A Martingale Version of a Theorem of Marcinkiewicz and Zygmund*

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1. Introduction. Suppose that $(x_n, n \geq 1)$ is an orthonormal sequence of independent random variables and $(a_n, n \geq 1)$ is a sequence of real numbers. Marcinkiewicz and Zygmund [5] proved that if $P[\Sigma a_k x_k$ converges$] = 1$, $\Sigma a_k^2 < \infty$. Recently, Gundy [3] extends their theorem to martingales as follows: Let $(d_n, \mathscr{F}_n, n \geq 1)$ be a sequence of martingale differences with $E(d_n^2 | \mathscr{F}_{n-1}) = 1$ a.e. and $P(|d_n| > \lambda | \mathscr{F}_{n-1}) \geq \gamma$ a.e. for some positive constants λ and γ, and let $(v_n, \mathscr{F}_{n-1}, n \geq 1)$ be a stochastic sequence, i.e., v_n is an $\mathscr{F}_n$-measurable random variable for each n. Then $\Sigma v_k^2 < \infty$ a.e. on the set $[\Sigma v_k d_k$ converges$]$.

Let $x_1, x_2, \ldots$ be independent, identically distributed random variables and $(a_{m,n}, m \geq 1, n \geq 1)$ be a double sequence of real numbers such that $\lim_m a_{m,n} = a_n$ for each n. In [6], Zygmund proved that if $\Sigma_{k=1}^{\infty} a_{m,k} x_k = T_m$ a.e. and $P[\sup_m |T_m| < \infty] = 1$, then $\Sigma a_k^2 < \infty$.

In this note, by stopping rules, we will extend Marcinkiewicz and Zygmund's theorem in a different direction and at the same time generalize Zygmund's theorem.

2. Main theorem. In this section, as well as in the following one, we will assume that $(d_k, \mathscr{F}_k, k \geq 1)$ is a sequence of martingale differences with $E(d_k^2 | \mathscr{F}_{k-1}) = 1$ $(\mathscr{F}_0 = \{\phi, \Omega\})$, $(a_{m,n}, m \geq 1, n \geq 1)$ is a double sequence

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of real numbers with $\lim_m a_{m,n} = a_n$ for each $n \geq 1$, and that $S_{m,n} = \Sigma_{k=1}^n a_{m,k} d_k$.

Theorem 1. Let

$$\inf_n E|d_n| \geq \delta > 0, \tag{1}$$

$$\lim_{K\to\infty} P[\sup_n |S_{m,n}| \geq K] = 0 \quad \text{uniformly in } m, \tag{2}$$

$$\sup_{m,n} |a_{m,n}| \leq M < \infty. \tag{3}$$

Then $\Sigma a_k^2 < \infty$.

Proof. For $K > \max(M, 2\delta^{-1})$ and $m = 1,2,\ldots$, put $b_n = b_n(m) = a_{m,n}$, $S_n = S_{m,n} = \Sigma_{k=1}^n b_k x_k$, and

$$t = t(m) = \inf \{n \geq 1 | S_n^2 > K^2\}. \tag{4}$$

For $j = 1,2,\ldots$, put $\tau = \min(t,j)$. It is easy to see that (for example, see [1])

$$E\, S_\tau^2 = E\, \Sigma_{k=1}^{\tau} b_k^2\, E(d_k^2 | \mathcal{F}_{k-1}) \geq P[t=\infty]\, \Sigma_{k=1}^{j} b_k^2. \tag{5}$$

On the other hand,

$$E\, S_\tau^2 = \int_{[t > j]} S_j^2 + \int_{[t \le j,\ b_t^2 d_t^2 \le K^4]} S_t^2 +$$

$$+ \int_{[t \le j,\ b_t^2 d_t^2 > K^4]} (S_{t-1}^2 + 2\, S_{t-1} b_t d_t + b_t^2\, d_t^2)$$

$$\le (K+K^2)^2 + \int_{[t \le j, b_t^2\, d_t^2 > K^4]} (2S_{t-1}\, b_t d_t + b_t^2\, d_t^2)$$

$$\le (K + K^2)^2 + (1+2K^{-1}) \int_{[t \le j, b_t^2\, d_t^2 > K^4]} b_t^2\, d_t^2.$$

Hence

$$(6) \qquad (K + K^2)^2 \ge \Sigma_{k=1}^j\, b_k^2\, \{P[t = \infty] - (1+2K^{-1}) \int_{[t=k, b_k^2 d_k^2 > K^4]} d_k^2\}.$$

Since $E\, d_k^2 = 1$ and $K > \max\, (M,\ 2\ \delta^{-1})$, we have

$$\int_{[b_k^2 d_k^2 > K^4]} |d_k| \le |b_k|\ K^{-2} \int_{[b_k^2 d_k^2 > K^4]} d_k^2 \le \delta/2,$$

$$\int_{[b_k^2 d_k^2 > K^4]} d_k^2 = 1 - \int_{[b_k^2 d_k^2 \le K^4]} d_k^2 \le 1 - (E|d_k| - \int_{[b_k^2 d_k^2 > K^4]} |d_k|)^2$$

$$\le 1 - \delta^2/4\ .$$

Choose K so large that $(1+2K^{-1})(1-\delta^2/4) \leq 1-\delta^2/8$. Then

(7) $$(K+K^2)^2 \geq \Sigma_{k=1}^{j} b_k^2 \{P[t=\infty]-(1+2K^{-1})(1-\delta^2/4)\}$$

$$\geq \Sigma_{k=1}^{j} b_k^2(P[t=\infty] - 1 + \delta^2/8).$$

The condition (2) implies that $P[t = \infty] > 1-\delta^2/16$ for all $m = 1,2,\ldots,$ if $K \geq K_0$ for some K_0. Let $K = K_0$. Then

(8) $$(K+K^2)^2 \geq (\delta^2/16)\ \Sigma_{k=1}^{j} b_k^2 = (\delta^2/16)\ \Sigma_{k=1}^{j} a_{m,k}^2.$$

Therefore $(K+K^2)^2 \geq (\delta^2/16)\ \Sigma_{k=1}^{\infty} a_k^2$, which completes the proof.

3. Some corollaries.

Corollary 1. If there exist positive constants λ and γ such that

(9) $$P(|d_k| > \lambda|\mathcal{F}_{k-1}) \geq \gamma \qquad \text{a.e.},$$

then $\Sigma a_k^2 < \infty$, provided that (2) is satisfied.

Proof. Obviously (9) implies (1). To prove (3), assume that there exists a subsequence k_m such that $|a_{n_{k_m},k_m}| > m$ for $m = 1,2,\ldots$. By Lévy's martingale version (for example, see [2], p. 324) of the Borel-Cantelli lemma, (9) implies that

(10) $$P[|d_{k_m}| > \lambda \text{ i.o.}] = 1.$$

Hence

$$P[|a_{n_{k_m},k_m} d_{k_m}| > m\,\lambda \text{ i.o.}] = 1,$$

which contradicts (2). Therefore (2) and (9) imply (3).

Corollary 2. Let $a_{m,n} = a_n$ for all $m \geq 1$ and $n \geq 1$. If

(11) $$P[\Sigma\, a_k\, d_k \text{ converges}] = 1,$$

then $\Sigma\, a_k^2 < \infty$, provided that (1) is satisfied.

Proof. Obviously (11) implies (2). We will prove that (1) and (11) imply that $\lim_n a_n = 0$. Assume that there exist $\epsilon > 0$ and a subsequence k_m such that $|a_{k_m}| \geq \epsilon$ for $m = 1,2,\ldots$. Then (11) implies that $P[\lim_m d_{k_m} = 0] = 1$. Since $E\, d_k^2 = 1$ implies that $(d_k,\ k \geq 1)$ is uniformly integrable, we obtain $\lim_m E|d_{k_m}| = 0$, which contradicts (1). Thus the proof is completed.

Corollary 2 reduces Gundy's condition (9) to condition (1), when the stochastic sequence $(v_n, \mathcal{F}_{n-1},\ n \geq 1)$ is a sequence of constants.

Corollary 3. Let $d_1, d_2, \ldots$ be orthonormal, independent random variables with zero median. If

(12) $$P[\lim_n S_{m,n} = T_m] = 1,$$

$$P[\sup_m |T_m| < \infty] = 1, \tag{13}$$

and if (1) holds, then $\Sigma\, a_k^2 < \infty$.

Proof. By Lévy's inequality (see, for example, [2], p. 106), (12) and (13) imply (2). Since d_n are independent and uniformly integrable, (1) implies (9) immediately. Therefore Corollary 3 follows from Corollary 1.

When $P[d_n = \pm 1] = 1/2$, Corollary 3 was proved by Zygmund [6].

4. Extension of a theorem of Kac and Steinhaus. Let $(d_k, \mathscr{F}_k,\ k \geq 1)$ be an orthonormal sequence of martingale differences such that $(d_k^2,\ k \geq 1)$ is uniformly integrable and let $a_{m,n}$ and $S_{m,n}$ be defined as in section 2.

Theorem 2. Under the conditions (2) and (3),

$$\Sigma\, a_k^2\, E(d_k^2 | \mathscr{F}_{k-1}) < \infty \quad \text{a.e.} \tag{14}$$

Proof. For $K > 0$ and $m, j = 1, 2, \ldots,$ define b_n, τ and τ as in section 2. Then, as before,

$$ES_\tau^2 = E\, \Sigma_{k=1}^{\tau}\, b_k^2\, d_k^2 \geq \Sigma_{k=1}^{j}\, b_k^2 \int_{[t=\infty]} d_k^2\,, \tag{15}$$

$$ES_\tau^2 \leq (K+K^2)^2 + (1+2K^{-1})\, \Sigma_{k=1}^{j}\, b_k^2 \int_{[t=k]} d_k^2\,. \tag{16}$$

By (2) and the uniformly integrability of $(d_k^2,\ k \geq 1)$, for all $K \geq K_0$ and

$k \geq k_0$, we have $\int_{[t=\infty]} d_k^2 > 1/2$ and $\int_{[t=k]} d_k^2 < 1/4$. Hence, as $j \to \infty$, $\Sigma_{k=k_0}^{j} b_k^2 = O(1)$. Therefore

$$O(1) = ES_\tau^2 = E\,\Sigma_{k=1}^{\tau} b_k^2\, E(d_k^2|\mathcal{F}_{k-1}) \geq \int_{[t=\infty]} \Sigma_{k=1}^{j} b_k^2\, E(d_k^2|\mathcal{F}_{k-1}).$$

Hence for all $K \geq K_0$,

$$\int_{[t=\infty]} \Sigma_{k=1}^{\infty} a_k^2\, E(d_k^2|\mathcal{F}_{k-1}) < \infty.$$

Since (2) implies that $\lim_{K\to\infty} P[t=\infty] = 1$, (14) follows immediately.

When $d_1, d_2, \ldots,$ are independent random variables and $a_{m,n} = a_n$ for $m,n = 1,2,\ldots,$ Theorem 2 was proved by Kac and Steinhaus [4].

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