On the Rate of Convergence in Renewal and Markov Renewal Processes

by

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Supported by the Air Force Office of Scientific Research

Contract AFOSR 955-65

Glen Baxter, Project Director

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Mimeograph Series No. 107

June 1967

NOTATIONS

1. Let $F(t)$ be a function of bounded variation in $[0,\infty]$. Its Laplace-Stieltjes transform (L.S.T.) will usually be denoted by the corresponding small letter, i.e.

$$f(s) = \int_0^\infty e^{-st}\, dF(t)$$

The halfplane of convergence of $f(s)$ is $\{s: \operatorname{Re} s > c\}$ for some real c. If $c \geq 0$, then we write $P(c)$ for the halfplane $\{s: \operatorname{Re} s > -c\}$.

2. The convolution of two functions of bounded variation $F(t)$, $G(t)$ $0 \leq t \leq \infty$ is denoted by

$$F * G(t) = \int_0^t F(t-x)\, dG(x)$$

if it exists. The n-fold convolution of $F(t)$ is denoted by $F^{(n)}(t) = F * F^{(n-1)}(t)$ and occasionally by $F^{n*}(t)$.

3. $U_c(t)$ denotes the unitstep function at c, i.e.

$$U_c(t) = 0 \quad \text{if} \quad t < c$$
$$= 1 \quad \text{if} \quad t \geq c.$$

If $c = 0$, we write $U(t)$.

4. The determinant of a square matrix A is denoted by $|A|$.

5. Theorems in the appendix are referred to as A. Thm.

CHAPTER I

EXPONENTIAL DECAY IN RENEWAL THEOREMS

For the special case of a distribution function that tends to one exponentially fast, the three basic renewal theorems are proved by the same method. Moreover an estimate of the remainder term is obtained. This extends results of M.R. Leadbetter [30].

Series expansions are obtained for the renewal function by using matrix methods. Similar methods apply equally well to the Φ-renewal moments as indicated in section 4.

Further generalizations are given in section 5.

1. Exponentially Bounded Distributions

Let $F(t)$ be the distribution function of a non-negative random variable. $F(t)$ is said to be <u>exponentially bounded</u> [at $+\infty$] if there exist constants $\lambda > 0$ and $0 \leq K < \infty$ such that for all $t \geq t_o \geq 0$

$$1 - F(t) \leq K e^{-\lambda t} \tag{1}$$

Theorems 4 and 5 of the appendix give equivalent conditions for (1), the former on $F(s)$ and the latter on the moments of $f(t)$.

The <u>decay parameter</u> λ^* of $F(t)$ is defined by

$$\lambda^* = \limsup_{t \to \infty} \{\lambda:\ 1-F(t) = o(e^{-\lambda t})\}. \tag{2}$$

It then follows immediately from A. Thm 3 that $-\lambda^*$ is the ''first'' singularity of $f(s)$.

We make the following assumptions:

(3) (i) $F(t)$ satisfies (1) with $\lambda = \lambda^*$

(4) (ii) $F(t)$ is strongly non-lattice i.e. $F(t)$ is non-lattice and $\liminf_{|t| \to \infty} |1-f(it)| > 0$, [43].

In case $F(t)$ is lattice, $f(s)$ behaves differently on the imaginary axis. Even if $F(t)$ is non-lattice, $|1-f(it)|$ can become arbitrarily small for large values of t, [31, p. 24]. A sufficient condition for (ii) is that $F(t)$ is absolutely continuous. We remark that (ii) implies in particular that $F(t)$ is non-degenerate at $t = 0$.

2. Basic Renewal Theorems

Let $\{X_k, k \geq 1\}$ be a sequence of independent, identically distributed non-negative random variables with common distribution function $F(t)$ satisfying (i) and (ii). Let $\mu = E[X_k] \neq 0$ and $\mu_2 = E[X_k^2]$.

Consider the partial sums $S_n = \sum_{k=1}^{n} X_k$, $n = 1,2,\ldots$ and the function $N(t) = \max\{n: S_n \leq t\}$. The process $\{X_k, k \geq 1\}$ is called an ordinary renewal process, and its renewal function is defined by

(5) $$H(t) = E[N(t)].$$

It is well known [36, p. 158] that

$$h(s) = f(s)\,[1-f(s)]^{-1}.$$

By A. Thm 1, $f(s)$ converges in $P(\lambda)$ so that $f(s)$ and $1-f(s)$ are analytic in $P(\lambda)$. At $s = 0$, $1-f(s)$ has an isolated zero of order one since $f(0) = F(\infty) = 1$ and for $|s| < \epsilon$, $f(s) = 1-\mu s+o(s)$ where $\mu \neq 0$ by (ii).

Since $F(t)$ is non-lattice, $1-f(s)$ has no zeros on the imaginary axis. An application of the Borel covering theorem yields that for every finite $T > 0$, there exists a $\lambda_1 > 0$ such that $1-f(s) \neq 0$ for all $s \neq 0$ for which $\text{Re } s > -\lambda_1$ and $|\text{Im } s| \leq T$.

By (ii) we can choose T so large that for $t \geq T$ $|1-f(it)| \geq K\eta$ where η is some positive quantity and K is defined by (1). Since $\lambda > 0$, there exists a real number λ_2 such that $0 < \lambda_2 < \lambda - \frac{\lambda}{1+\eta}$. From (1) it easily follows that for all t

$$|f(-\lambda_2 + it) - f(it)| \leq \frac{\lambda_2 K}{\lambda - \lambda_2}.$$

Let $t \geq T$, then

$$|1-f(-\lambda_2+it)| \geq |1-f(it)| - |f(-\lambda_2 + it) - f(it)|$$

$$\geq K\eta - \frac{\lambda_2 K}{\lambda-\lambda_2} = K[\eta - \frac{\lambda_2}{\lambda-\lambda_2}] > 0.$$

Hence for $-\lambda_2 < \text{Re } s \leq 0$ and $|\text{Im } s| > T$, also $1 - f(s) \neq 0$. If we put $\lambda' = \min(\lambda_1, \lambda_2)$ then we have proved:

<u>Lemma 1.1.</u>

If $F(t)$ satisfies (i) and (ii), then there exists a real number $\lambda' > 0$, such that $1-f(s) = 0$ has no solution in $P(\lambda')$ except $s = 0$.

Define $\rho = \min(\lambda,\lambda') > 0$. We prove the three basic renewal theorems in the case where $F(t)$ is exponentially bounded. To indicate the restricted character of the theorems, we put ''R-'' in front of the statements.

Theorem 1.1. R-elementary renewal theorem.

Let $F(t)$ satisfy (i) and (ii). Then there exist constants $\rho > 0$ and $0 \leq K < \infty$ such that for $t \geq t_o \geq 0$

$$(7) \qquad \left|H(t) - \frac{t}{\mu} - \frac{\mu_2 - 2\mu^2}{2\mu^2}\right| \leq K\, e^{-\rho t}\,.$$

Proof:

Consider the function $G(t) = H(t) - \frac{t}{\mu}$. Then $g(s)$ exists and equals $h(s) - \frac{1}{\mu s}$. But $h(s)$ has a simple pole at $s = 0$ with residue μ^{-1}. By lemma 1.1, $h(s)$ has no other singularities in $P(\rho)$. From A. Thm 2 then as $t \to \infty$

$$G(\infty) - G(t) = O[e^{-\rho t}].$$

This in turn implies the existence of constants t_o and $0 \leq K < \infty$ such that for all $t \geq t_o$

$$|G(t) - G(\infty)| \leq K\, e^{-\rho t},$$

where $G(\infty)$ is obtained from the equality $G(\infty) = g(0) = \frac{\mu_2}{2\mu^2} - 1$.

This proves the theorem.

We proceed to prove

Theorem 1.2. R-Blackwell's theorem.

Let $F(t)$ satisfy (i) and (ii). Then for every $a > 0$ there exist constants $\rho > 0$ and $0 \leq K < \infty$ such that for $t \geq t_o \geq 0$

$$(8) \qquad \left|\frac{H(t+a) - H(t)}{a} - \frac{1}{\mu}\right| \leq Ke^{-\rho t} .$$

Proof:

Let $a > 0$ be fixed and define $G(t) = a^{-1}\{H(t+a)-H(t)\}$. Then $g(s)$ exists and satisfies

$$(9) \qquad g(s) = \frac{1}{a} h(s)[e^{as} - 1] - \frac{e^{as}}{a} \int_o^a e^{-st} \, dH(t).$$

The first term on the right hand side of (9) converges in $P(\rho)$ by the same argument as before.

The second term is an entire function of s. Hence A. Thm 2 and the dominated convergence theorem imply that for all $t \geq t_o \geq 0$

$$|G(t) - G(\infty)| \leq Ke^{-\rho t}$$

where $G(\infty) = G(0) + g(0) = \frac{1}{\mu}$. This proves the theorem.

Finally we prove a restricted version of the key renewal theorem originally due to W.L. Smith [41]. If we require that $Q(t)$ of the theorem does not destroy the analyticity of $sh(s)$ in a neighborhood of the origin, then A. Thm 2 still applies. See [11, p. 41].

Theorem 1.3. R-key renewal theorem.

Let $F(t)$ satisfy (i) and (ii). Let $Q(t)$ be a real valued function satisfying the following conditions:

1. $Q(t) = 0$ for $t < 0$;
2. $Q(t) \geq 0$ for $t \geq 0$;
3. $Q(t)$ is non-increasing for $t \geq 0$;
4. For some $\rho' > 0$, $\int_0^\infty e^{\rho' t} Q(t)\, dt < \infty$.

Then there exist constants $\sigma = \min(\rho, \rho') > 0$ and $0 \leq K < \infty$ such that for $t \geq t_0 \geq 0$

$$(10) \qquad \left| \int_0^t Q(t-u)\, dH(u) - \frac{1}{\mu} \int_0^\infty Q(t)\, dt \right| \leq K e^{-\sigma t} .$$

Proof:

Define $G(t) = \int_0^t Q(t-u)\, dH(u)$. By a well-known theorem [48, p.88] $g(s)$ exists and is equal to

$$(11) \qquad g(s) = sh(s) \int_0^\infty e^{-st}\, Q(t)\, dt.$$

But $sh(s)$ converges in $P(\rho)$ and assumptions 2-4 together with A. Thm 3 imply that the second factor in (11) converges in $P(\rho')$. The remaining part of the proof follows as before.

It is worthwhile to indicate that the above theorems are in some sense best-possible. For if $F(t)$ is not exponentially bounded then

$\lambda = 0$. Obviously $f(o+) = 1$, but $\lim_{|s| \to 0} f(s)$ does not exist. This means that the origin is an essential singularity of $f(s)$, so that neither $f(s)$ nor $h(s)$ can be continued as a meromorphic function in the left half plane [11,p. 153]. This implies that an exponential decay as indicated by the above theorems is possible only if $F(t)$ is exponentially bounded. More specifically we prove

Lemma 1.2.

If $t(s) \equiv h(s) - \frac{1}{\mu s}$ converges in $P(\lambda)$ then for some $\bar{\lambda} > 0$, $f(s)$ converges in $P(\bar{\lambda})$.

Proof:

It follows from (6) that

$$f(s) = [1+\mu st(s)][1+\mu s + \mu st(s)]^{-1} . \tag{12}$$

By assumption, the numerator and the denominator of (12) are both analytic in $P(\lambda)$. The only possible singularities of $f(s)$ in $P(\lambda)$ are at zeros of the denominator. On the other hand A. Thm 3 states that the first singularity of $f(s)$ lies on the real line. But at $s = 0$ $1 + \mu s + \mu s\, t(s) = 1$, hence by continuity this function is non-zero for $|s| < \epsilon$ for some $\epsilon > 0$. This proves the lemma.

Combining the above lemma with theorem 3, we obtain the interesting

Corollary 1.1.

Let $Q(t)$ be defined as in R-theorem 1.3. Suppose that there exist constants $\lambda > 0$, $0 \leq K < \infty$ such that for $t \geq t_o \geq 0$ $|H(t) - \frac{t}{\mu}| \leq Ke^{-\lambda t}$. Then there exist constants $\lambda^* > 0$, $0 \leq K^* < \infty$

such that for $t \geq t_o^* \geq 0$ relation (10) holds, with K and σ replaced by K^* and λ^* respectively.

We remark that any one of the ergodicity conditions (7), (8) and (10) is sufficient to obtain the other two. We chose this version for future use.

In many cases one can improve the rate of convergence in the R-renewal theorems. This is indicated by Leadbetter in [30]. If one can compute the zeros of $f(s) - 1$, then the function $sh(s)$ can be further continued in the left halfplane as a meromorphic function. This is possible since we stay inside the region of convergence of $f(s)$. See Cox [10] for more specific examples.

Also if $1-F(t) = A\, e^{-\lambda t}\, t^{\alpha} + O[e^{-\mu t}]$ for $\mu > \lambda$ then a similar procedure improves the decay parameters, [11, p. 467]. By using Mittag-Leffler series, analogous results can be derived if $h(s)$ has an infinite number of poles in the negative halfplane.

3. Series Expansion of the Renewal Function

By A. Thm 4 $f(s)$ has a Maclaurin expansion for $|s| < \lambda$ so that $h(s)$ has a Laurent expansion in $0 < |s| < \rho$ by Thm. 1.1.

We derive a matrix procedure to obtain the coefficients in the formal expansion

$$h(s) = \frac{A_{-1}}{s} + A_o^* + A_1 s + A_2\, s^2 + \dots \tag{13}$$

using

$$f(s) = 1 + B_1\, s + B_2\, s^2 + B_3\, s^3 + \dots \tag{14}$$

where

$$B_i = (-1)^i \frac{\mu_i}{i!} \quad \text{and} \quad \mu_i = E[X_k^i].$$

Clearly $A_{-1} = \frac{1}{\mu}$. We also define $A_o = 1 + A_o^*$. Using (6), (13) and (14) the following chain of equalities is easily established:

$$\begin{aligned} &A_{-1}\, B_1 && = -1 \\ &A_{-1}\, B_2 + A_o\, B_1 && = 0 \\ &A_{-1}\, B_3 + A_o\, B_2 + A_1 B_1 && = 0 \\ &\cdots \quad \cdots \quad \cdots && \cdots \end{aligned} \tag{15}$$

Although this system can be solved recursively, we can find an explicit solution by introducing an infinite matrix $\mathcal{B}$, called the <u>moment matrix</u>:

$$\mathcal{B} = \begin{pmatrix} B_1 & 0 & 0 & \cdots \\ B_2 & B_1 & 0 & \cdots \\ B_3 & B_2 & B_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

The matrix $\mathcal{D}$ obtained from $\mathcal{B}$ by deleting the first row is called the <u>auxiliary matrix</u> of $\mathcal{B}$. For $n > 0$, let $\mathcal{B}_n$ be the submatrix of $\mathcal{B}$ obtained by deleting from $\mathcal{B}$ all rows and columns with index greater than n. Clearly $|\mathcal{B}_n| = B_1^n$. The submatrix $\mathcal{D}_n$ of $\mathcal{D}$ is obtained by dropping from $\mathcal{D}$ all rows and columns with index greater than or equal $n > 1$. In addition $\mathcal{D}_1 \equiv 1$.

We truncate the system (15) at the nth equation and solve for A_i, $i = -1,0,1,\ldots,n-2$. We then get

$$(16) \qquad |\mathcal{B}_n| A_i = (-1)^i \; |\mathcal{D}_{i+2}| B_1^{n-i-2}.$$

Since $B_1 = -\mu$, $|\mathcal{B}_n| = (-1)^n \mu^n$. So (16) reduces to

$$(17) \qquad A_i = |\mathcal{D}_{i+2}| \; \mu^{-(i+2)} \qquad i = -1,0,1,\ldots$$

for all i, since the right hand side is independent of n. We obtain immediately:

$$A_{-1} = \mu^{-1}$$

$$A_o^* = \frac{1}{2}\mu^{-2} [\mu_2 - 2\mu^2]$$

$$A_1 = \frac{1}{12}\mu^{-3} [3 \mu_2^2 - 2 \mu_1\mu_3]$$

$$A_2 = \frac{1}{24}\mu^{-4} [3 \mu_2^3 - 4\mu \; \mu_2 \; \mu_3 + \mu^2 \; \mu_4]$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad .$$

5. Series Expansion for the Renewal Moments

The fact that $f(s)$ has a convergent Maclaurin expansion may be further exploited to derive expansions for the renewal moments. This is somewhat similar to results of Leadbetter [29]. It turns out that it is sufficient to take powers of the triangular moment matrix $\mathcal{B}$.

W.L. Smith defines the nth renewal moment $\Phi_n(t)$ for $n \geq 1$ by

(18) $$\Phi_n(t) = E\{[N(t)+1][N(t)+2] \ldots [N(t)+n]\}.$$

It is easy to prove [29] that the L.S.T's are given by

(19) $$\varphi_n(s) = n!\ [1-f(s)]^{-n} \qquad n \geq 1.$$

Let

(20) $$\psi_n(s)\ n! = \varphi_n(s) \qquad n \geq 1$$

$$\psi_o\ (s) \equiv 1$$

then $\psi_n(s)$ has a Laurent expansion in $0 < |s| < \rho$ of the form

(21) $$\psi_n(s) = A_{n,-n}\ s^{-n} + A_{n,-n+1}\, s^{-n+1} + \ldots + A_{n,o} + A_{n,1} s + \ldots$$

where the first index of $A_{i,j}$ refers to the renewal moment under consideration; the second indicates the power of s in the expansion. A more compact notation is acquired by introducing the vectors

$$A_n' = (A_{n,-n}, A_{n,-n+1}, \ldots, A_{n,o}, \ldots) \qquad n \geq 1$$

$$A_o' = (1\ , 0\ , \ldots, 0\ , \ldots) \qquad n = 0$$

$$\bar{s}, = (1\ , s\ , \ldots, s^n\ , \ldots)$$

where A' is the transpose of A. Denoting by (a,b) the scalar product of a and b, (21) becomes

(22) $$\psi_n(s) = s^{-n}\ (A_n, \bar{s}) \qquad n = o,1,2,\ldots$$

The computation of the vectors A_n is simplified by using

Lemma 1.3.

$$\text{(i)} \quad \text{For every } n \geq 1, \quad \mathcal{B}\, A_{n+1} = -A_n \; ; \tag{23}$$

$$\text{(ii)} \quad \text{For every } n \geq 1, \quad \mathcal{B}^n A_n = (-1)^n A_o .$$

Proof:

To obtain (i) equate coefficients of equal powers of s on both sides of the equality $[1-f(s)]^{-(n+1)}\,[1-f(s)] = [1-f(s)]^{-n}$. Repeated use of (i) yields (ii).

Once the powers of the moment matrix are calculated, (23) can be solved explicitly for the vector A_n. The powers of $\mathcal{B}$ however have a simple form as shown in the next lemma, the proof of which is easy.

Lemma 1.4.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two (possible infinite) matrices, such that AB is well defined. Suppose that

$$a_{ij} = b_{ij} = 0 \qquad \text{if } j > i$$

$$a_{ij} = A_{i-j} \text{ and } b_{ij} = B_{i-j} \qquad \text{if } j \leq i.$$

Then $C = (c_{ij}) \equiv AB$ satisfies

$$c_{ij} = 0 \qquad \text{if } j > i$$

$$c_{ij} = C_{i-j} \qquad \text{if } j \leq i$$

for some constants C_k, $k = 0,1,2,\ldots$.

The matrix $\mathcal{B}$ is of the form required by the lemma. So there exist constants $B_i^{(n)}$, $i = 1,2,3,\ldots$ for all $n = 0,1,2,\ldots$ such that

$$\mathcal{B}^n = \begin{pmatrix} B_1^{(n)} & 0 & 0 & \cdots \\ B_2^{(n)} & B_1^{(n)} & 0 & \cdots \\ B_3^{(n)} & B_2^{(n)} & B_1^{(n)} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

To obtain the vectors A_n from (23), we repeat the procedure of the previous section.

Let $\mathcal{B}_m^n$ be the submatrix of $\mathcal{B}^n$ obtained by deleting from $\mathcal{B}^n$ every row and column with index greater than $m > 0$. Then $|\mathcal{B}_m^n| = \{B_1^{(n)}\}^m$. The auxiliary matrix of $\mathcal{B}^n$ is denoted by $\mathcal{D}^n$. The submatrix $\mathcal{D}_m^n$ is obtained by dropping all rows and columns from $\mathcal{D}^n$ with index greater than or equal $m > 1$. In addition $\mathcal{D}_1^n \equiv 1$ for every $n \geq 0$.

The infinite system $\mathcal{B}^n A_n = (-1)^n A_o$ is truncated at the mth equation and solved for $A_{n,j}$, where $j = -n, -n+1, \ldots, -n+m$.

$$|\mathcal{B}_m^n| A_{n,-n+k} = (-1)^{n+k} |\mathcal{D}_{k+1}^n| \{B_1^{(n)}\}^{m-k-1}$$

But $B_1^{(n)} = B_1^n = (-1)^n \mu^n$, so that we obtain

$$(24) \qquad A_{n,-n+k} = (-1)^{(n+1)k} |\mathcal{D}_{k+1}^n| \mu^{-n(k+1)} \quad k = 0,1,2,\ldots$$

since the right hand side is independent of m.

By subtracting the principal part of (21) from $\psi_n(s)$, the remaining function is convergent in $P(\rho)$ where ρ is defined in Thm 1.1. This is clear since the singularities of $\psi_n(s)$ are the same as those of h(s) by (19) and (20). From A. Thm 2 we easily get

Theorem 1.4.

Let F(t) satisfy conditions (i) and (ii) of §1. Then for every $n \geq 1$ there exists a constant $0 \leq K_n < \infty$ such that for $t \geq t_n \geq 0$

$$(25) \qquad \left|\frac{\Phi_n(t)}{n!} - \sum_{k=0}^{n} (-1)^{k(n+1)} \mu^{-n(k+1)} |\mathcal{B}^n_{k+1}| \frac{t^{n-k}}{(n-k)!}\right| \leq K_n e^{-\rho t}.$$

As an application, take $n = 2$. Then

$$\mathcal{B}_3^2 = \begin{pmatrix} B_1^2 & 0 & 0 \\ 2 B_1 B_2 & B_1^2 & 0 \\ 2 B_1 B_3 + B_2^2 & 2 B_1 B_2 & B_1^2 \end{pmatrix}$$

Corollary 1.2.

Let F(t) satisfy (i) and (ii). Then there exists a constant $0 \leq K_2 < \infty$ such that for $t \geq t_2 \geq 0$

$$(26) \qquad \left|\Phi_2(t) - \frac{t^2}{\mu^2} - 4 \frac{\mu_2}{\mu^3} t - \frac{1}{6} \frac{1}{\mu^4} (9 \mu_2^2 - 4 \mu \mu_3)\right| \leq K_2 e^{-\rho t}$$

5. Extensions of the R-renewal Theorems

A. Modified Renewal Process

A modified renewal process is defined as a sequence of independent nonnegative random variables $\{X_k, k \geq 1\}$ where $F_1(t) = P\{X_1 \leq t\}$ and

$F(t) = P\{X_k \leq t\}$ for $k = 2,3,\ldots$.

The renewal function (5) is determined by its L.S.T. $h(s) = f_1(s)\,[1-f(s)]^{-1}$ as shown in [36, p. 158]. All theorems from before are still valid if we assume that $F_1(t)$ satisfies (i) of §1 with an appropriate decay parameter. The minor changes in the results are obvious and will be omitted.

B. Generalized Renewal Process of the Chung-Pollard Type

Assume that a sequence of independent random variables is given, with common distribution function $F(t) = P[X_k \leq t]$, $k = 1,2,\ldots,$ $-\infty \leq t \leq \infty$. Define the partial sums as before. Define the function

$$H(t) = \sum_{n=1}^{\infty} P\{S_n \leq t\} \tag{27}$$

A process $\{X_k, k \geq 1\}$, satisfying the above conditions, and with ''renewal function'' $H(t)$ of (27) is called a <u>generalized renewal process of the Chung-Pollard type</u> [7].

Although $H(t)$ is no longer the expected value of the random variable $N(t) = \max\{n: S_n \leq t\}$, its study is closely related to that of the renewal function of section 2. Several authors discussed its asymptotic properties, [7], [36].

Let us denote by $\tilde{f}(s)$ and $\tilde{h}(s)$ the bilateral L.S.T's of $F(t)$ and $H(t)$ respectively. It is well-known [48, p. 257] that

$$\tilde{h}(s) = \tilde{f}(s)\,[1-\tilde{f}(s)]^{-1}. \tag{28}$$

The region of convergence of $\widetilde{f}(s)$ is a strip, parallel to the imaginary axis. This strip contains an open neighborhood of the origin, if and only if $F(t)$ is exponentially bounded at both tails, i.e.

(i) for some $\lambda_1 > 0$, $1 - F(t) = O[e^{-\lambda_1 t}]$ as $t \to +\infty$;

(ii) for some $\lambda_2 > 0$, $F(t) = O[e^{\lambda_2 t}]$ as $t \to -\infty$.

The proof of this statement, the analogue of A. Thm 1,2, can be found in [48, p. 239].

We assume in addition that

(iii) $F(t)$ is strongly non-lattice.

In this context lemma 1.1 becomes

Lemma 1.5.

Let $F(t)$ satisfy (i), (ii) and (iii). Then there exist two real numbers $\lambda_1' > 0$, $\lambda_2' > 0$ such that $\widetilde{f}(s) - 1 = 0$ has no solution for which $-\lambda_2' < \text{Re } s < \lambda_1'$, except $s = 0$. If $\mu \neq 0$ then $s = 0$ is a simple zero of $1 - \widetilde{f}(s)$. Otherwise this zero is double.

Define $\rho_1 = \min(\lambda_1, \lambda_1')$ and $\rho_2 = \min(\lambda_2, \lambda_2')$. Then

Theorem 1.5.

Let $F(t)$ satisfy the conditions (i), (ii) and (iii). Then there exist constants $0 \leq K_1 < \infty$, $0 \leq K_2 < \infty$ such that for every $a > 0$ holds that:

(i) If $\mu > 0$, then $\left|\frac{H(t+a)-H(t)}{a} - \frac{1}{\mu}\right| \leq K_1 e^{-\rho_1 t}$ for $t \geq t_1 \geq 0$

(29)

$$\left|\frac{H(t+a)-H(t)}{a}\right| \leq K_2 e^{\rho_2 t} \qquad \text{for } t \leq -t_2 \leq 0.$$

(30)

(ii) If $\mu < 0$, then $\left|\frac{H(t+a)-H(t)}{a}\right| \leq K_1 e^{-\rho_1 t}$ for $t \geq t_1 \geq 0$

$$\left|\frac{H(t+a)-H(t)}{a} + \frac{1}{\mu}\right| \leq K_2 e^{\rho_2 t} \quad \text{for } t \leq -t_2 \leq 0.$$

(iii) If $F(t)$ is symmetric then $\rho_1 = \rho_2 = \rho$ and

(31)

$$\left|\frac{H(t+a)+H(t-a)-2H(t)}{2a^2} - \frac{1}{\mu_2}\right| \leq K_1 e^{-\rho t} \quad \text{for } t \geq t_1 \geq 0$$

$$\left|\frac{H(t+a)+H(t-a)-2H(t)}{2a^2}\right| \leq K_2 e^{\rho t} \quad \text{for } t \leq -t_2 \leq 0.$$

Proof:

(i) Suppose $\mu > 0$ and let $a > 0$ be fixed.

Consider $G(t) = \frac{1}{a}\{H(t+a)-H(t)\}$, then $\tilde{g}(s)$ exists and is equal to $\frac{1}{a}\{e^{sa}-1\}\,\tilde{h}(s)$.

By lemma 1.5 we deduce that

$$G(\infty) - G(t) = O[e^{-\rho_1 t}] \quad \text{as } t \to +\infty$$

$$G(t) - G(-\infty) = O[e^{\rho_2 t}] \quad \text{as } t \to -\infty.$$

From the theorem of Chung-Pollard [7], [36, p. 215] we know that $G(-\infty) = 0$. On the other hand $G(+\infty) - G(-\infty) = \frac{1}{\mu}$. This proves both parts of (i).

(ii) The proof is similar for $\mu < 0$.

(iii) Assuming that $F(t)$ is symmetric, we obtain that $\tilde{f}(s) = \tilde{f}(-s)$. Introduce

$$G(t) = \frac{1}{a^2}\{H(t+a) + H(t-a) - 2H(t)\}.$$

Hence

$$\tilde{g}(s) = \frac{1}{a^2}\{e^{sa} + e^{-sa} - 2\}\,\tilde{h}(s).$$

By the same argument as before it follows that $\tilde{g}(s)$ converges in $-\rho < \text{Re } s < \rho = \rho_1 = \rho_2$. Since $\tilde{h}(s) = \tilde{h}(-s)$, $\tilde{g}(s) = \tilde{g}(-s)$ so that $G(-\infty) = 0$ and $G(+\infty) = \frac{2}{\mu_2}$. The result follows as before.

The other renewal theorems can be generalized to renewal processes of the above type with obvious changes.

One can assume that only one tail of the distribution is exponentially bounded. Results of this type, were obtained by Ch. Stone in [44], [45].

C. Generalized Renewal Process of the Heyde Type

In [17], C.C. Heyde defines a general renewal process where $H(t)$ preserves the character of an expectation. Let $\{X_k,\ k \geq 1\}$ be a sequence of independent random variables with common distribution $F(t)$, $-\infty \leq t \leq \infty$. Let $N_n = \max\{0, S_1, S_2, \ldots, S_n\}$ for $n \geq 1$. If we put $N(t) = \max\{k : N_k \leq t\}$ then a ''renewal function'' is defined by $H(t) = E[N(t)]$.

The process $\{X_k,\ k \geq 1\}$ with $H(t)$ as renewal function is called a <u>generalized renewal process of the Heyde type</u>.

We introduce the mass function $\tilde{F}(t) = U(t)\,F(t)$. The right tail $F(t)$ does not change, so that $\tilde{F}(t)$ is exponentially bounded if and only if $F(t)$ has an exponentially bounded tail for $t \to \infty$. Theorem 1 of [17] asserts that

(32) $$H(t) = \widetilde{F}(t) + U(t)\ F * H(t)$$

So all R-theorems apply to this case.

6. The Constant K

In every R-theorem there appears a constant K. In [30] Leadbetter gives an elegant way to compute K, that we include for completeness.

Assume that $G(\infty) - G(t) = O[e^{-\lambda t}]$ where $\lambda > 0$, and that $G(t)$ is normalized. For $0 < c < \lambda$ the inversion formula [48, p. 69] gives that for $t > 0$

(33) $$G(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \frac{g(s)}{s}\, ds.$$

Integrating $e^{st}\ g(s)\ s^{-1}$ along the rectangle with vertices $[c-iR,\ c+iR,\ -c+iR,\ -c-iR]$ and letting $R \to \infty$, it easily follows that for all $t > 0$

(34) $$e^{\lambda t} |G(\infty)-G(t)| \leq \inf_{0< c \leq \lambda} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\frac{g(-c+iu)}{c-iu}\right| du.$$

If the integral on the right converges, then it is a possible value of K. This happens for example if $|g(-c+iu)| = O(|u|^{\alpha})$ as $u \to \infty$ for some $\alpha < 0$. Another possibility occurs if $G(t)$ has a bounded derivative $p(t)$. For then Parceval's formula yields

(35) $$|G(\infty)-G(t)| \leq \frac{e^{-\lambda t}}{\sqrt{2\lambda}} \int_{0}^{\infty} e^{2\lambda x} [p(x)]^2\, dx.$$

In other cases different contours might be more appropriate. Cox [10, p. 51] gives an example of a semi-circular path for the case of a gamma distribution.

7. Examples

The above theory applies to many practical examples. General criteria for condition (i) of §1 are given by the theorems in the appendix, while (ii) is usually satisfied in practice.

In particular we mention all gamma distributions and any Weibull distribution with parameter not less than one. The latter example is quite interesting: Let X_k have a Weibull distribution with parameter $\alpha > 0$,

$$F(t) = P[X_k \leq t] = 1 - \exp\{-t^{\alpha}\}. \tag{36}$$

It is easy to show that $\mu_k = \Gamma(\frac{k}{\alpha} + 1)$ which is finite for all k. By using Stirling's formula, one shows that

$$\sqrt[k]{\frac{\mu_k}{k!}} \sim \exp\{1 - \frac{1}{\alpha}\} \cdot k^{\{\frac{1}{\alpha} - 1\}}$$

so that $\limsup_{k \to \infty} \sqrt[k]{\frac{\mu_k}{k!}} < \infty$ if and only if $\alpha \geq 1$. This example illustrates that (ii) in A. Thm 5 cannot be dropped.

Another example to which the above theory is not applicable is furnished by the log-normal distributions, i.e.

$$F(t) = P[X_k \leq t] = \frac{1}{\sqrt{2\pi\alpha}} \int_0^t \exp\{-\frac{[\log cx]^2}{\alpha}\} \frac{dx}{x}, \tag{37}$$

where $\alpha > 0$, $c > 0$. An easy computation shows that $\mu_k = c^{-k} \exp\{\frac{k^2\alpha}{2}\}$ so that (ii) of A. Thm 5 is again violated.

CHAPTER II

EXPONENTIAL DECAY IN MARKOV RENEWAL PROCESSES

In [22] D.G. Kendall proved a solidarity theorem for irreducible denumerable discrete time Markov chains. D. Vere-Jones refined Kendall's theorems by obtaining uniform estimates [47] while J.F.C. Kingman proved analogous results for an irreducible continuous time Markov chain [24, 25].

We derive similar solidarity theorems for an irreducible Markov renewal process. The transient case is discussed in section 3, and section 4 deals with the positive recurrent case.

Section 5 contains some more specific results on a finite Markov renewal process. An application to the $M|G|1$ queue is included in the last section.

1. Definitions and Basic Relations for a M.R.P.

We briefly recall the definitions and relations in R. Pyke's papers [37,38,39].

Let $K(t)$ and $L(t)$ be matrix functions with indices in the statespace I, and with $t \in [0,\infty]$. The matrix convolution product $K * L(t)$ is defined by

$$\{K * L(t)\}_{ij} = \sum_{k\in I} K_{ik} * L_{kj}(t) \quad \text{for } i,j \in I.$$

$Q(t)$ is the matrix of transition probabilities $Q_{ij}(t)$. For all $i,j \in I$

(1)
$$\begin{aligned} &\text{(i)} \quad Q_{ij}(t) \text{ is a mass function;} \\ &\text{(ii)} \quad Q_{ij}(t) = 0 \quad \text{for } t \leq 0; \\ &\text{(iii)} \quad \sum_{j \in I} Q_{ij}(\infty) = 1. \end{aligned}$$

A two dimensional process $\{(J_n, X_n),\ n \geq 0\}$ is defined on a complete probability space by means of

$$\text{(i)} \quad P[J_n = k,\ X_n \leq t \,|\, J_o, J_1, \ldots, J_{n-1}, X_1, X_2, \ldots, X_{n-1}] = Q_{J_{n-1},k}(t);$$

$$\text{(ii)} \quad P[X_o = 0] = 1;$$

$$\text{(iii)} \quad P[J_o = k] = a_k \quad \text{where} \quad a_k \geq 0,\ \sum_{k \in I} a_k = 1.$$

If J_n is the state after the nth transition and X_n the time spent in that state, then $Q_{ij}(t)$ is the probability that the next transition is into state j from i, and occurs before time t. For all $i \in I$ define $H_i(t) = \sum_{j \in I} Q_{ij}(t)$. By (1) $H_i(t)$ is a probability distribution. We denote by $\eta_i \leq \infty$ the first moment of $H_i(t)$. The process $\{(J_n, X_n),\ n \geq 0\}$ is called a <u>semi-Markov sequence</u>. The process $\{J_n,\ n \geq 0\}$ is a Markov chain with transition matrix $(p_{ij}) = (Q_{ij}(\infty))$.

Let $S_n = \sum_{i=1}^{n} X_i$. We define integer valued stochastic processes $\{N(t),\ t \geq 0\}$ where $N(t) = \sup\{n | S_n \leq t\}$ and $\{N_i(t),\ t \geq 0,\ i \in I\}$ where $N_i(t)$ is the number of times $J_k = i$ for $k = 1,2,\ldots,N(t)$, $N(t)+1$. The vector stochastic process $\{N_1(t), N_2(t), \ldots\}$ is called a <u>Markov Renewal Process</u> (M.R.P.). Clearly $N(t) = \sum_{i \in I} N_i(t)$ a.s.

Related to the ''counting'' process defined above is the ''state'' process $\{Z_t, t \geq 0\}$ where $Z(t) = J_{N(t)}$ which is called a Semi-Markov Process (S.M.P.).

R. Pyke has shown that $N(t)$ is an a.s. finite random variable if I is finite. If I is denumerable we assume that $\{N(t), t \geq 0\}$ is regular [37] so that $N(t)$ is finite a.s.

Let $G(t) = \{G_{ij}(t)\}$ where $G_{ij}(t) = P[Z_t = j | Z_o = i]$ if $t \geq 0$ and zero elsewhere. Then $G_{ij}(t)$ is the distribution of the time that the process $\{Z_t, t \geq 0\}$ visits state j for the first time, starting in state i. The first moment of $G_{ij}(t)$ is denoted by $\mu_{ij} \leq \infty$.

Let $M_{ij}(t) = E[N_j(t)|Z_o = i]$ for $t \geq 0$ and zero elsewhere and let $M(t) = \{M_{ij}(t)\}$. The functions $M_{ij}(t)$ are called the renewal functions, i.e. $M_{ij}(t)$ is the mean number of visits of the process $\{Z_t, t \geq 0\}$ to state j up to time t starting at i.

Finally let $P(t)$ be the matrix $\{P_{ij}(t)\}$ where $P_{ij}(t)=P[Z_t=j|Z_o=i]$ for $t \geq 0$ and zero elsewhere. $P_{ij}(t)$ is the probability that the S.M.P. is in state j at time t starting in state i.

Introducing taboo probabilities analogous to those used in the theory of Markov processes, we can define mass functions like ${}_kG_{ij}(t)$ and ${}_kM_{ij}(t)$. The following relations are frequently used in the sequel. To avoid repetitions we only state them for L.S.T.'s.

(2a) $$m_{ij}(s) = \sum_{k \in I} m_{ik}(s)\, q_{kj}(s) + q_{ij}(s)$$

(2b) $$= \sum_{k \in I} q_{ik}(s)\, m_{kj}(s) + q_{ij}(s)$$

(3) $$m_{ij}(s) = {}_k m_{ij}(s) + g_{ik}(s)\, m_{kj}(s)$$

(4) $$m_{ij}(s) = g_{ij}(s) + g_{ij}(s)\, m_{jj}(s)$$

(5) $$g_{ij}(s) = {}_i g_{ij}(s)\; {}_j m_{ii}(s)$$

(6) $$g_{ij}(s) = {}_k g_{ij}(s) + {}_j g_{ik}(s)\, g_{kj}(s)$$

(7) $$g_{ii}(s) = {}_k g_{ii}(s) + {}_k g_{ki}(s)\; {}_i m_{ik}(s)$$

(8) $$g_{ij}(s) = \sum_{k \in I} g_{kj}(s)\, q_{ik}(s) + [1-g_{jj}(s)]\, q_{ij}(s)$$

(9a) $$\pi_{ij}(s) = [1-h_j(s)][1-g_{jj}(s)]^{-1} \quad \text{if } i = j$$

(9b) $$= g_{ij}(s)\, \pi_{jj}(s) \qquad \text{if } i \neq j \,.$$

The mass functions $G_{ij}(t)$ are used to describe classification properties of the states. States i and j are said to communicate if and only if $i = j$ or $G_{ij}(\infty)\, G_{ji}(\infty) > 0$. The M.R.P. is called irreducible if all states are communicating. State i is called recurrent if and only if $G_{ii}(\infty) = 1$, otherwise it is called transient. A recurrent state i is said to be positive [null] if μ_{ii} is finite [infinite]. Part of the relationship between the M.R.P. and its corresponding Markov chain (C.M.C.) is expressed in

Lemma 2.1.

(i) A M.R.P. is irreducible if and only if its C.M.C. is irreducible;

(ii) State i is recurrent [transient] in the M.R.P. if and only if i is recurrent [transient] in the C.M.C.;

(iii) If I is finite, then i is positive in the M.R.P. if and only if i is positive in the C.M.C. and for all $j \in I$, $\eta_j < \infty$;

(iv) If a M.R.P. is irreducible then the μ_{jj} are all finite or all infinite.

It is well known that in an irreducible Markov chain all states are of the same type. Lemma 2.1 states that the same property holds in an irreducible M.R.P. This is an example of a so-called solidarity property of a M.R.P. The main object of this chapter is to prove the following ''solidarity theorem'': Let a M.R.P. be irreducible. If for some fixed state $i \in I$, $M_{ii}(t)$ has a ''certain'' exponential decay as $t \to \infty$, then for every $k \in I$ $M_{kk}(t)$ has a similar exponential decay as $t \to \infty$.

Analogous properties can be derived for the $P_{ij}(t)$.

2. A Lemma

The proofs of the solidarity theorem in the case of a Markov chain depend highly on the behavior of generating functions with nonnegative coefficients. In the M.R.P. case the proofs involve properties of L.S.T's of nondecreasing functions.

In addition to A. Thm 3 which has its analogue for powerseries, we need a lemma which is of a similar type.

Lemma 2.2.

Let $A(t) \geq 0$, $B(t) \geq 0$ be nondecreasing.

(i) If $C(t) = A(t) + B(t)$ and $c(s)$ converges in $P(\lambda)$, then $a(s)$ and $b(s)$ converge in $P(\lambda)$;

(ii) If $C(t) = A * B(t)$ and $B(t) \not\equiv 0$ and $c(s)$ converges in $P(\lambda)$, then $a(s)$ converges in $P(\lambda)$.

Proof:

(i) Suppose s real. Since $A(t)$ is nondecreasing, $a(s) \geq 0$. But then $c(s) = a(s) + b(s) \geq b(s)$. Hence $b(s)$ converges in $P(\lambda)$. Similarly for $a(s)$.

(ii) By A. Thm 3 the real point of the abscissa of convergence of $a(s)$, α, is a singularity of $a(s)$. If $\lambda < \alpha$ then α should be a zero of $b(s)$ since $c(s) = a(s)\, b(s)$. But $b(s) > 0$. Hence $\alpha \leq \lambda$.

This lemma will be used repeatedly in the proofs of the solidarity theorems.

3. The Transient Case

Consider an irreducible regular M.R.P. Assume that for some fixed pair of states $i, j \in I$, $M_{ij}(t)$ has an exponential decay, i.e. there exist constants $0 \leq K_{ij} < \infty$, $\lambda_{ij} > 0$, $0 < L_{ij} < \infty$ such that for all $t \geq t_{ij} \geq 0$

$$(E_o) \qquad |M_{ij}(t) - L_{ij}| \leq K_{ij}\, e^{-\lambda_{ij} t} .$$

By A. Thm 1 $m_{ij}(s)$ converges in $P[\lambda_{ij}]$ and $m_{ij}(0) < \infty$. From (4) we get that

$$m_{jj}(0) = g_{jj}(0)\, [1-g_{jj}(0)]^{-1}$$

so that $G_{jj}(\infty) < 1$. Lemma 2.1. (ii) together with a standard Tauberian argument implies:

If (E_o) holds then:

(10) 1. The M.R.P. is transient;

(11) 2. $L_{k\ell} = M_{k\ell}(\infty) = g_{k\ell}(o)[1-g_{\ell\ell}(o)]^{-1} < \infty$

Theorem 2.1.

Let the M.R.P. be irreducible and regular. Suppose that for some fixed pair of states i and j (E_o) holds. Then for any pair of states k and ℓ there exist constants $0 \leq K_{k\ell} < \infty$, $\lambda_{k\ell} > 0$, $0 < L_{k\ell} < \infty$ such that for $t \geq t_{k\ell} \geq 0$

(12) $$|M_{k\ell}(t) - L_{k\ell}| \leq K_{k\ell}\, e^{-\lambda_{k\ell} t}.$$

Moreover there exists a constant $\beta \geq \lambda_{k\ell}$ such that (12) holds where $\lambda_{k\ell}$ is replaced by β.

Proof:

We first show that (12) holds for $\ell = j$ (fixed) and arbitrary k. This follows immediately from (3), lemma 2.2 and the fact that $G_{ik}(\infty) \not\equiv 0$ by irreducibility.

To prove (12) for arbitrary ℓ, use (2.a) with $i = k$. In terms of mass functions we can write (2.a) in the form

(13) $$M_{kj}(t) = M_{k\ell} * Q_{\ell j}\,(t) + F_{k,\ell,j}(t)$$

where

$$F_{k,\ell,j}(t) = Q_{kj}(t) + \sum_{i \neq \ell} M_{ki} * Q_{ij}(t).$$

Iterating (13) if necessary, there exists an integer $n \geq 1$ such that

$$M_{kj}(t) = M_{k\ell} * Q_{\ell j}^{(n)}(t) + \widetilde{F}_{k,\ell,j}(t)$$

for an appropriate nondecreasing function $\widetilde{F}_{k,\ell,j}(t)$ and where $Q_{\ell j}^{(n)}(t) \not\equiv 0$. This is possible again by irreducibility.

Lemma 2.2 applies and an appeal to A. Thm 2 proves (12) for arbitrary k and ℓ.

To prove that there exists a ''best-possible'' decay parameter β, we note that $\lambda_{k\ell} \geq \lambda_{ij}$. By fixing k and ℓ as new original states we can go through the same proof as before. Hence if $\beta = \sup_{\ell,k\epsilon I} \lambda_{k\ell}$ then all $m_{k\ell}(s)$ are convergent in $P(\beta)$ but not in any larger halfplane.

Note that $L_{k\ell}$ follows from (11). The proof is terminated.

4. The Positive Recurrent Case

Consider an irreducible regular M.R.P. Assume that for a fixed pair of states $i,j\epsilon I$, there exist constants $0 \leq K_{ij} < \infty$, $\lambda_{ij} > 0$, $0 < \mu_{jj} < \infty$ and $|L_{ij}| < \infty$ such that for all $t \geq t_{ij} \geq 0$

$$\text{(E)} \qquad \left|M_{ij}(t) - \frac{t}{\mu_{jj}} - L_{ij}\right| \leq K_{ij}\, e^{-\lambda_{ij} t}$$

We first prove that:

(14) If (E) holds then the M.R.P. is positive.

For (E) implies by A. Thm 1 that $m_{ij}(s) - [\mu_{jj}s]^{-1}$ converges in $P[\lambda_{ij}]$. But by (4) $m_{ij}(s) = g_{ij}(s)[1-g_{jj}(s)]^{-1}$ so that $g_{jj}(0) = 1$. Hence state j is recurrent. Moreover j is positive since $\mu_{jj} < \infty$ and hence all states are positive by lemma 2.1 (iv). In order to prove that (E) holds for any pair of states, we introduce the functions

(15) $$N_{k\ell}(t) = M_{k\ell}(t) - \frac{t}{\mu_{\ell\ell}}$$

and their L.S.T.'s

(16) $$n_{k\ell}(s) = m_{k\ell}(s) - \frac{1}{s\mu_{\ell\ell}} .$$

The functions $N_{k\ell}(t)$ are not necessarily nondecreasing so that lemma 2.2 is no longer applicable. Nevertheless there exists a family of distribution functions $\{G_{k\ell}(t);\ k,\ell\epsilon I\}$ which is closely related to the functions $\{N_{k\ell}(t);\ k,\ell\epsilon I\}$. Lemma 2.2 applies to the functions $G_{k\ell}(t)$, and since (E) supplies more information about the states i and j we prove

Lemma 2.3.

When (E) holds then:

(i) $1-g_{jj}(s)$ has as its only zero in $P(\lambda_{ij})$ a simple isolated zero at $s=0$;

(ii) $\{g_{kj}(s),\ k\epsilon I\}$ have a common halfplane of convergence $P(\beta_j)$ where $\beta_j > 0$.

Proof:

We first prove (ii).

From (6) with $k=j$ we obtain that

$$g_{ij}(s) = {}_jg_{ij}(s)\,[1+g_{jj}(s)].$$

Therefore ${}_jG_{ij}(t) \not\equiv 0$ and $g_{jj}(s)$ converges wherever $g_{ij}(s)$ converges by lemma 2.2. If we set $i=j$ in (6) then lemma 2.2 shows that $g_{kj}(s)$ converges in the same halfplane as $g_{jj}(s)$, if ${}_jG_{jk}(t) \not\equiv 0$. The latter fact follows from (5) with $i = j$, $j=k$ and $s=0$.

Repeating the same argument starting with $g_{kj}(s)$, we find that for some $\beta_j \geq 0$ all $g_{kj}(s)$ converge in $P(\beta_j)$. From this argument it follows incidentally that $n_{ij}(s)$ and $n_{jj}(s)$ converge in the same halfplane. This proves (ii) for $\beta_j \geq 0$. By (E) and A. Thm 1 we know that $n_{ij}(s)$ converges in $P(\lambda_{ij})$. So does $n_{jj}(s)$. But by (3) and (16)

$$n_{jj}(s) = g_{jj}(s)\,[1-g_{jj}(s)]^{-1} - [s\,\mu_{jj}]^{-1}$$

so that by lemma 1.2 there exists a value $\beta_j > 0$ such that $g_{jj}(s)$ converges in $P(\beta_j)$ and $g_{jj}(s) \neq 1$ if $s \neq 0$. To prove that the root $s = 0$ is simple, we finally note that $g_{jj}(s)$ is analytic in $P(\beta_j)$ by A. Thm 3, $g_{jj}(0) = 1$, but $g'_{jj}(0) = -\mu_{jj} \neq 0$. This proves the lemma.

We now show that the previous lemma remains valid for arbitrary ℓ. An auxiliary formula will prove to be useful. In (6) we put $i=\ell$, $j=\ell$ and $k=j$, and then $i=\ell$, $j=j$ and $k=\ell$. By eliminating ${}_\ell g_{\ell j}(s)$ from the two equations obtained we get

$$(17) \qquad [g_{\ell\ell}(s) - {}_jg_{\ell\ell}(s)][1-{}_jg_{\ell\ell}(s)]^{-1} = 1-g_{\ell j}(s)\ g_{j\ell}(s).$$

The right hand side of (17) is symmetric in j and ℓ so that it also equals the left hand side with ℓ and j reversed. After some manipulations we obtain

$$(18) \qquad [1-g_{\ell\ell}(s)] = \{\frac{1-{}_jg_{\ell\ell}(s)}{1-{}_\ell g_{jj}(s)}\}[1-g_{jj}(s)].$$

Lemma 2.4.

When (E) holds, then for every $\ell \epsilon I$:

(i) $1-g_{\ell\ell}(s)$ has as its only zero in $P(\beta_j)$ a simple isolated zero at $s=0$;

(ii) $\{g_{k\ell}(s), k\epsilon I\}$ have a common halfplane of convergence $P(\beta_\ell)$ where $\beta_\ell > 0$.

Proof:

In view of (18) and lemma 2.3 we first prove

(iii) $1-{}_jg_{\ell\ell}(s)$ is analytic and nonzero in $P(\beta_j)$;

(iv) $1-{}_\ell g_{jj}(s)$ is analytic and nonzero in $P(\beta_\ell)$ for some $\beta_\ell > 0$.

To prove (iii) we start from (6) with $i=\ell$, $k=\ell$. Then

$$(19) \qquad g_{\ell j}(s) = {}_\ell g_{\ell j}(s) + {}_jg_{\ell\ell}(s)\ g_{\ell j}(s),$$

so that $1-{}_jg_{\ell\ell}(s)$ is analytic in $P(\beta_j)$ by lemma 2.2. From (19) is also follows that $1-{}_jg_{\ell\ell}(s) = 0$ in $P(\beta_j)$ if and only if ${}_\ell g_{\ell j}(s) = 0$. However ${}_\ell g_{\ell j}(s) \neq 0$, as follows from (5) with $i=\ell$.

In proving (iv) we note that (6) with $i=j$ and $k=\ell$ implies that

$1-{}_{\ell}g_{jj}(s)$ in analytic in $P(\beta_j)$. Since $G_{\ell\ell}(t)$ is nondecreasing A. Thm 3 states that $g_{\ell\ell}(s)$ has its first singularity on the real axis, say at $-\lambda_{\ell\ell} \leq 0$. If we show that $1-{}_{\ell}g_{jj}(s) \neq 0$ at $s = 0$ then by continuity there is a neighborhood of s=0 in which $1-{}_{\ell}g_{jj}(s)$ does not vanish. Hence $-\lambda_{\ell\ell} < 0$.

To show that ${}_{\ell}g_{jj}(0) \neq 1$, put $i=j$, $j=\ell$, $k=j$ and $s=0$ in (6) and $i=\ell$, $k=j$ and $s=0$ in (7). We obtain

$$1 = {}_{j}g_{j\ell}(0) + {}_{\ell}g_{jj}(0)$$

$$1-{}_{j}g_{\ell\ell}(0) = {}_{j}g_{j\ell}(0)\ {}_{\ell}m_{\ell j}(0).$$

But by (iii) ${}_{j}g_{j\ell}(0) > 0$, so that $1 > {}_{\ell}g_{jj}(0)$. This proves (iv) for $\beta_{\ell} = \lambda_{\ell\ell} > 0$.

Clearly (iii) and (iv) imply (i) by (18), and (ii) follows since the (ii)-part of the proof of lemma 2.3 shows that $g_{k\ell}(s)$ converges in the same halfplane as $g_{\ell\ell}(s)$. This proves the lemma.

We observe that if $1-{}_{\ell}g_{jj}(s) \neq 0$ for $-\beta_j \leq s < 0$, one can take $\beta_{\ell} \geq \beta_j$. If there is a solution $s = s_o$ with $-\beta_j \leq -s_o < 0$ then we take $\beta_{\ell} = -s_o$. Another way of defining β_{ℓ} is obtained from the relations (5) and (6):

$$[1-{}_{\ell}g_{jj}(s)]^{-1} = {}_{\ell}m_{jj}(s) \tag{20}$$

which is even more intuitive: The real point of the abscissa of convergence of ${}_{\ell}m_{jj}(s)$ is the first zero of $1-{}_{\ell}g_{jj}(s)$.

We prove the solidarity theorem in the positive recurrent case.

Theorem 2.2.

Let the M.R.P. be irreducible and regular. Suppose that for some fixed pair of states i and j (E) holds. Then for any pair of states k and ℓ there exist constants $0 \leq K_{k\ell} < \infty$, $\beta_\ell > 0$, $0 < \mu_{\ell\ell} < \infty$ and $|L_{k\ell}| < \infty$ such that for $t \geq t_{k\ell} \geq 0$

$$(21) \qquad |M_{k\ell}(t) - \frac{t}{\mu_{\ell\ell}} - L_{k\ell}| \leq K_{k\ell}\, e^{-\beta_\ell t}$$

Moreover there exists a common β such that (21) holds with β_ℓ replaced by β.

The constant $L_{k\ell}$ is given by

$$(22) \qquad L_{k\ell} = [2\,\mu_{\ell\ell}^2]^{-1}\,\{\mu_{\ell\ell}'' - 2\mu_{k\ell}\,\mu_{\ell\ell}\}$$

where $\mu_{\ell\ell}''$ is the second moment of $G_{\ell\ell}(t)$.

Proof:

By (16), (E) and lemma 2.3, $n_{kj}(s)$ converges in the same half-plane as $n_{ij}(s)$ so that (21) follows for $\ell=j$.

By lemma 2.4. $g_{k\ell}(s)$ is convergent in $P(\beta_\ell)$ and $[1-g_{\ell\ell}(s)]^{-1}$ has only a simple pole with residue $\mu_{\ell\ell}^{-1}$ in $P(\beta_\ell)$, so that (21) is proved for arbitrary k and ℓ.

The common value can be taken as $\beta = \inf_\ell \beta_\ell$ and (21) still holds.

The value $L_{k\ell}$ can be obtained from the equality

$$N_{k\ell}(\infty) = n_{k\ell}(0) = \lim_{s\to 0} \left\{\frac{g_{k\ell}(s)}{1-g_{\ell\ell}(s)} - \frac{1}{\mu_{\ell\ell}s}\right\}$$

by a standard dominated convergence argument in $P[\beta_\ell]$. This finishes the proof.

The value β of the theorem is not necessarily best-possible for all pairs of states.

Combining the above theorem with Cor. 1.1 we obtain immediately

Corollary 2.1.

Let the conditions of theorem 2.2 be satisfied. Assume that for all $i,j\in I$ $R_{ij}(t)$ is a function on $[0,\infty]$, nonnegative, nondecreasing and such that there exists $\rho_j > 0$ for which

$$\int_0^\infty e^{\rho_j t} \, d\, R_{ij}(t) < \infty \quad \text{for all } i\in I.$$

Then there exist appropriate constants such that for $t \geq t_{k\ell} \geq 0$

$$(23) \qquad \left|\int_0^t R_{k\ell}(t-u)\, dM_{k\ell}(u) - \frac{1}{\mu_{\ell\ell}} \int_0^\infty R_{k\ell}(t)dt\right| \leq K_{k\ell}\, e^{-\sigma_\ell t}.$$

As a special case of the above corollary, we derive an asymptotic decay for $P_{ij}(t)$.

It follows easily from (9) and (4) that

$$(24) \qquad \pi_{ij}(s) = [1-h_j(s)]\, m_{ij}(s) + \delta_{ij}[1-h_j(s)].$$

Lemma 2.5.

If (E) holds, then $h_\ell(s)$ converges in $P(\tau_\ell)$ for $\tau_\ell > 0$.

Proof:

We prove the lemma only for $\ell=j$, since lemma 2.4 implies it then for arbitrary $\ell\in I$.

Since $H_j(t)$ is a distribution function, A. Thm 3 shows that we only have to prove that $|h(s)| < \infty$ for $|s| < \tau_j$ for $\tau_j > 0$.

By lemma 2.3 $g_{kj}(s)$ converges in $P(\beta_j)$. Choose a real s in (8) with $i=j$ such that $-\beta_j < s < 0$ and $g_{kj}(s) \geq 1$. Since also $q_{jj}(s) \leq h_j(s)$ for all s, we obtain $g_{jj}(s) \geq [2-g_{jj}(s)]\, h_j(s)$ for $-\beta_j < s \leq 0$. By continuity of $g_{jj}(s)$ in $P(\beta_j)$ there exists $\tau_j > 0$ such that $2-g_{jj}(s) \neq 0$ for $-\tau_j < s \leq 0$ and hence $h_j(s)$ is bounded for $-\tau_j < s$. This proves the lemma.

<u>Corollary 2.2</u>.

Let the conditions of theorem 2.2 be satisfied. Then there exist appropriate constants such that for $t \geq t_{ij} \geq 0$

$$(25) \qquad \left|P_{ij}(t) - \frac{\eta_j}{\mu_{jj}}\right| \leq K_{ij} e^{-\sigma_j t}$$

for every pair of states $i,j \in I$.

<u>Proof</u>:

Put $R_{k\ell}(t) = 1-H_\ell(t)$ in Cor. 2.1. Then by lemma 2.5
$\int_0^\infty e^{\tau_j t}\, dR_{ij}(t) < \infty$ for all i.

If $i \neq j$ then the proof is immediate from (24); if $i=j$ then

$$\left|P_{jj}(t) - \frac{\eta_j}{\mu_{jj}}\right| \leq \left|P_{jj}(t)-1+H_j(t)- \frac{\eta_j}{\mu_{jj}}\right|+\left|1-H_j(t)\right|.$$

The corollary applies to the first term and lemma 2.5 to the second. Hence the proof is finished.

We indicate briefly how analogous results can be proved for the second moments of $N_j(t)$. Introduce

$$V_{ij}(t) = E[N_j^2(t)|Z_o = i].$$

Then it is easy to show [38] that

Lemma 2.6.

(26) $$V_{ij}(t) = G_{ij} * V_{jj}(t) - 2\,G_{ij}(t) + 2M_{ij}(t)$$

In the transient case (E_o) implies that for appropriate constants

(27) $$|V_{k\ell}(t) - g_{k\ell}^2(0)\,[1-g_{\ell\ell}(0)]^{-2}| \le K_{k\ell}\,e^{-\sigma_\ell t}.$$

Similarly (E) implies in the positive case that

(28) $$|V_{k\ell}(t)-t^2\mu_{\ell\ell}^{-2} - tA_{k\ell}\mu_{\ell\ell}^{-3}-B_{k\ell}\mu_{\ell\ell}^{-4}| \le K_{k\ell}\,e^{-\sigma_\ell t}$$

where

$$A_{k\ell} = \mu_{\ell\ell}'' - \mu_{k\ell}\,\mu_{\ell\ell} - \mu_{\ell\ell}^2$$

and

$$12\,B_{k\ell} = 9\mu_{\ell\ell}''^2 - 6\mu_{\ell\ell}^2[\mu_{\ell\ell}'' - \mu_{k\ell}''] - 4\mu_{\ell\ell}\,\mu_{\ell\ell}''' + 12\mu_{k\ell}\mu_{\ell\ell}[\mu_{\ell\ell}^2 - \mu_{k\ell}\mu_{\ell\ell}''].$$

5. Some Results on a Finite M.R.P.

Let us consider a finite irreducible M.R.P. with m different states. An obvious consequence of lemma 2.1 is

Lemma 2.7.

If $\max_{j\in I} \eta_j < \infty$, then the M.R.P. is positive.

It is well-known [35, p. 272] that a finite irreducible Markov chain is geometrically ergodic. By using skeleton chains, as J.F.C. Kingman did in [25], this property extends to conservative irreducible continuous Markov chains over a finite statespace. A M.R.P. does not possess this property in general. For example define a two state M.R.P. by $q_{ij}(s) = \frac{1}{2} f(s)$ for $i=1,2, j=1,2$. Let $|f'(0)| < \infty$. But if $f(s)$ does not converge in $P(\lambda)$ for some $\lambda > 0$ then the M.R.P. is not exponentially ergodic.

For brevity let $R(t) = \{R_{ij}(t)\}$ $i,j\epsilon I$ where $R_{ij}(t)$ is $Q_{ij}(t)$, $\delta_{ij} H_i(t)$, $G_{ij}(t)$ or $P_{ij}(t)$. Let also $r(s) = \{r_{ij}(s)\}$ be the matrix of L.S.T.'s of $R_{ij}(t)$.

Definition 1. $R(t)$ is said to be exponentially bounded (written R^{ϵ}), if and only if $\inf\{\lambda > 0 \,|\, |R_{ij}(t) - R_{ij}(\infty)| \leq Ke^{-\lambda t}, 0 \leq K < \infty, \text{ all } i,j\epsilon I\} > 0$.

Definition 2. $r(s)$ is said to be analytic (written r^{ϵ}), if and only if $\inf\{\rho > 0 \,|\, r_{ij}(s)$ converges in $P(\rho)$ for all $i,j\epsilon I\} > 0$.

The two definitions are equivalent by an obvious generalization of A. Thm 4. Introduce

Condition (i): $Q_{ij}(t)$ is strongly non-lattice

Condition (ii): $\max_j \eta_j < \infty$

We intend to show that under (i) and (ii)

$$q^{\epsilon} \Leftrightarrow h^{\epsilon} \Leftrightarrow g^{\epsilon} \Rightarrow \pi^{\epsilon}.$$

Lemma 2.8.

a. $q^{\epsilon} \Leftrightarrow h^{\epsilon}$

b. $g^{\epsilon} \Rightarrow q^{\epsilon}$

Proof:

a. This is obvious since $h_i(s) = \sum_{j\epsilon I} q_{ij}(s)$ and I is finite.

b. This implication was proved in lemma 2.5 for fixed second index. Since m is finite, it follows for all states.

Lemma 2.9.

If (i) holds, then $g^{\epsilon} \Rightarrow \pi^{\epsilon}$

Proof:

By the irreducibility of the M.R.P. $G_{jj}(t)$ is nondegenerate at $t = 0$. Since g^{ϵ} holds, lemma 1.1 together with (i) implies that $s = 0$ is the only zero of $1-g_{jj}(s)$ in $P(\beta)$ for some $\beta > 0$ and that this zero is simple. From the previous lemma we obtain that $1-h_j(s)$ is analytic in $P(\gamma)$ for some $\gamma > 0$ and has a simple zero at $s = 0$.

Hence $[1-h_j(s)][1-g_{jj}(s)]^{-1}$ is convergent in $P[\min(\beta,\gamma)]$. By (9.a) then, π^{ϵ} holds for the diagonal elements. However (9.b) and g^{ϵ} imply π^{ϵ} for the off-diagonal elements. This proves the lemma.

The next lemma is slightly more complicated.

Lemma 2.10.

$$g^{\epsilon} \Rightarrow g^{\epsilon}$$

Proof:

Assume that s is real. It follows from (8) that g satisfies the equation

$$g = q(I-q)^{-1} \{{}_d[(I-q)^{-1}]\}^{-1}$$

where ${}_dA$ denotes the diagonal part of A.

Let $u_{ij} = \{(I-q)^{-1}\}_{ij}$ and v_{ji} be the adjoint of the element $\delta_{ij} - q_{ij}$ in $|I-q|$. Then clearly

$$u_{ij} = \frac{1}{|I-q|} v_{ji}$$

and

$$\{_d[(I-q)^{-1}]\}^{-1}_{ij} = \frac{|I-q|}{\delta_{ij} v_{jj}} .$$

Hence for $i,j \epsilon I$,

(29) $$g_{ij}(s) = \sum_{\ell \epsilon I} q_{i\ell}(s)\, v_{j\ell}(s)\, [v_{jj}(s)]^{-1}.$$

Assume now that q^{ϵ} is satisfied. Since $v_{j\ell}(s)$ is a finite linear combination of products of elements of the analytic matrix $q(s)$, $\sum_{\ell \epsilon I} q_{i\ell}(s)\, v_{j\ell}(s)$ is analytic in the same domain as $q(s)$.

At $s = 0$, $|I-q(0)| = |I-P|$ where P is the transition matrix of the C.M.C. The Perron-Frobenius theorem for stochastic matrices [15, p.50] implies that $v_{ij}(0) > 0$ for all $i,j \epsilon I$. By analyticity of $v_{jj}(s)$ at $s = 0$, (29) is well defined in a neighborhood of $s=0$.

But $G_{ij}(t)$ is a distribution function. Hence by A. Thm 3 the first singularity of $g_{ij}(s)$ lies on the real axis. Hence g^{ϵ} holds, which proves the lemma.

By a combination of the last four lemmas, we obtain

Theorem 2.3.

Consider a finite irreducible M.R.P. that satisfies (i) and (ii). Assume that any one of the conditions $q^{\epsilon}, h^{\epsilon}, g^{\epsilon}, Q^{\epsilon}, H^{\epsilon}, G^{\epsilon}$ is satisfied. Then there exist constants $0 \leq K_i < \infty$, $\lambda_i > 0$ $i = 1,2$, such that

a. for $t \geq t_1 \geq 0$

(30) $$\left|P_{ij}(t) - \frac{\eta_j}{\mu_{jj}}\right| \leq K_1\, e^{-\lambda_1 t};$$

b. for $t \geq t_2 \geq 0$

(31) $$|M_{ij}(t) - \frac{t}{\mu_{jj}} - \frac{\mu''_{jj} - 2\mu_{ij}\mu_{jj}}{2\mu_{jj}^2}| \leq K_2 e^{-\lambda_2 t}.$$

Proof:

The proof of (30) is immediate from the lemmas.

Relation (31) follows for example from g^ϵ and the R-basic renewal theorem of chapter I.

Note that (29) implies that $\mu_{jj} = \frac{1}{v_{jj}(0)} \sum_{i\in I} v_{ii}(0)\, \eta_i$. It might be of interest to compare this relation with [40, p. 1456] where R. Pyke obtains $\mu_{jj} = \sum_{i\in I} {}_jM_{ji}(\infty)\, \eta_i$.

Definition 3: An irreducible M.R.P. is called doubly stochastic if $\sum_{i\in I} Q_{ij}(\infty) = \sum_{j\in I} Q_{ij}(\infty) = 1$ for all $i,j\in I$.

Corollary 2.3.

If an irreducible M.R.P. is positive and doubly stochastic, and if $\sum_{j\in I} \eta_j < \infty$, then for all $i,j\in I$

(i) $P_{ij}(\infty) = \eta_j\{\sum_{j\in I} \eta_j\}^{-1}$ and

(ii) $\lim_{t\to\infty} \frac{M_{ij}(t)}{t} = \{\sum_{j\in I} \eta_j\}^{-1}$.

Proof:

We know from the limit theorem for $P_{ij}(t)$ that $P_{ij}(\infty) = \eta_j \mu_{jj}^{-1}$. From (8) we obtain that

$$\mu_{ij} = \sum_{k\in I} \mu_{kj}\, p_{ik} + \eta_i - \mu_{jj}\, p_{ij}.$$

Summation over i yields $\mu_{jj} = \sum_{i\in I} \eta_i$ which proves the corollary.

6. An Application to the M|G|1 Queue

Let $H(t)$ be the distribution function of a nonnegative random variable with finite positive mean α. Assume that $h(s)$ converges in $P(\eta)$ for some $\eta \geq 0$.

Consider a single server queue with Poisson input at rate $\lambda > 0$, and service times, distributed according to $H(t)$. We define the random times $T_0, T_1, \ldots,$ as follows:

(i) $T_0 = 0$ a.s.

(ii) T_{n+1} is the time instant in which all customers, if any, present at T_n complete service. If there are no customers at T_n, then T_{n+1} is the instant in which the first customer to arrive after T_n completes service.

If $\xi(t)$ denotes the queue length at $t+$, then the process $\{\xi(T_n), T_{n+1}-T_n, n \geq 0\}$ is a S.M.P. on the nonnegative integers. The transition matrix $Q(t)$ defined in §1 can be evaluated:

(32)
$$Q_{ij}(X) = \begin{cases} \int_0^x e^{-\lambda y} \frac{[\lambda y]^j}{j!}\, dH^{(i)}(y) & \text{if } i > 0,\ j \geq 0 \\ \\ \int_0^x [1-e^{-\lambda(x-y)}]\, dQ_{1j}(y) & \text{if } i=0,\ j \geq 0, \end{cases}$$

in which $H^{(i)}(y)$ is the i-fold convolution of $H(y)$.

It is easy to verify that this S.M.P. is irreducible and regular. This approach to M|G|1 is due to M.F. Neuts in [33].

Define the mass function

$$G(t) = P\{T_n \leq t,\ \xi(T_n) = 0,\ \xi(T_m) \neq 0\ \ m = 1,2,\ldots,n-1$$

$$\text{for some}\ \ n \mid \xi(T_o) = 0\}.$$

Then $G(t)$ gives the mass distribution of the length of the busy period. If $\gamma(s)$ is the L.S.T. of $G(t)$ then $\gamma(s)$ is the unique solution in $P(0)$ to the equation

$$(33) \qquad z = h[s + \lambda - \lambda z]$$

which lies in $|z| < 1$ for all s in $P(0)$, [46, p. 47].

Let $M_{ij}(t)$ be the expected number of visits to state j in $(0,t]$, given that $\xi(T_o) = i$, then for $i = j = 0$

$$(34) \qquad m_{oo}(s) \equiv m(s) = \frac{\lambda}{\lambda+s}\,\gamma(s)[1-\frac{\lambda}{\lambda+s}\,\gamma(s)]^{-1}.$$

Assume that $\gamma(s)$ converges in $P(\theta)$ for $\theta > o$. The next lemma gives information about $\gamma(s)$.

<u>Lemma 2.11.</u>

A. If $1-\alpha\lambda > 0$ then:

(i) $\gamma(s)$ is the L.S.T. of a probability distribution;

(ii) $-\gamma'(0) = \alpha[1-\alpha\lambda]^{-1}$;

(iii) if $\eta > \lambda$ then $\theta \geq [\sqrt{\eta} - \sqrt{\lambda}]^2 > 0$;

(iv) the S.M.P. is positive recurrent.

B. If $1-\alpha\lambda = 0$ then:

(i) $\gamma(s)$ is the L.S.T. of a probability distribution;

(ii) $-\gamma'(0) = \infty$;

(iii) $\theta = 0$;

(iv) the S.M.P. is null recurrent.

C. If $1-\alpha\lambda < 0$ then:

(i) $\gamma(s)$ is not the L.S.T. of a probability distribution;

(ii) if $0 < \eta < K\lambda$ then $\theta > 0$;

(iii) the S.M.P. is transient.

<u>Proof</u>:

Except for A(iii), B(iii) and C(ii) all parts of the lemma are well-known. For example [33,46]. If $1-\alpha\lambda = 0$, then $-\gamma'(0) = \infty$ so that $\theta = 0$.

Assume $\eta > 0$. In proving A(iii) and C(ii) we note that $1-H(t) \leq Ke^{-\eta t}$ by A. Thm. 4. Let K be such that this inequality holds for all $t \geq 0$. Then

(35)
$$h(s) \geq \{s[1-K] + \eta\} \{s+\eta\}^{-1} \quad \text{if} \quad s \geq 0$$
$$h(s) \leq \{s[1-K] + \eta\} \{s+\eta\}^{-1} \quad \text{if} \ -\eta < s \leq 0$$

Let

(36) $$f[s,\gamma(s)] \equiv \lambda\gamma^2(s)-s\gamma(s)-[\lambda+\eta+\lambda(1-K)]\gamma(s)+s(1-K)+\lambda(1-K)+\eta \ .$$

Then from (33), (35) and (36) we obtain

$$f(s,\gamma(s)) \geq 0 \qquad \text{if} \quad -\eta < s+\lambda-\lambda\gamma(s) \leq 0;$$

$$f(s,\gamma(s)) \leq 0 \qquad \text{if} \quad s + \lambda \geq \lambda\gamma(s).$$

A consideration of the graph of the hyperbola $f(s,\gamma(s)) = 0$ in the prescribed regions yields:

(a) if $\eta \geq K\lambda$ then $\theta \geq [\sqrt{\eta} - \sqrt{K\lambda}]^2$;

(b) if $\eta < K\lambda$ then $\theta > 0$.

This proves the lemma.

Since α is the mean of the distribution $H(t)$ we have by (35)

$$-\alpha = h'(0) = \lim_{s \to 0+} \frac{h(s)-1}{s} \geq -\frac{K}{\eta},$$

and hence $\alpha\eta \leq K$. If additionally $\eta \geq K\lambda$ then $1-\alpha\lambda \geq 0$. This simplifies the statement of

Theorem 2.4.

The S.M.P. defined above is:

(i) transient and exponentially ergodic if $1-\alpha\lambda < 0$;

(ii) positive and exponentially ergodic if $\eta > \lambda K$.

Proof:

(i) By Thm. 2.1 and (34) we have to show that $m(s)$ converges in $P(\beta_o)$ for some $\beta_o > 0$. For then (E_o) of §3 is satisfied for $i = j = 0$.

If $1-\alpha\lambda < 0$ then the S.M.P. is transient. Moreover by C(ii) of the lemma $\gamma(s)$ converges in $P(\theta)$ for some $\theta > 0$; $\gamma(0) \neq 1$ by C(i), and hence $\gamma(s) \neq 1$ in a neighborhood of $s = 0$. By A. Thm. 3 $m(s)$

converges then in $P(\beta_o)$ where $\beta_o = \min(\lambda,\theta) > 0$.

(ii) By Thm. 2.2 we need to prove that (E) holds for $i=j=0$, or that $m(s) - \lambda[1-\alpha\lambda]s^{-1}$ converges in $P(\beta_o)$ for some $\beta_o > 0$. This follows from Thm. 1.1 since $\frac{\lambda}{\lambda+s}\gamma(s)$ converges in $P\{\min(\lambda,\theta)\} = P(\theta)$ where $\theta \geq [\sqrt{\eta} - \sqrt{K\lambda}]^2 > 0$, and is the L.S.T. of a strongly non-lattice distribution. This proves the theorem.

The above theorem together with lemma 2.11 should be compared with a result obtained by S. Karlin and J. McGregor in [21, p. 102].

Remark.

C.K. Cheong, a student of D. Vere-Jones, independently obtained several of the results in this chapter as discussed in his preliminary report [5].

CHAPTER III

RENEWAL THEOREMS WHEN THE FIRST OR THE SECOND MOMENT IS INFINITE

The classical renewal theorems do not tell much about the renewal function if the mean renewal lifetime is infinite.

To obtain more accurate results we prove a theorem that can be considered as the analogue of Smith's key renewal theorem [41] if $1-F(t) \sim t^{-\alpha}L(t)$ for $t \to \infty$ where $L(t)$ is slowly varying and $0 < \alpha \leq 1$.

In section 3 we consider $1 < \alpha < 2$. An application of the main theorem yields precise estimates for the renewal function in that case.

1. Regularly Varying Functions

In this section we collect a number of results that will be applied throughout the entire chapter. For a general discussion, see W. Feller [12].

Definition 1: A function $L(t)$ is called slowly varying if $L(t)$ is defined for $t > 0$, positive, continuous and if $\lim_{t\to\infty} \frac{L(xt)}{L(t)} = 1$ for all $x > 0$. We write $L(t)$ is s.v.

Definition 2: A distribution function $F(t) \in V_\alpha$ for $\alpha \geq 0$ if there exists a slowly varying function $L(t)$ such that

$$(1) \qquad 1-F(t) \sim t^{-\alpha} L(t) \quad \text{as} \quad t \to \infty$$

The real number α is the exponent of $F(t)$, and $F(t)$ is said to be a <u>regularly varying distribution</u> with exponent α. It is easy to show that if (1) holds for some $\alpha \geq 0$, then this α is unique. Moreover. if $F \in V_\alpha$ then $1-F(t) = c(t^{-\alpha+\epsilon})$ for every $\epsilon > 0$, and if $1-F(t) = O(t^{-\alpha})$ but $1-F(t) \neq o(t^{-\alpha})$ then $F \in V_\alpha$.

The class V_α is a subclass of the family of regularly varying functions as defined by Feller [12], K. Knopp [26] and others. If $\alpha = 0$ then we assume that $F(t) < 1$ for every $t \geq 0$. V_0 reduces to a class of slowly varying functions. A paper by S. Aljančić, R. Bojanič and M. Tomić [1] (later on referred to as A.B.T.) contains a number of important results, that will be used later.

<u>Lemma 3.1</u>.

Let $L(t)$ be slowly varying. Then

(i) $\frac{L(ut)}{L(t)} \to 1$ as $t \to \infty$ uniformly in every finite interval;

(ii) For every γ, $t^\gamma L(t) \to \infty$ if $\gamma > 0$ and $t^\gamma L(t) \to 0$ if $\gamma < 0$;

(iii) If $L_1(t)$ and $L_2(t)$ are slowly varying, so are $L_1(t)\, L_2(t)$ and $\frac{L_1(t)}{L_2(t)}$.

Also the next lemma is stated without proof. The first part is due to Parameswaran [34] and W.L. Smith [42]; the second part was proved by J. Lamperti [27] and W. Feller [12].

<u>Lemma 3.2</u>.

a. If $L(t)$ is s.v. for $t \geq a$, so is $M(t) = \int_a^t \frac{L(x)}{x}\, dx$

b. Let $f(x) > 0$, and suppose that $f'(x)$ exists for large x. If $\lim_{x\to\infty} \frac{xf'(x)}{f(x)} = \alpha$, then $f(x) = x^\alpha L(x)$ where $L(x)$ is s.v.

Corollary 3.1.

If $F(t)$ has a derivative for large t, and if for some $\alpha > 0$, $\lim_{t\to\infty} t^{1+\alpha} F'(t) = c(\neq 0, \neq\infty)$, then $F \epsilon V_\alpha$.

Proof:

For a fixed $\epsilon > 0$ choose $t \geq t_o$ such that

(2) $$(c-\epsilon)t^{-\alpha-1} \leq F'(t) \leq (\epsilon+c)\, t^{-\alpha-1},$$

and since $\alpha > 0$ this implies

(3) $$(c-\epsilon)\,\frac{t^{-\alpha}}{\alpha} < 1-F(t) \leq (c+\epsilon)\,\frac{t^{-\alpha}}{\alpha}\,.$$

From (2) and (3) we get that for $t \geq t_o$

$$\alpha\,\frac{c-\epsilon}{c+\epsilon} \leq \frac{tF'(t)}{1-F(t)} \leq \alpha\,\frac{c+\epsilon}{c-\epsilon}$$

The result follows now immediately from lemma 3.2.b.

One of the main properties of s.v. functions is expressed in the following lemma, which combines an Abelian and Tauberian theorem. An elementary proof is given by Feller in [12, p. 421].

Lemma 3.3.

If $L(t)$ is s.v. and $G(t)$ is a positive, monotone and right hand continuous function on $[0,\infty]$ and if $0 \leq \alpha < \infty$, then each of the relations

$$g(s) \sim s^{-\alpha}\, L(\tfrac{1}{s}) \qquad s \to 0+$$

and

$$G(t) \sim \frac{t^{\alpha}}{\Gamma(\alpha+1)}\, L(t) \qquad t \to \infty$$

implies the other.

Although slowly varying functions have important applications in limit theorems of sums of independent random variables, we do not need them here. We refer to the standard treatises on the subject [12,16,31]. An application similar to those contained herein is given in Lamperti, [28].

Next we derive a number of lemmas involving integrals of regularly varying distributions. We assume that if $F \epsilon V_\alpha$, then $1-F(t) = t^{-\alpha}L(t)$ for all $t > 0$, and that as $t \to 0+$ $L(t)$ is so defined that $t^{-\alpha}L(t) \to 1$. Moreover if $F \epsilon V_1$ then we define

(4) $$L^*(t) = \int_0^t [1-F(x)]\, dx.$$

Lemma 3.4.

(i). Let $F \epsilon V_\alpha$, $0 \leq \alpha < 1$. Then

$$\int_0^\infty e^{-st}[1-F(t)]dt \sim s^{\alpha-1}\,\Gamma(1-\alpha)\; L(\tfrac{1}{s}) \quad \text{as} \quad s \to 0+$$

(ii) Let $F \epsilon V_1$. Then

$$\int_0^\infty e^{-st}\; L^*(t)\; dt \sim \frac{1}{s}\, L^*(\tfrac{1}{s}) \quad \text{as} \quad s \to 0+$$

Proof:

(i). $$\int_0^\infty e^{-st}\,[1-F(t)]dt = s^{\alpha-1} \int_0^\infty e^{-u}\, u^{-\alpha}\, L(\tfrac{u}{s})\, du.$$

Now $\alpha < 1$. Hence A.B.T. Thm. 5 is applicable. There results that the integral on the right is asymptotically equal to $s^{\alpha-1}L(\frac{1}{s}) \int_0^\infty e^{-u}u^{-\alpha}du$ as $s \to 0+$.

(ii) By lemma 3.2.a $L^*(t)$ of (4) is s.v. and part (i) applies for $\alpha = 0$.

The next lemma evaluates some integrals asymptotically. The first part follows directly from A.B.T. Thm. 2, while in the second part A.B.T. Thm. 1 applies.

Lemma 3.5.

(i) Let $F \epsilon V_\alpha$. Then for all $k < \alpha - 1$

$$\int_t^\infty x^k [1-F(x)]\, dx \sim \frac{t^{k-\alpha+1}}{-k+\alpha-1} L(t) \quad \text{as} \quad t \to \infty$$

(ii) Let $F \epsilon V_\alpha$, $p > 0$ and $q > \alpha$. Then

$$\int_0^t (t-x)^{p-1} x^{q-1} [1-F(x)]dx \sim t^{p+q-1-\alpha} B(p, q-\alpha)\, L(t) \text{ as } t \to \infty$$

2. Renewal Theorems for the Case Where No First Moment Exists

In this section we prove analogues of renewal theorems that are classical in the case where a finite first moment exists.

An elementary renewal theorem was proved by Feller [13] in connection with fluctuation theory of recurrent events. He considered $1-F(t) \sim t^{-\alpha}$ where $0 < \alpha < 1$. W. L. Smith proved a result for the two boundary cases $\alpha = 0$, $\alpha = 1$, [42].

Assume from now on that $F \epsilon V_\alpha$, $0 \leq \alpha \leq 1$. Since $h(s) = \frac{f(s)}{1-f(s)}$, where $h(s)$ is the L.S.T. of the renewal function $H(t)$, and

$$f(s) = 1-s \int_0^\infty e^{-st} [1-F(t)]\, dt \tag{5}$$

it follows from lemma 3.4 that

(6) $$\text{if } \quad 0 \leq \alpha < 1 \quad \text{then} \quad h(s) \sim \frac{s^{-\alpha}}{\Gamma(1-\alpha)\ L(\frac{1}{s})} \quad \text{as } s \to 0+$$

(7) $$\text{and if } \alpha = 1 \quad \text{then} \quad h(s) \sim \frac{s^{-1}}{L^*(\frac{1}{s})} \quad \text{as } s \to 0+$$

Theorem 3.1.

If $F \epsilon V_\alpha$, then as $t \to \infty$

$$H(t) \sim \begin{cases} \dfrac{1}{L(t)} & \text{if } \alpha = 0 \\ \dfrac{t^\alpha}{L(t)} \dfrac{\sin \alpha\pi}{\alpha\pi} & \text{if } 0 < \alpha < 1 \\ \dfrac{t}{L^*(t)} & \text{if } \alpha = 1 \end{cases}$$

Proof:

Take first $0 < \alpha < 1$. Since $L(t) \nrightarrow 0$, $\frac{1}{L(\frac{1}{s})}$ is s.v. as $s \to 0+$. By (6) and lemma 3.3

$$H(t) \sim \frac{t^\alpha}{\Gamma(1-\alpha)\Gamma(1+\alpha)L(t)} \quad \text{as} \quad t \to \infty .$$

But $\Gamma(1-\alpha)\ \Gamma(1+\alpha) = \alpha\Gamma(\alpha)\Gamma(1-\alpha) = \alpha\pi[\sin \alpha\pi]^{-1}$ hence the theorem follows for $0 < \alpha < 1$.

If $\alpha = 0$, again (6) and lemma 3.3 give the result.

If $\alpha = 1$, then (7) and lemma 3.3 yield that

(8) $$H(t) \sim \frac{t}{L^*(t)} = \frac{t}{\int_0^t [1-F(x)]dx}$$

which finishes the proof.

Formula (8) shows that if the first moment should be finite, then $H(t) \sim \frac{t}{\mu}$ as $t \to \infty$ which is the elementary renewal theorem.

Besides the above theorem, two other renewal theorems are also useful: Blackwell's theorem and Smith's key renewal theorem. Since $F \epsilon V_\alpha$, where $\alpha \leq 1$ neither one of them gives more information then a o(1) relation.

From Thm. 3.1 we always can find a s.v. function $L_2(t)$ such that

$$(9) \qquad H(t) \sim t^\alpha L_2(t) .$$

To indicate the dependence of $H(t)$ on α, we write $H_\alpha(t) = t^\alpha L_2(t)$ as before.

If $\mu < \infty$ then the key renewal theorem essentially states that a function $Q(t)$, which has the same growth properties as $1-F(t)$ may be used to obtain a finite limit for the convolution $Q*H(t)$ as $t \to \infty$. If $F \epsilon V_\alpha$, then an appropriate choice of $Q(t)$ could be $Q(t) \sim t^{-\beta} L_1(t)$ where $0 < \beta < 1$ and $L_1(t)$ slowly varying as $t \to \infty$.

We know from (1) and (9) that both $L(t)$ and $L_2(t)$ are of bounded variation. By Jordan's theorem there exist two functions $\overline{L}_2(t)$ and $\underline{L}_2(t)$, which are nondecreasing and such that

$$(10) \qquad L_2(t) = \overline{L}_2(t) - \underline{L}_2(t).$$

Assume now that

$$(S) \qquad \varlimsup_{t\to\infty} \frac{\overline{L}_2(t) + \underline{L}_2(t)}{L_2(t)} < \infty .$$

Lemma 3.6.

If (S) holds then $\overline{L}_2(t)$ and $\underline{L}_2(t)$ are s.v.

Proof:

The only requirement we have to check is $\lim_{t\to\infty} \frac{\overline{L}_2(xt)}{\overline{L}_2(t)} = 1$ for all $x > 0$.

It follows from (S) that $\lim_{t\to\infty} \frac{\overline{L}_2(t)}{L_2(t)} = c < \infty$. Moreover $c \geq 1$ since $L_2(t) > 0$. Hence

$$\lim_{t\to\infty} \frac{\overline{L}_2(xt)}{\overline{L}_2(t)} = \lim_{t\to\infty} \frac{\overline{L}_2(xt)}{L_2(xt)} \cdot \frac{L_2(xt)}{L_2(t)} \cdot \frac{L_2(t)}{\overline{L}_2(t)} = c \cdot 1 \cdot \frac{1}{c} = 1.$$

If $\lim_{t\to\infty} \frac{\underline{L}_2(t)}{L_2(t)} > 0$, then the same argument shows that $\underline{L}_2(t)$ is s.v.

We state an analogue of Smith's renewal theorem. The proof is based on a number of lemmas.

Theorem 3.2.

Let $0 < \alpha \leq 1$, $F \in V_\alpha$. Assume that $L_2(t)$ satisfies (S).

For $0 \leq \beta < 1$, let $Q_\beta(t) = t^{-\beta} L_1(t)$ where $L_1(t)$ is s.v. and $Q_\beta(t)$ is nonincreasing.

Then as $t \to \infty$

$$(11) \qquad \int_0^t Q_\beta(t-x)\, dH_\alpha(x) \sim C(\alpha,\beta) \frac{\int_0^t Q_\beta(x)dx}{\int_0^t [1-F(x)]dx}$$

where $[C(\alpha,\beta)]^{-1} = (2-\beta)\, B(\alpha-\beta+1, 2-\alpha)$ for $0 < \alpha \leq 1$.

Proof:

We first prove the theorem for $L_2(t)$ nondecreasing (Part A); then we discuss the case when L_2 satisfies condition (S), (Part B).

Part A: $L_2(t)$ nondecreasing.

Let ϵ be a fixed positive real number, $0 < \epsilon < \frac{1}{4}$. Then

$$I(t) \equiv \int_0^t Q_\beta(t-x)\ dH_\alpha(x) = \{\int_0^{\epsilon t} + \int_{\epsilon t}^{t-\epsilon t} + \int_{t-\epsilon t}^{t}\}\ Q_\beta(t-x)\ dH_\alpha(x)$$

$$\equiv I_1(t) + I_2(t) + I_3(t).$$

Since $H_\alpha(x) = x^\alpha L_2(x)$ we can break up $I_2(t)$ and $I_3(t)$ into two parts

$$I_2(t) = \alpha \int_{\epsilon t}^{t-\epsilon t} Q_\beta(t-x)\ x^{\alpha-1}\ L_2(x)dx + \int_{\epsilon t}^{t-\epsilon t} Q_\beta(t-x)x^\alpha dL_2(x) \equiv I_{21}(t)+I_{22}(t)$$

Similarly $I_3(t) = I_{31}(t) + I_{32}(t)$.

We show that $I(t)$ is approximated by $I_{21}(t)$ for large values of t. For this reason, we first estimate $I_{21}(t)$.

Lemma 3.7.

For $t \to \infty$

$$\int_\epsilon^{1-\epsilon} (1-u)^{-\beta}\ u^{\alpha-1} L_1[t(1-u)]\ L_2[ut]du \sim L_1(t)\ L_2(t) \int_\epsilon^{1-\epsilon} (1-u)^{-\beta} u^{\alpha-1} du.$$

Proof:

Consider (compare lemma 3.1 (iii))

$$\left|\frac{1}{L_1(t)L_2(t)}\int_\epsilon^{1-\epsilon}(1-u)^{-\beta}\,u^{\alpha-1}\,L_1[t(1-u)]L_2(ut)du-\int_\epsilon^{1-\epsilon}(1-u)^{-\beta}u^{\alpha-1}du\right|$$

$$\leq \int_\epsilon^{1-\epsilon}(1-u)^{-\beta}\,u^{\alpha-1}\left|\frac{L_1[t(1-u)]L_2(ut)}{L_1(t)\;L_2(t)}-1\right|\,du.$$

Since $L_1(t)$ and $L_2(t)$ are s.v. there exists a constant δ, independent of u (lemma 3.1 (i)), such that for $i=1,2$, and $t \geq t_o$

$$\left|\frac{L_i(ut)}{L_i(t)}-1\right| \leq \delta \quad \text{for all } u \in (\epsilon,1-\epsilon).$$

Hence the above integral is majorized by

$$\delta(1+\delta)\int_\epsilon^{1-\epsilon}(1-u)^{-\beta}\,u^{\alpha-1}\,du.$$

It $t \to \infty$, then $\delta \to 0$, and hence the lemma follows.

For brevity, let us write

$$B(\epsilon) = \int_\epsilon^{1-\epsilon}(1-u)^{-\beta}\,u^{\alpha-1}\,du.$$

Lemma 3.7 shows that

$$I_{21}(t) \sim \alpha B(\epsilon)\;t^{\alpha-\beta}\;L_1(t)\;L_2(t)$$

$$\sim \alpha B(\epsilon)\;Q_\beta(t)\;H_\alpha(t).$$

We have to compare the integrals I_1, I_{22}, I_{31} and I_{32} with I_{21}: this is done in the next four lemmas.

Lemma 3.8. $c_1\epsilon^\alpha \le \lim_{t\to\infty} \frac{I_1(t)}{I_{21}(t)} \le c_2\epsilon^\alpha$, where c_1 and c_2 are constants independent of t and ϵ.

Proof: Since $Q_\beta(t)$ is nonincreasing we obtain from the definition of $I_1(t)$ that

$$Q_\beta(t)\ H_\alpha(\epsilon t) \le I_1(t) \le Q_\beta[t(1-\epsilon)]\ H_\alpha(\epsilon t)$$

Now as $t \to \infty$

$$\frac{Q_\beta(t)H_\alpha(\epsilon t)}{I_{21}(t)} \sim \frac{\epsilon^\alpha L_2(\epsilon t)}{\alpha B(\epsilon)L_2(t)} \sim \frac{\epsilon^\alpha}{\alpha B(\epsilon)}$$

and

$$\frac{Q_\beta[t(1-\epsilon)]H_\alpha(\epsilon t)}{I_{21}(t)} \sim \frac{(1-\epsilon)^{-\beta}\epsilon^\alpha}{\alpha B(\epsilon)} .$$

But $B(\epsilon) \ge B(\frac{1}{4})$ since $0 < \epsilon < \frac{1}{4}$. Hence for some constants c_1, c_2 independent of ϵ the lemma will follow.

Lemma 3.9. $I_{22}(t) = o(I_{21}(t))$ as $t \to \infty$.

Proof:

Clearly: $$\frac{I_{22}(t)}{I_{21}(t)} \sim \frac{1}{\alpha B(\epsilon)L_1(t)L_2(t)} \int_\epsilon^{1-\epsilon} (1-u)^{-\beta}u^\alpha L_1[t(1-u)]dL_2(ut)$$

which by the fact that L_1 is s.v. is majorized by

$$\frac{(1+\delta)}{\alpha B(\epsilon)} \frac{1}{L_2(t)} \int_\epsilon^{1-\epsilon} (1-u)^{-\beta}\ u^\alpha\ dL_2(ut)$$

since $L_2(t)$ is nondecreasing.

To estimate the latter integral, we apply a mean value theorem: there exists $c \in [\epsilon, 1-\epsilon]$ such that

$$\frac{1}{L_2(t)} \int_{\epsilon}^{1-\epsilon} (1-u)^{-\beta} u^{\alpha} dL_2(ut) = (1-c)^{-\beta} c^{\alpha} \{\frac{L_2[t(1-\epsilon)]}{L_2(t)} - \frac{L_2[\epsilon t]}{L_2(t)}\}.$$

But $L_2(t)$ is s.v. So the expression on the right tends to zero as $t \to \infty$. This proves the lemma.

Lemma 3.10. There exists a constant c_3 independent of ϵ and t such that

$$0 \leq \lim_{t\to\infty} \frac{I_{31}(t)}{I_{21}(t)} \leq c_3 \epsilon^{1-\beta}.$$

Proof:

Obviously as $t \to \infty$

$$\frac{I_{31}(t)}{I_{21}(t)} \sim \frac{1}{\alpha B(\epsilon) L_1(t) L_2(t)} \int_{1-\epsilon}^{1} (1-u)^{-\beta} u^{\alpha-1} L_1[t(1-u)] L_2(ut)\, du$$

$$\leq \frac{(1+\delta)^2}{\alpha B(\epsilon)} \int_{1-\epsilon}^{1} (1-u)^{-\beta} u^{\alpha-1}\, du$$

where δ was defined similarly as in lemma 3.7. The latter integral is majorized by

$$\frac{(1+\delta)^2}{\alpha B(\epsilon)} (1-\epsilon)^{\alpha-1} \int_{1-\epsilon}^{1} (1-u)^{-\beta} \leq c_3 \epsilon^{1-\beta}$$

This proves the lemma.

Finally

Lemma 3.11. $I_{32}(t) = o(I_{21}(t))$ as $t \to \infty$.

Proof:

As before

$$0 \le \frac{I_{32}(t)}{I_{21}(t)} \le \frac{(1+\delta)}{\alpha B(\epsilon)} \frac{1}{L_2(t)} \int_{1-\epsilon}^{1} (1-u)^{-\beta} u^{\alpha} dL_2(ut)$$

$$\le \frac{1+\delta}{\alpha B(\epsilon)} \frac{1}{L_2(t)} \int_0^{\epsilon} v^{-\beta} dL_2[t(1-v)].$$

But the last integral is an improper integral. Since $L_2(t)$ is nondecreasing, the existence of this integral is proved as follows: let $0 < \eta < \epsilon$, then

$$\int_0^{\epsilon} v^{-\beta} dL_2[t(1-v)] = \lim_{\eta \downarrow 0} \int_{\eta}^{\epsilon} v^{-\beta} dL_2[t(1-v)]$$

$$= \lim_{\eta \downarrow 0} \{\frac{1}{\epsilon^{\beta}} L_2[t(1-\epsilon)] - \frac{1}{\eta^{\beta}} L_2[t(1-\eta)] + \beta\int_{\eta}^{\epsilon} L_2[t(1-v)]v^{-\beta-1}dv\}$$

and since in (η,ϵ) $L_2[t(1-v)] \le L_2[t(1-\eta)]$

$$0 \le \frac{I_{32}(t)}{I_{21}(t)} \le \frac{1+\delta}{\alpha B(\epsilon)} \frac{1}{\epsilon^{\beta}} \{\frac{L_2[t(1-\epsilon)]}{L_2[t]} - 1\}$$

which tends to zero as $t \to \infty$. This proves the lemma.

Combining the last five lemmas, we obtain that for every $\epsilon > 0$

$$1+c_1\epsilon^{\alpha} \le \lim_{t\to\infty} \frac{I(t)}{I_{21}(t)} \le 1+c_2\epsilon^{\alpha} + c_3\epsilon^{1-\beta}.$$

Hence for $t \to \infty$ by lemma 3.7

$$I(t) \sim \alpha\, Q_\beta(t)\, H_\alpha(t) \int_0^1 (1-u)^{-\beta} u^{\alpha-1}\, du$$

or

(12) $$I(t) \sim \alpha B(1-\beta,\alpha)\; Q_\beta(t)\; H_\alpha(t).$$

To finish the proof of part A, we have to show that (12) and (11) are asymptotically equal. This is proved by using

Lemma 3.12. For $t \to \infty$

(i) $$\int_0^t [1-F(x)]\, dx \sim \begin{cases} \dfrac{t}{H_0(t)} & \text{if } \alpha = 0 \\[2ex] \dfrac{\sin \alpha\pi}{\alpha(1-\alpha)\pi}\, \dfrac{t}{H_\alpha(t)} & \text{if } 0 < \alpha < 1 \\[2ex] \dfrac{t}{H_1(t)} & \text{if } \alpha = 1 \end{cases}$$

(ii) $$\int_0^t Q_\beta(x)\, dx \sim \frac{t}{1-\beta}\; Q_\beta(t).$$

Proof:

Let $\alpha = 0$, then by lemma 3.5 (ii) with $p=1$ and $q=1 > 0$, and Thm. 3.1

$$\int_0^t [1-F(x)]\, dx \sim tL(t) \sim \frac{t}{H_0(t)}\, .$$

A similar proof using (4) gives the relation for $\alpha = 1$.

If $0 < \alpha < 1$, then by putting $p=1$, $q=1 > \alpha$ in (ii) of lemma 3.5

$$\int_0^t [1-F(x)]\, dx \sim t^{1-\alpha}\, B(1,1-\alpha)\; L(t).$$

But by Thm. 3.1 we also know that

$$L(t) \sim \frac{t^{\alpha} \sin \alpha\pi}{\alpha\pi\ H_{\alpha}(t)}$$

which proves (i) of the lemma.

Part (ii) is proved similarly.

An elementary computation shows then that

$$\alpha B(1-\beta,\alpha)\ Q_{\beta}(t)\ H_{\alpha}(t) \sim B(1-\beta,\alpha)\ \frac{\sin \alpha\pi}{\pi}\ \frac{1-\beta}{1-\alpha}\ \frac{\int_0^t Q_{\beta}(x)dx}{\int_0^t [1-F(x)]dx} \quad \text{as } t \to \infty$$

which agrees with (11).

This finishes the proof of part A.

Part B: $L_2(t)$ satisfies condition (S).

Let $L_2(t) = \overline{L}_2(t) - \underline{L}_2(t)$ and put $\overline{J}(t) = \int_0^t Q_{\beta}(t-x)\ d\{x^{\alpha}\ \overline{L}_2(x)\}$

$\underline{J}(t) = \int_0^t Q_{\beta}(t-x)\ d\{x^{\alpha}\ \underline{L}_2(x)\}$.

Since both $\overline{L}_2(t)$ and $\underline{L}_2(t)$ are nondecreasing and s.v. by lemma 3.6

$$\overline{J}(t) \sim \alpha B(1-\beta,\alpha)\ Q_{\beta}(t)\ t^{\alpha}\ \overline{L}_2(t) \quad \text{as } t \to \infty \tag{1}$$

$$\underline{J}(t) \sim \alpha B(1-\beta,\alpha)\ Q_{\beta}(t)\ t^{\alpha}\ \underline{L}_2(t) \quad \text{as } t \to \infty. \tag{2}$$

If we denote the right hand side of (1) and (2) by $\overline{K}(t)$ and $\underline{K}(t)$ respectively, then there exists a $\delta > 0$ such that for all $t \geq t_o$

$$1 - \delta \le \frac{\overline{J}\ (t)}{\overline{K}\ (t)} \le 1 + \delta$$

$$1 - \delta \le \frac{\underline{J}\ (t)}{\underline{K}\ (t)} \le 1 + \delta.$$

Hence

$$\frac{I(t)}{\alpha B(1-\beta,\alpha)\ Q_\beta(t) t^\alpha L_2(t)} = \frac{\overline{J}\ (t)}{\overline{K}\ (t)} \cdot \frac{\overline{L}_2\ (t)}{L_2\ (t)} - \frac{\underline{J}\ (t)}{\underline{K}\ (t)} \cdot \frac{\underline{L}_2\ (t)}{L_2\ (t)}$$

or for $t \ge t_o$

$$1-\delta\{\frac{\overline{L}_2(t)+\underline{L}_2(t)}{L_2(t)} \le \frac{I(t)}{\alpha B(1-\beta,\alpha) Q_\beta(t) H_\alpha(t)} \le 1+\delta\{\frac{\overline{L}_2(t)+\underline{L}_2(t)}{L_2(t)}\}.$$

By (S) we obtain that for $t \to \infty$

(13) $$I(t) \sim \alpha B(1-\beta,\alpha)\ Q_\beta(t)\ H_\alpha(t).$$

However lemma 3.12 was not based on the assumption that $L_2(t)$ was nondecreasing. Henceforth it implies the asymptotic equality of (11) and (13).

This finishes the proof of Theorem 3.2.

In the next theorem we derive an asymptotic result for $E[N^2(t)]$. Let $\operatorname{Var} N(t) = V(t)$.

<u>Theorem 3.3.</u>

If $F \epsilon V_\alpha$, $0 \le \alpha \le 1$ then as $t \to \infty$

$$V(t) \sim \begin{cases} \dfrac{1}{L^2(t)} & \text{if } \alpha = 0 \\ \dfrac{\sin^2 \alpha\pi}{\alpha^2\pi^2} \left\{ \dfrac{\sqrt{\pi}\, 2^{1-2\alpha}\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})} - 1 \right\} \dfrac{t^{2\alpha}}{L^2(t)} & \text{if } 0 < \alpha < 1 \\ o\left[\dfrac{t^2}{L^{*2}(t)}\right] & \text{if } \alpha = 1 \end{cases}$$

<u>Proof</u>:

It is well-known [41], that

$$E[N^2(t)] = H(t) + 2H * H(t)$$

so that the L.S.T. of the left hand side equals $h(s) + 2h^2(s)$. The proof for $\alpha=0$ is obvious by an appeal to Thm. 3.1 and lemma 3.3. Also $\alpha=1$ follows quickly from the same statements.

If $0 < \alpha < 1$ then we obtain that

$$E[N^2(t)] \sim \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)[\Gamma(1-\alpha)]^2 L^2(t)} \quad \text{as } t \to \infty . \tag{14}$$

Combining (14) with Thm. 3.1 the given result is immediate in view of the identities $\Gamma(\alpha)\,\Gamma(1-\alpha) = \frac{\pi}{\sin \alpha\pi}$ and $\Gamma(2\alpha+1) = \frac{1}{\sqrt{\pi}} 2^{2\alpha}\Gamma(\alpha+\frac{1}{2})\Gamma(\alpha+1)$. This finishes the proof.

The above theorem complements a result of Feller [13] where $F(t)$ is supposed to satisfy the relation $1-F(t) \sim At^{-\alpha}$ as $t \to \infty$ for $0 < \alpha < 1$.

3. Renewal Theorems for the Case Where Only the First Moment Exists

Assume that $F(t)$ has finite first moment μ. To apply theorem 3.2 we assume that $F \epsilon V_\alpha$ for $1 < \alpha < 2$, so that $\mu < \infty$ but $\mu_2 \leq \infty$. Define

(15) $$F_2(t) = \frac{1}{\mu} \int_0^t [1-F(x)]dx$$

(16) $$G(t) = H(t) - \frac{t}{\mu} + F_2(t)$$

(17) $$H(t) = \frac{t}{\mu} \bar{L}(t)$$

The importance of the above functions is illustrated in

Lemma 3.13.

(i) $F_2(t)$ is the distribution function of a nonnegative random variable. Its L.S.T. $f_2(s)$ is given by $f_2(s) = [\mu s]^{-1} [1-f(s)]$

(ii) If $F(t) \epsilon V_\alpha$ for $1 < \alpha \leq 2$, then $F_2(t) \epsilon V_{\alpha-1}$

(iii) $G(t) = (1-F_2) * H(t)$

(iv) $\bar{L}(t)$ is slowly varying.

Proof:

The first part of the lemma is well-known [41], and (ii) is a consequence of lemma 3.5 (i) with k=0. Let $g(s)$ be the L.S.T. of $G(t)$ then (16) implies that $g(s) = h(s)-\frac{1}{\mu s}\{1-\mu s\ f_2(s)\}=h(s)-\frac{f_2(s)}{1-f(s)}\cdot f(s)$ by applying (i) twice. From the last equality (iii) follows immediately. To prove (iv) we recall that the elementary renewal theorem applies: $\frac{H(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$. Hence $H(t) \sim \frac{t}{\mu} \bar{L}(t)$ for some s.v. $\bar{L}(t)$. This proves the lemma.

Theorem 3.4.

Let $\overline{L}(t)$ satisfy (S). If $F \in V_\alpha$ for $1 < \alpha < 2$ then as $t \to \infty$

$$H(t) - \frac{t}{\mu} \sim \frac{t^{2-\alpha}}{\mu^2(\alpha-1)(2-\alpha)} L(t).$$

Proof:

From (16) and (iii) of the lemma we obtain

$$(18) \qquad \int_0^t [1-F_2(t-x)]\, dH(x) = H(t) - \frac{t}{\mu} + F_2(t)$$

Let $Q_\beta(x) = 1-F_2(x)$ with $\beta=\alpha-1$ and $H_\alpha(t) = \frac{t}{\mu}\overline{L}(t)$; then Thm. 3.2 yields

$$(19) \qquad H(t) - \frac{t}{\mu} + F_2(t) \sim \frac{\int_0^t [1-F_2(x)]dx}{\int_0^t [1-F(x)]dx}$$

By (i) of the lemma, $F_2(t) = O(1)$ as $t \to \infty$ and $\int_0^t [1-F(x)]dx \to \mu$ as $t \to \infty$. By lemma 3.5 (ii) for p=q=1 and α replaced by $\alpha-1$ we obtain $\int_0^t [1-F_2(x)]dx \sim \frac{t^{2-\alpha}}{\mu(\alpha-1)} B(1,2-\alpha)\ L(t)$.

Using these expressions in (19) the stated result follows.

The theorem complements another result of Feller [13]. From lemma 3.1 (ii) we see that under the given conditions $H(t) - \frac{t}{\mu}$ still tends to infinity as $t \to \infty$.

Corollary 3.2.

Let $\overline{L}(t)$ satisfy (S). If $F \in V_\alpha$ for $1 < \alpha < 2$ then as $t \to \infty$

$$V(t) \sim \frac{2t^{3-\alpha}}{\mu^3(3-\alpha)(2-\alpha)} L(t)$$

We outline a possible generalization of the above procedure to the case where the mth moment of F exists $(m \leq 2)$ but not the m+1st moment.

Define a sequence of distribution functions $\{F_k(t), k=1,2,\ldots,m\}$ as follows

$$F_1(t) = F(t) \qquad k = 1$$

$$F_k(t) = \alpha_{k-1} \int_0^t [1-F_{k-1}(x)]dx \qquad 1 < k \leq m+1$$

where $\alpha_k^{-1} = \int_0^\infty [1-F_k(x)]dx,\ k=1,2,\ldots,m.$

The mass function $F_{m+1}(t)$ is not necessarily finite for $t \to \infty$ since $\mu_{m+1} = \infty$. By using L.S.T.'s it is easy to show that

$$\alpha_k = \frac{k\mu_{k-1}}{\mu_k} \qquad k = 1,2,\ldots,m.$$

In addition to the functions $F_k(t)$, we define

$$H_j(t) = \int_0^t [1-F_j(t-x)]\, dH(x) \qquad j=1,2,\ldots,m+1$$

The analogue of lemma 3.13 (iii) yields that for $1 < k \leq m+1$

$$H_k(t) = F_k(t) + \sum_{j=1}^{k-1} A_{k,j} \int_0^t (t-x)^{j-1}\, dH(x) + \sum_{j=1}^{k-1} A_{k,j+1}\, t^j \tag{20}$$

where

$$A_{k,j} = (-1)^{j-1} \binom{k-1}{j-1} \frac{\mu_{k-j}}{\mu_{k-1}} \qquad j = 2,3,\ldots,k$$

The system (20) can be solved recursively for the functions

$$\int_0^t (t-x)^{j-1}\, dH(x) .$$

An asymptotic value for $H_k(t)$ is obtainable from Smith's theorem for $k=2,3,\ldots,m$ and from Thm. 3.2. for $k = m+1$. Moreover precise bounds are known for the distributions $F_k(t)$ in terms of the moments. We refer to papers by R. Barlow, A. Marshall and F. Proschan [2,3].

For another approach when $m \geq 2$ we refer the reader to the important papers by Ch. Stone [43,44,45]. There precise bounds are given on the renewal function and in Blackwell's theorem, even for the generalized renewal process of the Chung-Pollard type [7]. His methods however are completely different from those outlined above.

Another way of obtaining estimates on the error term in the limit theorems is to apply Tauberian remainder theorems of S. Freud and A.E. Ingham. We refer to [19] where a very general version of such a theorem is given. Tauberian remainder theorems of a different nature are obtained by T. Ganelius, A. Beurling, S. Lyttkens and L. Frennemo, [14].

CHAPTER IV

ERGODICITY IN DERIVED MARKOV CHAINS

A general theorem is proved stating that the ergodicity of a stationary Markov chain is preserved for derived Markov chains as defined by J.W. Cohen, [8,9].

The four possible combinations that occur are discussed in separate sections.

In section 9 we apply the theorem to a certain type of continuous time Markov chains by using theorems of H. Miller, [32].

1. Definition of a Derived Markov Chain

In [8,9] J.W. Cohen describes a method to derive a new Markov chain ${}_2M$ from a given Markov chain ${}_1M$. Both chains have the same state space I and the Markov chain ${}_2M$ is derived from ${}_1M$ by using a deriving function $b(s,t)$ of two variables.

Let ${}_1M$ be determined by the stationary transition matrix ${}_1P(t) = ({}_1p_{ij}(t))$ and the initial distribution $\{p_i,\ i \in I\}$, which is irrelevant for our purposes. The time parameter t takes values in $[0,\infty]$ or in $[nt_o,\ n=0,1,\ldots$ for some $t_o > 0]$.

The deriving function $b(s,t)$ satisfies the following conditions:

(i) For every fixed $s_o > 0$, $b(s_o,t)$ is a probability distribution of a nonnegative random variable;

(ii) For every fixed t and all $s_1, s_2 \geq 0$

(1) $$b(s_1,t) * b(s_2,t) = b(s_1+s_2,t).$$

(iii) a. For $s=0$, $b(0,t) = U(t)$

b. If s varies continuously, then $b(s,t)$ converges completely to $U(t)$ if $s \downarrow 0$.

We define a new matrix ${}_2P(s) = ({}_2p_{ij}(s))$ by

(2) $${}_2P(s) = \int_0^\infty {}_1P(t)\, d_t\, b(s,t).$$

The properties of the deriving function imply that ${}_2P(s)$ is a stationary transition matrix. Since the statespace I remains unchanged as well as the initial distribution $\{p_i, i \in I\}$, there exists a stationary Markov chain ${}_2M$ that has ${}_2P(s)$ as transition matrix and $\{p_i, i \in I\}$ as initial distribution. Moreover, if s varies continuously then ${}_2P(s)$ is standard [6].

Depending on the range of s and t, four different cases can arise:

a. ${}_1M$ is a continuous chain, ${}_2M$ is a discrete chain;

b. ${}_1M$ is a continuous chain, ${}_2M$ is a continuous chain;

c. ${}_1M$ is a discrete chain, ${}_2M$ is a discrete chain;

d. ${}_1M$ is a discrete chain, ${}_2M$ is a continuous chain.

In the case that ${}_2M$ is discrete, s takes values in the set $[ns_o, n=0,1,2,\ldots$ for some $s_o > 0]$ and (2) reduces to a sum.

2. Properties of the Deriving Function

Much more can be said about $b(s,t)$ if we consider the cases of discrete and continuous s separately.

a. s discrete

Since $s = ns_o$ for some n, (1) implies that

$$(3)\quad \begin{aligned} b(s,t) &= b(ns_o,t) = a^{n*}(t) && \text{for } n \geq 1 \\ b(0,t) &= a^{o*}(t) \equiv U(t) && \text{for } n = 0 \end{aligned}$$

where $a^{n*}(t)$ is the n-fold convolution of the distribution function $a(t) = b(s_o,t)$.

b. s continuous

By (1), $b(s,t)$ is an infinitely divisible distribution of a nonnegative random variable for every fixed s. Denoting by $\beta(s,\lambda)$ the L.S.T. of $b(s,t)$ with respect to t, a theorem by Phillips [18,p. 660] yields that for real $\lambda \geq 0$

$$(4)\qquad \log \beta(s,\lambda) = s[-m\lambda + \int_0^\infty [e^{-\lambda t}-1]\, d\,\psi\,(t)]$$

where m is a nonnegative constant and $\psi(t)$ is defined by

$$(5)\qquad \psi(t) = -\int_t^\infty \frac{1+x^2}{x^2}\, dG(x) \quad \text{for } t > 0.$$

Here $G(x)$ is a real bounded nondecreasing function, which vanishes for negative argument. Moreover, $b(s,t)$ is completely determined by m and $\psi(t)$.

If in addition to the fact that s varies continuously, t takes on only discrete values, one can apply a theorem by Blum and Rosenblatt

[4] to obtain that $m=0$ and $\psi(0+) > -\infty$. Using these conditions, Cohen inverts the L.S.T. $\beta(s,\lambda)$:

$$(6) \qquad b(s,t) = e^{s\psi(0+)} \sum_{n=0}^{\infty} \frac{[-s\psi(0+)]^n}{n!} b^{n*}(t)$$

where

$$(7) \qquad b(t) = 1 - \frac{\psi(t)}{\psi(0+)}$$

is again the distribution function of a nonnegative random variable.

3. Ergodicity Conditions

Let ${}_iM$ $(i=1,2)$ be either one of the Markov chains under consideration.

Definition 1: Let ${}_iM$ be a continuous stationary Markov chain on I. A state $j \in I$ is said to be exponentially ergodic if there exist constants $\lambda_j > 0$ and $0 \leq K_j < \infty$ such that for all $t \geq 0$

$$(8) \qquad |{}_ip_{jj}(t) - \lim_{t\to\infty} {}_ip_{jj}(t)| \leq K_j\, e^{-\lambda_j t}.$$

Usually, (8) is only required for $t \geq t_o \geq 0$ for some t_o. By changing K_j however the above relation holds everywhere.

Definition 2: Let ${}_iM$ be a discrete stationary Markov chain on I. An aperiodic state $j \in I$ is said to be geometrically ergodic if there exist constants $0 \leq \rho_j < 1$ and $0 \leq K_j < \infty$ such that for all $n \geq 0$

$$(9) \qquad |{}_ip_{jj}^{(n)} - \lim_{n\to\infty} {}_ip_{jj}^{(n)}| \leq K_j(\rho_j)^n.$$

If state $j \epsilon I$ is periodic with period ${}_i v_j$, then j is said to be geometrically ergodic if there exist constants $0 \leq \rho_j < 1$ and $0 \leq K_j < \infty$ such that for all $n \geq 0$

$$(10) \qquad \left| {}_i p_{jj}^{(n\,{}_i v_j)} - \lim_{n\to\infty} {}_i p_{jj}^{(n\,{}_i v_j)} \right| \leq K_j (\rho_j)^{n\,{}_i v_j}.$$

The constant λ_j (ρ_j) will be called the decay parameter of state j.

Since it will be clear from the context, we usually will drop the terms ''geometrically'' and ''exponentially'' when we talk about ergodicity. An ergodic state should not be confused with a positive recurrent state. To simplify the notation even more, we always drop the index referring to the state.

Throughout we assume that for all $s > 0$, $b(s,t) \not\equiv U(t)$ since otherwise ${}_2P(s) \equiv I$ for all s, which is an uninteresting case.

The theorem we prove essentially states that if $j \epsilon_1 M$ is ergodic, then $j \epsilon_2 M$ is also ergodic. We can restrict our attention to transient and positive recurrent states, as was indicated by D.G. Kendall [22]. The constants involved in the ergodicity condition for the first chain determine the constants for the ergodicity condition in the second chain.

It is clear from (2) that some proofs might depend on a mean value theorem on $[0,\infty]$. We mention one that will suffice; the proof is analogous to Th. 10b. [48, p. 17].

Lemma 4.1.

Let $F(x)$ be the distribution function of a nonnegative random variable. Let $m(x)$ be a positive, bounded function continuously decreasing to zero as x tends to infinity. Assume that $\int_0^\infty m(x)dF(x) < \infty$.

Then there exists a constant ξ, neither zero nor infinity, such that

$$\int_0^\infty m(x)\ dF(x) \leq m(0)\ F(\xi).$$

We now state our

Main Theorem 4.1.

Let ${}_1M$ be a stationary Markov chain with state space I. Let ${}_2M$ be a stationary Markov chain derived from ${}_1M$ by the deriving function $b(s,t) \neq U(t)$ for all $s > 0$.

If the state $j \epsilon I$ is ergodic in ${}_1M$, then j is also ergodic in ${}_2M$.

We prove the theorem by establishing it for each of the four cases indicated in §1.

4. Proof for ${}_1M$ Continuous and ${}_2M$ Discrete

Let j be a fixed (exponentially) ergodic state in I, i.e. by (8)

$$(11)\qquad |{}_1p(t) - \epsilon\, {}_1\pi| \leq Ke^{-\lambda t}$$

where $\epsilon = 0$ if j is transient in ${}_1M$ and $\epsilon=1$ if j is positive. In the latter case $\{{}_1\pi_j;\ j\epsilon I^*\}$ is the stationary distribution over the positive states $I^* \subset I$.

Since ${}_2M$ is discrete, (3) implies that

$$(12)\qquad {}_2p^{(n)} = \int_0^\infty {}_1p(t)\ da^{n*}(t).$$

From (12) and the fact that ${}_1p_{jj}(t) > 0$ [6, p. 121] we see that $j\epsilon\ {}_2M$ is aperiodic.

Moreover, it can be shown (Thm. 5.1 in [9]) that

$$\text{(A)}\quad \begin{cases} \text{(i)}\ \lim_{s\to\infty} {}_2p(s) = \lim_{t\to\infty} {}_1p(t); \\ \text{(ii) If } j \text{ is positive in } {}_1M, \text{ then } j \text{ is positive in } {}_2M; \\ \text{(iii) If } j \text{ is transient in } {}_1M, \text{ then } j \text{ is transient in } {}_2M. \end{cases}$$

Hence, by (A), (12) and (11)

$$|{}_2p^{(n)} - \epsilon\, {}_2\pi| = |\int_0^\infty {}_1p(t)\, da^{n*}(t) - \epsilon_1\pi \int_0^\infty da^{n*}(t)| \leq K \int_0^\infty e^{-\lambda t} da^{n*}(t).$$

An application of lemma 4.1 yields a finite $\xi > 0$ such that

$$(13)\qquad |{}_2p^{(n)} - \epsilon_2\pi| \leq K\, a^{n*}(\xi).$$

If we put $0 < a(\xi) = \rho$, then $\rho < 1$ since $b(s,t) \neq U(t)$ for $s > 0$. Hence finally

$$|{}_2p^{(n)} - \epsilon_2\pi| \leq K\, a^{n*}(\xi) \leq K[a(\xi)]^n = K\rho^n \quad \text{for } n \geq 0,$$

which proves the theorem in this case.

We remark that we did not need the exponential ergodicity in (11). Suppose indeed that

$$(14)\qquad |{}_1p(t) - \epsilon_1\pi| \leq K\, f(t)$$

where $0 < f(0) < \infty$ and $f(t)$ decreases continuously to zero. Then exactly the same argument shows that for some $\rho' < 1$ and all n,

(15) $$|{}_2p^{(n)} - \epsilon_2\pi| \leq [Kf(0)]\ \rho'^n.$$

Cases where (14) applies are given by S. Karlin, [25].

5. Proof for ${}_1M$ Continuous and ${}_2M$ Continuous

Let $j \epsilon I$ be a fixed ergodic state in ${}_1M$ such that (11) holds. By Thm. 6.1 in [9] the conditions (A) are still valid.

Going through the same operations as before, (2) and (11) give

(16) $$|{}_2p(s) - \epsilon_2\pi| \leq K\ \beta(s,\lambda)$$

where $\beta(s,\lambda)$ is defined by (4). Since $\beta(s,\lambda)$ is the L.S.T. of a nonnegative random variable, $\beta(s,\lambda)$ converges for all $\lambda \geq 0$. If we define

(17) $$\lambda^* = m\lambda - \int_0^\infty [e^{-\lambda t} - 1]\ d\psi(t)$$

then $\lambda^* > 0$ since $b(s,t) \not\equiv U(t)$ for $s > 0$. Hence for $s > 0$

(18) $$|{}_2p(s) - \epsilon_2\pi| \leq K\ e^{-\lambda^* s}\ .$$

In the case that $\psi(0+) > -\infty$, lemma 4.1 applies again: there exists a $\xi \in (0,\infty)$ such that

(19) $$\lambda^* = m\lambda - [\psi(\xi) + \psi(0+)]$$

and (18) holds for this value of λ^*.

6. Proof for ${}_1M$ Discrete and ${}_2M$ Discrete

The proofs becomes slightly more complicated since periodicities may occur in both chains. Since the aperiodic case is the most important, we discuss it first. Let again j be a fixed ergodic state in ${}_1M$.

A. State $j \epsilon\ {}_1M$ is transient or aperiodic and positive.

Since j is ergodic, (9) implies that

(20) $$|{}_1p^{(n)} - \epsilon_1\pi| \leq K\rho^n$$

where ϵ has the same meaning as before.

We obtain from (2) and (3) that

(21) $${}_2p^{(m)} = \int_0^\infty {}_1p(t)\, da^{m*}(t).$$

But t takes on values in the set $\{nt_o, n=0,1,2,\ldots\}$ for some $t_o > 0$. Hence

$$a(t) = \sum_{n=0}^{\infty} a_n U(t-nt_o)$$

where $0 \leq a_o < 1$ (since $b(s,t) \not\equiv U(t)$), $0 \leq a_n \leq 1$ for $n > 0$ and $\sum_{n=0}^{\infty} a_n = 1$. We introduce the constants $\{a_n^{(m)}, n=0,1,2,\ldots,m=1,2,\ldots\}$ by the equation

(22) $$a^{m*}(t) = \sum_{n=0}^{\infty} a_n^{(m)} U(t-nt_o)$$

so that (21) becomes

(23) $$ {}_2p^{(m)} = \sum_{n=0}^{\infty} a_n^{(m)} \, {}_1p^{(n)}. $$

Since the conditions (A) of §4 are still valid the proof is now easy. For

(24) $$ |{}_2p^{(m)} - \epsilon\, {}_2\pi| = |\sum_{n=0}^{\infty} a_n^{(m)} \, {}_1p^{(n)} - \epsilon\, {}_1\pi \sum_{n=0}^{\infty} a_n^{(m)}| \leq K \sum_{n=0}^{\infty} a_n^{(m)} \rho^n. $$

Introduce the generating function

(25) $$ f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n \qquad \text{for } |\lambda| < 1 $$

then (22) implies

(26) $$ f^m(\lambda) = \sum_{n=0}^{\infty} a_n^{(m)} \lambda^n. $$

By assumption, $\rho < 1$ so that Abel's theorem yields that $\rho^* \equiv f(\rho) < 1$. Combining (24) and (26) we obtain that

(27) $$ |{}_2p^{(m)} - \epsilon\, {}_2\pi| \leq K(\rho^*)^m $$

which proves the theorem in this case.

B. State $j \epsilon\, {}_1M$ is periodic but aperiodic in ${}_2M$.

We assume that $j \epsilon\, {}_1M$ has period ${}_1v$ and is ergodic. Hence by (10) and a well-known theorem [6, p. 27]

(28) $$ |{}_1p^{(n\, {}_1v)} - {}_1v\, {}_1\pi| \leq K \rho^{n\, {}_1v}. $$

The conditions (A) change into (Thm. 7.2 [9])

(A)'

(i) $\lim_{m\to\infty} {}_2p^{(m)} = \frac{s}{{}_1v} \lim_{n\to\infty} {}_1p^{(n\,{}_1v)}$ where s is defined below;

(ii) If j is positive in ${}_1M$, then j is positive in ${}_2M$.

Since j is periodic, (25) may be rewritten as

$$f(\lambda) = \sum_{i=0}^{{}_1v-1} \lambda^i \{ \sum_{n=0}^{\infty} a_{n\,{}_1v+i} \lambda^{n\,{}_1v} \}. \tag{29}$$

Let k_o denote the smallest value of the integers $i \in \{0,1,\ldots,{}_1v-1\}$ for which $\sum_{n=0}^{\infty} a_{n\,{}_1v+i} > 0$. The integers $k_1, k_2, \ldots, k_r$, all belonging to the set $\{0,1,\ldots,{}_1v-1-k_o\}$ are defined in such a way that $k_1 < k_2 < \ldots < k_r$ and $\sum_{n=0}^{\infty} a_{n\,{}_1v+k_o+k_h} > 0$ for all $h \in \{1,2,\ldots,r\}$ where r is the number of values taken on by i. We denote by d the highest common divisor of ${}_1v$ and $k_1, k_2, \ldots, k_r$ and by s that of d and k_o.

It follows easily from the periodicity of j that

$${}_2p^{(m)} = \sum_{n=0}^{\infty} a^{(m)}_{n\,{}_1v} \; {}_1p^{(n\,{}_1v)} \tag{30}$$

so that we have to introduce

$$\tilde{f}(\lambda) = \sum_{n=0}^{\infty} a_{n\,{}_1v} \lambda^{n\,{}_1v}. \tag{31}$$

Using (A)' and (30) we obtain

$$(32)\qquad |{}_2p^{(m)} - \lim_{m\to\infty} {}_2p^{(m)}| = \Big|\sum_{n=0}^{\infty} a^{(m)}_{n\,{}_1v}\, {}_1p^{(n\,{}_1v)} - \frac{s}{{}_1v}\lim_{n\to\infty} {}_1p^{(n\,{}_1v)}\Big|$$

$$\le \sum_{n=0}^{\infty} a^{(m)}_{n\,{}_1v}\, |{}_1p^{(n\,{}_1v)} - {}_1v\,{}_1\pi| + {}_1v\,{}_1\pi\Big|\sum_{n=0}^{\infty} a^{(m)}_{n\,{}_1v} - \frac{s}{{}_1v}\Big| = I_1 + I_2.$$

From (28) and (31) it immediately follows that

$$(33)\qquad I_1 \le K\,\tilde{f}^m(\rho).$$

To estimate I_2 we use a well-known property of the roots of unity

$$(34)\qquad \sum_{n=0}^{\infty} a^{(m)}_{n\,{}_1v} = \frac{1}{{}_1v}\sum_{h=0}^{{}_1v-1} f^m\Big[\exp\Big(\frac{2\pi i h}{{}_1v}\Big)\Big].$$

There are exactly s terms in the sum on the right of (34) that are one, namely those that correspond to the s roots of the equation $\lambda^s = 1$. Let $\Omega_s = \{\lambda_1,\ldots,\lambda_s\}$ be the set of these roots. Let $N = \{\exp(\frac{2\pi i h}{{}_1v}),\ h = 0,1,2,\ldots,{}_1v-1\}$. Also let $B = N \cap \Omega_s^c$. Then if λ is a complex number such that $f(\lambda) \neq 1$, $|\lambda| = 1$, and $\lambda \in B$ then Abel's theorem yields that $|f(\lambda)| < 1$.

Define

$$\sigma = \max_{\lambda\in B} |f(\lambda)| < 1$$

Then

$$(35)\qquad I_2 = {}_1v\,{}_1\pi\Big|\sum_{n=0}^{\infty} a^{(m)}_{n\,{}_1v} - \frac{s}{{}_1v}\Big| = {}_1\pi\Big|\sum_{\lambda\in B} f^m(\lambda)\Big| \le {}_1\pi[{}_1v - s]\,\sigma^m.$$

If we put $\rho^* = \max\{\tilde{f}(\rho), \sigma\} < 1$ and $K^* \geq K + {}_1\pi[{}_1v - s]$ then (32), (33) and (35) together imply that

$$(36) \qquad |{}_2p^{(m)} - \lim_{m\to\infty} {}_2p^{(m)}| \leq K^*(\rho^*)^m$$

which shows that the result holds also in this case.

C. State j is periodic in ${}_1M$ and periodic in ${}_2M$.

Since the argument is analogous to the previous case, we only outline the result.

The condition (A)' (i) is replaced by

$$(A)'' \quad (i) \qquad \lim_{m\to\infty} {}_2p^{(m\,{}_2v)} = \frac{d}{{}_1v} \lim_{n\to\infty} {}_1p^{(n\,{}_1v)}$$

where ${}_2v$ is the period of j in ${}_2M$. (A)' (ii) is still valid. We obtain that

$$(37) \qquad |{}_2p^{(m\,{}_2v)} - \lim_{m\to\infty} {}_2p^{(m\,{}_2v)}| \leq K^*(\rho^*)^{m\,{}_2v}$$

where

$$\rho^* = \max\,[\tilde{f}(\rho), \delta]$$

$$\delta = \max_{\lambda\in C} |f(\lambda)|$$

$$C = N \cap \Omega_d^c$$

$$\Omega_d = \{\lambda_1, \lambda_2, \ldots, \lambda_d, \ (\lambda_k)^d = 1 \quad \text{for } k=1,\ldots,d\}$$

$$K^* \geq K + {}_1\pi[{}_1v - d]$$

7. Proof for $_1M$ Discrete and $_2M$ Continuous

Let j be a fixed ergodic state in $_1M$.

A. State $j \epsilon {_1M}$ is transient or aperiodic and positive.

By (6), since t varies discretely

$$b(t) = \sum_{n=0}^{\infty} a_n U(t-nt_o) \tag{38}$$

where $a_o = 0$, $0 \leq a_n \leq 1$ for $n > 0$ and $\sum_{n=0}^{\infty} a_n = 1$. The generating function of the $\{a_n\}$ is given by (25). For brevity, let $\mu = -s\psi(0+) \geq 0$ for $s \geq 0$.

It is easy to show that

$$_2p(s) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} e^{-\mu} \frac{\mu^n}{n!} a_k^{(n)} \; {_1p^{(k)}}. \tag{39}$$

The procedure employed in §6. Case A leads to

$$|{_2p(s)} - \epsilon \, {_2\pi}| \leq K e^{-\lambda s} \tag{40}$$

where

$$\lambda = -\psi(0+) \, [1-f(\rho)] \tag{41}$$

That $\lambda > 0$ follows from Abel's theorem and $\psi(0+) \neq 0$ (since $b(s,t) \neq U(t)$).

B. State j is periodic in $_1M$.

Since this case is analogous to §6. Case B we only outline the result.

Let s be the highest common divisor of ${}_1v$ and those numbers k for which $a_k > 0$. Let $\tilde{f}(\lambda)$, Ω_s and N be defined as before. Then

(42) $$|{}_2p(s) - \lim_{s\to\infty} {}_2p(s)| \le K^* e^{-\lambda^* s}$$

where

$$\lambda^* = \max\{-\psi(0+)[1-\tilde{f}(\rho)], -\psi(0+)[1-\tau]\}$$

$$\tau = \max_{\lambda\in N\cap\Omega_s^c} |f(\lambda)|$$

$$K^* \ge K + {}_1\pi({}_1v-s)$$

This case finishes the proof of our main theorem. It is interesting to observe that in almost all proofs ${}_1p_{jj}(t)$ or ${}_2p_{jj}(s)$ could be replaced by ${}_1p_{kj}(t)$ or ${}_2p_{kj}(s)$ where k belongs to the same class as j. For sake of uniformity we have chosen to take the states equal since otherwise the periodic cases would have become notationally involved.

8. A Theorem on Solidarity Between a Chain and Its Derived Chain

Although easy to prove, the following theorem might be of importance.

Theorem 4.2.

Suppose that ${}_1M$ is an irreducible stationary Markov chain on I. Let ${}_2M$ be the stationary Markov chain derived from ${}_1M$ by the deriving function $b(s,t) \not\equiv U(t)$ for $s > 0$. Suppose that any state

$j \in {}_1M$ is ergodic, then all states of ${}_1M$ and all states of ${}_2M$ are ergodic.

Proof:

The proof follows from Thm. 4.1 and the solidarity theorems of D.G. Kendall and D. Vere-Jones in the case of geometric ergodicity, [22,23,47] and of J.F.C. Kingman in the case of exponential ergodicity [24,25].

9. An Application

Consider an irreducible conservative continuous time Markov chain M with infinitesimal generator Q of the form

$$Q = \begin{pmatrix} \alpha_0 & c_1 & c_2 & c_3 & \cdots \\ \alpha_{-1} & c_0 & c_1 & c_2 & \cdots \\ \alpha_{-2} & c_{-1} & c_0 & c_1 & \cdots \\ \alpha_{-3} & c_{-2} & c_{-1} & c_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

where $\alpha_{-k} = -\sum_{n=-k+1}^{\infty} c_n$ for $k = 0,1,2,\ldots$.

Assume that

$$\begin{cases} \text{(i)} \ \sum_{-\infty}^{\infty} k\, c_k < 0 \\ \text{(ii) for } k > 0,\ c_k \leq C\lambda^k \text{ for some } C > 0 \text{ and } 0 \leq \lambda < 1. \end{cases}$$

We prove that under the above conditions M is positive recurrent and exponentially ergodic.

Consider an irreducible aperiodic discrete time Markov chain $\tilde{M}$ defined by its one step transition matrix

$$\tilde{P} = \begin{pmatrix} 1+\alpha_0 & c_1 & c_2 & c_3 & \cdots \\ \alpha_{-1} & 1+c_0 & c_1 & c_2 & \cdots \\ \alpha_{-2} & c_{-1} & 1+c_0 & c_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

A theorem by H. Miller [32] yields that $\tilde{M}$ is positive and geometrically ergodic if the conditions (i) and (ii) are satisfied. Moreover the common decay parameter is

$$\rho = 1 + \min_{t>0} \sum_{-\infty}^{\infty} c_k e^{kt}, \ K = 1$$

We show that M is derived from $\tilde{M}$. Let ${}_2M$ be the continuous Markov chain derived from $\tilde{M}$ by using as deriving function

$$b(t) = U(t-1), \ \psi(0+) = -1.$$

It follows from (38) and (39) that

(43) $${}_2P(s) = \sum_{n=0}^{\infty} e^{-s} \frac{s^n}{n!} \tilde{P}^{(n)}.$$

But an easy computation shows that the infinitesimal generator ${}_2Q$ of ${}_2M$ satisfies

(44) $${}_2Q = \lim_{s \downarrow 0} \frac{{}_2P(s) - I}{s} = \tilde{P} - I = Q$$

Identifying ${}_2M$ with M [6, p. 237] we obtain from Thm. 4.1 that M is positive and exponentially ergodic.

Since $b(t) = U(t-1)$, $a_1 = 1$ and $a_n = 0$, $n \neq 1$ in (25). Hence $f(\lambda) = \lambda$. By (41) the decay parameter of M is

$$\lambda = - \min_{t > 0} \sum_{-\infty}^{\infty} c_k e^{kt}, \; k = 1. \tag{45}$$

Similarly we can prove that if

$$\begin{cases} \text{(i)}' & \sum_{-\infty}^{\infty} k\, c_k < 0 \\ \text{(ii)}' & \text{for } k < 0,\; c_k \le D\, \mu^{-k} \text{ for some } D > 0 \text{ and } 0 \le \mu < 1 \end{cases}$$

then M is transient and exponentially ergodic with decay parameter

$$\lambda = - \min_{t < 0} \sum_{-\infty}^{\infty} c_k e^{kt}, \; K = 1.$$

We remark here that H. Miller obtains many other interesting results that can be applied along the same lines.

We finish the discussion with a simple example from queueing theory. Although the matrix Q, defined below, is not exactly of the form given above, another theorem in [32] assures that the same conclusions are still valid.

Consider a queueing model with exponential input at rate λ, exponential service times at rate μ, and n servers in parallel. It is easily shown that the infinitesimal generator in this case is of the form [20, p. 433].

$$\begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \mu & -\lambda-\mu & \lambda & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 2\mu & -\lambda-2\mu & \lambda & \cdots & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -\lambda-n\mu+\mu & \lambda & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & n\mu & -\lambda-n\mu & \lambda & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & n\mu & -\lambda-n\mu & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Here $p_{ij}(t)$ is the probability that at time t there are j customers waiting in line or being served, given that there were i customers in line at time 0.

Clearly $\sum_{-\infty}^{\infty} k\, c_k = \lambda - n\mu$, while (ii) and (ii)' are always satisfied. Moreover

$$E(t) \equiv \sum_{-\infty}^{\infty} c_k\, e^{kt} = -\lambda - n\mu + \lambda e^{t} + n\mu\, e^{-t}.$$

Suppose $\lambda > n\mu$, then the process is transient and for all i and j and all $t \geq 0$

$$p_{ij}(t) \leq \exp\{-[\sqrt{\lambda} - \sqrt{n\mu}]^2\, t\}. \tag{46}$$

Suppose $\lambda < n\mu$, then the process is positive recurrent and for all i and j and all $t \geq 0$

$$|p_{ij}(t) - \lim_{t\to\infty} p_{ij}(t)| \leq \exp\{-[\sqrt{n\mu} - \sqrt{\lambda}]^2\, t\}. \tag{47}$$

We prove (46). We only have to find the decay parameter. But for $t < 0$

$$E'(t) = 0 \quad \text{if} \quad t_o = -\frac{1}{2} \log \frac{\lambda}{n\mu}$$

and hence

$$\min_{t < 0} E(t) = E[-\frac{1}{2} \log \frac{\lambda}{n\mu}] = -\lambda - n\mu + 2\sqrt{n\mu\lambda} .$$

The above results (46) and (47) should be compared with [21, p. 102], and for $n = 1$ with Thm. 2.4.

Acknowledgments

The author would like to thank Professor Marcel F. Neuts for his advice and encouragement and to Professor Baxter and the Air Force Office of Scientific Research for their financial assistance.

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APPENDIX

Theorem 1: If $A(t)$ is a real valued function satisfying $A=O[e^{\alpha t}]$ for $t \to \infty$ and some real number α, then $a(s)$ converges for all s such that $\text{Re } s > \alpha$. [48, p. 38].

Theorem 2: (i) If $a(s)$ converges for $s = \alpha+i\delta$, $\alpha > 0$, then $A(t) = o[e^{\alpha t}]$ as $t \to \infty$;

(ii) If $a(s)$ converges for $s=\alpha+i\delta$, $\alpha < 0$, then $A(+\infty)$ exists and $A(t) - A(\infty) = o(e^{\alpha t})$ as $t \to \infty$. [48,p. 40].

Theorem 3: If $A(t) \geq 0$ is nondecreasing then the real point of the axis of convergence of $a(s)$ [called the first singularity of $a(s)$] is a singularity of $a(s)$. [48, p. 57].

Theorem 4: A function $F(t)$ defined on $[0,\infty]$ is exponentially bounded if and only if $f(s)$ converges in $P(\lambda)$ for some $\lambda > 0$.

Theorem 5: A L.S.T. $f(s)$ converges in $P(\lambda)$ for some $\lambda > 0$ if and only if

(i) $|f^{(n)}(0)| < \infty$ for all n, and

(ii) $\limsup_{n \to \infty} \{\frac{1}{n!} |f^{(n)}(0)|\}^{1/n} < \infty$.

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DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Purdue University	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

On the Rate of Convergence in Renewal and Markov Renewal Processes

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Report

5. AUTHOR(S) *(Last name, first name, initial)*

Teugels, Jozef L.

6. REPORT DATE June 1967	7a. TOTAL NO. OF PAGES 92	7b. NO. OF REFS 48

8a. CONTRACT OR GRANT NO. AFOSR 955-65 b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Mimeograph Series Number 107 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Air Force Office of Scientific Research

13. ABSTRACT

$\{X_k, k=1,2,\ldots\}$ is a sequence of independent identically distributed random variables with distribution function $F(t)$ and mean μ. Two problems are: (1) Is it possible to give an estimate on the rate of convergence in the renewal theorems? (2) What can be said if $\mu = \infty$? In the first problem we assume that for some $\lambda > 0$, $1-F(t) \leq e^{-\lambda t}$ for large t. We then prove the classical renewal theorems with a negative exponential error bound. In the second problem a class of distributions is formed for which for $t \to \infty$ $1-F(t) \sim t^{-\alpha}L(t)$ where $0 \leq \alpha \leq 1$ and $L(t)$ is a slowly varying function. A theorem, analogous to Smith's key renewal theorem is obtained for this case. It gives an accurate estimate on the renewal function if $1 < \alpha < 2$.

In Chapter II an irreducible Markov Renewal Process with renewal functions $M_{ij}(t)$ is studied, and it is shown that all of the functions $M_{ij}(t)$ converge exponentially to their asymptote if and only if one of them does.

This solidarity theorem extends the results of D.G. Kendall and D. Vere-Jones for discrete time Markov chains and those of J.F.C. Kingman for continuous parameter Markov chains.

The exponential ergodicity for Markov chains (Chapter II) is preserved by the derived chains of a given Markov chain. The concept of derived chains is due to J.W. Cohen.

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renewal process						
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