On Bishop's Upcrossing Inequality

by

Ta-Feng Lin

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Glen Baxter, Project Director

Department of Statistics

Division of Mathematical Sciences

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Bishop used a simple but elegant combinatorial lemma to derive an upcrossing inequality (unpublished), from which the Chacon-Ornstein Ergodic Theory follows. Here we shall use the same method to derive a somewhat different upcrossing inequality.

Before we state the upcrossing inequality, we would like to introduce the combinatorial lemma.

For fixed n, let $a(0), a(1), \ldots, a(n); b(0), \ldots, b(n)$ be real numbers, consider such N that there exists $u_1, v_1, \ldots, u_N, v_N$ with

(1) $$0 \le u_1 < v_1 < u_2 < v_2 < \cdots < u_N < v_N \le n .$$

(2) $$a(u_i) \le b(v_i) \qquad 1 \le i \le N .$$

(3) $$a(u_{i+1}) \le b(v_i) \qquad 1 \le i \le N-1 .$$

Define ω_n be the maximum of such N, ω_n is called the upcrossing number from sequence $\{a(i)\}_1^n$ to sequence $\{b(i)\}_1^n$.

Let $\mathcal{P}$ be the collection of empty or finite sequences $P = \{s_1, t_1, \ldots, s_m, t_m\}$ with $0 \le s_1 < t_1 \le s_2 < t_2 \le \cdots \le s_m < t_m \le n$.

Define $m_P = m$, and $SP = \sum_1^m [b(t_i) - a(s_i)]$ for $P = \{s_1, t_1, \ldots s_m, t_m\} \in \mathcal{P}$.

<u>Lemma</u>: For $P \in \mathcal{P}$, there exists $Q \in \mathcal{P}$ such that $m_Q \ge \omega_n$ and $SQ \ge SP$.

Proof:

It suffices to prove the case $m_P < \omega_n$ (=N say).

(a). If $v_1 \leq s_1$, let $Q_1 = \{u_1 < v_1 \leq s_1 < t_1 \leq \cdots \leq s_m < t_m\}$.

(b). If $v_1 > s_1$, let $s_{m+1} = n$.

Then m intervals $(s_1, s_2], \ldots (s_m, s_{m+1}]$ contain $N(> m)$ points; hence there exists $(s_i, s_{i+1}]$, $1 \leq i \leq m$, such that $\{v_k, v_{k+1}\} \subset (s_i, s_{i+1}]$

If $t_i \leq u_{k+1}$, let $Q_1 = \{s_1 < t_1 \leq \cdots \leq s_i < t_i \leq u_{k+1} < v_{k+1} \leq s_{i+1} < \cdots < t_m\}$; if $t_i > u_{k+1}$, let $Q_1 = \{s_1 < t_1 \leq \cdots \leq s_i < v_k \leq u_{k+1} < t_i \leq s_{i+1} < \cdots < t_m\}$. In any case, $SQ_1 \geq SP$ and $m_{Q_1} = m_P + 1$.

Repeating the same procedure, one will get the result.

Let T be a positive contraction linear operator on L_1 and let $f = \{f_0, f_1, \ldots, f_n\}$, $p = \{p_0, p_1, \ldots, p_n\}$ be sequences of measurable functions with

(4) $f_i^+ \in L_1$, $0 \leq i \leq n$ and $T(\sum_\Omega f_i)^+ \geq \sum_\Omega f_{i+1}$ for any $\Omega \subset \{0,1,\ldots(n-1)\}$.

(5). $p_i \geq 0$, $0 \leq i \leq n$, and if $h \in L_1$, $|h| \leq p_i$, $0 \leq i \leq n-1$,

then $T|h| \leq p_{i+1}$.

We shall use here a convention that the summation over empty sets is zero.

Let

$$\begin{cases} a(u,x) = \sum_0^u f_i(x) \\ b(v,x) = \sum_0^v (f_i - p_i)(x) \end{cases} \quad 0 \leq u,v \leq n .$$

Define $\omega_n(x)$ to be the upcrossing number from sequence $\{a(i,x)\}_{i=0}^n$ to sequence $\{b(i,x)\}_{i=0}^n$, and $\bar{\omega}_n(x)$ to be the upcrossing number from sequence $\{o, a(o,x), \ldots a(n,y)\}$ to the sequence $\{o, b(o,x), \ldots b(n,x)\}$.

<u>Theorem</u> 1: (Bishop's upcrossing inequality). $\omega_n(x)$ and $\bar{\omega}_n(x)$ are measurable and

$$\int \bar{\omega}_n(x)\, p_o(x)\, d\mu \leq \int f^+_{o(x)}\, d\mu$$

<u>Theorem</u> 2:

$$\int \omega_n(x)\, p_o(x)\, d\mu \leq \int T\, f_o^+\, (x)\, d\mu \, .$$

We shall prove only Theorem 2.

Proof: In order to prove theorem 2, we introduce

$$a'\,(u,x) = \sum_1^u f_i(x) = a(u,x) - f_o(x)\,,$$

$$b'\,(v,y) = \sum_1^v (f_i - p_i)(x) = b(v,x) - (f_o - p_o)(x)\,. \qquad o \leq u,v \leq n\,.$$

$$(6).\quad SP(x) = \sum_1^m [b(t_i,x) - a(s_i,x)] = \left(\sum_{s_1+1}^{t_1} + \ldots + \sum_{s_m+1}^{t_m}\right) f_i(x) - \left(\sum_o^{t_1} + \ldots + \sum_o^{t_m}\right) p_i(x)\,,$$

$$(7).\quad S'P(x) = \sum_1^m [b'(t_i,x) - a'(s_i,x)] = \left(\sum_{s_1+1}^{t_1} + \ldots + \sum_{s_m+1}^{t_m}\right) f_i(x) - \left(\sum_1^{t_1} + \ldots + \sum_1^{t_m}\right) p_i(x)$$

$$= SP(x) + mp_o(x)\,.$$

$$\text{for}\quad P = \{t_1 < s_1 \leq \ldots \leq s_m < t_m\} \in \mathcal{P}\,.$$

Let $\lambda(x) = \max_{P \in \mathcal{P}} SP(x)\ (\geq 0)$,

$$\lambda'(x) = \max_{P \in \mathcal{P}} S'P(x)\ (\geq 0)\,.$$

For fixed x, from the definition of $\omega_n(x)$, it is possible to choose $P \in \mathcal{P}$ such that $m_P = m \geq \omega_n(x)$. Then, using (7), we have

$$\omega_n(x)\, p_o(x) \leq m\, p_o(x) = S'P(x) - SP(x) \leq \lambda'(x) - SP(x) .$$

This inequality is true for all $P \in \mathcal{P}$ such that $m_P \geq \omega_n(x)$; hence

$$\omega_n(x)\, p_o(x) \leq \lambda'(x) - \max_{m_P \geq \omega_n(x)} SP(x) = \lambda'(x) - \lambda(x) .$$

The last equal sign follows from the lemma. Hence, for any x,

(8). $$\omega_n(x)\, p_o(x) \leq \lambda'(x) - \lambda(x) .$$

If we prove that

(9). $$\lambda'(x) \leq T\,\lambda(x) + T\, f_o^+(x) ,$$

then we are done.

To prove (9), consider any $P = \{s_1 < t_1 \leq \dots \leq s_m < t_m\} \in \mathcal{P}$.

(a) If $s_1 \geq 1$ let $P_1 = \{s_1 - 1 < t_1 - 1 \leq \dots \leq s_m - 1 < t_m - 1\}$; then

$$SP_1(x) = \Big(\sum_{s_1}^{t_1-1} + \dots + \sum_{s_m}^{t_m-1}\Big) f_i(x) - \Big(\sum_{o}^{t_1-1} + \dots + \sum_{o}^{t_m-1}\Big) p_i(x) ,$$

and by (4), (5) and (7), we have

$$\begin{aligned} T\,[\lambda(x) + f_o^+(x)] &\geq T\,[(SP)^+(x) + f_o(x)] \geq T\,[(SP)^+(x)] \\ &\geq T\Big(\sum_{s_1}^{t_1-1} + \dots + \sum_{s_m}^{t_m-1} f_i\Big)^+(x) - \Big(\sum_{o}^{t_1-1} + \dots + \sum_{o}^{t_m-1}\Big) Tp_i(x) \\ &\geq \Big(\sum_{s_1+1}^{t_1} + \dots + \sum_{s_m+1}^{t_m}\Big) f_i(x) - \Big(\sum_{1}^{t_1} + \dots + \sum_{1}^{t_m}\Big) p_i(x) \\ &= S'P(x) . \end{aligned}$$

(b) If $s_1 = 0$, $t_1 = 1$ let $P_1 = \{s_2-1 < t_2-1 \leq \cdots \leq s_m-1 < t_m-1\}$.

(c) If $s_1 = 0$, $t_1 \geq 2$ let $P_1 = \{s_1 < t_1-1 \leq s_2-1 < t_2-1 \leq \cdots \leq s_n-1 < t_m-1\}$.

One can prove, as in (a), that

$$T\,[\lambda(x) + f_o^+(x)] \geq T[SP_1)^+(x) + f_o^+(x)] \geq S'P(x) .$$

In any case, $P_1 \in \mathcal{P}$ and $T\,\lambda(x) + T\,f_p^+(x) \geq S'P(x)$. for any $P \in \mathcal{P}$.
Hence

$$T\,[\,\lambda(x) + f_o^+(x)] \geq \lambda'(x) .$$

This concludes the proof.