banerjee [1], Harter [21], Klerk-Grobben and Sandberg [27], Mitra [30], Resnikoff [39] and Terpstra [43]. Rather than give references to the many interesting tests of hypotheses involving the range, it seems best to mention only a recent book by Sarhan and Greenberg [41] in which many references may be found.

Previous papers have not shed much light on the effects of ordinary departures from normality, being largely confined to specialized non-normal curves, such as, the exponential, rectangular, logistic, and log χ^2. Cox [11] comes the closest to generality, since he uses two normal distributions to obtain desired α_3, α_4 combinations, but the two normal distributions are taken to be so far apart as to virtually not overlap, thus giving a bimodal population.

Using a general system of distributions, Burr [5] and Hatke [24], it was possible to calculate the first four moments of the distribution of the range for 64 non-normal populations, using sample sizes from $n = 2$ to 10. This sketches out a picture of the effect of non-normality on the distribution of the range.

2. THE SYSTEM OF DISTRIBUTIONS

The simplest form of this system is the distribution function $F(x)$ given by Burr [5]:

$$\begin{aligned} F(x) &= 1 - (1+x^c)^{-k} \qquad & x \geq 0 \\ &= 0 & x < 0 \quad , \end{aligned} \tag{1}$$

where c and k are positive real numbers. The values of c and k readily yield moments as beta functions. Then by a linear transformation on x, μ and σ and standardized moments α_3 and α_4 may be fitted to data. As is shown in a paper on medians, submitted to the <u>Journal of the American Statistical Association</u>, the α_3, α_4 combinations covered by (1) include a very large part of that for the main Pearson System Types IV and VI and an important part of that of the other main Type I, that is the beta distribution. Also it includes the α_3, α_4 curve for bell shaped Type III (or gamma) distributions, and others.

Figure 1 shows the α_3, α_4 combinations for all but the last population of Tables 1 to 3.

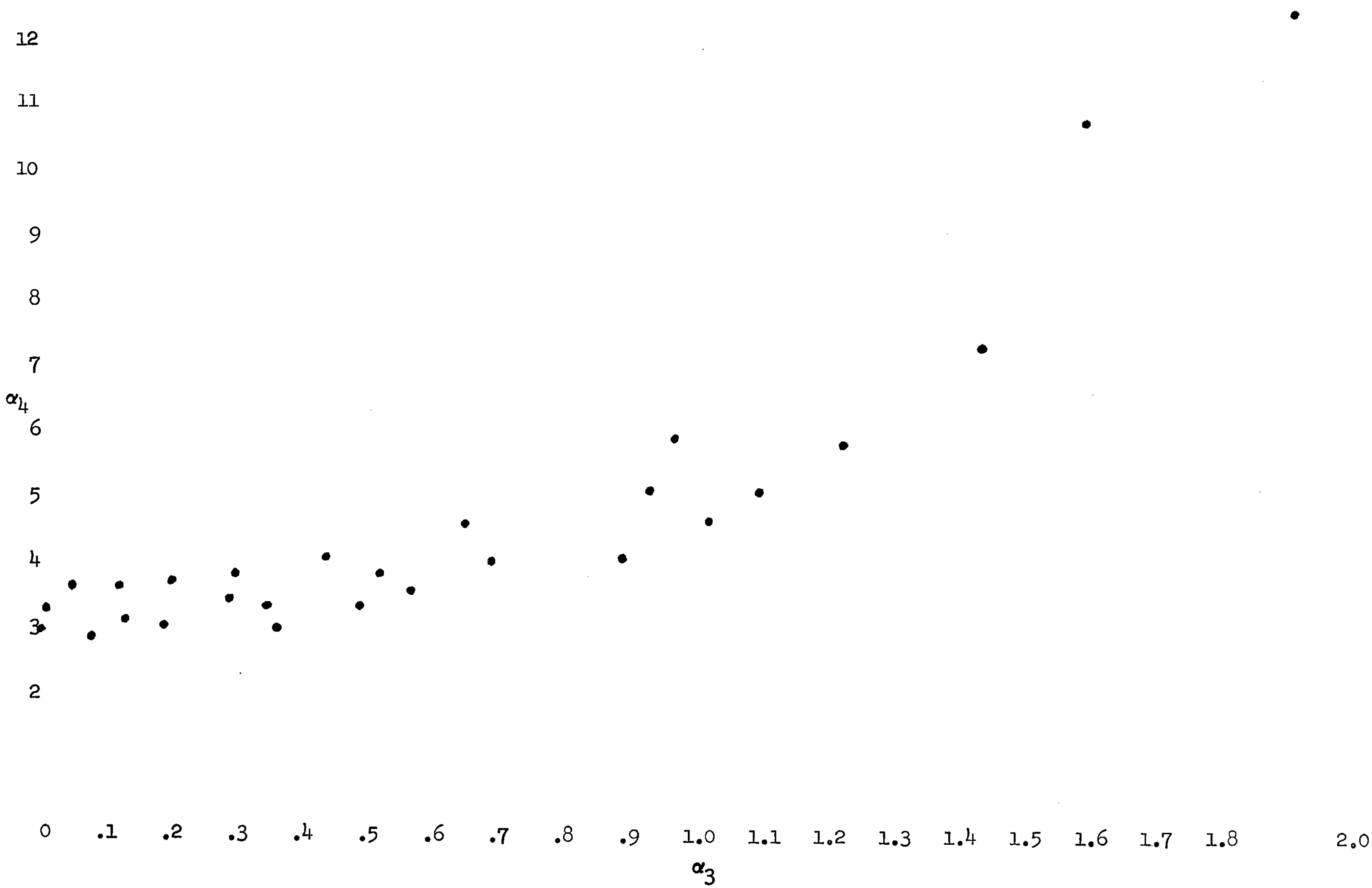

Fig. 1. Curve shape characteristics α_3, α_4 for populations (1) for which distribution of range is given in Tables 1 to 3.

3. CALCULATION OF THE DISTRIBUTION OF THE RANGE

The joint density function for the extreme variates $x_1 < x_n$ of a random sample of n from a population with continuous distribution function $F(x)$ is

$$f(x_1,x_n) = n(n-1)\, f(x_1)\, [F(x_n) - F(x_1)]^{n-2}\, f(x_n) \quad . \tag{2}$$

Since the range $R = x_n - x_1$, a change of variable gives

$$h(x_1,R) = n(n-1)\, f(x_1)\, [F(x_1+R) - F(x_1)]^{n-2}\, f(x_1+R) \quad . \tag{3}$$

Then to obtain the density function of R, say $g(R)$ one integrates out x_1. Thus distributions for which $F(x)$ is explicit and not an integral are intrinsically easier to handle than are those like the normal where $F(x)$ is an integral.

Substituting (1) into (3) and letting x_1 be x :

$$h(x,R) = n(n-1)c^2k^2\, x^{c-1}(x+R)^{c-1} \cdot \{[1+x^c]^{-k} - [1+(x+R)^c]^{-k}\}^{n-2} \quad . \tag{4}$$

Integrating x from 0 to ∞ would yield $g(R)$. Then

$$\mu'_{j:R} = E[R^j] = \int_0^\infty R^j\, h(R)\, dR = \int_0^\infty \int_0^\infty R^j\, h(x,R)\, dx\, dR \quad . \tag{5}$$

Direct use of the last expression was made by substituting in $h(x,R)$ from (4). Since the integration was not tractable, the double integrals were evaluated by approximation. Previous work Burr [5] sufficed to choose appropriate intervals for x and R so that 200 increments in each direction gave adequate coverage.

Double precision on an IBM 7094 was used to minimize errors due to heavy cancellation, and a check of $\int_0^\infty \int_0^\infty h(x,R)dx\, dR$ against 1 was also used. Further the results for a nearly normal case of 1 checked normal curve results very closely.

The cases of (1) calculated were the 81 combinations of c and k running 2,3,...,10 each. Since the sample sizes n were also 2 to 10, this gave 729 distributions of the range. For each of these, $\mu_R, \sigma_R, \alpha_{3:R}$ and $\alpha_{4:R}$ were found. Already available Burr [5] were $\mu_x, \sigma_x, \alpha_{3:x}$ and $\alpha_{4:x}$ for the 81 populations. From the foregoing were found, for all cases, the following quantities of interest:

$$d_2 = \mu_R \,/\, \mu_x = \text{standardized mean range} \tag{6}$$

$$d_3 = \sigma_R \,/\, \sigma_x = \text{standardized sigma for range} \tag{7}$$

$$V = \sigma_R \,/\, \mu_R = \text{coefficient of variation of range} \tag{8}$$

$$d_2 \,/\, d_{2,\text{normal}} = \text{comparison of } E(R) \text{ to normal value} \quad . \tag{9}$$

Those in (6) and (7) are of much interest in quality control charts.

The results calculated for $c = 2, k = 2$ are somewhat unreliable, since the check sum for the double integral with R^o was not exactly 1, due probably to the extreme tailing out. The total was only .9718 for $n = 10$. There was also a very slight deficiency for $c = 2, k = 3$ and $c = 3, k = 2$ the lowest sum being .9985.

4. CHARACTERISTICS OF THE DISTRIBUTION OF THE RANGE

Figure 1 shows the collection of curve shape parameters α_3, α_4 for the population whose range distribution characteristics are given in Tables 1 to 3. As may be seen quite a substantial path is sketched out in α_3, α_4 space, by the collection.

Table 1 gives the standardized mean and standard deviation of the range, that is, so that the population σ_x is adjusted to unity. The d_2 and d_3 values are such that if multiplied by any desired σ_x they would provide μ_R and σ_R, as for example in quality control work for the R chart with standard σ_x given. It has been pointed out in the literature, for example Cox [11], that d_2 values vary but little. Table 1 certainly confirms this. Moreover, d_3 is also quite stable, gradually increasing as the population curve shapes become more non-normal, while d_2 is decreasing. Sample size has much less effect on d_3 that is, σ_R, than upon d_2, that is, $E(R)$.

Table 2 presents a comparison of d_2 values for the collection of populations relative to the normal curve values of d_2 (often called d_n in the literature). There is high stability until the skewness reaches about 1.0, with most d_2 values below the normal d_n.

The coefficient of variation V_R of the range is quite important because it indicates the relative variability of R's around their mean. Thus the V_R's indicate the relative error in estimating σ_x for a population when using R/d_2 as estimate. Such coefficients thus have a bearing on the efficiency of the point estimate. The coefficients tend to increase as the populations become more non-normal, but decrease steadily as n increases to 10. See Table 2.

Table 3 presents the curve shape characteristics of the range distribution for the collection of populations. For only slightly non-normal populations, α_3 for the range decreases as n increases from 2 to 10. But for moderately non-normal populations there is a minimum α_3 at some n, beyond which α_3 again increases. The normal curve has the same trend, Harter and Clemm [22], with the

TABLE 1. VALUES OF STANDARDIZED MEAN $\mu_R / \sigma_x = d_2$ AND STANDARD DEVIATION $\sigma_R / \sigma_x = d_3$ FOR RANGES FOR VARIOUS POPULATIONS, FOR $n = 2,3,4,5,8,10$

Population				d_2						d_3					
α_3	α_4	c	k	2	3	4	5	8	10	2	3	4	5	8	10
-.01	3.01	5	6	1.13	1.69	2.06	2.33	2.85	3.08	.85	.89	.88	.87	.82	.80
.00	3.33	7	3	1.12	1.68	2.05	2.32	2.85	3.09	.86	.91	.91	.90	.88	.86
.04	3.65	10	2	1.11	1.67	2.04	2.31	2.86	3.10	.87	.93	.94	.93	.92	.91
.07	2.88	4	10	1.13	1.70	2.06	2.33	2.84	3.07	.85	.88	.87	.85	.79	.77
.11	3.67	9	2	1.14	1.67	2.04	2.31	2.86	3.10	.87	.93	.94	.93	.92	.91
.12	3.19	5	4	1.12	1.69	2.05	2.32	2.85	3.08	.86	.90	.90	.88	.85	.83
.18	3.05	4	6	1.13	1.69	2.06	2.32	2.84	3.07	.85	.89	.88	.86	.82	.80
.19	3.74	8	2	1.11	1.67	2.04	2.31	2.85	3.10	.87	.93	.94	.94	.92	.92
.28	3.48	5	3	1.12	1.68	2.05	2.31	2.85	3.09	.86	.91	.91	.91	.88	.87
.29	3.86	7	2	1.11	1.67	2.04	2.31	2.85	3.10	.87	.93	.94	.94	.93	.92
.34	3.36	4	4	1.12	1.68	2.05	2.32	2.84	3.07	.86	.90	.90	.89	.86	.84
.35	3.04	3	10	1.13	1.69	2.06	2.32	2.83	3.05	.85	.89	.87	.86	.81	.78
.43	4.11	6	2	1.11	1.66	2.03	2.30	2.84	3.09	.88	.94	.95	.95	.94	.94
.48	3.38	3	6	1.12	1.68	2.05	2.31	2.83	3.05	.86	.90	.90	.88	.85	.83
.51	3.87	4	3	1.11	1.67	2.04	2.30	2.83	3.07	.87	.92	.93	.92	.91	.90
.56	3.60	3	5	1.12	1.68	2.04	2.30	2.82	3.05	.87	.91	.91	.90	.87	.86
.64	4.63	5	2	1.10	1.66	2.02	2.29	2.83	3.08	.88	.95	.96	.97	.97	.97
.68	4.04	3	4	1.11	1.67	2.03	2.29	2.82	3.05	.88	.93	.93	.93	.91	.90
.88	4.12	2	10	1.10	1.66	2.01	2.27	2.78	3.01	.88	.94	.94	.93	.90	.89
.92	5.13	3	3	1.10	1.64	2.00	2.27	2.80	3.04	.89	.96	.97	.98	.98	.98
.96	5.94	4	2	1.09	1.64	2.00	2.26	2.80	3.05	.90	.97	1.00	1.01	1.03	1.03
1.01	4.71	2	7	1.09	1.64	2.00	2.26	2.77	3.00	.90	.97	.97	.97	.95	.95
1.09	5.12	2	6	1.09	1.63	1.99	2.25	2.76	3.00	.90	.97	.98	.99	.98	.98
1.22	5.83	2	5	1.08	1.62	1.97	2.23	2.75	2.99	.91	.99	1.01	1.02	1.02	1.03
1.43	7.36	2	4	1.06	1.60	1.95	2.21	2.73	2.97	.93	1.01	1.04	1.06	1.09	1.10
1.59	10.81	3	2	1.06	1.58	1.93	2.19	2.73	2.97	.93	1.02	1.05	1.08	1.12	1.14
1.91	12.46	2	3	1.03	1.55	1.89	2.15	2.67	2.91	.95	1.05	1.09	1.12	1.16	1.19
4.09	∞	2	2	.92	1.37	1.66	1.89	2.33	2.53	.89	.99	1.03	1.06	1.12	1.16
Normal				1.13	1.69	2.06	2.33	2.85	3.08	.85	.89	.88	.86	.82	.80

TABLE 2. VALUES OF COEFFICIENT OF VARIATION AND AND OF $d_{2,popn.} / d_{2,normal}$ FOR RANGES FOR VARIOUS POPULATIONS AND SAMPLE SIZES 2 TO 10

Population				$d_{2,popn.} / d_{2,normal}$				Coeff. of Variation, V_R					
α_3	α_4	c	k	2	4	5	10	2	3	4	5	8	10
-.01	3.01	5	6	1.000	1.000	1.000	1.001	.756	.526	.428	.372	.288	.259
.00	3.33	7	3	.993	.995	.997	1.006	.769	.541	.445	.389	.307	.278
.04	3.65	10	2	.998	.990	.994	1.009	.781	.554	.459	.404	.323	.294
.07	2.88	4	10	1.003	1.002	1.001	.997	.750	.518	.420	.364	.280	.250
.11	3.67	9	2	.988	.990	.994	1.008	.781	.554	.459	.404	.323	.295
.12	3.19	5	4	.997	.997	.998	1.002	.763	.533	.436	.381	.298	.269
.18	3.05	4	6	1.000	.999	.999	.998	.756	.525	.428	.372	.288	.259
.19	3.74	8	2	.987	.990	.993	1.008	.782	.555	.460	.405	.324	.296
.28	3.48	5	3	.992	.993	.995	1.003	.772	.543	.447	.392	.310	.282
.29	3.86	7	2	.986	.989	.992	1.007	.785	.557	.462	.407	.326	.298
.34	3.36	4	4	.994	.995	.996	.999	.767	.537	.440	.385	.302	.274
.35	3.04	3	10	1.000	.999	.998	.993	.756	.523	.425	.369	.285	.256
.43	4.11	6	2	.983	.986	.989	1.004	.790	.562	.467	.412	.332	.304
.48	3.38	3	6	.994	.994	.993	.992	.768	.536	.439	.383	.300	.272
.51	3.87	4	3	.987	.989	.990	.998	.782	.552	.456	.401	.320	.292
.56	3.60	3	5	.990	.991	.991	.992	.776	.544	.447	.391	.309	.281
.64	4.63	5	2	.978	.981	.984	1.000	.801	.572	.477	.423	.343	.316
.68	4.04	3	4	.984	.985	.986	.991	.789	.557	.460	.405	.323	.295
.88	4.12	2	10	.978	.978	.978	.979	.802	.566	.467	.410	.325	.294
.92	5.13	3	3	.972	.974	.976	.987	.814	.582	.486	.431	.351	.324
.96	5.94	4	2	.966	.969	.973	.991	.825	.595	.499	.445	.366	.339
1.01	4.71	2	7	.970	.970	.971	.975	.819	.583	.484	.428	.344	.315
1.09	5.12	2	6	.965	.966	.967	.973	.830	.594	.495	.439	.355	.327
1.22	5.83	2	5	.957	.959	.961	.970	.846	.609	.511	.455	.372	.344
1.43	7.36	2	4	.944	.946	.949	.964	.873	.635	.536	.480	.398	.370
1.59	10.81	3	2	.935	.938	.943	.966	.878	.642	.545	.490	.409	.382
1.91	12.46	2	3	.915	.918	.923	.945	.918	.677	.577	.520	.437	.408
4.09	∞	2	2	.812	.808	.811	.823	.971	.724	.621	.564	.482	.456
Normal								.756	.525	.427	.371	.288	.259

TABLE 3. THIRD AND FOURTH STANDARD MOMENTS α_3, α_4
FOR RANGE DISTRIBUTION FOR VARIOUS POPULATIONS
AND SAMPLE SIZES 2 TO 10

Population				α_3						α_4					
α_3	α_4	c	k	2	3	4	5	8	10	2	3	4	5	8	10
-.01	3.01	5	6	1.00	.65	.52	.46	.40	.39	3.88	3.29	3.19	3.17	3.19	3.22
.00	3.33	7	3	1.08	.76	.65	.60	.56	.55	4.26	3.64	3.54	3.52	3.55	3.59
.04	3.65	10	2	1.17	.86	.76	.72	.68	.68	4.65	4.00	3.89	3.87	3.91	3.95
.07	2.88	4	10	.96	.60	.47	.41	.34	.33	3.74	3.17	3.09	3.08	3.11	3.15
.11	3.67	9	2	1.17	.87	.77	.73	.70	.71	4.71	4.06	3.96	3.95	4.02	4.07
.12	3.19	5	4	1.05	.71	.60	.55	.51	.51	4.12	3.53	3.45	3.44	3.51	3.56
.18	3.05	4	6	1.01	.66	.54	.49	.45	.45	3.95	3.38	3.30	3.30	3.37	3.42
.19	3.74	8	2	1.19	.88	.79	.76	.74	.75	4.82	4.18	4.10	4.10	4.21	4.28
.28	3.48	5	3	1.13	.81	.71	.68	.67	.68	4.54	3.94	3.88	3.90	4.05	4.15
.29	3.86	7	2	1.22	.92	.83	.80	.80	.81	5.03	4.40	4.33	4.36	4.53	4.63
.34	3.36	4	4	1.09	.77	.67	.63	.62	.64	4.39	3.80	3.74	3.77	3.92	4.01
.35	3.04	3	10	1.01	.66	.54	.49	.45	.46	3.95	3.38	3.31	3.32	3.40	3.46
.43	4.11	6	2	1.27	.99	.91	.89	.90	.93	5.43	4.79	4.76	4.82	5.08	5.23
.48	3.38	3	6	1.10	.77	.67	.64	.63	.64	4.39	3.79	3.74	3.76	3.91	4.01
.51	3.87	4	3	1.22	.92	.84	.82	.84	.87	5.11	4.51	4.48	4.55	4.83	4.99
.56	3.60	3	5	1.16	.84	.75	.72	.73	.75	4.70	4.09	4.04	4.08	4.28	4.40
.64	4.63	5	2	1.38	1.11	1.05	1.04	1.08	1.12	6.22	5.56	5.56	5.67	6.03	6.23
.68	4.04	3	4	1.26	.96	.89	.87	.90	.93	5.30	4.66	4.64	4.71	4.99	5.16
.88	4.12	2	10	1.27	.96	.86	.84	.87	.94	5.15	4.40	4.30	4.30	4.40	4.41
.92	5.13	3	3	1.48	1.21	1.15	1.15	1.20	1.24	6.77	6.02	6.01	6.11	6.45	6.63
.96	5.94	4	2	1.60	1.34	1.30	1.30	1.34	1.38	7.74	6.91	6.88	6.96	7.23	7.36
1.01	4.71	2	7	1.40	1.10	1.02	1.00	1.02	1.06	5.91	5.09	4.99	5.02	5.19	5.29
1.09	5.12	2	6	1.48	1.19	1.11	1.10	1.12	1.16	6.45	5.59	5.49	5.52	5.73	5.86
1.22	5.83	2	5	1.61	1.32	1.26	1.24	1.27	1.31	7.36	6.40	6.29	6.33	6.56	6.70
1.43	7.36	2	4	1.81	1.53	1.47	1.45	1.47	1.49	8.85	7.65	7.44	7.41	7.46	7.50
1.59	10.81	3	2	1.93	1.65	1.57	1.53	1.48	1.45	9.62	8.11	7.67	7.44	7.01	6.79
1.91	12.46	2	3	2.07	1.75	1.66	1.61	1.53	1.49	10.31	8.47	7.89	7.57	7.01	6.74
4.09	∞	2	2	2.04	1.61	1.40	1.25	.90	.70	8.68	6.42	5.52	4.97	4.11	3.82
Normal				1.00	.65	.52	.47	.41	.40	3.87	3.29	3.19	3.17	3.18	3.20

minimum α_3 occurring at $n = 13$. For the extremely non-normal populations at the bottom of Table 3, the leveling off, if it does occur is beyond $n = 10$.

The kurtosis α_4 for R shows a minimum between $n = 2$ and 10 in all cases but the most non-normal. Also it tends to increase with increasing non-normality. For the smallest sample sizes the range is typically more non-normal than the population.

The quantities tabulated in Tables 1 and 3 can be of direct use in problems involving the distribution of $\bar{R}$, since for k R's in $\bar{R}$:

$$E(\bar{R}) = d_2 \sigma_x \qquad \sigma_{\bar{R}} = \sigma_R / \sqrt{k}$$

$$\alpha_{3:\bar{R}} = \alpha_{3:R} / \sqrt{k} \qquad \alpha_{4:R} = 3 + (\alpha_{4:R} - 3) / k ,$$

exactly as for $\bar{x}$'s .

5. SUMMARY

The results obtained show a considerable stability or robustness in the distribution of the range from non-normal distributions. Thus normal curve constants can be justifiably be used until non-normality becomes quite marked. In particular normal curve control chart constants appear to be quite generally applicable.

6. ACKNOWLEDGEMENT

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