# The Queue with Poisson Input and General Service Times, treated as a Branching Process*

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Abstract

The $M|G|1$ queue is treated as a sequence of branching processes, the duration of which constitutes a busy period. The first generation in each branching process consists of the customers present at the beginning of the busy period, the second generation consists of all customers, who arrive during the service time of the first generation, etc. When the queue becomes idle, the branching process becomes extinct.

This approach permits a more elementary treatment of the $M|G|1$ queue, without use of Rouché's theorem. It provides a natural sequence of approximants to the distributions, which we consider and it provides a simple derivation of the virtual waitingtime.

The paper also considers two random variables of interest, which have not been considered hitherto. One is the total number of customers, served in $(o,t]$, the other is the virtual age or the time already spent in the queue, by the customer is service at time $t$.

We further consider a new imbedded semi-Markov process and study its asymptotic behavior.

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## I. Introduction

This paper presents a alternate approach to the study of the single server queue with Poisson input and independent, identically distributed service times, the $M|G|1$ queue. The basic structure of the $M|G|1$ queue is the following: There is a sequence of alternating busy and idle periods, all of which are independent random variables. Each busy period has the structure of a type of branching process, described as follows. At the beginning of a busy period there are $i$ customers present, ($i$ is equal to one, except for the initial busy period) and one of these customers enters service. These customers form the first generation. The second generation consists of all those, who arrive during the $i$ service times of the first generation. The third generation consists of all arrivals during the service time of a generation, then at the end of its service time the queue becomes empty and there is a negative exponential idle period, until a new customer arrives, who initiates a new such branching process.

This approach explains the similarity between the functional equations of the theory of branching processes and Takács' functional equation in Queueing theory. We also obtain through it, a natural sequence of approximants to the L.S. transform $\gamma(s)$ of the busy period. In fact, throughout this paper $\gamma(s)$ will be defined as the limit of this sequence and Rouché's theorem will not be needed, except to show the uniqueness of the solution in Takács' equation.

We will further obtain easy derivations of the usual related stochastic processes, such as the queuelength in continuous time, the virtual waitingtime etc. by relating them to this imbedded sequence of branching processes.

Finally, the distribution of $N(t)$, the total number of customers, served in $(o,t]$ is obtained using Van Dantzig's method of collective marks for facility of derivation and presentation.

We will now prove some properties, which will be needed in the sequel. In all that follows, we will make the following convention: All generating functions in $z$ or $w$ are considered for $z$, or $w$, in $[0,1]$ or $[0,1)$ and all Laplace and Laplace-Stieltjes transforms are considered on $[0,\infty)$ or $(0,\infty)$. It will usually be obvious that such functions are transforms or generating functions, so that the extensions to the unit-disk or to the half-plane can be made uniquely through analytic continuation.

Let $H(\cdot)$ be the distribution of a nondegenerate, nonnegative random variable and let $h(s)$ be its L.S. transform. Let $\lambda$ be a positive number and let $\alpha$ be the mean of $H(\cdot)$, assumed finite. We define the sequence of functions $h_n(s,z)$, $n \geq 1$, as follows:

(1)
$$h_1(s,z) = h(s + \lambda - \lambda z) \quad ,$$

$$h_{n+1}(s,z) = h[s + \lambda - \lambda h_n(s,z)], \qquad n \geq 1 \quad .$$

Lemma 1

The sequence $h_n(s,0)$ converges increasingly to a function $\gamma(s)$, which is the L.S. transform of a probability mass-function.

Proof:

We will show below, that the function $h_n(s,0)$ is the L.S. transform of a mass-function $G_n(\cdot)$ on $[0,\infty)$, with the property that for all $n$ and $x$, we have:

$$G_n(x) \leq G_{n+1}(x) < 1 \quad .$$

This implies that $h_n(s,0)$ is increasing in $n$. This also follows from the fact that $h(s+\lambda-\lambda z)$ is strictly increasing in $z$ on $[0,1]$ for every $s$.

It follows therefore that

(2) $$\lim_{n \to \infty} h_n(s,0) = \gamma(s) ,$$

exists and is the L.S. transform of a mass-function on $[0,\infty)$, which is the limit in distribution of the functions $G_n(x)$ .

Lemma 2.

The function $\gamma(s)$ is the L.S. transform of a probability distribution if and only if $1 - \alpha\lambda \geq 0$. $\gamma(0)$ is the smallest positive root of the equation

(3) $$\theta = h(\lambda - \lambda\theta)$$

in $(0,1]$ .

Proof: If we consider the graphs $y = h(\lambda - \lambda x)$ and $y = x$ and consider the points, whose abcissae are the iterates $h_n(0,0)$, then it is clear that $\gamma(0)$ is the smallest positive root of equation (3).

Also, equation (3) has a root $\theta \leqslant 1$ if and only if $1 - \alpha\lambda \leqslant 0$ and it has only the root $\theta = 1$ if and only if $1 - \alpha\lambda \geq 0$. If $1 - \alpha\lambda = 0$ then $y = x$ and $y = h(\lambda - \lambda x)$ are tangent at $x = 1$.

Lemma 3

We have:

(4) $$\begin{aligned} \gamma'(0) &= -\alpha(1 - \alpha\lambda)^{-1}, && \text{if } 1 - \alpha\lambda > 0 \\ &= \infty , && \text{if } 1 - \alpha\lambda = 0 \end{aligned}$$

Proof:

From (1) it follows that:

(5) $$h'_{n+1}(s,0) + \lambda h'_n(s,0)\, h'[s + \lambda - \lambda h_n(s,0)] = h'[s + \lambda - \lambda h_n(s,0)] .$$

Setting $s = 0+$ and letting $n \to \infty$, the result follows, since $h(s)$ is differentiable at zero.

Lemma 4

The series

(6)
$$\sum_{n=1}^{\infty} \left[h_n^i(s,z) - h_n^i(s,0)\right], \quad i \geq 1,$$

converges uniformly for all $s \geq 0$, $0 \leq z \leq 1$ if $1 - \alpha\lambda > 0$. Moreover:

(7)
$$\sum_{n=1}^{\infty} \left[h_n^i(s,z) - h_n^i(s,0)\right] - \sum_{n=1}^{\infty} \left[h_{n+1}^i(s,z) - h_n^i(s,0)\right] =$$
$$h^i(s + \lambda - \lambda z) - \gamma^i(s), \quad i \geq 1 .$$

Proof:

The series in (6) is termwise dominated by the series

(8)
$$\sum_{n=1}^{\infty} \left[1 - h_n(0,0)\right]$$

which is obtained by setting $s = 0$, $z = 1$ in (6).

If $1 - \alpha\lambda > 0$, then all the functions $h_n(0,z)$ are convex increasing in $[0,1]$ and do not intersect the line $y = z$ in $[0,1)$. Their graphs lie entirely above the tangent at $z = 1$, which has an intercept $1 - \alpha^n\lambda^n$ with the ordinate axis. This shows that $h_n(0,0) > 1 - \alpha^n\lambda^n$, which implies the convergence of the series (8). Formula (7) follows from:

$$\sum_{n=1}^{N+1} \left[h_n^i(s,z) - h_n^i(s,0)\right] - \sum_{n=1}^{N} \left[h_{n+1}^i(s,z) - h_n^i(s,0)\right] =$$

$$h_1^i(s,z) - h_N^i(s,0)$$

and formula (2) .

Lemma 5

The analytic continuation $\gamma(s)$, $\mathrm{Re}\ s > 0$ of $\gamma(s)$, $s > 0$ is the unique solution to the functional equation

$$z = h(s + \lambda - \lambda z) \tag{9}$$

which lies in the unit disk $|z| < 1$ for all $s$, $\mathrm{Re}\ s > 0$ .

Remark:

This fact is well-known, Takacs [7] p.47, and is usually used to define $\gamma(s)$. Its proof involves Rouché's theorem. We will not use this property or equation (9) in the sequel and hence, show that most of the properties of the $M|G|1$ queue can be obtained without using Rouché's theorem.

## II. The Imbedded Branching Processes

We assume that at time $t = 0$, there are $i$ customers in the queue, one of who enters service at that time. We will assume $i > 0$ throughout, indicatine each time how a trivial modification yields the corresponding result in the case $i = 0$.

We define the random times $T_o$, $T_1, \ldots$ as follows: i) $T_o = 0$ and ii) $T_{n+1}$ is the time instant in which all customers, if any, present at $T_n$ complete service. If there are no customers at $T_n$, then $T_{n+1}$ is the instant in which the first customer to arrive after $T_n$ completes service.

Let $\xi(t)$ denote the queue-length at $t + 0$. Consider the bivariate sequence of random variables $\{\xi(T_n), T_{n+1} - T_n, \; n \geq 0\}$. It follows immediately that this sequence defines a semi-Markov process with the nonnegative integers as state space. Its transition matrix $Q(\cdot)$, defined by:

$$Q_{ij}(x) = P\{\xi(T_{n+1}) = j, \; T_{n+1} - T_n \leq x \mid \xi(T_n) = i\} ,$$

is given by:

(10)
$$Q_{ij}(x) = \int_0^x e^{-\lambda y} \frac{(\lambda y)^j}{j!} \, d\, H_i(y), \qquad i > 0, \quad j \geq 0 .$$

$$Q_{0j}(x) = \int_0^x \left[1 - e^{-\lambda(x-y)}\right] d \; Q_{1j}(y), \qquad j \geq 0.$$

in which $H_i(y)$ denotes the i-fold convolution of $H(\cdot)$ at $y$.

The L.S. transforms $q_{ij}(\cdot)$ of the $Q_{ij}(\cdot)$ are given by:

(11)
$$q_{ij}(s) = \int_0^\infty e^{-(\lambda+s)y} \frac{(\lambda y)^j}{j!} \, d\, H_i(y), \quad i > 0$$

$$q_{0j}(s) = \frac{\lambda}{\lambda+s} \; q_{1j}(s) ,$$

and they satisfy:

(12)
$$\sum_{j=0}^\infty q_{ij}(s) \, z^j = h^i(s + \lambda - \lambda z), \quad s \geq 0, \; 0 \leq z \leq 1, \; i > 0 .$$

a. The transition probabilities within a branching process.

We denote by ${}_oQ_{ij}^{(n)}(x)$ the probability that in a branching process of the type described above the n-th transition occurs before time $x$, that there are $j$ customers present at the time of the n-th transition and that of course the population has not become extinct before, given that there are $i$ persons in the original generation; formally:

$$P\{T_n \leq x,\ \xi(T_n) = j,\ \ \xi(T_\nu) \neq 0,\ \nu = 1,\ldots, n-1 \mid \xi(T_o) = i\} \quad .$$

The L.S. transforms ${}_oq_{ij}^{(n)}(s)$ of ${}_oQ_{ij}^{(n)}(s)$ satisfy the following recurrence relations:

(13)
$$ {}_oq_{ij}^{(1)}(s) = q_{ij}(s)$$

$$ {}_oq_{ij}^{(n+1)}(s) = \sum_{\nu=1}^{\infty} {}_oq_{i\nu}^{(n)}(s)\, q_{\nu j}(s), \quad n \geq 1 \quad .$$

If we set

(14)
$$ {}_oq_i^{(n)}(s,z) = \sum_{j=0}^{\infty} {}_oq_{ij}^{(n)}(s)\, z^j \ , \quad 0 \leq z \leq 1,\ i > 0$$

we get:

(15) a.
$$ {}_oq_i^{(1)}(s,z) = h^i(s + \lambda - \lambda z) = h_1^i(s,z) \quad ,$$

and

(15) b.
$$ {}_oq_i^{(n+1)}(s,z) = \sum_{\nu=1}^{\infty} {}_oq_{i\nu}^{(n)}(s)\, h^\nu(s + \lambda - \lambda z) =$$

$$ {}_oq_i^{(n)}[s,h(s + \lambda - \lambda z)] - {}_oq_i^{(n)}(s,0),\ n \geq 1 \ .$$

This leads to:

(16)
$$ {}_{o}q_i^{(1)}(s,z) = h_1^i(s,z) , $$
$$ {}_{o}q_i^{(n)}(s,z) = h_n^i(s,z) - h_{n-1}^i(s,o) , \quad n > 1 . $$

In particular, we obtain:

(17)
$$ \sum_{n=1}^{N} {}_{o}q_{io}^{(n)}(s) = \sum_{n=1}^{N} {}_{o}q_i^{(n)}(s,o) = h_N^i(s,o), \quad i \geq 1 . $$

This shows that $h_N(s,o)$ is the L.S. transform of the probability that a busy period consists of at most $N$ generations in the branching process and has a duration of at most $x$. If we call this probability $G_N(x)$, then it is clear that $G_N(x)$ has the properties used in the proof of lemma 1. It also follows that $\gamma(s)$ as defined in (2) is the L.S. transform of the distribution of the length of the busy period. It also shows that if the branching process starts out with $i$ individuals, the time till absorption is equidistributed to the sum of $i$ independent busy periods.

b. The Renewal functions of the Semi-Markov Process.

Most of the processes of interest to Queueing theory in the $M|G|1$ queue may be obtained and studied by reference to the imbedded semi-Markov process, $\{\xi(T_n), T_n - T_{n-1}\}$. We will now study this semi-Markov process in its own right. It is clearly irreducible and aperiodic and is non-lattice.

We denote by $M_{ij}(t)$ the expected number of visits to state $j$ in $(o,t]$, given that $\xi(T_o) = i$ and we denote by $m_{ij}(s)$ the L.S. transform of $M_{ij}(t)$. Since clearly:

$$ m_{oj}(s) = \frac{\lambda}{\lambda+s} m_{1j}(s), \quad j \geq 0 $$

we assume $i > 0$ henceforth.

For $j = 0$, we obtain:

(18) $$m_{io}(s) = \gamma^{i}(s)\left[1 - \frac{\lambda}{\lambda+s}\gamma(s)\right]^{-1} ,$$

since the visits to state 0 form a modified renewal process consisting of an initial busy period, starting with $i$ customers, followed by alternating periods of emptiness with a negative exponential distribution and ordinary busy periods.

For $j > 0$, we define the following functions:

(19) $$g_{ij}(s) = \sum_{n=1}^{\infty} {}_{o}q_{ij}^{(n)}(s), \qquad i \geq 0$$

which may be calculated as follows:

(20) $$g_{oj}(s) = \frac{\lambda}{\lambda+s} g_{1j}(s) ,$$

and for $i > 0$ .

(21) $$\sum_{j=1}^{\infty} g_{ij}(s) z^{j} = \sum_{n=1}^{\infty}\left[{}_{o}q_{i}^{(n)}(s,z) - {}_{o}q_{i}^{(n)}(s,o)\right] =$$

$$\sum_{n=1}^{\infty}\left[h_{n}^{i}(s,z) - h_{n}^{i}(s,o)\right] ,$$

by formulae (1) and (16).

We then have for $i > 0, j > 0$ that:

(22) $$m_{ij}(s) = m_{io}(s)\frac{\lambda}{\lambda+s} g_{1j}(s) + g_{ij}(s) ,$$

since a visit to state $j$ can occur either with or without intermediate visit to state $0$.

For future reference, we record the following relations:

(23) $$\mathfrak{M}_i(s,z) = \sum_{j=1}^{\infty} m_{ij}(s)z^j ,$$

by definition.

Then also:

(24) $$\mathfrak{M}_i(s,z) = m_{io}(s)\frac{\lambda}{\lambda+s} \sum_{n=1}^{\infty}\left[h_n(s,z) - h_n(s,o)\right] + \sum_{n=1}^{\infty}\left[h_n^i(s,z) - h_n^i(s,o)\right]$$

by (21) and (22).

Finally, by (7), (10) and (24), we obtain:

(25) $$\mathfrak{M}_i(s,z) - \mathfrak{M}_i\left[s,h(s+\lambda-\lambda z)\right] = \frac{\lambda}{\lambda+s}\ m_{io}(s)\left[h(s+\lambda-\lambda z)-\gamma(s)\right] + \left[h^i(s+\lambda-\lambda z) - \gamma^i(s)\right] = h^i(s+\lambda-\lambda z) - \gamma^i(s)\frac{s+\lambda-\lambda h(s+\lambda-\lambda z)}{s-\lambda-\lambda\gamma(s)} .$$

c. The Stationary Distribution of the Imbedded Markov Chain:

The imbedded Markov chain $\xi(T_n)$ is of less importance in itself than the usual imbedded Markov chain of D. G. Kendall, as it does not follow the fluctuations in queue-length as closely and as regularly. To a transition from state $i > 0$ in our chain there correspond $i$ transitions in the usual chain. However we will show, as a matter of academic interest how the generating function of the stationary transition probabilities can be expressed in terms of iterates of the function $h(\cdot)$.

Let us denote by $\beta_j$ the limit as $n \to \infty$ of the probability $P\{\xi(T_n) = j \mid \xi(T_o) = i\}$, then we have:

$$(26) \qquad \beta_o \int_o^\infty e^{-\lambda u} \frac{(\lambda u)^j}{j!} \, d\, H(u) + \sum_{i=1}^{\infty} \beta_i \int_o^\infty e^{-\lambda u} \frac{(\lambda u)^j}{j!} \, d\, H_i(u) = \beta_j$$

for all $j$.

Setting:

$$(27) \qquad B(z) = \sum_{j=0}^{\infty} \beta_j z^j , \quad 0 \leq z \leq 1 ,$$

we obtain:

$$(28) \qquad B[h(\lambda-\lambda z)] - B(z) = \beta_o[1-h(\lambda-\lambda z] .$$

If $1 - \alpha\lambda < 0$, we set $z = \gamma(o) < 1$ and obtain $\beta_o = 0$ and hence $B(z) = 0$, by lemma (2).

If $1 - \alpha\lambda \geq 0$, we replace $z$ in (28), successively by $h_n(o,z)$, $n \geq 1$ and add the resulting equations. Since $h_n(o,z)$ tends to 1 for every $z$ in $[0,1]$, we obtain:

$$(29) \qquad B(1) - B(z) = \beta_o \sum_{n=1}^{\infty} [1-h_n(o,z)], \quad 0 \leq z \leq 1 .$$

Since $B(1) = 1$, we obtain, when $1 - \alpha\lambda > 0$, that:

$$(30) \qquad B(z) = 1 - \beta_o \sum_{n=1}^{\infty} [1 - h_n(o,z)] ,$$

and:

(31) $$\beta_o = \left[1 + \sum_{n=1}^{\infty} (1 - h_n(o,o))\right]^{-1} .$$

The series converges by lemma 4.

We will show in the next section that $B(z) = 0$ when $\alpha\lambda = 1$. By differentiation in (30), we obtain that the limiting expected queue-length at the points of transition $T_n$ is given by $\beta_o\alpha\lambda(1-\alpha\lambda)^{-1}$.

d. The asymptotic behavior of the Semi-Markov Process.

We denote by $\mu_j$, $j \geq 0$, the mean recurrence time of the state $j$ in the semi-Markov process $\{\xi(T_n), T_n - T_{n-1}\}$. If $1 - \alpha\lambda \leq 0$ then $\mu_o = \infty$, since the expected length of the busy period is infinite. This implies $\mu_j = \infty$ for all $j \geq 0$.

Since the recurrence time between visits to state 0 consists of a negative exponential idle time followed by an independent busy period, we have $\mu_o = \lambda^{-1} - \gamma'(o) = \lambda^{-1}(1 - \alpha\lambda)^{-1}$. This may also be obtained using the classical Tauberian theorem:

(22) $$\mu_o^{-1} = \lim_{s \to o+} s\, m_{1o}(s) = \lambda(1 - \alpha\lambda)$$

when $1 - \alpha\lambda > 0$.

For $j \neq 0$, we have

(33) $$\mu_j^{-1} = \lim_{s \to o+} s\, m_{1j}(s) = \mu_o^{-1} \lim_{s \to o+} g_{1j}(s) = \mu_o^{-1} g_{1o}(o)$$

but we have

(34) $$\sum_{j=1}^{\infty} g_{1j}(o)\, z^j = \sum_{n=1}^{\infty} [h_n(o,z) - h_n(o,o)] =$$

$$\sum_{n=1}^{\infty} [1 - h_n(o,o)] - \sum_{n=1}^{\infty} [1 - h_n(o,z)] = \beta_o^{-1} B(z) - 1 = \sum_{j=1}^{\infty} \beta_j \beta_o^{-1} z^j ,$$

It follows that:

(35) $$\mu_j = \beta_o\, \beta_j^{-1}\, \lambda^{-1}(1 - \alpha\lambda)^{-1}, \qquad j \neq 0 \quad .$$

Let $P_{ij}(t)$ be the probability that the semi-Markov process is in state $j$ at time $t$, given that it started in state $i$. Then it is known that $P_j = \lim_{t \to \infty} P_{ij}(t)$ exists and is given by

(36) $$P_j = \eta_j\, \mu_j^{-1} , \qquad j \geq 0 .$$

where $\eta_j$ is the expected time spent in state $j$, before the next transition. Since:

(37) $$\eta_o = \lambda^{-1} + \alpha \qquad \eta_j = j\alpha, \qquad j > 0$$

it follows that:

(38) $$P_o = 1 - \alpha^2 \lambda^2$$

$$P_j = j\, \beta_j\, \beta_o^{-1}\, \lambda\alpha(1-\lambda\alpha), \qquad j > 0 \quad .$$

Direct verification shows that those probabilities sum to one and that the mean of this distribution is given by: $\beta_o \lambda^2 \alpha(1-\alpha^2\lambda^2)^{-1}[\alpha(1+\alpha\lambda) + \alpha_2]$ is the second moment of $H(\cdot)$ .

## III. The Queue-length in Continuous Time

Let $\pi_{ij}$ be the probability that at time $t$, there are $j$ customers in the queue, given that $\xi(T_o) = i$. We first note that if $i = o$, we have:

$$\text{(39)} \qquad \pi_{oj}(t) = \delta_{oj}\, e^{-\lambda t} + \int_o^t e^{-\lambda \tau}\, \lambda\, \pi_{1j}(t-\tau)dt\ , \qquad j \geq o$$

so that we may assume $i > o$ henceforth.

If there are customers present at time $t$, they either belong to the generation, which is then undergoing service, or they are new arrivals since the last transition in the branching process. We define $P_i(\nu, r, t)$ to be the probability, that at time $t$ there are $\nu$ customers left of the present generation and that there are $r$ new arrivals since the last transition, given that $\xi(T_o) = i$. We then have:

$$\text{(40)} \qquad P_i(o, o, t) = \pi_{io}(t) = \int_o^t e^{-\lambda(t-\tau)}\, d\, M_{io}(\tau)\ ,$$

and for $\nu > 1$:

$$\text{(41)} \qquad P_i(\nu, r, t) = \sum_{k=\nu}^{\infty} \int_o^t e^{-\lambda(t-\tau)} \frac{[\lambda(t-\tau)]^r}{r!} \left[H_{k-\nu}(t-\tau) - H_{k-\nu+1}(t-\tau)\right] d\left[M_{ik}(\tau) + \delta_{ik}\, U(\tau)\right]$$

Formula (41) is obtained as follows: The last visit before $t$ in the semi-Markov process must be to some state $k(k \geq \nu)$ at some time $\tau$ and in the interval $(\tau, t]$ exactly $k-\nu$ customers complete service and $r$ new arrivals occur. When $k = i$, there is the added possibility that no transitions have yet occurred in $(o,t]$. $U(\cdot)$ is the distribution degenerate at zero.

For $\nu = 1$, we obtain:

$$(42) \qquad P_i(1, r, t) = \sum_{k=1}^{\infty} \int_o^t e^{-\lambda(t-\tau)} \frac{[\lambda(t-\tau)]^r}{r!} \left[H_{k-1}(t-\tau) - H_k(t-\tau)\right] d\left[M_{ik}(\tau) + \delta_{ik}U(\tau)\right]$$

$$+ \int_o^t e^{-\lambda(t-\tau)} \, d\, M_{io} \int_o^{t-\tau} \left[1 - H(t-\tau-\nu)\right] \frac{[\lambda(t-\tau-\nu)]^r}{r!} \lambda \, dv \, ,$$

The extra term corresponds to the case, in which the member of the present generation if the first customer of a new busy period.

Denoting by $P_i^*(\nu, r, s)$ the Laplace-transform of $P_i(\nu, r, t)$ we obtain

(43) a. $P_i^*(o, o, s) = (s + \lambda)^{-1} m_{io}(s)$

b. for $\nu \geq 1$:

$$\sum_{r=o}^{\infty} P_i^*(\nu, r, s) z^r =$$

$$(s + \lambda - \lambda z)^{-1} \left[1 - h(s + \lambda - \lambda z)\right] \left\{ \sum_{k=\nu}^{\infty} \left[m_{ik}(s) + \delta_{ik}\right] h^{k-\nu}(s + \lambda - \lambda z) + \frac{\lambda}{\lambda + s} \, m_{io}(s) \, \delta_{1\nu} \right\}$$

and:

$$(44)\quad \sum_{\nu=o}^{\infty} \sum_{r=o}^{\infty} w^{\nu} z^{r} P_i^*(\nu, r, s) =$$

$$\frac{m_{io}(s)}{s+\lambda}\left[1 + \lambda w \frac{1 - h(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right] +$$

$$\frac{1 - h(s+\lambda-\lambda z)}{s+\lambda-\lambda z} \cdot \frac{1}{w - h(s+\lambda-\lambda z)} \left\{ w\, \mathcal{Q}_i(s,w) + w^{i+1} - w\, h^i(s+\lambda-\lambda z) - w\, \mathcal{Q}_i[s, h(s+\lambda-\lambda z)] \right\},$$

By setting $w = z$ in (44) we obtain the generating function on $j$ of the Laplace transforms $\pi_{ij}^*(s)$ of the probabilities $\pi_{ij}(t)$. We simplify the resulting expression, using (18) and (25) and obtain the well-known formula:

$$(45)\quad \sum_{j=o}^{\infty} \pi_{ij}^*(s)\, z^j = \frac{z^{i+1}}{s+\lambda-\lambda z} \cdot \frac{1 - h(s+\lambda-\lambda z)}{z - h(s+\lambda-\lambda z)} +$$

$$(z-1)\, \frac{\gamma^i(s)\, h(s+\lambda-\lambda z)}{[s+\lambda-\lambda\gamma(s)]\,[z - h(s+\lambda-\lambda z)]},$$

Takacs [7] p. 74.

It follows readily from (39) that (45) is also valid for $i = o$.

## IV. The Total Number of Customers served in $(o,\tau]$

We denote by $N(t)$, the total number of customers, who have completed service in $(o,t]$. The distribution of $N(t)$ was obtained for the Poisson queue by Harold and Irwin Greenberg [3]. The strong law of large numbers and the asymptotic normality of $N(t)$ for a more general class of bulk queues was established by Neuts [5]. We will now obtain the joint distribution of $N(t)$ and $\xi(t)$ for the $M|G|1$ queue, using the method of collective marks.

We first note that:

$$(46)\qquad P\{N(t) = r,\ \xi(t) = j|\xi(o) = o\} = \delta_{or}\ \delta_{oj} e^{-\lambda t} + \int_o^t e^{-\lambda(t-\tau)}\lambda\ P\{N(\tau) = r,\ \xi(\tau) = j|\xi(o) = 1\}d\tau$$

so that henceforth we assume that $i > o$.

### a. The method of collective marks.

D. Van Dantzing [1] showed how many formulae, involving generating functions and/or Laplace transforms can be proved directly by giving them a probabilistic interpretation. If $F(x)$ is the distribution of a nonnegative random variable $X$, then $f(s) = \int_o^\infty e^{-sx}dF(x)$, $s > o$, is the waitingtime until a first event in a Poisson process with parameter $s > o$. If $p_o, p_1, \dots$ are the probabilities associated with a discrete probability distribution on the nonnegative integers, then for $o \leq z \leq 1$, $\sum_{\nu=o}^{\infty} p_\nu z^\nu$ is a probability in the following sense; if the discrete variable takes on the value $\nu$, then we perform $\nu$ Bernoulli trials with probability $1 - z$ of success (marks). The generating function is then the probability that in the compound experiment no mark occurs. Formulae involving generating functions and Laplace transforms now often become simple consequences of probabilistic properties, provided one considers the more complicated events involving the "marking" processes.

A lucid account of the method may be found in Runnenburg [6].

b. The number of customers, served during a busy period.

We consider a busy period, starting with $i$ customers, again as a type of branching process. We consider three marking processes, independent of each other 1) a Poisson process with parameter $s > o$. 2) a sequence of Bernoulli trials which marks each departing customer with probability $1 - z$, $o \leq z \leq 1$ and 3) a sequence of Bernoulli trials, which marks the customers present at the beginning of each generation with probability $1 - w$, $o \leq w \leq 1$.

Let $h_n^i(s,z,w)$ denote the probability that if there are $n$ generations in a busy period, the $n$ - th generation completes service before the first $s$ - event, that none of the departing customers was marked in the $z$ - process and that none of the customers present at the end of the $n$ - th generation is marked in the $w$ - process, then we have:

$$(47) \qquad h_1^i(s,z,w) = z^i\, h^i(s + \lambda - \lambda w),$$

$$h_{n+1}^i(s,z,w) = z^i\, h^i[s + \lambda - \lambda h_n(s,z,w)]$$

The recurrence relation is obtained as follows: If there are at most $n + 1$ generations, then we require that none of the $i$ customers is marked in the first generation and none of their descendents are marked in the next $n$ generations.

Now $h_n(s,z,o)$ is the probability, that there are at most $n$ generations in a busy period, that the end of the $n$ - th generation occurs before the first $s$ - event, and that none of the departing customers is marked in the $z$ - process.

It follows, by monotonicity, that:

(48) $$\gamma(s,z) = \lim_{n} h_n(s,z,o)$$

exists and is equal to $E\{e^{-s\ L}\ z^{\ N}\}$ , where $L$ is the length of a busy period and $N$ the number of customers served during it.

The analytic continuation $\gamma(s,z)$, $R_e\ s \geq o$, $|z| \leq 1$ is also the only solution of the equation:

(49) $$\xi = z\ h(s + \lambda - \lambda z \xi)$$

which lies in the unit desk $|\xi| \leq 1$. This follows directly, using Rouché's theorem.

Using either (47), (48), or (49) we may show by direct differentiation that for $1 - \alpha\lambda \geq o$:

(50) $$\frac{d}{d\ z}\ \gamma(o,z)\Big|_{z=1} = \frac{1 + \alpha\lambda}{1 - \alpha\lambda}$$

which is the expected number of customers, served during a busy period. Also $\gamma(s,1) = \gamma(s)$ and $\gamma(o,z)$ is the generating function of the number of customers served during a busy period. An argument analogous to Lemma 3, shows that $\gamma(o,z)$ is an honest generating function if and only if $1 - \alpha\gamma \geq o$ .

c. The distribution of $N(t)$:

We define the following probabilities:

$$\pi_{ij}(\nu,t) = P\{N(t) = \nu,\ \xi(t) = j \mid \xi(o) = i\}\ , \qquad i > o\ . \tag{51}$$

and we consider the transform:

$$\Pi_i(s,z,w) = \sum_{\nu=o}^{\infty} \sum_{j=o}^{\infty} z^{\nu} w^{j} \int_o^{\infty} e^{-st}\, \Pi_{ij}(\nu,t)dt \tag{52}$$

For $s > o$, $o \leq z \leq 1$, $o \leq w \leq 1$, $\pi_i(s,z,w)$ is the probability that in a queue with $i$ customers initially, before the first $s$ - event, none of the departing customers have been marked by the $z$ - process and none of the customers who arrived between the first $s$ - event and the last departure from the queue before it, have been marked by a $w$ - process, which marks arriving customers with probability $1 - w$ .

We consider two cases; either the initial busy period has not yet ended at the first $s$ - event or one or more complete busy periods have elapsed by that time.

We interpret the following expressions:

$$_o m_{io}(s,z) = \gamma^{i}(s,z) \left[1 - \frac{\lambda}{\lambda + s}\ \gamma(s,z)\right]^{-1} \tag{53}$$

is the probability that no departing customer is marked during any of the busy periods, <u>completed</u> before the first $s$ - event.

The probability $\phi_{ij}(s,z) j > o$, is defined as follows: Assuming that the first $s$ - event occurs during some busy period (and not during an idle period), $\phi_{ij}(s,z)$ is the probability that, if the busy period started with $i$ customers, the last generation before the $s$ - event had $j$ customers in it.

For $j > o$, we have that:

$$(54) \qquad {}_o m_{ij}(s,z) = {}_o m_{io}(s,z) \cdot \frac{\lambda}{\lambda + s} \phi_{ij}(s,z) + \phi_{ij}(s,z)$$

is the probability that before the first $s$ - event, a generation of size $j$ is reached during some busy period and so that no customer departing before this occurs is marked by the $z$ - process.

We also have that:

$$(55) \qquad \sum_{j=1}^{\infty} \phi_{ij}(s,z)w^j = \sum_{n=1}^{\infty} \left[h_n^i(s,z,w) - h_n^i(s,z,o)\right]$$

This can also be proved by interpretation: If a generation of size $j$ occurs as described above in the definition of $\phi_{ij}(s,z)$ we mark each customer in it with probability $1 - w$.

Formula (55) expresses then the events described in the definition of $\phi_{ij}(s,z)$ occur and moreover that no customer is marked in the $w$ - process.

The right hand side of (55) is obtained by observing that the desired event may occur at the end of the first, second, ..., $n$ - th ... generation.

We have now the tools to describe the path-function probabilities up to the last complete generation before the first $s$ - event. We may also consider the $w$ - process as one which marks all arriving customers with the probability $1 - w$. The only probability that remains to be expressed is the following:

If $j$ is the size of the last generation before the first $s$ - event, what is the probability that none of the customers departing between the beginning of the service of the last generation and the first $s$ - event, will be marked in the process and also that none of all the customers present at the first $s$ - event was marked in the $w$ - process.

This probability is given by:

(56) If $j = o$:

$$\int_o^\infty e^{-(s+\lambda)u}du + w\int_o^\infty e^{-su}du\int_o^u e^{-\lambda\tau}[1 - H(w-\tau)]e^{-(\lambda-\lambda w)\ (w-\tau)}\lambda\ d\tau$$

$$= \frac{1}{\lambda + s} + \frac{\lambda w}{\lambda - s} \cdot \frac{1 - H(s+\lambda-\lambda w)}{s + \lambda - \lambda w},$$

and:

(57) If $j > o$:

$$\sum_{k=o}^{j-1}\int_o^\infty e^{-su}du\int_o^u e^{-(\lambda-\lambda w)\tau}\ z^k\ [1 - h(u-\tau)]w^{j-k}d\ H_k(\tau)$$

$$= w\,\frac{1 - h\ (s+\lambda-\lambda w)}{s + \lambda - \lambda w} \cdot \frac{w^j - z^j h^j(s+\lambda-\lambda w)}{w - z\ h(s+\lambda-\lambda w}$$

Applying the law of total probability, we obtain:

(58) $\Pi_i(s,z,w) =$

$$om_{ij}(s,z)\left[\frac{1}{\lambda + s} + \frac{\lambda w}{\lambda + s} \cdot \frac{1 - h(s+\lambda-\lambda w)}{s + \lambda - \lambda w}\right]$$

$$+ \sum_{j=1}^{\infty} om_{ij}(s,z)w\,\frac{1 - h(s+\lambda-\lambda w)}{s + \lambda - \lambda w} \cdot \frac{w^j - h_1^j(s,z,w)}{w - h_1\ (s,z,w)}$$

$$+ w\,\frac{1 - h(s+\lambda-\lambda z)}{s + \lambda - \lambda w} \cdot \frac{w^i h_1^i(s,z,w)}{w - h_1(s,z,w)}$$

We note that:

$$(59)\quad \sum_{j=1}^{\infty} om_{io}(s,z)w^j - \sum_{j=1}^{\infty} om_{ij}(s,z)h_1^j(s,z,w) =$$

$$\frac{\lambda}{\lambda + s}\, om_{io}(s,z)\Big[h_1(s,z,w) - \gamma(s,z)\Big] + h_1^i(s,z,w) - \gamma^i(s,z)$$

by (54) and (55).

Substituting in (58) and simplifying, we obtain:

$$(60)\quad \Pi_i(s,z,w) = \frac{w^{i+1}[1 - h(s+\lambda-\lambda w)]}{(s+\lambda-\lambda w)[w-zh(s+\lambda-\lambda w)]} + \frac{\gamma^i(s,z)}{s+\lambda-\lambda\gamma(s,z)} \quad .$$

From formula (60), we may obtain (45) by setting $z = 1$ and replacing $w$ by $z$. For $w = 1$, we obtain the generating function of the number of customers served in $(o,t]$.

$$(61)\quad \Pi_i(s,z,1) = \frac{1 - h(s)}{s[1 - zh(s)]} + \frac{\gamma^i(s,z)}{s+\lambda-\lambda\gamma(s,z)} \quad .$$

## V. The Virtual Waiting-time

The virtual waitingtime $\eta(t)$ is the length of time a (virtual) customer, who arrives at time $t$, will have to wait before entering service. The imbedded sequence of "branching processes" allows a very easy and natural derivation of the distribution of $\eta(t)$ .

If the queue is not empty at time $t$, then $\eta(t) \neq o$ and can be written as

$$(62)\quad \eta(t) = U_t + V_t$$

where $U_t$ is the length of time, till the next transition in the imbedded semi-Markov process and $V_t$ is the length of time required to serve the customers, if any, who have arrived since the beginning of the service time of the present generation.

We obtain $\eta(t)$ from the joint distribution of $U_t$ and $V_t$, where we define $U_t = V_t = o$ if $\eta(t) = o$.

The same probability arguments as used before, lead to:

$$(63) \quad \int_o^\infty e^{-\zeta t} dt \int_o^\infty \int_o^\infty e^{-s_1 u - s_2 v} d\, P\{U_t \le u,\ V_t \le v \mid \xi(o) = i\}$$

$$= \int_o^\infty e^{-\zeta t} dt \int_o^t e^{-\lambda(t-\tau)} d\, M_{io}(\tau) +$$

$$\int_o^\infty e^{-\zeta t} dt \int_o^\infty \int_o^\infty e^{-s_1 u - s_2 v} \int_o^t \sum_{r=o}^\infty e^{-\lambda(t-\tau)} \frac{[\lambda(t-\tau)]^r}{r!} \, d\, H_r(v)$$

$$d\, H(t+u-\tau)\, d\left[M_{io} * F(t)\right]$$

$$+ \int_o^\infty e^{-\zeta t} dt \int_o^\infty \int_o^\infty e^{-s_1 u - s_2 v} \int_o^t \sum_{r=o}^\infty \sum_{\nu=1}^\infty e^{-\lambda(t-\tau)} \frac{[\lambda(t-\tau)]}{r!} \, d\, H_r(v)$$

$$d\, H_r(v)\, d\, H_\nu(t+u-\tau)\, d\left[M_{i\nu}(t) + \delta_{i\nu} U(t)\right], \quad i > o$$

These three terms correspond respectively to the following three cases; the first term corresponds to $\eta(t) = o$, the second term corresponds to the case in which the last transition in the semi-Markov process is into state 0, but at time $t$ a new customer has already arrived and the third term corresponds to the case, in which the last state visited in the semi-Markov process was $\nu \neq o$.

Evaluating the transforms, setting $s_1 = s_2$, and simplifying we obtain:

$$(64) \qquad \int_0^\infty e^{-\zeta t} dt \int_0^\infty e^{-sx} d\, P\{\eta(t) \le x | \xi(0) = i\} =$$

$$\frac{h^i(s) - s\,\gamma^i(\zeta)\,[\xi + \lambda - \lambda\gamma(\zeta)]}{\xi - s + \lambda - \lambda h(s)}$$

for $i \ge 0$ . Takács [7].

## VI. The Virtual Age

We define the virtual age at time $t$ as zero if the server is idle and as the time already spent in the queue by the customer, who is in service at time $t$, if the queue is not empty. We define $\alpha(0) = 0$ .

We propose to calculate the transform:

$$(65) \qquad \int_0^\infty e^{-\zeta t} dt \int_0^t e^{-st} d\, P\{\alpha(t) \le \tau | \xi(0) = i\} ,$$

The result is obtained by considering the possible locations of $t$ relative to the imbedded branching processes.

We list below these possible configurations and the corresponding contributions to (65).

Case 1: $\alpha(t) = 0$

$$(66) \qquad I_1 = (\lambda+\zeta)^{-1} m_{i0}(\zeta),$$

Case 2: The initial group of $i$ customers has not yet completed service i.e. $\alpha(t) = t$ .

$$(67) \qquad I_2 = (\zeta+s)^{-1}\left[1 - h^i(\zeta+s)\right]$$

Case 3: At least one transition has occurred before $t$ in the semi-Markov process. At some time $\tau \geq o$, there is an ending of the service time of a generation and $\nu$ customers are in the queue. These $\nu$ customers terminate service at some time $t - u$. The customer who is in service at time $t$, arrives at some time $t - u - v \geq \tau$ and there are $r$ customers ahead of him in the generation to which he belongs, all of who must have arrived in the interval $(\tau, t - u - v)$. Expressing the probability of this event, we get the following contribution to (65)

(68) $$I_3 =$$

$$\int_o^\infty e^{-\zeta t} dt \int_o^t e^{-su} \int_o^{t-u} e^{-sv} \int_o^{t-u-v} \sum_{\nu=1}^{\infty} \sum_{r=o}^{\infty} e^{-\lambda(t-u-v-\tau)} \frac{[\lambda(t-u-v-\tau)]^r}{r!}$$

$$\left[H_r(u) - H_{r+1}(u)\right] d\, H_\nu(t-u-\tau)\, \lambda\, du\, d\left[M_{i\nu}(\tau) + \delta_{i\nu}\, U(\tau)\right] =$$

$$= \frac{1 - h(s+\zeta)}{s + \zeta} \cdot \frac{\lambda}{\lambda - s - \lambda h(s+\zeta)} \left[\mathfrak{M}_i[\zeta, h(\zeta+s)] + h^i(\zeta+s)\right.$$

$$\left. - \mathfrak{M}_i\left[\zeta, h[\xi+\lambda-\lambda h(\zeta+s)]\right] - h^i\left[\zeta + \lambda - \lambda h(\zeta+s)\right]\right] =$$

$$= \frac{\lambda}{\lambda - s - \lambda h(s+\zeta)} \cdot \frac{1 - h(s+\zeta)}{\zeta + s} \begin{array}{l} h^i(\zeta+s) - \gamma^i(\zeta) + \\ \lambda(\lambda+\zeta)^{-1} m_{io}(\zeta)[h(\zeta+\lambda-\lambda h(\zeta+s)) - \gamma(\zeta)] \end{array}$$

by formula (25).

Case 4: The second last transition before $t$ was into state $0$ and at time $\tau$ a new customer arrives. His service terminates at $t - u$, the customer in service at time $t$ arrives at $t - u - v \geq \tau$ and in $(\tau, t - u - v)$, $r$ customers have arrived.

(69) $I_4 =$

$$\int_0^\infty e^{-\zeta t}dt \int_0^t e^{-su} \int_0^{t-u} e^{-sv} \int_0^{t-u-v} \sum_{r=0}^{\infty} e^{-\lambda(t-u-v-\tau)} \frac{[\lambda(t-u-v-\tau)]^r}{r!} dH(t-u-\tau)$$

$$\left[H_r(u) - H_{r+1}(u)\right] \lambda \, du \, d\left[M_{io} * F(\tau)\right] =$$

$$\frac{\lambda}{\lambda+\zeta} m_{io}(\zeta) \cdot \frac{\lambda}{\lambda-s-\lambda h(s+\zeta)} \cdot \frac{1-h(s+\zeta)}{s+\zeta} \left[h(\zeta+s) - h[\zeta+\lambda-\lambda h(\zeta+s)]\right]$$

Case 5: The last transition before $t$ in the semi-Markov process was into state $0$. A new customer has arrived before $t$ at time $t-u$ and his service time is not yet finished at time $t$.

(70) $I_5 =$

$$\int_0^\infty e^{-\zeta t}dt \int_0^t e^{-su}\left[1 - H(u)\right] d\left[M_{io} * F(t-w)\right] =$$

$$= \frac{\lambda}{\lambda+\zeta} \cdot \frac{1-h(\zeta+s)}{\zeta+s} \; m_{io}(\zeta) \, ,$$

These five cases exhaust all possibilities. Summing and simplifying, we obtain:

(71) $$\int_0^\infty e^{-\zeta t}dt \int_0^t e^{-sx} d\, P\{\alpha(t) \leq x \,|\, \xi(o) = i\} =$$

$$= \frac{\lambda}{\zeta+s}\left[1 + \frac{sh^i(\zeta+s)}{\lambda-s-\lambda h(\zeta+s)}\right] - \frac{s\gamma^i(\zeta)}{[\lambda+\zeta-\lambda\gamma(\zeta)]\,[\lambda-s-\lambda h(\zeta+s)]}$$

This formula also holds for $i = o$.

Using the standard limit theorem for semi-Markov processes, we can show that as $t \to \infty$, the limiting distribution of $\alpha(t)$ is the same as the one for the virtual waitingtime $\eta(t)$, given by:

(72) $$\int_0^\infty e^{-sx} d\, P\{\eta(\infty) \leq x\} = \int_0^\infty e^{-sx} d\, P\{\alpha(\infty) \leq x\} =$$

$$\frac{(1-\lambda\alpha)s}{s-\lambda-\lambda h(s)}$$

for $1 - \lambda\alpha > 0$ .

VII. The functions $h_n(s,z)$ in the case of the Poisson queue.

The calculation of the functional iterates $h_n(s,z)$ is of course unwielding in general, but in the case of the Poisson queue they can be expressed in terms of Chebishev polynomials. If we set $\lambda = 1$ and $\alpha = \rho^{-1}$, then $h(s) = \rho(\rho+s)^{-1}$ .

Then we have:

(73) $$h_1(s,z) = \rho\left[\rho + 1 + s - z\right]^{-1} ,$$

$$h_{n+1}(s,z) = \rho\left[\rho + s + 1 - h_n(s,z)\right]^{-1} ,$$

We set

(74) $$h_n(s,z) = \frac{\rho\, P_{n-1}[\rho + 1 + s, z]}{Q_n[\rho + 1 + s, z]}$$

where $P_n$ and $Q_n$ are polynomials of degree $n$ in $\zeta + 1 + s$ . It follows that:

(75) $$P_n[\rho + 1 + s, z] = Q_n[\rho + 1 + s, z]$$

and

(76) $$Q_{n+1}[\rho + 1 + s, z] = (\rho + s + 1)\, Q_n[\rho + s + 1, z] - \rho\, Q_{n-1}[\rho + s + 1, z]$$

$$Q_0[\rho + 1 + s, z] = 1$$

$$Q_1[\rho + 1 + s, z] = \rho + 1 + s - z$$

Solving the difference equation, we find after some calculation that:

(77) $$h_n(s,z) = \rho^{\frac{1}{2}} \frac{(\rho+1+s-z)\ \sin\ (n-1)\ \theta + \sqrt{\rho}\ \sin\ (n-2)\ \theta}{(\rho+1+s-z)\ \sin\ n\ \theta + \sqrt{\rho}\ \sin\ (n-1)\ \theta}$$

with

$$\cos\ \theta = \frac{\rho + 1 + s}{2} \cdot \frac{1}{\sqrt{\rho}} \quad .$$

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13. ABSTRACT

The M|G|1 queue is treated as a sequence of branching processes, the duration of which constitutes a busy period. The first generation in each branching process consists of the customers present at the beginning of the busy period, the second generation consists of all customers, who arrive during the service time of the first generation, etc. When the queue becomes idle, the branching process becomes extinct.

This approach permits a more elementary treatment of the M|G|1 queue, without use of Rouché's theorem. It provides a natural sequence of approximants to the distributions, which we consider and it provides a simple derivation of the virtual waitingtime.

The paper also considers two random variables of interest, which have not been considered hitherto. One is the total number of customers, served in (o,t], the other is the virtual age or the time already spent in the queue, by the customer in service at time t.

We further consider a new imbedded semi-Markov process and study its asymptotic behavior.

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