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Some Limit Theorems on Branching Processes and Certain Related Processes*

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0. Summary.

Let $Z(t)$, $t \geq 0$, representing the number of particles alive at time t, be a continuous time (temporally homogeneous) Markov branching process as defined in Harris [5]. Let a positive constant b be the associated risk of death of a particle and let $h(s)$ be the probability generating function of the probabilities p_k, $k = 0, 2, 3, \ldots$, with $\sum_{k=0}^{\infty} p_k = 1$, where p_k is the probability that a particle is replaced on death by k new particles. Let $N(t)$ denote the number of particle deaths occuring during time interval $(0, t)$ and $Y(t) = \int_0^t Z(\tau)d\tau$. In this paper, results of Puri ([8], [9]) concerning the vector process $(Z(t), Y(t), N(t))$ obtained for the case of simple homogeneous birth and death process, have been generalized to branching processes. Also some limit theorems have been established concerning the behavior of the vector process.

1. Introduction.

Let $Z(t)$, $t \geq 0$, representing the number of particles present at time t, be a continuous time Markov branching process, where each particle existing at

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t has a probability $b\tau + o(\tau)$, $(b > 0)$ of dying in the interval $[t, t + \tau)$. Given that a particle dies at any time t, the probabilities are $p_0, p_2, p_3, \ldots$, that it is replaced by $0, 2, 3, \ldots$, particles, where $\sum_{i=0}^{\infty} p_i = 1$, and $0 \leq p_0 < 1$. It is assumed that different particles behave independently. Let $h(s) = \sum_{k=0}^{\infty} p_k s^k$, with $|s| \leq 1$. Here we shall be concerned with the temporally homogeneous case; consequently b and h are independent of t. Let $h'(1) < \infty$. Without loss of generality, it is assumed that $Z(0) = 1$. Let $N(t)$ denote the total number of particle-deaths occuring during $(0,t)$. Also, let $Y(t) = \int_0^t Z(\tau)d\tau$. Finally, let $\Psi(s_1,s_2,s_3;t)$ (Ψ for short) denote the Laplace quasi-probability generating function (q.p.g.f., for short) of the process $\{Z(t), Y(t), N(t)\}$ defined by

$$\Psi(s_1,s_2,s_3;t) = E\{s_1^{Z(t)}\ s_3^{N(t)} \exp[-\ s_2Y(t)]|Z(0) = 1\}, \tag{1}$$

for

$$0 \leq s_1 \leq 1\ ,\ 0 \leq s_3 \leq 1\ ,\ \text{and}\ \ s_2 \geq 0\ .$$

In two earlier papers (Puri [8], [9]) some limiting results concerning the joint distribution of the processes $Z(t)$, $Y(t)$, and $N(t)$ were established in the case of a simple homogeneous birth and death process. The present paper extends these results to a branching process characterized by b and $h(s)$. In particular, the results presented here include those due to Sevast'yanov [13] (see Harris [5], pages 108-110) as special cases where one is concerned only with the process $Z(t)$. Following Sevast'yanov we shall be exploiting a great deal the usual backward differential equation of the process $\{Z(t), Y(t), N(t)\}$.

The integral $Y(t)$ arises in several domains of application. For instance, in the study of response of host to injection of virulent bacteria, $Y(t)$ could be regarded as a measure of the total amount of toxins produced by the bacteria during time interval $(0, t)$, assuming a constant toxin-excretion rate per bacterium. Another problem, closely related to this, is the study of the distribution of the response time of the host, after infection with a certain dose of particles. The particles here are assumed to be self-reproducing entities such as bacteria or viruses. The response which the particles elicit from the host during the course of time may be death, development of a tumor or a local lesion, or a bacterial burst caused by the bacteriophages, etc.. (See Puri [11], [12] and Gani [4] for such applications.) More specifically, let

$$\Pr[X(t + \Delta t) = 0 | Z(t) = z,\ X(t) = 1] = \nu(z,t)\Delta t + o(\Delta t) , \tag{2}$$

where

$$X(t) = \begin{cases} 1 & \text{if the response has not occurred until time } t \\ 0 & \text{Otherwise,} \end{cases} \tag{3}$$

and $\nu(z,t)$ is a nonnegative measurable function of z and t. In particular, let $\nu(z,t) = \sigma z$, with $\sigma \geq 0$. Then starting with $Z(0) = 1$ and $X(0) = 1$, for a given realization ω of $Z(\tau)$ for all τ with $0 \leq \tau \leq t$, it is easy to establish that the probability of no response during $(0, t]$ is given by

$$\exp \{- \sigma \int_0^t Z(\tau;\omega)d\tau\} . \tag{4}$$

Multiplying this by $s_1^{Z(t;\,\omega)}\ s_2^{N(t;\,\omega)}$ and taking expectation of the product, over all the realizations $\{Z(\tau;\omega)\ ;\ 0 \leq \tau \leq t\}$, we have

$$E\{s_1^{Z(t)}\ s_3^{N(t)}\ \exp[-\ \sigma\ Y(t)]\}, \tag{5}$$

which is exactly the q.p.g.f. (1) with s_2 replaced by σ . Thus the integrals such as $Y(t)$ are of great practical importance particularly for the study of response time distribution arising in various live situations. Integrals more sophisticated than $Y(t)$ will naturally arise by varying the function $\nu(z,t)$ above. The distribution problems concerning such random variables as integrals of certain stochastic processes have been studied elsewhere by Bartlett [2] and by the author (see [8], [9]). One of the problems that we shall be concerned with here is that of obtaining the asymptotic expression of the q.p.g.f. (1) for large t . This, in turn, will exhibit the asymptotic behaviour of the probability of no response, namely $\Pr(X(t) = 1)$, for the simple case with $\nu(z,t) = \sigma z$. (See theorem 8, and lemma 3.)

Following the standard approach which utilizes the property of a Markov process that every epoch t is a point of regeneration (see for instance, Bartlett and Kendall [3]), Ψ can be shown to satisfy the integral equation (6) by considering whether or not a particle-death occurs in $(0,\ t)$; the epoch τ of the first death, if there is one; and the contribution to Ψ of what subsequently happens.

$$\Psi(s_1,s_2,s_3;t) = s_1\ \exp[-\ (b + s_2)t] + b\ s_3 \int_0^t \exp[-\ (b + s_2)\tau]h(\Psi(s_1,s_2,s_3;t-\tau))d\tau. \tag{6}$$

Equation (6) can be easily transformed into Kolmogorov differential equation

$$\Psi_t = b\ s_3\ h(\Psi) - (b + s_2)\ \Psi\ , \tag{7}$$

where Ψ_t denotes the partial derivative of Ψ with respect to t, and where the equation (7) is subject to the initial condition

$$\Psi(s_1, s_2, s_3; 0) = s_1\ . \tag{8}$$

It may be remarked that Ψ of (1) is defined and the equations (6) and (7) hold for all complex values of s_1 and s_3 with $|s_1| \leq 1$ and $|s_3| \leq 1$. However, unless otherwise mentioned we shall restrict ourselves to the real values of s_1, s_3 with $0 \leq s_1, s_3 \leq 1$. With $s_3 = 0$, from (6) it follows that

$$\Psi(s_1, s_2, 0; t) = s_1 \exp[-(b + s_2)t]\ . \tag{9}$$

This being of no interest, we shall henceforth restrict ourselves to the case with $s_3 > 0$, unless otherwise stated. As such we may define the function

$$f(x) = h(x) - \left(\frac{b+s_2}{bs_3}\right) x\ , \tag{10}$$

for $0 \leq x \leq 1$, so that (3) may be rewritten as

$$\Psi_t = b\ s_3\ f(\Psi)\ . \tag{11}$$

It may be remarked here that solution of (11) subject to (8) is unique with $\Psi(1,0,1;\ t) \equiv 1$. This follows from the fact that $h'(1) < \infty$ and that the right side of (11) satisfies a Lipschitz condition in the appropriate region for Ψ. For details reader may refer to Harris [5].

Next section deals with the moments of order one and two of the process $\{Z(t), Y(t), N(t)\}$, and section 3 with certain elementary results which will be needed in later sections. Section 4 is concerned with certain interconnections between the limiting distributions of $Z(t)$, $Y(t)$ and $N(t)$. Section 5 deals with theorems concerning a derived process $\{R(t),\ S(t),\ T(t)\}$ defined by (42) and its limiting behaviour as $t \to \infty$. Finally in section 6 we are concerned with certain asymptotic results for the process $\{Z(t),\ Y(t),\ N(t)\}$ when $h'(1) \leq 1$ and also with those of a derived process $\{\xi(t),\ \zeta(t),\ \eta(t)\}$ defined by (91), for the case when $h'(1) = 1$.

Before the closing of this section, we wish to add that the results of this paper have been in existence for sometimes in the form of a Mimeo-Series (see Puri [10]). After the paper was submitted for publication, the author's attention was drawn by the referee to a recent paper of Athreya and Karlin [1], where the reader may find some further results concerning the process $N(t)$ based on the exploitation of the relation (45). For this, the author is grateful to the referee. He is also grateful to Professor Karlin [6] for the argument, simpler than my original one, used in proving the a.s. convergence of the random variables $S(t)$ and $T(t)$ of section 5. Finally the reader may find very minor overlap between this paper and [1] concerning only the process $N(t)$, for example, the a.s. convergence of $T(t)$ (theorem 6) and the fact $R = T$ a.s. (theorem 7). This, of course, is only accidental and is retained here as such for the sake of completeness.

2. Moments of the Process $\{Z(t), Y(t), N(t)\}$.

While moments of $Z(t)$ are well known (see Harris [5]), we give in the following the moments of order one and two connected with the process $\{Z(t), Y(t), N(t)\}$, where a and A respectively stand for $b(h'(1) - 1)$ and $[h''(1) - h'(1) + 1]/[h'(1) - 1]$. These moments can be easily derived from (7).

$$E\,Z(t) = e^{at};\ E\,Y(t) = \frac{1}{a}(e^{at}-1)\ ;\ E\,N(t) = \frac{b}{a}(e^{at}-1)\ . \tag{12}$$

$$\operatorname{Var} Z(t) = \begin{cases} A\,e^{at}(e^{at}-1)\,, & a \neq 0 \\ h''(1)bt\,, & a = 0 \end{cases} \tag{13}$$

$$\operatorname{Var} Y(t) = \begin{cases} \frac{A}{a^2}\,[(e^{2at}-1) - 2\,at\,e^{at}]\,, & a \neq 0 \\ h''(1)\,\frac{bt^3}{3}\,, & a = 0 \end{cases} \tag{14}$$

$$\operatorname{Var} N(t) = \begin{cases} \frac{e^{at}-2at\,e^{at}-1}{1-h'(1)} + \frac{1+2at\,e^{at}-e^{2at}}{(1-h'(1))^2} + \frac{h''(1)}{(1-h'(1))^3}(2at\,e^{at}+1-e^{2at}), & a \neq 0 \\ b\,t + h''(1)\,\frac{(bt)^3}{3}\,, & a = 0 \end{cases} \tag{15}$$

$$\operatorname{Cov}(Z(t), Y(t)) = \begin{cases} \frac{A}{a}\,e^{at}\,(e^{at} - 1 - at)\,, & a \neq 0 \\ h''(1)\,\frac{bt^2}{2}\,, & a = 0 \end{cases} \tag{16}$$

$$\mathrm{Cov}(Z(t),\ N(t)) = \begin{cases} \dfrac{be^{at}}{a}\left[(1-\dfrac{b}{a}h''(1))(1-e^{at})+t(ah'(1)-b\ h''(1))\right], & a \neq 0 \\ h''(1)\,\dfrac{(bt)^2}{2}, & a = 0 \end{cases} \tag{17}$$

$$\mathrm{Cov}(Y(t),\ N(t)) = \begin{cases} \left[\dfrac{(h'(1))^2-1-2\ h''(1)}{(h'(1)-1)^2}\right] t\ e^{at} - \dfrac{e^{at}}{h'(1)-1} + \dfrac{b\ A}{a^2}e^{2at} \\ \qquad + \dfrac{b}{a^2}\left[\dfrac{(h'(1))^2-h'(1)-h''(1)}{h'(1)-1}\right], & a \neq 0 \\ h''(1)\,\dfrac{b^2t^3}{3}, & a = 0 \end{cases} \tag{18}$$

The limiting behavior (as $t \to \infty$) of various correlation coefficients ρ's are given as below:

$$\lim_{t\to\infty} \rho_{Z,Y}(t) = \lim_{t\to\infty} \rho_{Z,N}(t) = \begin{cases} 0 & a < 0 \\ \sqrt{3}/2 & a = 0 \\ 1 & a > 0 \end{cases} \tag{19}$$

$$\lim_{t\to\infty} \rho_{Y,N}(t) = \begin{cases} \sqrt{\dfrac{h''(1) + h'(1)(1-h'(1))}{h''(1) + (1-h'(1))}} & a < 0 \\ 1 & a \geq 1 \end{cases} \tag{20}$$

Also when $a = 0$, $\rho_{Z,Y}(t) \equiv \sqrt{3}/2$.

3. Some Elementary Results.

It is easy to establish the following properties of the function $f(x)$ of (10).

(a) There exists exactly one root of the equation $f(x) = 0$ lying between zero and one. This root treated as a function of s_2 and s_3 is continuous in its arguments for all $s_2 \geq 0$ and $0 < s_3 \leq 1$. Denote this root by $q(s_2,s_3)$, which is taken to be zero when $s_3 = 0$. Also when $p_0 = 0$, this root is zero for all values of s_2 and s_3. In all other cases, this is positive.

(b) Let $0 < p_0 < 1$, then $q(s_2,s_3) \uparrow q(s_2,1)$ as $s_3 \uparrow 1$, and $q(s_2,1) \uparrow q$ as $s_2 \downarrow 0$, so that $0 < q(s_2,s_3) \leq q(s_2,1) \leq q \leq 1$, where q is the well known probability of extinction of the process $Z(t)$ and is the smallest nonnegative root of the equation $h(x) = x$. (q equals one if $h'(1) \leq 1$ and is strictly less than one if $h'(1) > 1$.)

(c) For $0 \leq x \leq 1$, $f(x) \gtreqless 0$ according as $x \lesseqgtr q(s_2,s_3)$. Again,

$$f'(q(s_2,s_3)) \equiv h'(q(s_2,s_3)) - \frac{b+s_2}{bs_3} \tag{21}$$

is strictly negative for all $s_2 \geq 0$, $0 \leq s_3 \leq 1$, except when $s_2 = 0$, $s_3 = 1$ and $h'(1) = 1$, where it is equal to zero.

(d) $q(s_2,s_3)$ is the smallest nonnegative root for x of the equation

$$\Psi(x,s_2,s_3;t) = x \ ; \ 0 \leq x \leq 1 , \tag{22}$$

where t may have any positive value and $s_2 \geq 0$, $0 \leq s_3 \leq 1$.

Proof of (d) follows from (31) and the argument used for proving theorem 1 below. Again it follows easily from (6) that the q.p.g.f. Ψ satisfies the inequality

$$|\Psi(s_1,s_2,s_3;t)| \leq \frac{b|s_3|}{b+s_2} + (|s_1| - \frac{b|s_3|}{b+s_2}) \exp [-(b+s_2)t] , \tag{23}$$

where s_1 and s_3 are allowed to take complex values with $|s_1| \leq 1$, and $|s_3| \leq 1$.

Let for $k = 0, 1, 2, \ldots,$

$$\varphi_k(s_2,s_3;t) = \sum_{m=0}^{\infty} s_3^m \int_0^{\infty} e^{-s_2 y} d_y \Pr[Z(t) = k,\ Y(t) \leq y,\ N(t) = m] , \tag{24}$$

We then have the identity

$$\Psi(s_1,s_2,s_3;t) \equiv \sum_{k=0}^{\infty} s_1^k \varphi_k(s_2,s_3;t) . \tag{25}$$

In particular it follows from (6) that φ_0 satisfies

$$\Psi(0,s_2,s_3;t) = \varphi_0(s_2,s_3;t) = b\ s_3 \int_0^t e^{-(b+s_2)\tau} h(\varphi_0(s_2,s_3;\ t-\tau))d\tau , \tag{26}$$

which yields $d\ \varphi_0/dt = b\ s_3\ h(\varphi_0) - (b+s_2)\ \varphi_0$, with $\varphi_0(s_2,s_3;0) = 0$.

From (26) we immediately have

$$\frac{bp_0 s_3}{b+s_2}(1-e^{-(b+s_2)t}) \le \varphi_0(s_2,s_3;t) \le \frac{bs_3}{b+s_2}(1-e^{-(b+s_2)t}) \quad . \tag{27}$$

Finally we have

Theorem 1. *For every fixed* (s_2,s_3), $\varphi_0(s_2,s_3;t) \uparrow q(s_2,s_3)$ as $t \uparrow \infty$.

Proof. We first note that for $t,\tau \ge 0$,

$$\Psi(s_1,s_2,s_3;\ t+\tau) = \Psi\ (\Psi(s_1,s_2,s_3;\ \tau),\ s_2,s_3;\ t)\ . \tag{28}$$

Putting $s_1 = 0$, it follows that

$$\varphi_0(s_2,s_3;\ t+\tau) = \Psi(\varphi_0(s_2,s_3;\ \tau),\ s_2,s_3;\ t)\ , \tag{29}$$

so that

$$\varphi_0(s_2,s_3;\ t+\tau) \ge \Psi(0,s_2,s_3;\ t) = \varphi_0(s_2,s_3;\ t)\ . \tag{30}$$

Thus $\varphi_0(s_2,s_3;t)$, for every given s_2 and s_3 , is a monotone increasing function of t and being bounded between 0 and 1 has a limit as $t \to \infty$, say $\pi(s_2,s_3)$. Letting $\tau \to \infty$ in (28) we have for all t

$$\pi(s_2,s_3) = \Psi(\pi(s_2,s_3),\ s_2,s_3;\ t)\ . \tag{31}$$

Again from (11) we have

$$\frac{\Psi(s_1,s_2,s_3;\ t+\tau) - \Psi(s_1,s_2,s_3;\ t)}{\tau} = b\ s_3\ f(\Psi(s_1,s_2,s_3;t)) + \frac{o(\tau)}{\tau}\ ,$$

where on replacing s_1 by $\pi(s_2,s_3)$ and using (31) we find for any $\tau > 0$

$$0 = b\ s_3\ f(\pi(s_2,s_3)) + \frac{o(\tau)}{\tau}\ .$$

Letting $\tau \to 0$, we find that $\pi(s_2,s_3)$ satisfies the equation $f(x) = 0$, which by virtue of property (a) of $f(x)$ establishes that $\pi(s_2,s_3) = q(s_2,s_3)$.

Corollary 1. $$\frac{bp_0s_3}{b+s_2} \leq q(s_2,s_3) \leq \frac{bs_3}{b+s_2}\ . \tag{32}$$

Corollary follows from theorem 1 and (27). Again, it can be seen that the right side equality of (32) holds only when $p_0 = 1$, a case which we have excluded by assumption.

4. Limiting Behaviour of the Process $\{Z(t),\ Y(t),\ N(t)\}$.

Following theorem gives the limiting behaviour of the process $\{Z(t),Y(t),N(t)\}$ as $t \to \infty$.

Theorem 2. Let $p_0 > 0$, then for every fixed (s_1,s_2,s_3) such that $0 \leq s_1 \leq 1$, $0 < s_3 \leq 1$, $s_2 \geq 0$, as $t \to \infty$, $\Psi(s_1,s_2,s_3;t) \uparrow q(s_2,s_3)$ or $\Psi(s_1,s_2,s_3;\ t) \downarrow q(s_2,s_3)$ according as $s_1 < q(s_2,s_3)$ or $s_1 > q(s_2,s_3)$ respectively.

Outlines of Proof. When $s_1 = 0$, the theorem holds by virtue of theorem 1. Thus, let us first consider the case when $0 < s_1 < q(s_2,s_3)$. From(11) we have

$\Psi_t|_{t=0} = bs_3 f(s_1)$, which is positive by virtue of property (c) of $f(x)$ as given in the last section. Therefore, there exists an $\epsilon > 0$, such that for all $0 < \tau \leq \epsilon$, $\Psi(s_1,s_2,s_3;\ \tau) > s_1$. Now for any given $t > 0$, choose τ and a positive integer n such that $0 < \tau \leq \epsilon$ and $t = n\tau$. Then

$$\Psi(s_1,s_2,s_3;\ t) = \Psi(s_1,s_2,s_3;\ n\tau) = \Psi(\Psi(s_1,s_2,s_3;\ \tau),s_2,s_3;\ (n-1)\tau)$$

$$> \Psi(s_1,s_2,s_3;\ (n-1)\tau) > \ldots > \Psi(s_1,s_2,s_3;\ \tau) > s_1 \quad .$$

Thus for all $t > 0$, $\Psi(s_1,s_2,s_3;\ t) > s_1$. From this and (28) it follows that $\Psi(s_1,s_2,s_3;\ t)$ is an increasing function of t . The remaining argument is similar to the one used in proof of theorem 1. Also, the proof for the case with $s_1 > q(s_2,s_3)$ follows along similar lines and is therefore omitted.

From theorem 2 follows the convergence in law of the process $\{Z(t), Y(t), N(t)\}$ to the random vector $\{Z,Y,N\}$, where

$$E\ \{s_1^Z\ e^{-s_2 Y}\ s_3^N\} = q(s_2,s_3)\ . \tag{33}$$

However, it is well known that if $h'(1) \leq 1$, $Z(t) \to 0$ a.s. as $t \to \infty$. On the other hand if $h'(1) > 1$, $Z(t)$ tends to zero with probability q which is strictly less than one and to ∞ with probability $1-q$. Also it may be noted from property (b) of $f(x)$ that $q(0,1) = q$ is equal to one or strictly less than one according as $h'(1) \leq 1$ or $h'(1) > 1$. Therefore when $h'(1) \leq 1$, $\{Z,Y,N\}$ is an honest set of random variables. On the other hand when $h'(1) > 1$ they are finite only with probability $q < 1$. Restricting only to the joint distribution of Y and N we have

$$E\{e^{-s_2 Y} s_3^N\} = q(s_2, s_3) \ , \tag{34}$$

and the following theorem.

<u>Theorem</u> 3: <u>For every</u> $k = 1,2,\ldots,$ <u>with</u> $P(N = k) > 0$,

$$(Y|N = k) = \frac{1}{2b} \chi^2_{2k} \ . \tag{35}$$

<u>Proof</u> From the fact that $q(s_2, s_3)$ is the unique root of $f(x) = h(x) - (\frac{b+s_2}{bs_3}) x = 0$, lying between 0 and 1 , it is easy to show that

$$q(s_2, 1) = q(0, \frac{b}{b+s_2}) \ , \tag{36}$$

which on using (34) with $s_3 = 1$ yields

$$E(e^{-s_2 Y}) = E\{\frac{1}{1+(s_2/b)}\}^N = \sum_{k=1}^{\infty} P(N=k)\ E\{\exp[-s_2\ \chi^2_{2k}/2b]\} \ , \tag{37}$$

where in the end we have used the fact that $P(N=0) = 0$. The theorem now easily follows.

<u>Theorem</u> 4. <u>For</u> $k = 1,2,\ldots,$

$$P(N=k) = \frac{1}{k!} \frac{d^{k-1}}{dx^{k-1}} (h(x))^k \Big|_{x=0} \ . \tag{38}$$

Proof. Let $0 < s_3 \leq 1$ and $s_2 \geq 0$ be such that $\frac{bs_3}{b+s_2} < 1$, so that $h(z)$ treated as a function of a complex variable z is regular on and inside the closed contour $C = \{z: |z| \leq \frac{bs_3}{b+s_2}\}$. Then since $|(\frac{bs_3}{b+s_2})\, h(z)| < |z|$ at all points z on the perimeter of C , by Langrange's theorem (see Whittaker and Watson [14]) the equation $x = (\frac{bs_3}{b+s_2})h(x)$ has one root in the interior of C which in our case is denoted by $q(s_2,s_3)$. Furthermore by the same theorem we have

$$q(s_2,s_3) = \sum_{k=1}^{\infty} \frac{1}{k!} (\frac{bs_3}{b+s_2})^k \frac{d^{k-1}}{dx^{k-1}} [h(x)]^k \Big|_{x=0} . \tag{39}$$

In this letting $s_2 = 0$ but keeping $s_3 < 1$, we have

$$q(0,s_3) = E(s_3^N) = \sum_{k=0}^{\infty} s_3^k \cdot \frac{1}{k!} \frac{d^{k-1}}{dx^{k-1}} [h(x)]^k \Big|_{x=0} , \tag{40}$$

and from this the result follows.

Remark. If $h'(1) \leq 1$, $q(0,1) = 1$ so that the probabilities of theorem 4 add up to one. If on the other hand $h'(1) > 1$, they add up to q which is strictly less than one, the remaining probability $(1-q)$ being equal to $P(N = \infty)$.

Note that since the random variables $Y(t)$ and $N(t)$ are nondecreasing functions of t , $Y(t) \uparrow Y$ a.s. and $N(t) \uparrow N$ a.s., where q.p.g.f. of Y and N is given by (34). Also when $h'(1) < 1$ and $h''(1) < \infty$, one can easily establish that both $Y(t)$ and $N(t)$ converge respectively to Y and N in mean square (m.s.). For instance, in the case of $Y(t)$, by elementary calculations (see Puri [8]) one can find the expression for $E(Y(t+\tau) - Y(t))^2$, where on letting $\tau \to \infty$, one can show (we omit the messy algebra here) that with $a = b(h'(1)-1)$,

$$\lim_{\tau\to\infty} E(Y(t+\tau)-Y(t))^2 = E(Y-Y(t))^2 = \frac{1}{a^2} e^{at}[(e^{at}-2)\frac{h''(1)}{h'(1)-1} + 2], \quad (41)$$

which tends to zero as $t \to \infty$. This establishes convergence of $Y(t)$ to Y in m.s. .

Let

$$R(t) = \frac{Z(t)}{EZ(t)}, \quad S(t) = \frac{Y(t)}{EY(t)}, \quad T(t) = \frac{N(t)}{EN(t)}. \quad (42)$$

In the next section we shall discuss the limiting behavior of the process $\{R(t), S(t), T(t)\}$.

5. Convergence of the Process $\{R(t), S(t), T(t)\}$.

For $u, v, w \geq 0$, let

$$\Phi(u,v,w;\ t) = E\{\exp(-uR(t) - v\, S(t) - w\, T(t))\}. \quad (43)$$

On using (1) this becomes equal to

$$\psi(\exp(-\frac{u}{EZ(t)}), \frac{v}{EY(t)}, \exp(-\frac{w}{EN(t)});\ t)$$

We then have

Theorem 5. *If* $h'(1) < 1$, $h''(1) < \infty$, *then*

$$\lim_{t\to\infty} \Phi(u,v,w;\ t) = q(bv(1-h'(1)), \exp[-w(1-h'(1))]). \quad (44)$$

Proof is omitted as it follows along the lines of proof of theorem 9 of section 6.2 and the facts that

$$\lim_{t\to\infty} E\, Y(t) = [b(1-h'(1))]^{-1}, \quad \lim_{t\to\infty} E\, N(t) = (1-h'(1))^{-1} .$$

Whereas the behavior of $R(t)$ is already well known (see Harris [5]), we give in the following those of $S(t)$ and $T(t)$.

Theorem 6. $S(t)$ and $T(t)$ tend to S and T respectively as $t \to \infty$, both with probability one. Furthermore, if $h'(1) \neq 1$ and $h''(1) < \infty$, then both these convergences are in m.s.. More specifically, both $E(S(t) - S)^2$ and $E(T(t) - T)^2$ are $O(\exp[-|1-h'(1)|bt])$ as $t \to \infty$.

Proof. The a.s. convergence of $S(t)$ is an immediate consequence of the fact that $Z(t)\, e^{-at}$ being a nonnegative martingale converges with probability one. Regarding the a.s. convergence of $T(t)$, we note that

$$Z(t) = 1 + \sum_{i=1}^{N(t)} \theta_i \quad , \tag{45}$$

where $(\theta_i + 1)$ denotes the number of particles replacing the ith particle-death. Now the a.s. convergence of $T(t)$ follows immediately from (45) by invoking the strong law and using the a.s. convergence of $Z(t)e^{-at}$. (See Athreya and Karlin [1], for further exploitation of relation (45)). For the case of convergence in m.s., we shall consider only the case of $S(t)$, the case of $T(t)$ being analogous. After elementary computation, one first finds an expression for $E(S(t+\tau) - S(t))^2$. This expression being somewhat lengthy is not reproduced here.

However in this for fixed $\tau > 0$, letting $t \to \infty$ it is easy to verify that when $h'(1) \neq 1$, $E(S(t+\tau) - S(t))^2 \to 0$ uniformly in τ , from which follows that $S(t) \to S$ in m.s.. In the same expression, if instead we let $\tau \to \infty$, one obtains

$$E(S(t)-S)^2 = \begin{cases} \dfrac{2A\ e^{at}}{(1-e^{at})^2}\,[(e^{at}-1) - at\ e^{at}]\ ,\ h'(1) < 1 \\[2ex] \dfrac{2A\ e^{-at}}{(1-e^{-at})^2}\,[1-e^{-at} - at\ e^{-at}]\ ,\ h'(1) > 1\ , \end{cases} \tag{46}$$

where A and a are as defined in section 2. From (46) it is clear that

$$E(S(t) - S)^2 = 0(\exp[-|1-h'(1)|b\ t])\ ,\ (t \to \infty)\ . \tag{47}$$

This completes the proof.

Again, when $h'(1) > 1$ and $h''(1) < \infty$, it is well known (see Harris [5]) that $R(t)$ converges in m.s. and with probability one to a random variable R . If $\varphi(u)$ is the moment-generating function (m.g.f.) of R , $\varphi(u) = E[\exp(-uR)]$, we have its functional inverse given by

$$\varphi^{-1}(s) = (1-s)\ \exp\{\int_1^s [\frac{h'(1)-1}{h(x)-x} + \frac{1}{1-x}]\ dx\};\ q < s \leq 1\ . \tag{48}$$

This result is due to Sevast'yanov [13]. It is also known that the distribution function corresponding to φ has a density except for a possible discontinuity at zero. Following theorem connects these results with those of $S(t)$ and $T(t)$.

<u>Theorem</u> 7. <u>If</u> $h'(1) > 1$, $h''(1) < \infty$, <u>then</u>

$$R = S = T \text{ , a.s. ,} \tag{49}$$

<u>where the functional inverse of m.g.f. of the common distribution is given by</u> (48).

<u>Proof</u>. Easy calculations yield that

$$\lim_{\tau \to \infty} E(R(t+\tau) - S(t))^2 = E(R-S(t))^2 = \frac{2A}{(e^{at}-1)^2}(e^{at}-1-at) \quad , \tag{50}$$

which tends to zero as $t \to \infty$. Thus $S(t) \to R$ in m.s.. From this and theorem 6, it follows that $R = S$ a.s.. Proof of $T = R$ a.s. is analogous.

<u>Remark</u>. It is not surprising to find under the conditions of the theorem 7 that $R = S$ a.s., keeping in mind the facts that $Z(t)$ tends to a constant (zero with probability q and ∞ with probability $1 - q$) and both $E\,Z(t)$ and $EY(t)$ tend to ∞ as $t \to \infty$, so that as $t \to \infty$ one would expect

$$R(t) \sim S(t) \quad \text{a.s.} \tag{51}$$

6. Some Asymptotic Results:

Whereas asymptotic results for the case with $h'(1) > 1$ have already been given along with others in theorems 6 and 7, in the following subsections we shall deal with results corresponding to the cases with (i) $h'(1) < 1$ and (ii) $h'(1) = 1$.

6.1. Case with $h'(1) < 1$.

Following theorem gives an asymptotic expression for the q.p.g.f. $\Psi(s_1,s_2,s_3;t)$ for large t . With regards to the response-time distribution problem discussed in section 1, one can easily obtain the asymptotic expression for the probability of no response by simply putting $s_1 = s_3 = 1$ and $s_2 = \sigma$, in expression (52) below.

It is clear that for large t, the theoretical no-response curve is approximately exponential in its behaviour.

Theorem 8. *If* $h'(1) < 1$, $h''(1) < \infty$, *then for* $s_2 \geq 0$, $0 < s_3 \leq 1$, *as* $t \to \infty$

$$\Psi(s_1,s_2,s_3;t) = q(s_2,s_3) \pm \exp[A(s_1,t)]\{1+O(\exp[A(s_1,t)])\} \tag{52}$$

where $+$ *or* $-$ *signs are taken according as* s_1 *is greater or less than* $q(s_2,s_3)$ *and*

$$A(s_1,t) = f'(q(s_2,s_3))(K(s_1) + bs_3t) , \tag{53}$$

and where for $s \neq q(s_2,s_3)$, $0 \leq s \leq 1$,

$$K(s) = \int_{q(s_2,s_3)}^{s} B(x)dx + \frac{\log|q(s_2,s_3)-s|}{f'(q(s_2,s_3))} , \tag{54}$$

with

$$B(x) = \frac{1}{f(x)} - \frac{1}{f'(q(s_2,s_3))(x-q(s_2,s_3))} . \tag{55}$$

In particular, when $s_1 = q(s_2,s_3)$,

$$\Psi(s_1, s_2, s_3; t) = q(s_2, s_3) . \tag{56}$$

The proof follows along standard lines (see Karlin [7]) and is therefore omitted. An eager reader may, however, find its details in the Mimeo-Series (Puri [10]).

Corollary 2. If $h'(1) < 1$, $h''(1) < \infty$, *then*

$$\lim_{t\to\infty} \frac{q(s_2,s_3) - \varphi_0(s_2,s_3; t)}{\exp [A(0,t)]} = 1 , \tag{57}$$

where $A(0,t)$ *is defined by* (53) *with* $s_1 = 0$.

Proof follows from theorem 8 taking $s_1 = 0$ and the fact that $f'(q(s_2,s_3)) < 0$.

Corollary 3. Under the conditions of the theorem 8, for $0 < s_1 < 1$,

$$\lim_{t\to\infty} E(s_1^{Z(t)} e^{-s_2 Y(t)} s_3^{N(t)} \mid Z(t) > 0) = 0 , \tag{58}$$

$$\lim_{t\to\infty} E(e^{-s_2 Y(t)} s_3^{N(t)} \mid Z(t) = 0) = q(s_2,s_3) , \tag{59}$$

and for $s_1 < q(s_2,s_3)$,

$$\lim_{t\to\infty} \frac{\Psi(s_1,s_2,s_3;\, t) - \varphi_0(s_2,s_3;\, t)}{q(s_2,s_3) - \varphi_0(s_2,s_3;\, t)} = 1 - \exp[f'(q(s_2,s_3)) \int_0^{s_1} \frac{dx}{f(x)}] . \tag{60}$$

The proof of (60) follows from theorem 8; those of (58) and (59) follow from theorem 8 and the well known asymptotic expression for $P(Z(t) = 0)$ for large t (see Harris [5]), given by

$$P(Z(t) = 0) = 1-\exp[-(1-h'(1))K + at] + O(\exp[2\, at]) , \tag{61}$$

where $a = b(h'(1)-1)$, and

$$K = \int_0^1 [\frac{1}{(1-h'(1))(1-x)} - \frac{1}{h(x) - x}] \; dx .$$

Finally it may be remarked that (58) and (59) with its right side replaced by $q(s_2,s_3)/q$ hold even when $h'(1) > 1$.

6.2. Case with $h'(1) = 1$.

Before we prove the main theorem of this section we require lemmas 1-4 given below. Lemma 1 holds irrespect of whether $h'(1) = 1$ or not. Lemmas 2 and 3 hold not only when $h'(1) = 1$ but also when $h'(1) < 1$. Lemma 3 is by itself important in that the expression (79) of lemma 3 with $s_1 = s_3 = 1$ and $s_2 = \sigma$ yields an improvement over (52) giving an asymptotic expression for the probability of no response as discussed in section 1.

Lemma 1. If $h''(1) < \infty$ and $0 < p_0 < 1$, then for every arbitrary but fixed ϵ_1, ϵ_2 such that $\epsilon_1 > 0$ and $0 < \epsilon_2 \leq 1$, $L(s_2,s_3;\, t) = q(s_2,s_3) - \varphi_0(s_2,s_3;\, t) \downarrow 0$ as $t \to \infty$ uniformly for all (s_2,s_3) with $0 \leq s_2 \leq \epsilon_1$ and $\epsilon_2 \leq s_3 \leq 1$.

Proof. Taking partial Taylor expansion of right side of $d\varphi_0/dt = bs_3h(\varphi_0)-(b+s_2)\varphi_0$, around point $\varphi_0 = q(s_2,s_3)$, it is easy to show that $L(s_2,s_3;\, t)$ satisfies the differential equation,

$$\frac{dL}{dt} = b\, s_3\, f'(q(s_2,s_3))\, L - \frac{bs_3}{2} h''(*)L^2 , \tag{62}$$

where $\varphi_0 < * < q(s_2,s_3)$. Also, since $\varphi_0(s_2,s_3;0) = 0$, we have $L(s_2,s_3;0) = q(s_2,s_3)$. Let for an arbitrarily given $0 < \epsilon < 1$, t_0 be such that for $t > t_0$, $\exp(-bt) < \epsilon$. Then using (27) we find that $\varphi_0(s_2,s_3;\, t) \geq \delta$, uniformly for all (s_2,s_3) satisfying $0 \leq s_2 \leq \epsilon_1$, $\epsilon_2 \leq s_3 \leq 1$ and for $t \geq t_0$, where

$$\delta = \frac{bp_0\, \epsilon_2}{b+\epsilon_1} (1-\epsilon) > 0 . \tag{63}$$

Hence for $t \geq t_0$

$$h''(*) \geq h''(\varphi_0) \geq h''(\delta) > 0 . \tag{64}$$

Now since $f'(q(s_2,s_3))$ is negative (property (c) of function $f(x)$) we have from (62)

$$\frac{dL}{dt} \leq - \frac{b\, s_3}{2} h''(*)\, L^2 .$$

Using (64) this implies that

$$\frac{1}{L^2}\, \frac{dL}{dt} \leq - \frac{b\epsilon_2}{2} h''(\delta) \tag{65}$$

uniformly for all $t \geq t_0$, $0 \leq s_2 \leq \epsilon_1$ and $\epsilon_2 \leq s_3 \leq 1$.

Integrating (65) on both sides from t_0 to t , we have

$$\frac{1}{L(s_2,s_3;\ t)} - \frac{1}{L(s_2,s_3;\ t_0)} \geq \frac{b\epsilon_2}{2}\ h''(\delta)(t - t_0)\quad , \tag{66}$$

which yields

$$L(s_2,s_3;\ t) \leq \frac{2}{b\epsilon_2\ h''(\delta)(t-t_0)}\ ;\ (t > t_0)\ . \tag{67}$$

Since the right side of (67) does not depend upon s_2 and s_3 and it tends to zero as $t \to \infty$, the lemma follows in view of theorem 1.

Lemma 2. Let $H(s_1,s_2,s_3;t) = \Psi(s_1,s_2,s_3;t) - q(s_2,s_3)$, $h'(1) \leq 1$, $h'''(1) < \infty$. Then for every arbitrary but fixed ϵ_1, ϵ_2 such that $0 < \epsilon_2 \leq 1$ and $\epsilon_1 > 0$, as $t \to \infty$

$$\frac{1}{t}\int_0^t H(s_1,s_2,s_3;\ \tau)\ d\tau \to 0\ , \tag{68}$$

uniformly for all (s_2,s_3) with $0 \leq s_2 \leq \epsilon_1$ and $\epsilon_2 \leq s_3 \leq 1$.

Proof. Note that the condition $h'(1) \leq 1$ implies that $p_0 > 0$. Let $Q(t) = 1 - P(Z(t) = 0)$, then from (25) we have

$$|H(s_1,s_2,s_3;\ t)| = |\sum_{k=1}^{\infty} s_1^k\ \varphi_k(s_2,s_3;t) + \varphi_0(s_2,s_3) - q(s_2,s_3)|$$

$$\leq Q(t) + L(s_2,s_3;\ t)\quad , \tag{69}$$

where $L(s_2,s_3;\ t)$ is as defined in lemma 1. As in (62), it can be easily shown that

$$\frac{dL}{dt} = b\ s_3\ f'(q(s_2,s_3))\ L - \frac{bs_3}{2}\ h''(q(s_2,s_3))\ L^2 + \frac{bs_3}{6}\ h'''(*)L^3\ , \tag{70}$$

where $\varphi_0 < * < q(s_2,s_3)$. Thus we have for $t \geq 0$,

$$\frac{1}{L^2}\frac{dL}{dt} \leq -\frac{bs_3}{2} h''(q(s_2,s_3)) + \frac{bs_3}{6} h'''(q(s_2,s_3)) L . \tag{71}$$

Again, by virtue of lemma 1, it is possible to find $t(\epsilon_1,\epsilon_2) = t_1$ (say), such that for all $t \geq t_1$, $0 \leq s_2 \leq \epsilon_1$ and $\epsilon_2 \leq s_3 < 1$,

$$L(s_2,s_3;t) \leq \frac{3h''(q(\epsilon_1,\epsilon_2))}{2\ h'''(1)} \quad . \tag{72}$$

On the other hand by virtue of property (b) of $f(x)$ as given in section 2, for $0 \leq s_2 < \epsilon_1$, $\epsilon_2 \leq s_3 \leq 1$, the right side of (72) is less than or equal to $3h''(q(s_2,s_3))/2\ h'''(q(s_2,s_3))$ and hence from (71) we have

$$\frac{1}{L^2}\frac{dL}{dt} \leq -\frac{bs_3}{4} h''(q(s_2,s_3)) \leq \frac{-b\epsilon_2}{4} h''(q(\epsilon_1,\epsilon_2)) , \tag{74}$$

which holds for all $t \geq t_1$, $0 \leq s_2 \leq \epsilon_1$ and $\epsilon_2 \leq s_3 \leq 1$. Integrating (74) on both sides from t_1 to t , we have after some simplification

$$L(s_2,s_3;\ t) \leq \frac{4}{b\ \epsilon_2\ h''(q(\epsilon_1,\epsilon_2))(t-t_1)},\ (t > t_1) \quad . \tag{75}$$

Again from (69) we have for $t_2 > t_1$,

$$\left|\int_0^t H(s_1,s_2,s_3;\ \tau)d\tau\right| \leq \int_0^t |H(s_1,s_2,s_3;\tau)|d\tau \leq t_2 + \int_{t_2}^t (Q(\tau) + L(s_2,s_3;\tau))\ d\ \tau. \tag{76}$$

Now it is well known that

$$Q(t) \sim \exp\ [-(1 - h'(1))(K + bt)],\ (t \to \infty)\ , \tag{77}$$

when $h'(1) < 1$, and

$$Q(t) \sim \frac{2}{b\ h''(1)t}\ ,\ (t \to \infty)\ , \tag{78}$$

when $h'(1) = 1$. These results are due to Sevast'yanov [13] and can be found in Harris [5]. In particular (77) follows from (61). From (77) and (78) it follows that when $h'(1) \leq 1$, $\int_{t_2}^{t} Q(\tau)d\tau = o(\tau)$. Furthermore it is clear from (75) that $\int_{t_2}^{t} L(s_2,s_3;\ \tau)d\tau = o(t)$. The lemma now follows from (76) using these facts.

Lemma 3. Under the conditions of lemma 2,

$$H(s_1,s_2,s_3;t) = \frac{c(s_1-q(s_2,s_3))\ \exp[-ct + o(t)]}{c-e(s_1-q(s_2,s_3))(1 - \exp[-ct + o(t)])}\ , \tag{79}$$

where as $t \to \infty$, $o(t)/t$ tends to zero uniformly for all (s_1,s_2,s_3) with $0 \leq s_1 \leq 1$, $0 \leq s_2 \leq \epsilon_1$, $\epsilon_2 \leq s_3 \leq 1$, for every fixed (ϵ_1,ϵ_2) with $\epsilon_1 > 0$, $1 \geq \epsilon_2 > 0$, and where

$$c = -b\ s_3\ f'(q(s_2,s_3));\ e = \frac{b\ s_3}{2}\ h''(q(s_2,s_3))\ . \tag{80}$$

Proof. We shall prove (79) for the case when $s_1 > q(s_2,s_3)$; proof for the case when $s_1 < q(s_2,s_3)$ follows in a similar manner and is therefore omitted. We are thus given that $s_1 > q(s_2,s_3)$, $0 \leq s_2 \leq \epsilon_1$, and $0 < \epsilon_2 \leq s_3 \leq 1$. As before, the analogue of (70) for the present case is given by

$$\frac{dH}{dt} = -\ cH + eH^2 + \frac{b\ s_3}{6}\ h'''(*)\ H^3\ , \tag{81}$$

where $q(s_2,s_3) < * < \Psi(s_1,s_2,s_3;t)$, and H is as defined in lemma 2. Here H is positive for $t > 0$, since $s_1 > q(s_2,s_3)$. Dividing both sides of (81) by the sum of the first two terms on its right side and integerating we have

$$\int_{s_1-q(s_2,s_3)}^{H} \frac{dH}{H(-c + eH)} = t + \frac{b\ s_3}{6} \int_0^t \frac{h'''(*)H^2\ dt}{(-c + eH)} . \tag{82}$$

Note that $-c + eH = bs_3 \cdot [f'(q(s_2,s_3)) + \frac{1}{2} h''(q(s_2,s_3))\ H] < 0$ and since $h'(1) \leq 1$ and $H < 1 - q(s_2,s_3)$, we have

$$\begin{aligned} |-c+eH| &= (b+s_2) - bs_3 h'(q(s_2,s_3)) - \frac{bs_3}{2} h''(q(s_2,s_3))H \\ &= (b+s_2) - bs_3[h'(1)+h''(q^*)(q(s_2,s_3)-1)] - \frac{bs_3}{2} h''(q(s_2,s_3))H \\ &= b(1-s_3h'(1))+s_2+ bs_3h''(q^*)(1-q(s_2,s_3)) - \frac{bs_3}{2} h''(q(s_2,s_3))H \\ &\geq \frac{bs_3}{2} h''(q(s_2,s_3))\ (1-q(s_2,s_3)) , \end{aligned} \tag{83}$$

where $q(s_2,s_3) < q^* < 1$. Thus if I denotes the integral on the right side of (82), then using the fact that $H < 1 - q(s_2,s_3)$ in (83), we have

$$|I| \leq \frac{h'''(1)}{3h''(q(\epsilon_1,\epsilon_2))} \int_0^t H(s_1,s_2,s_3;\ \tau)\ d\tau = o(t) \tag{84}$$

where the last step follows from lemma 2. (82) now reduces to

$$\int_{s_1-q(s_2,s_3)}^{H} \frac{dH}{H(c-eH)} = -\ t + o(t) , \tag{85}$$

which on solving for H in a straight forward manner yields the desired result (79).

Lemma 4. For fixed $u,v,w > 0$, if $h'(1) = 1$ and $h''(1) < \infty$, then

(a) $$\lim_{t\to\infty} q(\frac{2v}{bt^2h''(1)} , \exp[\frac{-2w}{b^2t^2h''(1)}]) = q(0,1) = 1 \tag{86}$$

(b) $$\lim_{t\to\infty} t\{b+\frac{2v}{bt^2h''(1)} - b\ \exp[\frac{-2w}{b^2t^2h''(1)}]\cdot h'(q(\frac{2v}{bt^2h''(1)}, \exp[\frac{-2w}{b^2t^2h''(1)}]))\} = 2\sqrt{v+w} . \tag{87}$$

$$\text{(c)} \quad \lim_{t\to\infty} t\{\exp[\frac{-2u}{bt\, h''(1)}] - q(\frac{2v}{bt^2h''(1)}, \exp[\frac{-2w}{bt^2h''(1)}])\} = \frac{-2(u+\sqrt{v+w})}{bh''(1)} . \tag{88}$$

Proof is elementary and makes use of the property (b) of $f(x)$ and the fact that $q(s_2,s_3)$ satisfies the equation $h(x) - \frac{(b+s_2)x}{bs_3} = 0$.

We shall assume from here on that $h'(1) = 1$. Let

$$M_{Z(t)} = EZ(t)/Q(t);\ M_{Y(t)} = EY(t)/Q(t);\ M_{N(t)} = EN(t)/Q(t) , \tag{89}$$

where $Q(t) = P(Z(t) > 0)$. By virtue of (78) and (12) we have for $t \to \infty$,

$$M_{Z(t)} \sim \frac{bth''(1)}{2} ;\ M_{Y(t)} \sim \frac{bt^2h''(1)}{2} ;\ M_{N(t)} \sim \frac{b^2t^2h''(1)}{2} . \tag{90}$$

Finally let

$$\xi(t) = \frac{Z(t)}{M_{Z(t)}} ;\ \zeta(t) = \frac{Y(t)}{M_{Y(t)}} ;\ \eta(t) = \frac{N(t)}{M_{N(t)}} ; \tag{91}$$

then we have

Theorem 9. If $u,v,w \geq 0$, $h'(1) = 1$, and $h'''(1) < \infty$, then we have

$$\lim_{t\to\infty} E\{\exp[-u\xi(t) - v\zeta(t) - w\eta(t)] \mid Z(t) > 0\}$$

$$= \frac{4(v+w)\exp[-2\sqrt{v+w}]}{(1-\exp[-2\sqrt{v+w}])[2\sqrt{v+w} - (\sqrt{v+w} - u)(1-\exp[-2\sqrt{v+w}])]} . \tag{92}$$

Proof. It is easy to see that the left side of (92) is equal to the limit as $t \to \infty$ of the expression

$$\frac{[q(s_2,s_3) - \varphi_0(s_2,s_3;\ t)] - [q(s_2,s_3) - \Psi(s_1,s_2,s_3;\ t)]}{Q(t)} , \tag{93}$$

with s_1, s_2 and s_3 respectively replaced by $\exp[-u/M_{Z(t)}]$, $v/M_{Y(t)}$ and $\exp[-w/M_{N(t)}]$. Since $H(0,s_2,s_3;t) = \varphi_0(s_2,s_3;t) - q(s_2,s_3)$, on putting $s_1 = 0$ in (79) we have

$$\varphi_0(s_2,s_3;t) = q(s_2,s_3) - \frac{cq(s_2,s_3)\exp[-ct + o(t)]}{c+e\, q(s_2,s_3)(1-\exp[-ct+o(t)])} \ . \tag{94}$$

The proof of the theorem follows by substituting (78), (79), (90) and (94) in (93), and taking the limit of (93) as $t \to \infty$. While so doing we use lemma 4 and the fact that $o(t)$ of (79) and (94) is such that $o(t)/t$ tends to zero uniformly for all (s_1,s_2,s_3) ranging over an appropriate region.

Remark 1. From theorem 9, it follows that as $t \to \infty$

$$(\xi(t),\ \zeta(t),\ \eta(t) \mid Z(t) > 0) \overset{\mathscr{L}}{\to} (\xi,\zeta,\eta) \ , \tag{95}$$

where the m.g.f. of the vector random variable (ξ,ζ,η) is given by (92). Again in (92) since v and w occur as $(v+w)$ it follows that $\zeta = \eta$ a.s..

Remark 2. Putting $v = w = 0$ in (92) we have

$$\lim_{t\to\infty} E\{\exp[-u\xi(t)] | Z(t) > 0\} = \frac{1}{1+u} \ . \tag{96}$$

This is a well known result and is due to Sevast'yanov [13]. From this it follows that ξ is exponentially distributed. Again, the m.g.f. of the common distribution of ζ and η is given by

$$E\{\exp(-v\ \zeta)\} = \frac{4\sqrt{v}}{\exp(2\sqrt{v}) - \exp(-2\sqrt{v})} \ . \tag{97}$$

Unfortunately the m.g.f. of either (ξ,ζ,η) or of the common distribution of ζ and η appear quite involved and we shall not explore them further here.

References

[1] Athreya, K.B. and Karlin, S. (1967). Limit theorems for the split times of branching processes. J. Math and Mechanics, 17, No. 3, pp. 257-278.

[2] Bartlett, M.S. (1961). Equations for stochastic path integrals. Proc. Camb. Phil. Soc., 57, pt. 3, pp. 568-73.

[3] Bartlett, M.S. and Kendall, D. G. (1950). "On the use of characteristic functional in the analysis of some stochastic processes occurring in Physics and Biology". Proc. Camb. Phil. Soc., Vol. 47, pt. 1, pp. 65-76.

[4] Gani, J. (1965). Stochastic models for bacteriophage, J. Appl. Prob., 2, pp. 225-268.

[5] Harris, T.E. (1963). The theory of branching processes. Springer-Verlag, Berlin.

[6] Karlin, S. (1968). Personal Communication.

[7] Karlin, S. (1966). A first course in stochastic processes. Academic press, New York and London.

[8] Puri, P.S. (1966). "On the homogeneous birth-and-death process and its integral". Biometrika, 53, pp. 61-71.

[9] Puri, P.S. (1966). "Some further results on the birth-and-death process and its integral". Proc. Camb. Phil. Soc.. (In Press).

[10] Puri, P.S. (1967). Some limit theorems on branching processes and certain related processes, Mimeo-Series No. 135, Jan. 1967, Dept. of Statistics, Purdue University, Lafayette, Indiana.

[11] Puri, P.S. (1966). "Class of stochastic models of response to injection of virulent bacteria in the absence of defense mechanism". Proceedings of fifth Berkeley Symposium on Mathematical Statistics and Probability, 4, pp. 511-535.

[12] Puri, P.S. (1968). Some new results in the mathematical theory of phage-reproduction. Submitted for publication to J. Appl. Prob..

[13] Sevast'yanov, B.A.(1951). "The theory of branching random processes". Uspehi - Matemat. Nauk 6, pp. 47-99 [Russian].

[14] Whittaker, E.T. and Watson, G.N. (1952). A course of modern analysis. Cambridge University Press, Cambridge.