# On the distribution of the largest root of a matrix in multivariate analysis*

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1. Introduction and Summary. Distribution problems in multivariate analysis are often related to the joint distribution of the characteristic roots of a matrix derived from sample observations. This well-known Fisher-Girshick-Hsu-Mood-Roy distribution (under certain null hypotheses) of $s$ non-null characteristic roots can be expressed in the form

$$f(\theta_1,\ldots,\theta_s) = C(s,m,n) \prod_{i=1}^{s} \theta_i^m (1-\theta_i)^n \prod_{i>j} (\theta_i - \theta_j) \tag{1.1}$$

$$0 < \theta_1 \leq \cdots \leq \theta_s < 1 \quad ,$$

where

$$C(s,m,n) = \pi^{\frac{1}{2}s^2} \Gamma_s(m+n+s+1) \,/\, \{\Gamma_s(\tfrac{1}{2}(2m+s+1))\Gamma_s(\tfrac{1}{2}(2n+s+1))\Gamma_s(\tfrac{1}{2}s)\} \quad , \tag{1.2}$$

$\Gamma_s(\cdot)$ is the multivariate gamma function defined in [2], and $m$ and $n$ are defined differently for various situations described in [4], [6]. Pillai [3], [5] has given the density function of the larger of two roots, i.e. when $s = 2$, as a hypergeometric function. In this paper, the result is extended to the

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general case giving the density function of the largest of $s$ roots as a generalized hypergeometric function [1],[2]. The density function of the largest root of a sample covariance matrix derived by Sugiyama [7] can be obtained from the one derived here by considering the transformation $\frac{1}{2}\lambda_s = n\theta_s$ and making $n$ tend to infinity.

2. The distribution of the largest root. Let us recall first the definition of the hypergeometric function of matrix argument [2]. If $\underset{\sim}{S}$ and $\underset{\sim}{T}$ are $(p \times p)$ symmetric matrices, then

$$(2.1) \qquad {}_2F_1(a_1,a_2;\, b;\, \underset{\sim}{S})$$

$$= \frac{\Gamma_p(b)}{\Gamma_p(a_1)\Gamma_p(b-a_1)} \int_{\underset{\sim}{O}}^{\underset{\sim}{I}} |\underset{\sim}{I}-\underset{\sim}{S}\,\underset{\sim}{T}|^{-a_2} |\underset{\sim}{T}|^{a_1-\frac{1}{2}(p+1)} |\underset{\sim}{I}-\underset{\sim}{T}|^{b-a_1-\frac{1}{2}(p+1)} (d\underset{\sim}{T}).$$

Now make the following transformation [3],[5],[7] in (1,1):

$$(2.2) \qquad \theta_s = \theta_s, \quad \ell_i = \theta_i/\theta_s, \qquad i = 1,\ldots, s-1 \ .$$

We get

$$(2.3) \qquad f_1(\ell_1,\ell_2,\ldots, \ell_{s-1},\theta_s)$$

$$= C(s,m,n)\theta_s^{ms+(s-1)(1+\frac{1}{2}s)}(1-\theta_s)^n \prod_{i=1}^{s-1} \{\ell_i^m(1-\ell_i\theta_s)^n(1-\ell_i)\} \prod_{i>j=1}^{s-1} (\ell_i-\ell_j).$$

Now for the integration of (2.3) with respect to the $\ell$'s in the range $0 < \ell_1 \leq \cdots \leq \ell_{s-1} < 1$, note that in the multivariate beta function form the $(s-1)$-fold integral will reduce to (2,1) with $a_1 = m+\frac{1}{2}s$, $a_2 = -n$, $b = m+s+1$, $\underset{\sim}{S} = \theta_s\underset{\sim}{I}_{s-1}$, $p = s-1$, and the limits $\underset{\sim}{O}$ and $\underset{\sim}{I}_{s-1}$ except that the result thus obtained should be multiplied by $\Pi^{-\frac{1}{2}(s-1)^2}\,\Gamma_{s-1}(\frac{s-1}{2})$ since the integrand of (2.1) is equivalent to that of (2.3) only after an orthogonal transformation to diagonalize the matrix $\underset{\sim}{T}$ and integrating out the elements of this orthogonal matrix. Thus we get

$$(2.4)\qquad f_2(\theta_s) = \{C(s,m,n)\,\Gamma_{s-1}(m+\tfrac{1}{2}s)\,\Gamma_{s-1}(1+\tfrac{1}{2}s)\,\Gamma_{s-1}(\tfrac{s-1}{2})/\Pi^{\frac{1}{2}(s-1)^2}\,\Gamma_{s-1}(m+s+1)\}$$

$$\theta^{ms+(s-1)(1+\frac{1}{2}s)}(1-\theta_s)^n\ {}_2F_1(m+\tfrac{1}{2}s,-n;m+s+1;\ \theta_s\,\underset{\sim}{I}_{s-1}).$$

When $s = 2$, (2.4) reduces to the result given by Pillai [3],[5]. Further, the density function of the smallest root, $\theta_1$, can be obtained from (2.4) by changing $1-\theta_s$ to $\theta_1$ and $m$ to $n$ [3],[5]. In addition, the density function of the largest root of a sample covariance matrix [7] can be obtained from (2.4) by the method given in the last section.

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