Some Results on Almost Sure and Complete Convergence in the Independent and Martingale Cases

by

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CHAPTER I

SOME RESULTS ON THE CONVERGENCE OF LINEAR COMBINATIONS OF INDEPENDENT RANDOM VARIABLES

1. Introduction

Throughout Chapter I, we let $(D_k, k \geq 1)$ be a sequence of independent random variables, a_{nk} be a matrix of real numbers, $A_n = \Sigma_{k=1}^{\infty} a_{nk}^2$, $T_{nm} = \Sigma_{k=1}^{m} a_{nk} D_k$ and T_n be the a.s. limit of T_{nm} as $m \to \infty$ whenever it exists. T_n is said to converge completely to zero in the sense of Hsu and Robbins [15] if $\Sigma_{n=1}^{\infty} P[|T_n| > \epsilon] < \infty$ for all $\epsilon > 0$. The purpose here is to present various sets of conditions for the complete or a.s. convergence of T_n to zero. It should be noted that T_n converging completely to zero implies that T_n converges a.s. to zero and that the two types of convergence are equivalent if the T_n's form an independent sequence of random variables. The results given below extend or improve results given by Hsu and Robbins [15], Erdos [10], Pruitt [24], and Chow [6]. Work done by Franck and Hanson [11] and Chow [6] is closely related to that presented here. The double truncation method of proof developed by Erdos [10] and improved by other authors ([6], [24], and [11]) is fundamental to the present investigation. Theorems 1.1 and 1.2 give general sets of conditions to insure the complete convergence of T_n to zero, Theorem 1.1 treating the independent identically distributed case and Theorem 1.2 treating the more general situation where the assumption that the D_k's are identically distributed is dropped. Several important applications to more specific situations are stated in Corollaries 1.1 - 1.3. Corollary 1.4 is of special interest since it shows that Erdos's double truncation method used in [10] to get sharp results on complete convergence can sometimes also be used to obtain sharp results on almost sure convergence.

2. Preparatory Lemmas

LEMMA 1.1. Let $ED_k^2 \leq K < \infty$, $ED_k = 0$, $a_{nk} > 0$, $A_n < \infty$ and $\Sigma_{n=1}^{\infty} \exp(-t/A_n) < \infty$ for all $t > 0$. Let $D'_{nk} = D_k \, I[a_{nk} \, D_k \leq n^{-\rho}]$ for some $\rho > 0$ and $T'_{nm} = \Sigma_{k=1}^{m} a_{nk} D'_{nk}$. Then T'_{nm} converges a.s. to a random variable T'_n as $m \to \infty$ and

$$\Sigma_{n=1}^{\infty} P[T'_n > \epsilon] < \infty \quad \text{for all} \quad \epsilon > 0 . \tag{1}$$

PROOF. According to the Kolmogorov convergence theorem, ([22], p.236) $\underset{m\to\infty}{\text{Lim}}\, T_{nm} = T_n$ exists a.s. since $\Sigma_{k=1}^{\infty} a_{nk}^2 \, ED_k^2 < \infty$. Since $\Sigma_{k=1}^{\infty} P[D_k \neq D'_{nk}] = \Sigma_{k=1}^{\infty} P[D_k > n^{-\rho}/a_{nk}] \leq KA_n \, n^{2\rho} < \infty$, it then follows that T'_{nm} converges a.s. to a random variable T'_n as $m \to \infty$. Fix $\epsilon > 0$. Let $t = \min(\epsilon/(2A_n), n^{\rho})$. Since $a_{nk} \, D'_{nk} \, t \leq 1$, it then follows that $E \exp(a_{nk} \, D'_{nk} \, t) \leq \exp(E \, a_{nk} \, D'_{nk} \, t + E \, a_{nk}^2 \, D'^2_{nk} \, t^2)$ using the easily established fact ([6], p. 1488) that $E \exp Y \leq \exp(EY + EY^2)$ for a random variable $Y \leq 1$. Since $E \, a_{nk} \, D'_{nk} \, t \leq 0$, we obtain $E \exp(a_{nk} \, D'_{nk} \, t) \leq \exp(a_{nk}^2 \, t^2 \, ED'^2_{nk})$. Assuming without loss of generality that $ED_k^2 \leq 1$, it then follows by the independence of the D'_{nk}'s in k that $E \exp(tT'_n) \leq \exp(t^2 A_n)$. By the Chebychev inequality, $P[T'_n > \epsilon] \leq \exp(-\epsilon t) \, E \exp(tT'_n) \leq \exp(-\epsilon t) \exp(t^2 A_n)$. If $\epsilon/(2A_n) > n^{\rho}$, we obtain $P[T'_n > \epsilon] \leq \exp(-\epsilon n^{\rho}/2)$. On the other hand, if $\epsilon/(2A_n) \leq n^{\rho}$, then $P[T'_n > \epsilon] \leq \exp(-\epsilon^2/(4A_n))$. Hence $\Sigma_{n=1}^{\infty} P[T'_n > \epsilon] < \infty$ for all $\epsilon > 0$.

LEMMA 1.2. Let $(Z_k, k \geq 1)$ be a sequence of independent random variables and $|a_{nk}| \leq Kn^{-\alpha}$ for some $\alpha > 0$. Let either $Z''_{nk} = Z_k \, I[|a_{nk}| \, Z_k > \epsilon/N]$ or $Z''_{nk} = Z_k \, I[|a_{nk} \, Z_k| > \epsilon/N]$ for fixed $\epsilon > 0$ and positive integer N. Let $T''_{nm} = \Sigma_{k=1}^{m} a_{nk} Z''_{nk}$ and assume T''_{nm} converges a.s. to a random variable T''_n as $m \to \infty$. Let $f_n(j)$ be the number of subscripts k such that $|a_{nk}| > \epsilon/(Nj)$ for integers $n \geq 1$ and $j \geq 1$. Let $g_j = [(jKN/\epsilon)^{1/\alpha}]$ where $[\cdot]$ is the greatest integer function. Then

$$(2) \qquad \Sigma_{n=1}^{\infty} P[|T''_n| > \epsilon] \leq \Sigma_{j=1}^{\infty} \Sigma_{n=1}^{g_j} f_n(j) \sup_k P[j-1 \leq |Z_k| < j] \quad \text{and}$$

$$(3) \quad \Sigma_{n=1}^{\infty} P[|T''_n| > \epsilon] \leq \Sigma_{j=1}^{\infty} \Sigma_{n=1}^{g_j} (f_n(j) - f_n(j-1)) \sup_k P[|Z_k| \geq j-1].$$

PROOF. T''_n is well defined by hypothesis.

$$P[|T''_n| > \epsilon] \leq P[\exists\, k \ni |a_{nk} \, Z_k| > \epsilon/N] \leq \Sigma_{k=1}^{\infty} P[|Z_k| > \epsilon/(N|a_{nk}|)]$$

$$\leq \Sigma_{j=1}^{\infty} (f_n(j) - f_n(j-1)) \sup_k P[|Z_k| \geq j-1] \quad .$$

Now $|a_{nk}| \leq Kn^{-\alpha}$ implies that $f_n(j) = 0$ for $n > g_j$. Thus

$$\Sigma_{n=1}^{\infty} P[|T''_n| > \epsilon] \leq \Sigma_{j=1}^{\infty} \Sigma_{n=1}^{g_j} (f_n(j) - f_n(j-1)) \sup_k P[|Z_k| \geq j-1] \quad .$$

Similarly, $P[\exists\, k \ni |a_{nk} \, Z_k| > \epsilon/N] \leq \Sigma_{j=1}^{\infty} \sup_k P[j-1 \leq |Z_k| < j] \, f_n(j)$.

Hence $\Sigma_{n=1}^{\infty} P[|T''_n| > \epsilon] \leq \Sigma_{j=1}^{\infty} \Sigma_{n=1}^{g_j} f_n(j) \sup_k P[j-1 < |Z_k| \leq j]$.

LEMMA 1.3. Let $(Z_k, k \geq 1)$ be a sequence of independent random variables with $E|Z_k|^\upsilon \leq 1$ for some $\upsilon > 0$. Let either $Z'''_{nk} = Z_k \; I[n^{-\rho}/|a_{nk}| < Z_k \leq \epsilon/(N|a_{nk}|)]$ or $Z'''_{nk} = Z_k \; I[n^{-\rho}/|a_{nk}| < |Z_k| \leq \epsilon/(N|a_{nk}|)]$ for $\rho > 0$, $\epsilon > 0$, and positive integer N. Let $T'''_{nm} = \Sigma_{k=1}^m a_{nk} Z'''_{nk}$ and assume that T'''_{nm} converges a.s. to a random variable T'''_n as $m \to \infty$. Then

$$P[|T'''_n| > \epsilon] \leq (\Sigma_{k=1}^\infty |a_{nk}|^\upsilon n^{\rho\upsilon})^N \quad . \tag{4}$$

PROOF. T'''_n is well defined by hypothesis.

$$P[|T'''_n| > \epsilon] \leq P[\exists \text{ N k's } \ni |Z_k| > n^{-\rho}/(|a_{nk}|)] \leq (\Sigma_{k=1}^\infty P[|Z_k| > n^{-\rho}/(|a_{nk}|)])^N \leq (\Sigma_{k=1}^\infty |a_{nk}|^\upsilon n^{\rho\upsilon})^N$$

using the fact that $P[|Z_k|^\upsilon > x] \leq x^{-\upsilon}$ for any $x > 0$.

LEMMA 1.4. Let $E|D_k|^\upsilon \leq K < \infty$ for some $1 \leq \upsilon < 2$, $|a_{nk}| \leq Kn^{-\alpha}$ for some $\alpha > 0$, and $\Sigma_{k=1}^\infty |a_{nk}|^\upsilon \leq Kn^{-\lambda}$ for some $\lambda > 0$. Let $D'_{nk} = D_k \; I[|a_{nk} D_k| \leq n^{-\rho}]$ for some $0 < \rho < \text{Min}(\alpha, \lambda/\upsilon)$ and $T'_{nm} = \Sigma_{k=1}^m a_{nk} D'_{nk}$. Let either $\Sigma_{k=1}^\infty |a_{nk}| \leq K$ and $ED_k = 0$ or $\Sigma_{k=1}^\infty |a_{nk}| \to 0$ as $n \to \infty$. Then T'_{nm} converges a.s. to a random variable T'_n as $m \to \infty$ and T'_n converges completely to zero.

PROOF. Since $\Sigma_{k=1}^\infty |a_{nk}| E|D'_{nk}| \leq \Sigma_{k=1}^\infty |a_{nk}| E|D_k| < \infty$, it follows that T'_{nm} converges a.s. to a random variable T'_n so $m \to \infty$. $E|D_k|^\upsilon \leq K$ for $\upsilon > 1$ implies that the D_k's are uniformly integrable. Thus, if $ED_k = 0$ and $\upsilon > 1$, then $|ED'_{nk}| = |ED_k I[|D_k| > n^{-\rho}/|a_{nk}|]$ $\leq E[|D_k| I[|D_k| > n^{-\rho+\alpha}/K]] \to 0$ as $n \to \infty$ uniformly in k. Thus

$|\Sigma_{k=1}^{\infty} a_{nk} ED'_{nk}| \leq \Sigma_{k=1}^{\infty} |a_{nk}||ED'_{nk}| \to 0$ as $n \to \infty$ for the case $ED_k = 0$ and $\upsilon > 1$. Since $|ED'_{nk}| \leq E|D_k| \leq K$, it follows that $\Sigma_{k=1}^{\infty} a_{nk} ED'_{nk} \to 0$ as $n \to \infty$ for the case $\Sigma_{k=1}^{\infty} |a_{nk}| \to 0$ as $n \to \infty$. Since these are the only two cases which may occur, it follows that $\Sigma_{k=1}^{\infty} a_{nk} ED'_{nk} \to 0$ as $n \to \infty$. Let $Y'_{nk} = D'_{nk} - ED'_{nk}$ and $t = n^{\rho}/2$. Since $EY'_{nk} = 0$ and $|a_{nk} Y'_{nk} t| \leq 1$, it follows by a lemma of Chow ([6], p. 1482) that $E \exp(a_{nk} Y'_{nk} t) \leq \exp(t^2 a_{nk}^2 EY'^2_{nk})$. We decompose $\exp(t^2 a_{nk}^2 E(Y'_{nk})^2) = \exp(t^2 |a_{nk}|^{\upsilon} E|a_{nk} Y'_{nk}|^{2-\upsilon} |Y'_{nk}|^{\upsilon})$. $2 - \upsilon > 0$ and $|a_{nk} Y'_{nk}| \leq 2n^{-\rho}$ together imply $|a_{nk} Y'_{nk}|^{2-\upsilon} \leq n^{-\rho(2-\upsilon)} 2^{2-\upsilon}$. We assume without loss of generality that $E|D_k|^{\upsilon} \leq 1$. Then $E|D'_{nk}| \geq |ED'_{nk}|$ and the c_r inequality ([22], p. 155) yields $E|Y'_{nk}|^{\upsilon} \leq 2^{\upsilon}$. Combining the above yields $\exp(t^2 a_{nk}^2 E(Y'_{nk})^2) \leq \exp(t^2 |a_{nk}|^{\upsilon} n^{-\rho(2-\upsilon)} 4)$. Hence $E \exp(t \Sigma_{k=1}^{\infty} a_{nk} Y'_{nk}) \leq \exp(t^2 \Sigma_{k=1}^{\infty} |a_{nk}|^{\upsilon} n^{-\rho(2-\upsilon)} 4)$ $\leq \exp(t^2 n^{-\rho(2-\upsilon)-\lambda} 4K)$. Fix $\epsilon > 0$. A Chebychev argument yields $P[|\Sigma_{k=1}^{\infty} a_{nk} Y'_{nk}| > \epsilon] \leq 2 \exp(-\epsilon t) \exp(t^2 n^{-\rho(2-\upsilon)-\lambda} 4K)$. Since $t = n^{\rho}/2$, it follows that $\Sigma_{n=1}^{\infty} P[|\Sigma_{k=1}^{\infty} a_{nk} Y'_{nk}| > \epsilon] < \infty$. Since $\Sigma_{k=1}^{\infty} a_{nk} ED'_{nk} \to 0$ as $n \to \infty$, it follows that $\Sigma_{n=1}^{\infty} P[|T'_n| > 2\epsilon] < \infty$.

3. Convergence in the Identically Distributed Case

THEOREM 1.1. Let $|a_{nk}| \leq Kn^{-\alpha}$ for some $\alpha > 0$ and $E|D_k|^{(1+\alpha+\beta)/\alpha} < \infty$ where the D_k are identically distributed and $\beta > -1 - \alpha$.

i) If $(1+\alpha+\beta)/\alpha \geq 2$, $A_n \leq Kn^{\beta-\alpha}$, $\Sigma_{n=1}^{\infty} \exp(-t/A_n) < \infty$ for all $t > 0$, $ED_k^2 \log^+ |D_k| < \infty$ and $ED_k = 0$, then T_n converges completely to zero.

ii) If $(1+\alpha+\beta)/\alpha = 2$, $\Sigma_{k=1}^{\infty} |a_{nk}|^{\delta} \leq Kn^{\alpha(2-\delta)-1}$ for some $0 < \delta < 2$ and $ED_k = 0$, then T_n converges completely to zero.

iii) If $1 \leq (1+\alpha+\beta)\alpha < 2$, $A_n \leq Kn^{\beta-\alpha}$, $\Sigma_{k=1}^{\infty}|a_{nk}|^{(1+\alpha+\beta)/\alpha} \leq Kn^{-\gamma}$ for some $\gamma > 0$ and either $\Sigma_{k=1}^{\infty} |a_{nk}| \leq K$ and $ED_k = 0$ or $\Sigma_{k=1}^{\infty} |a_{nk}| \to 0$ as $n \to \infty$, then T_n converges completely to zero.

iv) If $0 < (1+\alpha+\beta)/\alpha < 1$, $A_n \leq Kn^{\beta-\alpha}$, $\Sigma_{k=1}^{\infty} |a_{nk}|^{(1+\alpha+\beta)/\alpha} \leq Kn^{-\gamma}$ for some $\gamma > 0$, and $a_{nk} = 0$ for $k > n^{\zeta}$ where $\zeta < \gamma\alpha/(1+\alpha+\beta)$, then T_n converges completely to zero.

PROOF. (i) and (ii). Fix $\epsilon > 0$. According to the Kolmogorov convergence theorem, ([22], p. 236) $\lim_{m\to\infty} T_{nm} \equiv T_n$ exists a.s. since $\Sigma_{k=1}^{\infty} a_{nk}^2 ED_k^2 < \infty$. We may decompose $T_n = \Sigma_{k=1}^{\infty} a'_{nk} D_k - \Sigma_{k=1}^{\infty} a''_{nk} D_k$ where $a'_{nk} > 0$ and $a''_{nk} > 0$. $P[|T_n| > 2\epsilon] \leq P[|\Sigma_{k=1}^{\infty} a'_{nk} D_k| > \epsilon] + P[|\Sigma_{k=1}^{\infty} a''_{nk} D_k| > \epsilon]$. Hence, without loss of generality, we assume $a_{nk} > 0$ throughout the remainder of the proof of (i) and (ii). Let $D'_{nk} = D_k I[a_{nk} D_k \leq n^{-\rho}]$ where $\rho > 0$ will be chosen later. Let $T'_{nm} = \Sigma_{k=1}^{m} a_{nk} D'_{nk}$. By Lemma 1.1, T'_{nm} converges a.s. to a random variable T'_n as $m \to \infty$ and $\Sigma_{n=1}^{\infty} P[T'_n > \epsilon] < \infty$.

Let $D''_{nk} = D_k I[a_{nk} D_k > \epsilon/N]$ where N is a positive integer to be chosen later. Let $T''_{nm} = \Sigma_{k=1}^{m} a_{nk} D''_{nk}$. $\Sigma_{k=1}^{\infty} P[D''_{nk} \neq 0] = \Sigma_{k=1}^{\infty} P[D_k > \epsilon/(Na_{nk})] \leq C N^2 A_n/ \epsilon^2 < \infty$. Thus, by an application of the Borel Cantelli lemma, it follows that T''_{nm} converges a.s. to a random variable T''_n as $m \to \infty$. Applying (2) of Lemma 1.2 with $Z_k = D_k$ yields $\Sigma_{n=1}^{\infty} P[T''_n > \epsilon] \leq \Sigma_{j=1}^{\infty} \Sigma_{n=1}^{g_j} f_n(j) P[j-1 \leq |D_k| < j]$ where $f_n(j)$ and g_j are defined in the statement of Lemma 1.2. We now consider (i). By the definitions of A_n and $f_n(j)$, $A_n \geq f_n(j)\, \epsilon^2/(Nj)^2$. Since $A_n \leq Kn^{\beta-\alpha}$, it follows that $f_n(j) \leq Kn^{\beta-\alpha} N^2 j^2/\epsilon^2$. Thus $\Sigma_{n=1}^{\infty} P[T''_n > \epsilon] \leq (KN^2/\epsilon^2) \Sigma_{j=1}^{\infty} \Sigma_{n=1}^{g_j} n^{\beta-\alpha} j^2 P[j-1 \leq |D_k| < j]$. Elementary computation shows that $\Sigma_{n=1}^{g_j} n^{\beta-\alpha} \leq K' j^{(\beta-\alpha+1)/\alpha}$ if $\beta-\alpha \neq 1$ and

$\Sigma_{n=1}^{g_j} n^{\beta-\alpha} \leq K' \log j$ for $j \geq 2$ if $\beta-\alpha=1$ where K' is a fixed constant independent of j. $ED_k^2 \log^+|D_k| < \infty$ implies that $\Sigma_{j=1}^{\infty} j^2 \log j \, P[j-1 \leq |D_k| < j] < \infty$. Similarly, $E|D_k|^{(1+\alpha+\beta)/\alpha} < \infty$ implies that $\Sigma_{j=1}^{\infty} j^{(1+\alpha+\beta)/\alpha} P[j-1 \leq |D_k| < j] < \infty$. Hence $\Sigma_{n=1}^{\infty} P[T_n'' > \epsilon] < \infty$ in the case of (i). We now consider (ii). Since $\Sigma_{k=1}^{\infty} |a_{nk}|^{\delta} \leq Kn^{\alpha(2-\delta)-1}$, it follows that $f_n(j) \leq Kn^{\alpha(2-\delta)-1}(Nj/\epsilon)^{\delta}$. Thus $\Sigma_{n=1}^{\infty} P[T_n'' > \epsilon] \leq (KN^{\delta}/\epsilon^{\delta}) \Sigma_{j=1}^{\infty} \Sigma_{n=1}^{g_j} n^{\alpha(2-\delta)-1} j^{\delta} P[j-1 \leq |D_k| < j]$. Elementary computation shows that $\Sigma_{n=1}^{g_j} n^{\alpha(2-\delta)-1} \leq K' j^{2-\delta}$ where K' is a fixed constant independent of j. $ED_k^2 < \infty$ implies $\Sigma_{j=1}^{\infty} j^2 P[j-1 \leq |D_k| < j] < \infty$. Thus, $\Sigma_{n=1}^{\infty} P[T_n'' > \epsilon] < \infty$ in the case of (ii).

Let $D_{nk}''' = D_k - D_{nk}' - D_{nk}''$, i.e., $D_{nk}''' = D_k \, I[n^{-\rho}/a_{nk} < D_k \leq \epsilon/(Na_{nk})]$. Let $T_{nm}''' = \Sigma_{k=1}^{m} a_{nk} D_{nk}'''$. $\Sigma_{k=1}^{\infty} P[D_{nk}''' \neq 0] \leq \Sigma_{k=1}^{\infty} P[D_k > n^{-\rho}/a_{nk}] \leq C A_n n^{2\rho} < \infty$. Thus, by an application of the Borel Cantelli lemma, it follows that T_{nm}''' converges a.s. to a random variable T_n''' as $m \to \infty$. Without loss of generality we assume $E|D_k|^{(1+\alpha+\beta)/\alpha} \leq 1$ Then, applying Lemma 1.3 with $Z_k = D_k$ and $\upsilon = (1+\alpha+\beta)/\alpha$ implies that $\Sigma_{n=1}^{\infty} P[|T_n'''| > \epsilon] \leq \Sigma_{n=1}^{\infty} (\Sigma_{k=1}^{\infty} |a_{nk}|^{(1+\alpha+\beta)/\alpha} n^{\rho(1+\alpha+\beta)/\alpha})^N \leq \Sigma_{n=1}^{\infty} (n^{-1+\rho(1+\alpha+\beta)/\alpha} K')^N$ for some constant K'. By choosing ρ sufficiently small and N sufficiently large, the preceding sum becomes finite. Combining the above results for T_n', T_n'', and T_n''', it follows that $\Sigma_{n=1}^{\infty} P[T_n > 3\epsilon] < \infty$. Replacing D_k by $-D_k$ in the above arguments yields $\Sigma_{n=1}^{\infty} P[-T_n > 3\epsilon] < \infty$. This completes the proof of (i) and (ii).

(iii). Since $\Sigma_{k=1}^{\infty} |a_{nk}| E|D_k| < \infty$, it follows that T_{nm} converges a.s. to a random variable T_n as $m \to \infty$. Fix $\epsilon > 0$. Let $D_{nk}' = D_k \, I[|a_{nk} D_k| \leq n^{-\rho}]$ where $0 < \rho < \min(\alpha, \gamma\alpha/(1+\alpha+\beta))$. Let

$T'_{nm} = \Sigma_{k=1}^{m} a_{nk} D'_{nk}$. By Lemma 1.4, T'_{nm} converges a.s. to a random variable T'_n as $m \to \infty$ and T'_n converges completely to zero.

Let $D''_{nk} = D_k I[|a_{nk} D_k| > \epsilon/N]$ where N is a positive integer to be chosen later. Let $T''_{nm} = \Sigma_{k=1}^{m} a_{nk} D''_{nk}$. Since $\Sigma_{k=1}^{\infty} |a_{nk}| E|D_k| < \infty$, it follows that T''_{nm} converges a.s. to a random variable T''_n as $m \to \infty$. Applying (2) of Lemma 1.2 with $Z_k = D_k$ yields $\Sigma_{n=1}^{\infty} P[|T''_n| > \epsilon] \leq \Sigma_{j=1}^{\infty} \Sigma_{n=1}^{g_j} f_n(j) P[j-1 \leq |D_k| < j]$ where $f_n(j)$ and g_j are defined in the statement of Lemma 1.2. By the definitions of A_n and $f_n(j)$, $A_n \geq f_n(j) \epsilon^2/(Nj)^2$. Since $A_n \leq Kn^{\beta-\alpha}$, it follows that $f_n(j) \leq Kn^{\beta-\alpha} N^2 j^2/\epsilon^2$. Thus $\Sigma_{n=1}^{\infty} P[|T''_n| > \epsilon]$ $\leq (KN^2/\epsilon^2)\Sigma_{j=1}^{\infty} \Sigma_{n=1}^{g_j} n^{\beta-\alpha} j^2 P[j-1 \leq |D_k| < j]$. Elementary computation shows that $\Sigma_{n=1}^{g_j} n^{\beta-\alpha} \leq K' j^{(\beta-\alpha+1)/\alpha}$ where K' is a constant independent of j. $E|D_k|^{(1+\alpha+\beta)/\alpha} < \infty$ implies that $\Sigma_{j=1}^{\infty} j^{(1+\alpha+\beta)/\alpha} P[j-1 \leq |D_k| < j] < \infty$. Hence $\Sigma_{n=1}^{\infty} P[|T''_n| > \epsilon] < \infty$.

Let $D'''_{nk} = D_k - D'_{nk} - D''_{nk}$, i.e., $D'''_{nk} = D_k I[n^{-\rho}/|a_{nk}| < |D_k| \leq \epsilon/(N |a_{nk}|)]$. Let $T'''_{nm} = \Sigma_{k=1}^{m} a_{nk} D'''_{nk}$. T'''_{nm} converges a.s. to a random variable T'''_n since $\Sigma_{k=1}^{\infty} |a_{nk}| E|D_k| < \infty$. Without loss of generality, we may assume $E|D_k|^{(1+\alpha+\beta)/\alpha} \leq 1$. Applying Lemma 1.3 with $Z_k = D_k$ and $\upsilon = (1+\alpha+\beta)/\alpha$ implies that $P[|T'''_n| > \epsilon]$ $\leq (\Sigma_{k=1}^{\infty}|a_{nk}|^{(1+\alpha+\beta)/\alpha} n^{\rho(1+\alpha+\beta)/\alpha})^N$. Since $\Sigma_{k=1}^{\infty}|a_{nk}|^{(1+\alpha+\beta)/\alpha} \leq Kn^{-\gamma}$, it follows that $P[|T'''_n| > \epsilon] \leq (Kn^{-\gamma+(1+\alpha+\beta)\rho/\alpha})^N$. By choosing ρ sufficiently small and N sufficiently large, it follows that $\Sigma_{n=1}^{\infty} P[|T'''_n| > \epsilon] < \infty$. Combining the above results for T'_n, T''_n and T'''_n, it follows that T_n converges completely to zero thus completing the proof of (iii).

<u>(iv)</u>. Fix $\epsilon > 0$. Let $D'_{nk} = D_k I[|a_{nk} D_k| \le n^{-\rho}]$ for some $\rho > 0$ to be chosen later and let $T'_n = \Sigma_{k=1}^{\infty} a_{nk} D'_{nk}$ $\cdot |\Sigma_{k=1}^{\infty} a_{nk} D'_{nk}| \le \Sigma_{k=1}^{\infty} |a_{nk} D'_{nk}| \le n^{\zeta-\rho}$. Hence $\Sigma_{n=1}^{\infty} P[|T'_n| > \epsilon] < \infty$ by choosing $\rho > \zeta$. Let $D''_{nk} = D_k I[|a_{nk} D_k| > \epsilon/N]$ where N is a positive integer to be chosen later and let $T''_n = \Sigma_{k=1}^{\infty} a_{nk} D''_{nk}$. Applying (2) of Lemma 1.2 with $Z_k = D_k$ yields $\Sigma_{n=1}^{\infty} P[|T''_n| > \epsilon] \le \Sigma_{j=1}^{\infty} \Sigma_{n=1}^{g_j} f_n(j) P[j-1 \le |D_k| < j]$ where $f_n(j)$ and g_j are defined in the statement of Lemma 1.2. By the definitions of A_n and $f_n(j)$, $A_n \ge f_n(j)\, \epsilon^2/(Nj)^2$. Since $A_n \le Kn^{\beta-\alpha}$, it follows that $f_n(j) \le Kn^{\beta-\alpha} N^2 j^2/\epsilon^2$. Thus $\Sigma_{n=1}^{\infty} P[|T''_n| > \epsilon] \le (KN^2/\epsilon^2) \Sigma_{j=1}^{\infty} \Sigma_{n=1}^{g_j} n^{\beta-\alpha} j^2 P[j-1 \le |D_k| < j]$. Elementary computation shows that $\Sigma_{n=1}^{g_j} n^{\beta-\alpha} \le K' j^{(\beta-\alpha+1)/\alpha}$ where K' is a constant independent of j. $E|D_k|^{(1+\alpha+\beta)/\alpha} < \infty$ implies that $\Sigma_{j=1}^{\infty} j^{(1+\alpha+\beta)/\alpha} P[j-1 \le |D_k| < j] < \infty$. Hence $\Sigma_{n=1}^{\infty} P[|T''_n| > \epsilon] < \infty$. Let $D'''_{nk} = D_k - D'_{nk} - D''_{nk}$, i.e., $D'''_{nk} = D_k I[n^{-\rho}/|a_{nk}| < |D_k| \le \epsilon/(N|a_{nk}|)]$. Without loss of generality, we may assume $E|D_k|^{(1+\alpha+\beta)/\alpha} \le 1$. Applying Lemma 1.3 with $Z_k = D_k$ and $\upsilon = (1+\alpha+\beta)/\alpha$ yields $P[|T'''_n| > \epsilon] \le (\Sigma_{k=1}^{\infty} |a_{nk}|^{(1+\alpha+\beta/\alpha} n^{\rho(1+\alpha+\beta)/\alpha})^N \le (Kn^{-\gamma+\rho(1+\alpha+\beta)/\alpha})^N$. We now choose $\rho < (\gamma\alpha) / (1+\alpha+\beta)$ such that $\rho > \zeta$ is satisfied. It then follows for sufficiently large N that $\Sigma_{n=1}^{\infty} P[|T'''_n| > \epsilon] < \infty$. Combining the above results for T'_n, T''_n, and T'''_n, it follows that T_n converges completely to zero. The proof of (iv) and of the theorem is complete.

COROLLARY 1.1. Let D_k be identically distributed, $E|D_k|^{2/\alpha} < \infty$ for some $\alpha > 0$, $ED_k = 0$ if $0 < \alpha \le 1$, $|a_{nk}| \le Kn^{-\alpha}$, $a_{nk} = 0$ for $k > n$, and $\Sigma_{k=1}^{\infty} \exp(-t/A_n) < \infty$ for all $t > 0$. Then T_n converges completely to zero.

PROOF. Let $0 < \alpha < 1$. Then $|a_{nk}| < Kn^{-\alpha}$ and $a_{nk} = 0$ for $k > n$ together imply that $A_n \leq K^2 n^{1-2\alpha}$. Thus for $\beta = 1-\alpha$, the hypotheses of Theorem 1.1(i) are satisfied. Let $\alpha = 1$. Then for $\delta = 1$ the hypotheses of Theorem 1.1(ii) are satisfied. Let $1 < \alpha \leq 2$. Then, letting $\beta = 1-\alpha$, one set of hypotheses of Theorem 1.1(iii) are satisfied. Let $\alpha > 2$. Then for $\beta = 1-\alpha$ and $\zeta = \gamma = 1$, the hypotheses of Theorem 1.1(iv) are satisfied.

REMARKS. The fundamental result of the above type on complete convergence states that if the D_k are identically distributed $E|D_k|^{2/\alpha} < \infty$ for some $\infty > \alpha > 1/2$, $ED_k = 0$ if $1 \geq \alpha > 1/2$, $a_{nk} = n^{-\alpha}$ for $k \leq n$, $a_{nk} = 0$ for $k > n$, then $T_n \equiv \Sigma_{k=1}^{n} D_k / n^{\alpha}$ converges completely to zero. This result is due to Hsu and Robbins [15] for $\alpha = 1$ and Erdos [10] for $\alpha \neq 1$. Corollary 1.1 includes the above result and generalizes it to a triangular matrix of coefficients satisfying certain restrictions on the magnitude of its entries. Theorem 1.1 generalizes this result still further by replacing the hypothesis of triangularity by more general hypotheses on the a_{nk}'s. In [10] Erdos also proves that if the D_k are a sequence of identically distributed random variables, $ED_k^4 < \infty$, and $ED_k = 0$, then there exists an $r > 0$ such that for $a_{nk} = 1/(n^{\frac{1}{2}}(\log n)^r)$ when $k \leq n$ and $a_{nk} = 0$ for $k > n$, it follows that $T_n \equiv \Sigma_{k=1}^{n} D_k / (n^{\frac{1}{2}}(\log n)^r)$ converges completely to zero. Corollary 1.1 and Theorem 1.1 generalize this result also. From Corollary 1.1 it is easy to see that $r = \frac{1}{2} + \delta$ for $\delta > 0$ works in the statement of the Erdos result. Obviously the $(\log n)^r$ term in the denominator cannot be dropped entirely since $S_n/n^{\frac{1}{2}}$ obeys the central limit theorem ([22], p. 247). This shows that the condition

$\Sigma_{n=1}^{\infty} \exp(-t/A_n) < \infty$ for all $t > 0$ cannot be dropped from the statements of Theorem 1.1(i) and Corollary 1.1. For, if it could be dropped, $ED_k^4 < \infty$ would then imply $\Sigma_{k=1}^{n} D_k/n^{\frac{1}{2}}$ converges completely to zero. For use in applications of Theorem 1.1(i) and Corollary 1.1 it is interesting to note that $A_n = o\ ((\log n)^{-1})$ implies that $\Sigma_{n=1}^{\infty} \exp(-t/A_n) < \infty$ for all $t > 0$ and that $A_n = O\ ((\log n)^{-1})$ does not imply that $\Sigma_{n=1}^{\infty} \exp(-t/A_n) < \infty$ for all $t > 0$. The condition $A_n = o\ ((\log n)^{-1})$ is easy to verify in practice and by the previous remark only slightly stronger than $\Sigma_{n=1}^{\infty} \exp(-t/A_n) < \infty$ for all $t > 0$.

Recently, Chow ([6], p. 1488) has proved that if the D_k are a sequence of identically distributed random variables with $ED_k = 0$, $E|D_k|^{2/\alpha} < \infty$, $0 < \alpha \leq 1$, $|a_{nk}| \leq KA_n$ for $k \leq n$, $a_{nk} = 0$ for $k > n$, $A_n \leq Kn^{-\alpha}$, then T_n converges completely to zero. It was this particular result which motivated the present work. Corollary 1.1 improves and generalizes this result by replacing $|a_{nk}| \leq KA_n$ for $k \leq n$, and $A_n \leq Kn^{-\alpha}$ by weaker conditions on the a_{nk} matrix and by extending the result to the case where $E|D_k|^{2/\alpha} < \infty$ for some $\alpha > 1$ but $ED_k^2 = \infty$. Theorem 1.1 generalizes this result still further by replacing the hypothesis of triangularity by more general hypotheses on the a_{nk}'s.

Erdos [10] has established that if the D_k are identically distributed, then $\Sigma_{k=1}^{n} D_k/n^{\alpha}$ converging completely to zero implies that $E|D_k|^{2/\alpha} < \infty$ if $\alpha > \frac{1}{2}$ (and implies that $ED_k = 0$ if $\frac{1}{2} < \alpha < 1$). Hence Corollary 1.1 is sharp for $\alpha > \frac{1}{2}$. Even for $\alpha \leq \frac{1}{2}$, Corollary 1.1 is rather sharp since it says that if the D_k are identically distributed with $ED_k = 0$ and $ED_k^4 < \infty$, then $\Sigma_{k=1}^{n} D_k/(n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\delta})$ converges completely to zero for all $\delta > 0$. But by the Hartman and

Wintner iterated logarithm [12], $\Sigma_{k=1}^{n} D_k/(n^{\frac{1}{2}}(\log\log n)^{\frac{1}{2}})$ does not converge completely to zero.

COROLLARY 1.2. Let the D_k be identically distributed, $E|D_k|^{1+1/\alpha} < \infty$ and $|a_{nk}| \leq Kn^{-\alpha}$ for some $\alpha > 0$.

i) If $0 < \alpha < 1$, $ED_k = 0$, and $A_n \leq Kn^{-\alpha}$, then T_n converges completely to 0.

ii) If $\alpha = 1$, $ED_k = 0$, and $\Sigma_{k=1}^{\infty} |a_{nk}|^{\delta} \leq Kn^{1-\delta}$ for some $0 < \delta < 2$, then T_n converges completely to 0.

iii) If $\alpha > 1$ and either $\Sigma_{k=1}^{\infty} |a_{nk}| \leq K$ and $ED_k = 0$ or $\Sigma_{k=1}^{\infty} |a_{nk}| \to 0$ as $n \to \infty$, then T_n converges completely to 0.

PROOF. (i) is immediate from Theorem 1.1 (i) with $\beta = 0$. (ii) is immediate from Theorem 1.1 (ii) with $\alpha = 1$ and $\beta = 0$. (iii) is immediate from Theorem 1.1 (iii) with $\beta = 0$.

REMARKS. Pruitt [24] has proved that for matrix conditions somewhat stronger than regularity, i.e. $\Sigma_{k=1}^{\infty} a_{nk} \to 1$ as $n \to \infty$, $\Sigma_{k=1}^{\infty} |a_{nk}| \leq K$ and $|a_{nk}| \leq Kn^{-\alpha}$ for some $\alpha > 0$, it then follows for the D_k identically distributed with $E|D_k|^{1+1/\alpha} < \infty$ that T_n converges completely to ED_k. Corollary 1.2 implies and generalizes this result by replacing $\Sigma_{k=1}^{\infty} |a_{nk}| \leq K$ by a weaker condition when $\alpha \leq 1$. Pruitt gives examples to show his results are sharp. Hence, Corollary 1.2 (i) and (iii) are sharp and Corollary 1.2 (ii) is sharp for $\delta \geq 1$.

COROLLARY 1.3. Let the D_k be identically distributed and $E|D_k|^{2/\eta} < \infty$.

i) If $0 < \eta \leq 1$, $ED_k = 0$, $ED_k^2 \log^+|D_k| < \infty$, and $A_n \leq Kn^{-\eta}$, then T_n converges completely to 0.

ii) If $\eta = 1$, $ED_k = 0$, and $\Sigma_{k=1}^{\infty} |a_{nk}|^{\delta} \leq Kn^{-\delta/2}$ for some $0 < \delta < 2$, then T_n converges completely to 0.

iii) If $1 < \eta \leq 2$, $A_n \leq Kn^{-\eta}$, $\Sigma_{k=1}^{\infty} |a_{nk}|^{2/\eta} \leq Kn^{-\gamma}$ for some $\gamma > 0$, and either $\Sigma_{k=1}^{\infty} |a_{nk}| \leq K$ and $ED_k = 0$ or $\Sigma_{k=1}^{\infty} |a_{nk}| \to 0$ as $n \to \infty$, then T_n converges completely to 0.

iv) If $\eta > 2$, $\Sigma_{k=1}^{\infty} |a_{nk}|^{2/\eta} \leq Kn^{-\gamma}$ for some $\gamma > 0$, $A_n \leq Kn^{-\eta}$, and $a_{nk} = 0$ for $k > n^{\zeta}$ for some $\zeta < (\eta\gamma)/2$, then T_n converges completely to 0.

PROOF. Let $0 < \eta \leq 1$. $A_n \leq Kn^{-\eta}$ implies that $|a_{nk}| \leq K^{\frac{1}{2}} n^{-\eta/2}$. Hence for $\beta = -\eta/2$ and $\alpha = \eta/2$ the hypotheses of Theorem 1.1 (i) are satisfied and $(1 + \alpha + \beta)/\alpha = 2/\eta$. Let $\eta = 1$. Choosing $\alpha = \frac{1}{2}$, (ii) is then immediate from Theorem 1.1 (ii), noting that $\Sigma_{k=1}^{\infty} |a_{nk}|^{\delta} \leq Kn^{-\delta/2}$ implies that $|a_{nk}| \leq K^{1/\delta} n^{-\frac{1}{2}}$. Let $1 < \eta \leq 2$. Then, as above, $|a_{nk}| \leq K^{\frac{1}{2}} n^{-\eta/2}$. Hence for $\beta = -\eta/2$ and $\alpha = \eta/2$ the hypotheses of Theorem 1.1(iii) are satisfied and $(1 + \alpha + \beta)/\alpha = 2/\eta$. Let $\eta > 2$. Again $|a_{nk}| \leq K^{\frac{1}{2}} n^{-\eta/2}$. Hence for $\beta = -\eta/2$ and $\alpha = \eta/2$, the hypotheses of Theorem 1.1(iv) are satisfied and $(1 + \alpha + \beta)/\alpha = 2/\eta$.

REMARK. If we assume that the D_k's are identically distributed, $a_{nn} = 1/n^{\eta/2}$ for some $\eta > 0$, and $a_{nk} = 0$ for $k \neq n$, then $T_n \equiv D_n/n^{\eta/2}$ converging completely (a.s.) to zero implies that $E|D_k|^{2/\eta} < \infty$ since $\Sigma_{k=1}^{\infty} P[|D_k| > k^{\eta/2}] < \infty$ is equivalent to $E|D_k|^{2/\eta} < \infty$. Thus Corollary 1.3 (i) is sharp for $\eta < 1$, Corollary 1.3 (iii) and (iv) are sharp for $\gamma \leq 1$, and Corollary 1.3 (ii) is sharp.

COROLLARY 1.4. Let $A_n \leq Kn^{-1}$, $a_{nk}^2 \leq K/k$, $ED_k^2 < \infty$, and $ED_k = 0$ where the D_k are identically distributed. Then T_n converges a.s. to zero.

PROOF. Let $\beta = -\frac{1}{2}$ and $\alpha = \frac{1}{2}$. Then all hypotheses of Theorem 1.1(i) are satisfied except $ED_k^2 \log^+|D_k| = \infty$ is possible. An examination of the proof of (i) shows that $ED_k^2 \log^+|D_k| < \infty$ is used only in establishing that $\Sigma_{n=1}^{\infty} P[T_n'' > \epsilon] < \infty$. Hence we need only establish that T_n'' converges a.s. to zero, where $D_{nk}'' = D_k\, I[a_{nk}D_k > \epsilon/N]$ for some positive integer N and $\epsilon > 0$, $T_{nm}'' = \Sigma_{k=1}^{m} a_{nk}\, D_{nk}''$, and T_n'' is the almost sure limit of T_{nm}'' as $m \to \infty$. In the above we take $a_{nk} > 0$ without loss of generality.

By the Holder inequality, $|T_n''|^2 \leq A_n\, \Sigma_{k=1}^{\infty} (D_{nk}'')^2$ $= A_n\, \Sigma_{k=1}^{\infty} D_k^2\, I[D_k > \epsilon/(Na_{nk})] \leq (K/n)\, \Sigma_{k=1}^{\infty} D_k^2\, I[D_k^2 > \epsilon^2 k/(N^2K)]$ since $a_{nk}^2 \leq K/k$. But $X \equiv \Sigma_{k=1}^{\infty} D_k^2\, I[D_k^2 > \epsilon^2 k/(N^2K)] < \infty$ a.s. since $\Sigma_{k=1}^{\infty} P[D_k^2 > \epsilon^2 k/(N^2K)] < \infty$ follows from the fact that the D_k's are identically distributed with $ED_k^2 < \infty$. Thus $|T_n''|^2 \leq X/n$ a.s. and hence T_n'' converges a.s. to zero.

REMARKS. Chow ([6], p. 1484) has recently proved that if the D_k's are identically distributed with $ED_k = 0$, $ED_k^2 < \infty$, $\Sigma_{k=1}^{\infty} a_{nk}^2 \to 1/n$ as $k \to \infty$, and $a_{nk} = 0$ for $k > n$, then T_n converges a.s. to zero. Corollary 1.4 includes this result and generalizes it by replacing the assumption of triangularity of the a_{nk} matrix by a weaker condition. Unlike the proof given by Chow, the present proof makes no use of the strong law of large numbers.

It is interesting to compare corollaries 1.3 (i) and 1.4. Corollary 1.3 (i) does not imply Corollary 1.4. However if in Corollary 1.4 we were to assume the finiteness of a slightly higher moment than the second, i.e., $ED_k^2 \log^+|D_k| < \infty$, we could drop the assumption that $a_{nk}^2 \leq K/k$ and conclude that T_n converges completely to zero by

Corollary 1.3 (i). Chow [6] has established that his result mentioned above is sharp; hence, it follows that Corollary 1.4 is sharp also.

4. Convergence in the Non-identically Distributed Case

An examination of the proof of Theorem 1.1 shows that the identically distributed hypotheses may be dropped by slightly strengthening the moment condition assumed. This yields Theorem 1.2.

THEOREM 1.2. Let $|a_{nk}| \le Kn^{-\alpha}$ for some $\alpha > 0$.

i) If $E|D_k|^{(1+\alpha+\beta)/\alpha}(\log^+|D_k|)^{1+\xi} \le K$ for some $\xi > 0$, $(1 + \alpha + \beta)/\alpha \ge 2$, $ED_k^2(\log^+|D_k|)^{2+\xi} \le K$, $ED_k = 0$, $A_n \le Kn^{\beta-\alpha}$, and $\Sigma_{n=1}^{\infty} \exp(-t/A_n) < \infty$ for all $t > 0$, then T_n converges completely to zero.

ii) If $ED_k^2 \le K$, $ED_k = 0$, and $\Sigma_{k=1}^{\infty} |a_{nk}|^{\delta} \le Kn^{\alpha(2-\delta)-1-\xi}$ for some $0 < \delta < 2$ and $\xi > 0$, then T_n converges completely to zero.

iii) If $E|D_k|^{(1+\alpha+\beta)/\alpha}(\log^+|D_k|)^{1+\xi} \le K$ for some $\xi > 0$, $1 \le (1 + \alpha + \beta)/\alpha < 2$, $A_n \le Kn^{\beta-\alpha}$, $\Sigma_{k=1}^{\infty} |a_{nk}|^{(1+\alpha+\beta)/\alpha} \le Kn^{-\gamma}$ for some $\gamma > 0$ and either $\Sigma_{k=1}^{\infty} |a_{nk}| \le K$ and $ED_k = 0$ or $\Sigma_{k=1}^{\infty}|a_{nk}| \to 0$ as $n \to \infty$, then T_n converges completely to zero.

iv) If $E|D_k|^{(1+\alpha+\beta)/\alpha}(\log^+|D_k|)^{1+\xi} \le K$ for some $\xi > 0$, $0 < (1 + \alpha + \beta)/\alpha < 1$, $A_n \le Kn^{\beta-\alpha}$, $\Sigma_{k=1}^{\infty} |a_{nk}|^{(1+\alpha+\beta)/\alpha} \le Kn^{-\gamma}$ for some $\gamma > 0$ and $a_{nk} = 0$ for $k > n^{\zeta}$ where $\zeta < \gamma(1 + \alpha + \beta)/\alpha$ then T_n converges completely to zero.

PROOF. <u>(i) and (ii)</u>. Fix $\epsilon > 0$ and assume without loss of generality that $a_{nk} > 0$. Let $D'_{nk} = D_k\, I[a_{nk}D_k \le n^{-\rho}]$ where $\rho > 0$ is chosen later in the proof, $D''_{nk} = D_k\, I[a_{nk}D_k > \epsilon/N]$ where N is a positive integer to be chosen later, and $D'''_{nk} = D_k - D'_{nk} - D''_{nk}$. Let

$T'_{nm} = \Sigma_{k=1}^{m} a_{nk} D'_{nk}$, $T''_{nm} = \Sigma_{k=1}^{m} a_{nk} D''_{nk}$, and $T'''_{nm} = \Sigma_{k=1}^{m} a_{nk} D'''_{nk}$.

The proof of the facts that T_{nm} converges a.s. to T_n as $m \to \infty$, T'_{nm} converges a.s. to T'_n as $m \to \infty$, $\Sigma_{n=1}^{\infty} P[T'_n > \epsilon] < \infty$, T''_{nm} and T'''_{nm} each converge a.s. to random variables T''_n and T'''_n respectively as $m \to \infty$, and that $\Sigma_{n=1}^{\infty} P[T'''_n > \epsilon] < \infty$ used in the proof of Theorem 1.1 (i) and (ii) in no way depended on the additional assumption of the random variables D_k being identically distributed. Thus the proof of these facts is the same here and we therefore omit repeating their proofs. We thus need only to establish $\Sigma_{n=1}^{\infty} P[T''_n > \epsilon] < \infty$ to complete the proof of (i) and (ii). Applying (3) of Lemma 1.2 with $Z_k = D_k$ yields $\Sigma_{n=1}^{\infty} P[T''_n > \epsilon] \leq \Sigma_{j=1}^{\infty} \Sigma_{n=1}^{g_j} (f_n(j) - f_n(j-1)) \sup_k P[|D_k| \geq j - 1)]$, where $f_n(j)$ and g_j are defined in the statement of Lemma 1.2. We now consider (i). By the Chebychev inequality, $P[|D_k| \geq j-1]$ $\leq P[|D_k|^{(1+\alpha+\beta)/\alpha}(\log^+|D_k|)^{1+\xi} \geq (j-1)^{(1+\alpha+\beta)/\alpha}(\log^+(j-1))^{1+\xi}]$ $\leq K(j-1)^{-(1+\alpha+\beta)/\alpha}(\log^+(j-1))^{-(1+\xi)}$. Likewise $P[|D_k| \geq j-1]$ $\leq P[D_k^2 (\log^+|D_k|)^{2+\xi} \geq (j-1)^2(\log^+(j-1))^{2+\xi}] \leq K(j-1)^{-2} (\log^+(j-1))^{-(2+\xi)}$. To show that $\Sigma_{n=1}^{\infty} P[T''_n > \epsilon] < \infty$, it is sufficient to show that $\Sigma_{j=4}^{\infty} \Sigma_{n=1}^{g_j} (f_n(j) - f_n(j-1)) \sup_k P[|D_k| > j - 1] < \infty$ since $g_j < \infty$ for $j = 1, 2,$ and 3.

We now consider two cases. First, if $(1 + \alpha + \beta)/\alpha = 2$, then $\Sigma_{j=4}^{\infty} \Sigma_{n=1}^{g_j} (f_n(j)-f_n(j-1)) \sup_k P[|D_k| \geq j - 1] \leq K \Sigma_{j=4}^{\infty} \Sigma_{n=1}^{g_j}$ $(f_n(j)-f_n(j - 1))(j - 1)^{-2} (\log(j - 1))^{-(2+\xi)}$. By the definitions of A_n and $f_n(j)$, it is clear that $A_n \geq f_n(j) \epsilon^2/(Nj)^2$. Since $A_n \leq Kn^{\beta-\alpha}$ by hypothesis, we conclude that $f_n(j) \leq (Kn^{\beta-\alpha} N^2 j^2)/\epsilon^2$. Inversion of the order of summation and summation by parts thus yields

$\Sigma_{j=4}^{\infty} \Sigma_{n=1}^{g_j} (f_n(j)-f_n(j-1)) \sup_k P[|D_k| > j-1] \leq K^2N^2/\epsilon^2$
$\Sigma_{j=4}^{\infty} j^2 \quad ((j-1)^{-2}(\log(j-1))^{-(2+\xi)}-j^{-2}(\log j)^{-(2+\xi)}) \Sigma_{n=1}^{g_j} n^{\beta-\alpha}$.
Since $\Sigma_{n=1}^{g_j} n^{\beta-\alpha} = \Sigma_{n=1}^{g_j} n^{-1} \leq K' \log j$, where K' is independent of j , it then follows by elementary computation that the preceding sum is finite. Thus $\Sigma_{n=1}^{\infty} P[T_n'' > \epsilon] < \infty$ in Case 1. Secondly, if $(1 + \alpha + \beta)/\alpha > 2$, then $\Sigma_{j=4}^{\infty} \Sigma_{n=1}^{g_j}(f_n(j)-f_n(j-1)) \sup_k P[|D_k| > j-1]$
$\leq K\Sigma_{j=4}^{\infty} \Sigma_{n=1}^{g_j} (f_n(j)-f_n(j-1))(j-1)^{-(1+\alpha+\beta)/\alpha}(\log(j-1))^{-(1+\xi)} \leq K^2N^2/\epsilon^2$
$\Sigma_{j=4}^{\infty} j^2 ((j-1)^{-(1+\alpha+\beta)/\alpha}(\log(j-1))^{-(1+\xi)}-j^{-(1+\alpha+\beta)/\alpha}(\log j)^{-(1+\xi)})$
$\Sigma_{n=1}^{g_j} n^{\beta-\alpha}$. Since $\Sigma_{n=1}^{g_j} n^{\beta-\alpha} \leq K' j^{(\beta-\alpha+1)/\alpha}$, where K' is independent of j, it then follows by elementary computation that the preceding sum is finite. Hence $\Sigma_{n=1}^{\infty} P[T_n'' > \epsilon] < \infty$ in the case of (i).

We now consider (ii). By the Chebychev inequality, $P[|D_k| \geq j - 1] \leq K(j-1)^{-2}$. To show that $\Sigma_{n=1}^{\infty} P[T_n'' > \epsilon] < \infty$, it is sufficient to show that $\Sigma_{j=2}^{\infty} \Sigma_{n=1}^{g_j}(f_n(j)-f_n(j-1)) \sup_k P[|D_k| > j-1] \leq$

$$K\Sigma_{j=2}^{\infty} \Sigma_{n=1}^{g_j} (f_n(j) - f_n(j-1))(j-1)^{-2} < \infty .$$

Since $\Sigma_{n=1}^{\infty} |a_{nk}|^\delta \leq K n^{\alpha(2-\delta)-1-\xi}$, it follows that $f_n(j) \leq Kn^{\alpha(2-\delta)-1-\xi}(Nj/\epsilon)^\delta$.
Inversion of the order of summation and summation by parts shows it is sufficient to show $\Sigma_{j=2}^{\infty} j^\delta((j-1)^{-2} -j^{-2}) \Sigma_{n=1}^{g_j} n^{\alpha(2-\delta)-1-\xi} < \infty$. Since $\Sigma_{n=1}^{g_j} n^{\alpha(2-\delta) - 1 - \xi} \leq K' j^{2-\delta-\xi/\alpha}$, where K' is independent of j , it follows by elementary computation that the preceding sum is finite. Combining the above results yields $\Sigma_{n=1}^{\infty}P[T_n > 3\epsilon] < \infty$. By symmetry $\Sigma_{n=1}^{\infty} P[|T_n| > 3\epsilon] < \infty$.

<u>(iii) and (iv)</u>. Fix $\epsilon > 0$. Let $D'_{nk} = D_k I[|a_{nk} D_k| \leq n^{-\rho}]$ where $\rho > 0$ is to be chosen later, $D''_{nk} = D_k I[|a_{nk} D_k| > \epsilon/N]$ where N is

a positive integer to be chosen later, and $D'''_{nk} = D_k - D'_{nk} - D''_{nk}$. Let $T'_{nm} = \Sigma_{k=1}^{m} a_{nk} D'_{nk}$, $T''_{nm} = \Sigma_{k=1}^{m} a_{nk} D''_{nk}$, and $T'''_{nm} = \Sigma_{k=1}^{m} a_{nk} D'''_{nk}$. The proof of the facts that T_{nm} converges a.s. to T_n as $m \to \infty$, T'_{nm} converges a.s. to T'_n as $m \to \infty$, $\Sigma_{n=1}^{\infty} P[|T'_n| > \epsilon] < \infty$, T''_{nm} and T'''_{nm} each converge a.s. to random variables T''_n and T'''_n respectively as $m \to \infty$, and $\Sigma_{n=1}^{\infty} P[|T'''_n| > \epsilon] < \infty$ used in the proof of Theorem 1.1 (iii) and (iv) in no way depended on the additional assumption of the random variables being identically distributed. Thus the proofs of these facts are the same here and we therefore omit them. We thus need only to establish $\Sigma_{n=1}^{\infty} P[|T''_n| > \epsilon] < \infty$ to complete the proof of (iii) and (iv). Applying (3) of Lemma 1.2 with $Z_k = D_k$ yields $\Sigma_{n=1}^{\infty} P[|T''_n| > \epsilon] \leq \Sigma_{j=1}^{\infty} \Sigma_{n=1}^{g_j} (f_n(j) - f_n(j-1)) \sup_k P[|D_k| \geq j - 1]$, where $f_n(j)$ and g_j are defined in the statement of Lemma 1.2. By the Chebychev inequality, $P[|D_k| \geq j - 1] \leq P[|D_k|^{(1+\alpha+\beta)/\alpha}(\log^+|D_k|)^{1+\xi} \geq (j-1)^{(1+\alpha+\beta)/\alpha}(\log^+(j-1))^{1+\xi}] \leq K(j-1)^{-(1+\alpha+\beta)/\alpha}(\log^+(j-1))^{-(1+\xi)}$. To show that $\Sigma_{n=1}^{\infty} P[|T''_n| > \epsilon] < \infty$, it is sufficient to show that $\Sigma_{j=4}^{\infty} \Sigma_{n=1}^{g_j}(f_n(j)-f_n(j-1)) \sup_k P[|D_k| > j - 1] \leq K\Sigma_{j=4}^{\infty} \Sigma_{n=1}^{g_j}(f_n(j)-f_n(j-1)) (j-1)^{-(1+\alpha+\beta)/\alpha}(\log^+(j+1))^{-(1+\xi)} < \infty$ since g_j is finite. By the definitions of A_n and $f_n(j)$, $A_n \geq f_n(j)\, \epsilon^2/(Nj)^2$. Since $A_n \leq K\, n^{\beta-\alpha}$ by hypothesis, we conclude that $f_n(j) \leq (Kn^{\beta-\alpha}N^2 j^2)/\epsilon^2$. Inversion of the order of summation and summation by parts then show it is sufficient to show that $\Sigma_{j=3}^{\infty} j^2((j-1)^{-(1+\alpha+\beta)/\alpha}(\log(j-1))^{-(1+\xi)} - j^{-(1+\alpha+\beta)/\alpha} (\log j)^{-(1+\xi)}) \Sigma_{n=1}^{g_j} n^{\beta-\alpha} < \infty$. Since $\Sigma_{n=1}^{g_j} n^{\beta-\alpha} \leq K'\, j^{(1+\alpha+\beta)/\alpha}$ where K' is independent of j , it follows that the preceding sum is finite. Hence $\Sigma_{n=1}^{\infty} P[|T''_n| > \epsilon] < \infty$ and $\Sigma_{n=1}^{\infty} P[|T_n| > 3\,\epsilon] < \infty$. The result is established.

REMARKS. The assumption of the D_k's being identically distributed can obviously be dropped from the statement of Corollaries 1.1 - 1.3 in an analogous manner as done above in Theorem 1.2. Statements of these corollaries are therefore omitted. Theorem 1.2 was motivated by a result given by Chow ([6], p. 1489) for the non-identically distributed case. His result states that if $ED_k = 0$, $E|D_k|^{(1+\lambda)/\alpha} (\log^+|D_k|)^2 \leq K$, $|a_{nk}| \leq K A_n$ for $k \leq n^\lambda$, $a_{nk} = 0$ for $k > n^\lambda$, and $A_n \leq Kn^{-\alpha}$ where $\lambda \geq 1$ and $0 < \alpha \leq 1$, then T_n converges completely to zero. Theorem 1.2 extends this result by treating the case where rows of the a_{nk} matrix may have infinitely many non-zero entries and the case where the second moments of the D_k's may be infinite. The moment condition given in Theorem 1.2 is sharper (except for the special case $\alpha = \lambda = 1$ where the Chow result is sharper).

CHAPTER II

SOME CONVERGENCE PROPERTIES OF GENERALIZED GAUSSIAN RANDOM VARIABLES

1. Introduction

Throughout Chapter II, we let $(\Omega, \mathcal{F}, P)$ be a probability space with $(\mathcal{F}_{nk}, k \geq 1)$ an increasing sequence of σ-fields for each fixed $n \geq 1$ with $\mathcal{F}_{nk} \subset \mathcal{F}$. Let $(D_{nk}, \mathcal{F}_{nk}, k \geq 1)$ be a martingale difference sequence for each fixed $n \geq 1$, i.e., each D_{nk} is $\mathcal{F}_{nk}$ measurable and $E(D_{nk} | \mathcal{F}_{n,k-1}) = 0$ a.s. for all $k \geq 2$ for each $n \geq 1$. Let $(D_n, n \geq 1)$ be a sequence of independent random variables. Let a_{nk} and a_n be a matrix and a sequence of real numbers respectively. Let $A_n = \Sigma_{k=1}^{\infty} a_{nk}^2$, $T_{nm} = \Sigma_{k=1}^{m} a_{nk} D_{nk}$ and T_n be the a.s. limit of T_{nm} as $m \to \infty$ whenever it exists. As in Chapter I, T_n is said to converge completely to zero in the sense of Hsu and Robbins [5] if $\Sigma_{n=1}^{\infty} P[|T_n| > \epsilon] < \infty$ for all $\epsilon > 0$. According to Chow [6], a random variable D is generalized Gaussian if there exists an $\alpha \geq 0$ such that for every real t, $E \exp(tD) \leq \exp(t^2 \alpha^2/2)$. The minimum of these numbers α is denoted by $\tau(D)$. Special cases of generalized Gaussian random variables include normal and bounded random variables each with mean zero. (See [6], p. 1482). We denote the variance of a random variable D by Var (D), its symmetrized version by D^s ([22], p. 245), the nth derivative of a function f by $f^{(n)}$, and the indicator function of a set A by $I[A]$.

In this chapter, we prove a number of almost sure convergence results concerning sums of generalized Gaussian random variables. Lemmas 2.1 - 2.3 give some elementary properties of generalized Gaussian random variables which are useful in studying their convergence properties.

In Chapter I, certain convergence results were established for linear combinations of independent random variables. If in addition to independence the random variables are assumed to be generalized Gaussian, Chow ([6], p. 1483) has shown that a sharper analysis is possible. In Theorem 2.1, we extend a result of Chow about the complete convergence of linear combinations of generalized Gaussian random variables to the martingale case. The heart of the proof of Theorem 2.1 is the establishment of Lemma 2.4 which states that T_n is generalized Gaussian with $\tau^2(T_n) \leq 2A_n$ when each D_{nk} is generalized Gaussian in a certain conditional sense with $\tau(D_{nk}) \leq 1$.

Theorem 2.2 and Corollary 2.3 of Theorem 2.3 each give conditions under which $\Sigma_{n=1}^{\infty} a_n D_n$ converging a.s. implies that $\Sigma_{n=1}^{\infty} a_n^2 ED_n^2 < \infty$ where the D_n are each generalized Gaussian. Corollary 2.2 of Theorem 2.3 gives conditions under which $\Sigma_{n=1}^{\infty} R_n D_n$ converging a.s. implies that $\Sigma_{n=1}^{\infty} R_n^2 < \infty$ a.s. where the D_n are generalized Gaussian and $(R_n, n \geq 1)$ is a sequence of independent random variables independent of the D_n also. Theorem 2.3 is related to a result of Marcinkiewicz and Zygmund ([23], p. 72). Example 2.1 concluding the chapter indicates that the restrictions on $\tau(D_n)$ given in the hypotheses of Theorem 2.2 and Corollary 2.3 may not be dropped entirely, although these restrictions can possibly be relaxed somewhat.

2. Preparatory Lemmas and Elementary Properties of Generalized Gaussian Random Variables

LEMMA 2.1. Let D be a generalized Gaussian random variable with $\tau(D) \leq \alpha$. Then $ED = 0$, $ED^2 \leq \alpha^2$, and $|ED^n| \leq n^{-n/2} n!\, \alpha^n \exp(n/2)$.

PROOF. $E \exp(t\,D) \leq \exp(t^2\,\alpha^2/2)$ implies that $1 + t\,ED + (t^2/2!)\,ED^2 + \ldots + \leq 1 + t^2\,\alpha^2/2 + (t^2\,\alpha^2/2)^2/2! + \ldots +$. Subtracting 1 and dividing by t on both sides and then letting $t \downarrow 0$ and $t \uparrow 0$ yields $ED = 0$. Dividing both sides again by t and letting $t \to 0$ yields $ED^2 \leq \alpha^2$. Now $|ED^n| = |(E\exp(tD))^{(n)}|_{t=0}|$

$\leq (n!\rho\,/\,(2\Pi)) \int_0^{2\Pi} \frac{E|\exp(\rho\,e^{i\theta})\,D|}{\rho^{n+1}}\,d\theta$ since $E\exp(t\,D)$ is an entire function of t by hypothesis. $|\exp(\rho e^{i\theta}D)| = \exp(\rho\cos\theta)D$. Thus $E|\exp(\rho e^{i\theta})D| = E\exp(\rho\cos\theta D) \leq \exp(\rho^2\cos^2\theta\,\alpha^2/2) \leq \exp(\rho^2\,\alpha^2/2)$. Thus $|ED^n| \leq \rho^{-n}\,n!\,\exp(\rho^2\,\alpha^2/2)$ for all $\rho > 0$. The optimum choice of ρ is $\rho = \sqrt{n}/\alpha$ according to elementary calculus. Thus $|ED^n| \leq n^{-n/2}\,n!\,\alpha^n \exp(n/2)$.

LEMMA 2.2. Let D be a generalized Gaussian random variable with $\tau(D) \leq \alpha$ and $ED^2 = \beta^2$. Then

$$(1) \qquad (\beta^2 - 2\epsilon^2 - 4\alpha^2 \exp(-\gamma^2/(2\alpha^2)))\,/\,(2\gamma(\gamma-\epsilon)) \leq P[|D| > \epsilon] \leq 2\exp(-\epsilon^2/(2\alpha^2)) \text{ for all constants } \gamma \geq \epsilon > 0 .$$

PROOF. The right hand side of the inequality follows easily by a Chebychev argument and is stated in [6], p. 1483. $\beta^2 = ED^2 = 2\int_0^\infty x\,P[|D| > x]\,dx = 2[(\int_0^\epsilon + \int_\epsilon^\gamma + \int_\gamma^\infty)\,x\,P[|D| > x]\,dx] \leq 2\epsilon^2 + 2(\gamma-\epsilon)\gamma\,P[|D| > \epsilon] + 4\int_\gamma^\infty x\exp(-x^2/(2\alpha^2))\,dx = 2\epsilon^2 + 2(\gamma-\epsilon)\gamma\,P[|D| > \epsilon] + 4\alpha^2\exp(-\gamma^2/(2\alpha^2))$. Solving the inequality above for $P[|D| > \epsilon]$

LEMMA 2.3. $E \sup_{n\geq 1} D_n^2 < \infty$ if and only if $\Sigma_{n=1}^{\infty} \Sigma_{k=1}^{\infty} P[D_n^2 > k] < \infty$.

PROOF. $E[\sup_{n\geq 1} D_n^2] < \infty$ if and only if $\Sigma_{k=1}^{\infty} P[\sup_{n\geq 1} D_n^2 > k] < \infty$ if and only if $\Sigma_{k=1}^{\infty} (1- \prod_{n=1}^{\infty} P[D_n^2 \leq k]) < \infty$ if and only if $\prod_{k=1}^{\infty} \prod_{n=1}^{\infty} P[D_n^2 \leq k]$ converges if and only if $\Sigma_{k=1}^{\infty} \Sigma_{n=1}^{\infty} P[D_n^2 > k] < \infty$. Here we use the fact that $\Sigma_{n=1}^{\infty} a_n$ converges if and only if $\prod_{n=1}^{\infty} (1-a_n)$ converges for $a_n \geq 0$ ([18], p. 220).

LEMMA 2.4. Let $E(\exp(t\, D_{nk})|\mathcal{F}_{n,k-1}) \leq \exp(t^2)$ a.e. for every constant t and let $A_n < \infty$. Then T_n is generalized Gaussian with $\tau^2(T_n) \leq 2A_n$.

PROOF. For fixed t and m, let $Y_j = \exp(t\, T_{nj} + t^2 \Sigma_{k=j+1}^{m} a_{nk}^2)$ for $j = 1,2,\ldots, m$, using the convention $\Sigma_{k=m+1}^{m} (\cdot) = 0$. Choose $j \neq 1$. $E(Y_j|\mathcal{F}_{n,j-1}) = \exp(t\, T_{n,j-1} + t^2 \Sigma_{k=j+1}^{m} a_{nk}^2)\, E \exp(t\, a_{nj}\, D_{nj}|\mathcal{F}_{n,j-1})$ $\leq \exp(t\, T_{n,j-1} + \Sigma_{k=j}^{m} a_{nk}^2\, t^2) = Y_{j-1}$ a.e. Hence $EY_j \leq EY_{j-1}$. By induction, $EY_m \leq EY_1$, $EY_m = E \exp(t\, T_{nm})$ and $EY_1 = E \exp(t\, a_{n1}\, D_{n1})$ $\exp(t^2 \Sigma_{k=2}^{m} a_{nk}^2) \leq \exp(t^2 \Sigma_{k=1}^{m} a_{nk}^2) \leq \exp(t^2 A_n)$. Hence $E \exp(t\, T_{nm})$ $\leq \exp(t^2 A_n)$. By Lemma 2.1, $ED_{nk}^2 \leq 2$. $(E|T_{nm}|)^2 \leq E\, T_{nm}^2 + 1 =$ $\Sigma_{k=1}^{m} a_{nk}^2\, ED_{nk}^2 + 1 \leq 2A_n + 1$. Hence T_{nm} converges a.s. to T_n as $m \to \infty$ by the Doob martingale convergence theorem ([8], p. 319). It then follows by an application of the Fatou lemma that

$E \exp(t\, T_n) \leq \exp(t^2 A_n)$.

REMARK. The manner of constructing the Y_j's so that they form a supermartingale was learned from a paper of Dubins and Freedman ([9], p. 804). A slightly different proof can be given which does not use this technique.

3. Convergence

THEOREM 2.1. Let $E(\exp(t\, D_{nk})|\mathcal{F}_{n,k-1}) \leq \exp(t^2)$ a.e. for every constant t, $A_n < \infty$, and $\Sigma_{n=1}^{\infty} \exp(-\lambda/A_n) < \infty$ for all $\lambda > 0$. Then T_n converges completely to zero.

PROOF. By Lemma 2.4, T_n is generalized Gaussian with $\tau^2(T_n) \leq 2A_n$. Thus, by Lemma 2.2, $P[|T_n| > \epsilon] \leq 2 \exp(-\epsilon^2 / (4A_n))$ for all $\epsilon > 0$. Thus $\Sigma_{n=1}^{\infty} P[|T_n| > \epsilon] < \infty$ for all $\epsilon > 0$.

REMARK. If $(D_{nk}, k \geq 1)$ is an independent sequence for each fixed n, the result reduces to a theorem due to Chow ([6], p. 1483). The key step in the proof of Theorem 2.1 is the establishment of Lemma 2.4. After that, the proof is almost identical to that given by Chow in the independent case.

COROLLARY 2.1. Let $|D_{nk}| \leq K$ a.e. for some $K < \infty$, $A_n < \infty$, $\Sigma_{n=1}^{\infty} \exp(-\lambda/A_n) < \infty$ for all $\lambda > 0$. Then T_n converges completely to zero.

PROOF. Without loss of generality, we assume $K = 1$. It is sufficient to show that $E(\exp(t\, D_{nk})|\mathcal{F}_{n,k-1}) \leq \exp(t^2)$ a.e. for each t and $k \geq 2$ and $n \geq 1$. Consider $t > 0$. If $t \geq 1$, then $\exp(t\, D_{nk}) \leq \exp(t^2)$ a.e. Consider $0 < t < 1$. $\operatorname{Exp}(t\, D_{nk}) \leq 1 + t\, D_{nk} + \Sigma_{n=2}^{\infty} t^n/n!$ $\leq 1 + t\, D_{nk} + t^2$ a.e. Hence $E \exp(t\, D_{nk}) \leq 1 + t^2 \leq \exp(t^2)$ for $0 < t < 1$. Thus $E(\exp(t\, D_{nk})|\mathcal{F}_{n,k-1}) \leq \exp(t^2)$ a.e. for each $t > 0$, for each $t < 0$ by symmetry, and for $t = 0$ trivially.

REMARKS. As stated in Chapter I., it is easy to check that $A_n = o(\log^{-1} n)$ implies that $\Sigma_{n=1}^{\infty} \exp(-\lambda/A_n) < \infty$ for all $\lambda > 0$. It is also true that $A_n = O(\log^{-1} n)$ does not in general imply that

$\Sigma_{n=1}^{\infty} \exp(-\lambda/A_n) < \infty$ for all $\lambda > 0$. The condition $A_n = o(\log^{-1} n)$ is easy to verify in practice and by the above remark only slightly stronger than $\Sigma_{n=1}^{\infty} \exp(-\lambda/A_n) < \infty$ for all $\lambda > 0$. Corollary 2.1 has been proved by Hill ([4], p. 405) in the special case where $(D_{nk}, k \geq 1)$ are independent for each fixed n and $P[D_{nk} = 1] = P[D_{nk} = -1] = 1/2$. Even for this case, Erdos ([4], p. 404) gives an example which shows that $\Sigma_{n=1}^{\infty} \exp(-\lambda/A_n) < \infty$ for all $\lambda > 0$ cannot be replaced by $A_n = O(\log^{-1} n)$. Hence the statement of Theorem 1 is rather sharp. As an example of the application of Theorem 1, one may take $a_{nk} = 1/(n^{1/2}(\log n)^{1/2+\delta})$ for $k \leq n$ and $a_{nk} = 0$ for $k > n$, where $\delta > 0$. Then it follows that $E(\exp(t\, D_{nk})|\mathcal{F}_{n,k-1}) \leq \exp(t^2)$ a.e. implies that $T_n \equiv \Sigma_{k=1}^{n} D_{nk} / (n^{1/2}(\log n)^{1/2+\delta})$ converges completely to zero. This example is given by Chow in the independent case ([1], p. 1484). The papers by Hill [4] and Chow [1] may be referred to for other applications of Theorem 2.1 since the examples given there still apply in the general martingale case.

THEOREM 2.2. Let D_n be generalized Gaussian with $\tau(D_n) \leq \alpha_n < \infty$ and $\Sigma_{n=1}^{\infty} \exp(-1/\alpha_n^2) < \infty$. Then $\Sigma_{n=1}^{\infty} D_n$ converges a.s. implies that $\Sigma_{n=1}^{\infty} ED_n^2 < \infty$.

PROOF. According to a result of Doob ([8], p. 339), $\Sigma_{n=1}^{\infty} D_n$ converges a.s., $E \sup_{n\geq 1} |D_n|^2 < \infty$ implies that $\Sigma_{n=1}^{\infty} ED_n^2 < \infty$. Hence it suffices to show that $E \sup_{n\geq 1}|D_n|^2 < \infty$. By Lemma 2.3, it suffices to show that $\Sigma_{n=1}^{\infty} \Sigma_{k=1}^{\infty} P[D_n^2 > k] < \infty$. According to Lemma 2.2, $P[D_n^2 > k] \leq 2\exp(-k/(2\alpha_n^2))$. Hence it suffices to show that $\Sigma_{n=1}^{\infty} \Sigma_{k=1}^{\infty} \exp(-k/(2\alpha_n^2)) < \infty$. $\Sigma_{n=1}^{\infty} \Sigma_{k=1}^{\infty} \exp(-k/2\alpha_n^2)) = \Sigma_{n=1}^{\infty} \exp(-1/2\alpha_n^2))\, 1/(1-\exp(-1/(2\alpha_n^2)))$.

But $\alpha_n \to 0$ by hypothesis and hence $1/(1-\exp(-1/(2\alpha_n^2))) \to 1$. But $\Sigma_{n=1}^{\infty} \exp(-1/(2\alpha_n^2)) < \infty$ by hypothesis.

REMARK. As remarked above, $\alpha_n^2 = o(\log^{-1} n)$ implies that $\Sigma_{n=1}^{\infty} \exp(-1/(2\alpha_n^2)) < \infty$. This condition is easier to verify in practice.

THEOREM 2.3. Let the D_n be symmetric with $P[|D_n| > \delta] > \rho$ for some $\delta > 0$ and $\rho > 0$. Let $(R_n, n \geq 1)$ be a sequence of independent random variables independent of the D_n also. Then $\Sigma_{n=1}^{\infty} R_n D_n$ converging a.s. implies that $\Sigma_{n=1}^{\infty} R_n^2 < \infty$ a.s.

PROOF. It is easy to show that $\Sigma_{n=1}^{\infty} R_n^2 D_n^2 < \infty$ a.s. by the three series theorem (See Section 3 Chapter III of thesis). Now $R_n^2 D_n^2 \geq \delta^2 R_n^2 I[|D_n| > \delta]$. Thus $\Sigma_{n=1}^{\infty} R_n^2 I[|D_n| > \delta] < \infty$ a.s. Fix $c > 0$. By the three series theorem $\infty > \Sigma_{n=1}^{\infty} P[R_n^2 I[|D_n| > \delta] > c] = \Sigma_{n=1}^{\infty} P[R_n^2 > c, |D_n| > \delta] = \Sigma_{n=1}^{\infty} P[R_n^2 > c] P[|D_n| > \delta] > \rho \Sigma_{n=1}^{\infty} P[R_n^2 > c]$. By the three series theorem, $\infty > \Sigma_{n=1}^{\infty} E[R_n^2 I[|D_n| > \delta] I[R_n^2 I[|D_n| > \delta] < c]] = \Sigma_{n=1}^{\infty} E[R_n^2 I[|D_n| > \delta] I[R_n^2 < c]] = \Sigma_{n=1}^{\infty} E[R_n^2 I[R_n^2 < c]] P[|D_n| > \delta]$
$\geq \rho \Sigma_{n=1}^{\infty} E[R_n^2 I[R_n^2 < c]]$. By the three series theorem,

$$\infty > \Sigma_{n=1}^{\infty} \text{Var}\, [R_n^2 I[|D_n| > \delta] I[R_n^2 I[|D_n| > \delta] < c]]$$
$$= \Sigma_{n=1}^{\infty} E[R_n^4 I[|D_n| > \delta] I[R_n^2 < c]] - E^2 [R_n^2 I[|D_n| > \delta] I[R_n^2 < c]]$$
$$= \Sigma_{n=1}^{\infty} P[|D_n| > \delta] E[R_n^4 I[R_n^2 < c] - \Sigma_{n=1}^{\infty} P^2[|D_n| > \delta] E^2[R_n^2 I[R_n^2 < c]]$$
$$\geq \rho \Sigma_{n=1}^{\infty} E[(R_n^2)^2 I[R_n^2 < c] - P[|D_n| > \delta] E^2[R_n^2 I[R_n^2 < c]]$$
$$\geq \rho \Sigma_{n=1}^{\infty} E[(R_n^2)^2 I[R_n^2 < c] - E^2[R_n^2 I[R_n^2 < c]] = \rho \Sigma_{n=1}^{\infty} \text{Var}[R_n^2 I[R_n^2 < c]].$$

Hence by the three series theorem, $\Sigma_{n=1}^{\infty} R_n^2 < \infty$ a.s.

REMARKS. We cannot drop the assumption of symmetry even with $ED_n = 0$ as the following simple example shows. Let $(D_n, n \geq 1)$ be independent with $P[D_n = (-1)^n] = (n^2-1)/n^2$ and $P[D_n = (-1)^{n+1}(n^2-1)] = 1/n^2$.

Let $(R_n, n \geq 1)$ be independent with $(R_n, n \geq 1)$ independent of the D_n also and with $P[R_n = 1/n^{1/2}] = (n^2 - 1)/n^2$ and $P[R_n = -(n^2 - 1)/n^{1/2}] = 1/n^2$. Note $ED_n = ER_n = 0$, $\sum_{n=1}^{\infty} R_n^2 = \infty$, $P[|D_n| > 1/2] \geq P[|D_n| \geq 1] \geq 1/2$, and $\Sigma_{n=1}^{\infty} D_n R_n$ converges a.s. This establishes the example.

The following theorem is stated by Marcinkiewicz and Zygmund ([23], p. 72).

(2) Let $ED_n^2 = 1$, $ED_n = 0$, $P[|D_n| > \delta] > \rho$

for some $\delta > 0$ and $\rho > 0$ and $\Sigma_{n=1}^{\infty} a_n D_n$ converge a.s. Then $\sum_{n=1}^{\infty} a_n^2 < \infty$.

Theorem 2.3 is related to (2). The sequence of constants $(a_n, n \geq 1)$ occurring in (2) is replaced by the sequence of independent random variables $(R_n, n \geq 1)$ occurring in Theorem 2.3. Also, the assumption that $ED_n^2 = 1$ in (2) is replaced by the assumption that the random variables $(D_n, n \geq 1)$ are symmetric. As indicated, the example just given shows that the assumption of symmetry in Theorem 2.3 is essential. However, if we assume in addition in Theorem 2.3 that $ED_n^2 = 1$ then even in the non-symmetric case, the result follows immediately from (2) by a Fubini argument.

COROLLARY 2.2. Let the D_n be generalized Gaussian with $\tau(D_n) \leq C < \infty$ and $ED_n^2 = 1$. Let $(R_n, n \geq 1)$ be a sequence of independent random variables independent of the D_n also. Then $\Sigma_{n=1}^{\infty} R_n D_n$ converging a.s. implies that $\Sigma_{n=1}^{\infty} R_n^2 < \infty$ a.s.

PROOF. It suffices to show that $P[|D_n^s| > \delta] > \rho$ for some positive constants δ and ρ by Theorem 2.3. But this follows immediately from the left hand side of the inequality in Lemma 2.2 with $\beta = 1$, $\epsilon = 1/2$, and γ chosen sufficiently large since $(R_n^s, n \geq 1)$ are generalized Gaussian with $\tau(D_n^s) \leq \sqrt{2}C$.

REMARK. The special case of Corollary 2.2 where the D_n's are coin tossing random variables is well-known. Corollary 2.2 can also be gotten directly from (2) using a Fubini argument.

COROLLARY 2.3. Let D_n be generalized Gaussian with $ED_n^2 = 1$ and $\tau(D_n) \leq C < \infty$. Then $\Sigma_{n=1}^{\infty} a_n D_n$ converges a.s. implies that $\Sigma_{n=1}^{\infty} a_n^2 < \infty$.

PROOF. The result follows immediately from Corollary 2.2 with $R_n = a_n$.

REMARK. We may give an alternate proof of Corollary 2.3 by noting that $ED_n^4 \leq B < \infty$ for some constant $B < \infty$ by Lemma 2.1 which implies that $(D_n^2, n \geq 1)$ are uniformly integrable. The result then follows from a theorem of Kac and Steinhaus [17].

COROLLARY 2.4. Let D_n be generalized Gaussian with $\tau(D_n) \leq C\sqrt{ED_n^2}$ for some $C < \infty$. Then $\Sigma_{n=1}^{\infty} a_n D_n$ converges a.s. implies that $\Sigma_{n=1}^{\infty} a_n^2 ED_n^2 < \infty$.

PROOF. Corollary 2.4 is merely a restatement of Corollary 2.3. For, $E(D_n/\sqrt{ED_n^2})^2 = 1$, $\tau(D_n/\sqrt{ED_n^2}) = \tau(D_n)/\sqrt{ED_n^2} \leq C < \infty$, $\Sigma_{n=1}^{\infty} (a_n\sqrt{ED_n^2})(D_n/\sqrt{ED_n^2}) = \Sigma_{n=1}^{\infty} a_n D_n$ converges a.s., and $\Sigma_{n=1}^{\infty} (a_n\sqrt{ED_n^2})^2 = \Sigma_{n=1}^{\infty} a_n^2 ED_n^2$. The result is established.

We now give an example related to Corollary 2.3 and Theorem 2.2. Looked at from the viewpoint of Theorem 2.2, the example shows that D_n

generalized Gaussian with $\tau(D_n) \leq 1$ and $\Sigma_{n=1}^{\infty} D_n$ converging a.s. does not imply that $\Sigma_{n=1}^{\infty} D_n^2 < \infty$ a.s. Looked at from the viewpoint of Corollary 2.3 the example shows that D_n generalized Gaussian, $ED_n^2 = 1$, $\tau(D_n) \leq \sqrt{n \log n}$ and $\Sigma_{n=1}^{\infty} a_n D_n$ converging a.s. does not imply that $\Sigma_{n=1}^{\infty} a_n^2 < \infty$.

EXAMPLE 2.1. Let $P[D_n = + \log \log n] = P[D_n = - \log \log n] = p_n$ and $P[D_n = 0] = 1-2p_n$ where $p_n = 1/(n \log n (\log \log n)^{3/2})$ for $n \geq 2$. Fix $c > 0$. Then $\Sigma_{n=2}^{\infty} p_n < \infty$. Hence $\Sigma_{n=2}^{\infty} P[|D_n| > c] < \infty$. $\Sigma_{n=2}^{\infty} ED_n I[|D_n| < c] = 0$ by symmetry. Clearly $\Sigma_{n=2}^{\infty} \mathrm{Var}(D_n I(|D_n| < c)) < \infty$. Thus $\Sigma_{n=2}^{\infty} D_n$ converges a.s. $\Sigma_{n=2}^{\infty} ED_n^2 = \Sigma_{n=2}^{\infty} p_n(\log \log n)^2 = \infty$. We claim $E \exp(t D_n) \leq \exp(t^2/2)$ for all constants t for n sufficiently large, i.e., that D_n is generalized Gaussian with $\tau(D_n) \leq 1$ for n sufficiently large. $E \exp(t D_n) = 1-2p_n + p_n(\exp(t \log \log n) + \exp(-t \log \log n))$. Consider the case $t \geq 2$. Since $1 \leq \exp(t^2/4)$, $\exp(t^2/2) - \exp(t^2/4) \geq \exp(t^2/4)$, and $(\log n)^t > (\log n)^{-t}$, it suffices to prove that $2 \exp(t \log \log n) \leq p_n^{-1} \exp(t^2/4)$, i.e., that $\exp(t \log \log n - t^2/4) \leq n \log n(\log \log n)^{3/2}/2$. By differentiation, it suffices to show that $\exp[(\log \log n)^2] \leq n \log n(\log \log n)^{3/2}/2$, i.e. that $(\log \log n)^2 \leq \log(n \log n(\log \log n)^{3/2}/2)$. This clearly holds for n sufficiently large. Consider now the case $0 \leq t < 2$. $1-2p_n + 2p_n(\exp(t \log \log n) + \exp(-t \log \log n))/2 \leq 1-p_n + 2p_n(\exp(t^2(\log \log n)^2)$ since $\exp(x) + \exp(-x) \leq 2 \exp(x^2)$ for all x. Thus it suffices to show that $f(t) = 1-2p_n + 2p_n(\exp(t^2(\log \log n)^2) - \exp(t^2/2) \leq 0$ for $0 \leq t < 2$. $f(0) = 0$ $f'(t) = 4t\, p_n(\log \log n)^2 \exp(t^2(\log \log n)^2) - t \exp(t^2/2)$. Thus $f'(t) \leq 0$ if and only if $4p_n(\log \log n)^2 \exp(t^2(\log \log n)^2) \leq \exp(t^2/2)$, if and only if $4(\log\log n)^{1/2} \exp(t^2((\log\log n)^2 - 1/2)) \le n \log n$. Clearly, t sufficiently small implies that $f'(t) \le 0$. Let t be smallest positive value such that $f'(t) = 0$. Then $\exp(t^2((\log\log n)^2 - 1/2))$ $= n \log n \,/\, (4(\log\log n)^{1/2})$, i.e., $t^2(\log\log n)^2 - 1/2 =$ $\log(n \log n \,/\, (4(\log\log n)^{1/2})))$ which implies that $t^2 \ge \log(n \log n \,/\, (4(\log\log n)^{1/2}))) \,/\, (\log\log n)^2 \ge 4 \log n \,/$ $(\log\log n)^2$ for n sufficiently large. Thus, for sufficiently large n, $f'(t) \le 0$ for $0 \le t \le 2$ and hence $f(t) \le 0$ for $0 \le t \le 2$ for sufficiently large n. Hence $E \exp(t\, D_n) \le \exp(t^2/2)$ for all $t \ge 0$ for sufficiently large n. By symmetry, the result holds for all t. Let $D_n' = D_n \,/\, \sqrt{ED_n^2}$. Then $\tau(D_n') \le 1 \,/\, \sqrt{ED_n^2} = (n \log n)^{1/2}$ $(\log\log n)^{-1/4} \le (n \log n)^{1/2}$ for sufficiently large n. Let $a_n = \sqrt{ED_n^2}$. Then $\Sigma_{n=1}^{\infty} a_n D_n'$ converges a.s., $E(D_n')^2 = 1$ and $\tau(D_n') \le \sqrt{n \log n}$ for sufficiently large n and $\Sigma_{n=1}^{\infty} a_n^2 = \infty$. Thus the example is completed.

CHAPTER III

SOME MARTINGALE CONVERGENCE RESULTS

1. Introduction

Throughout Chapter III we let $(\Omega,\mathcal{F},P)$ be a probability space and $(\mathcal{F}_n, n \geq 0)$ be an increasing sequence of σ-fields contained in $\mathcal{F}$ with $\mathcal{F}_0 = (\phi,\Omega)$. We call $(X_n,\mathcal{F}_n,n \geq 1)$ a stochastic sequence provided each X_n is a random variable measurable with respect to $\mathcal{F}_n$. Given a martingale $(X_n,\mathcal{F}_n,n \geq 1)$ we let $D_1 = X_1$, $D_2 = X_2 - X_1,\ldots,$ $D_n = X_n - X_{n-1},\ldots$. We then call $(D_n,\mathcal{F}_n,n \geq 1)$ a martingale difference sequence. Alternately any stochastic sequence $(D_n,\mathcal{F}_n, n \geq 1)$ with $E(D_n|\mathcal{F}_{n-1}) = 0$ for $n \geq 2$ is a martingale difference sequence. By a stopping variable t is meant an extended positive integer valued random variable such that the set $[t = n] \in \mathcal{F}_n$ for each $n \geq 1$. We let $I(A)$ denote the characteristic function of a given set A and A' denote the compliment of A.

In Chapter II, where the emphasis was on the generalized Gaussian structure, one martingale convergence result was established. In Chapter III the main emphasis is on the martingale structure and several martingale convergence results are established.

Recently Szynal [25] (See also Levy's transmittal remarks [20]) has considered the relationship between the condition that

(1) $$\Sigma_{n=1}^{\infty} E(D_n^2 / (b_n^2 + D_n^2)) < \infty$$

and the a.s. convergence of $\Sigma_{n=1}^{m} D_n / b_m$ to zero where $(D_n, n \geq 1)$ is a sequence of independent random variables with $ED_n = 0$ and $(b_n, n \geq 1)$ is a sequence of positive constants increasing to infinity. (1) should be viewed as a weakening of the well known Kolmogorov condition that $\Sigma_{n=1}^{\infty} ED_n^2 / b_n^2 < \infty$, i.e., that the sum of the variances be finite. After stating an example (Example 3.1) which indicates an error in the results given in [25], we prove a martingale convergence theorem (Theorem 3.1) which when specialized yields a corrected version of the results given in [25]. In particular, Corollary 3.1 states that if $(D_n, \mathcal{F}_n, n \geq 1)$ is a stochastic sequence with $\Sigma_{n=1}^{\infty} E(|D_n|^{2r} / (|D_n|^{2r} + b_n^{2r}) | \mathcal{F}_{n-1}) < \infty$ a.s. on a set A for some $1/2 \leq r \leq 1$ and positive constants $(b_n, n \geq 1)$, it follows that $\Sigma_{n=1}^{\infty} (D_n - a_n) / b_n$ converges a.s. to a random variable on A where $a_n = E(D_n I(|D_n| \leq b_n) | \mathcal{F}_{n-1})$.

Recently Chow ([4],[2]) has established with the use of an inequality of Burkholder [1] that $(D_n, \mathcal{F}_n, n \geq 1)$ a martingale difference sequence such that $\Sigma_{n=1}^{\infty} E|D_n|^{2r} / n^{r+1} < \infty$ for some $r \geq 1$ implies that $\Sigma_{n=1}^{m} D_n/m$ converges a.s. to zero as $m \to \infty$. Related to this, Corollary 3.2 of Theorem 3.1 states that if $(D_n, \mathcal{F}_n, n \geq 1)$ is a stochastic sequence such that $\Sigma_{n=1}^{\infty} E(|D_n|^{2r}/(n^{r+1} + |D_n|^{2r})) < \infty$ for some $r \geq 1$ then $\Sigma_{n=1}^{m} (D_n - a_n) / m$ converges a.s. to zero where $a_n = E(D_n I(|D_n|^{2r} \leq n^{r+1}) | \mathcal{F}_{n-1})$.

In the case where $(D_n, n \geq 1)$ is a sequence of independent random variables and $(b_n, n \geq 1)$ is a sequence of positive constants, the easily established Proposition 3.2 characterizes condition (1) as being equivalent to the a.s. finiteness of the sum of squares $\Sigma_{n=1}^{\infty} D_n^2/b_n^2$.

Corollary 3.6 and Proposition 3.3 then give relationships between the a.s. convergence of $\Sigma_{n=1}^{\infty} D_n$ and of $\Sigma_{n=1}^{\infty} D_n^2$ in the independent case.

Let $f_n(r) \equiv f_n = \begin{cases} 1 & \text{if } 0 < r \leq 1 \\ c_n^{1-r} & \text{if } r > 1 \end{cases}$ where $(c_n, n \geq 1)$ is a sequence of positive constants such that $\Sigma_{n=1}^{\infty} c_n^{-1} < \infty$ and let $(D_n, \mathcal{F}_n, n \geq 1)$ be a martingale difference sequence. It is known that $\Sigma_{n=1}^{\infty} E|D_n|^{2r} / f_n < \infty$ for some $r > 0$ implies that $\Sigma_{n=1}^{\infty} D_n$ converges a.s. The result is due to Levy ([19], Theorem 68) for $r = 1$, due to Loeve ([21], p. 286) for $1/2 < r < 1$, due to Chow ([3], p. 555) for $r > 1$, and is trivial for $0 < r \leq 1/2$. Recently Burkholder [1] has proved that $E(\Sigma_{n=1}^{\infty} D_n^2)^{1/2} < \infty$ implies that $\Sigma_{n=1}^{\infty} D_n$ converges a.s. This is an essential improvement of the Levy $(r = 1)$ result mentioned above. Theorem 3.2 states that if $E(\Sigma_{n=1}^{\infty} |D_n|^{2r} / f_n)^{1/2r} < \infty$ for some $r > 0$, then $\Sigma_{n=1}^{\infty} D_n$ converges a.s. Since the result for $0 < r \leq 1/2$ is trivial, for $1/2 < r < 1$ follows immediately from the Burkholder result, and coincides with the Burkholder result for $r = 1$, Theorem 3.2 really only treats the case $r > 1$. For the case $r > 1$, Theorem 3.2 sharpens the Chow result mentioned above. Corollary 3.8 gives a local version of Theorem 3.2.

If $(D_n, n \geq 1)$ is a sequence of independent random variables with $ED_n = 0$ and $E|D_n|^{2r}(\log^+|D_n|)^{1+\epsilon} \leq M$ for constants $M < \infty$, $\epsilon > 0$, and $1/2 \leq r < 1$, then $\Sigma_{n=1}^{m} D_n / m^{1/(2r)}$ converges a.s. to zero according to a result of Chung [7]. Theorem 3.4 and Corollary 3.10 provide an extension of this result to the martingale case.

Given a sequence of random variables $(D_n, n \geq 1)$, let $X_m = \Sigma_{n=1}^m D_n$ define an associated sequence $(X_n, n \geq 1)$. It is known that if $(D_n, n \geq 1)$ is a sequence of independent symmetric random variables with $\sup_{n\geq 1}|X_n| < \infty$ a.s., then X_n converges a.s. to a random variable as $n \to \infty$ ([8], p. 121). According to Corollary 3.11 of Theorem 3.5, if $(D_n, \mathcal{F}_n, n \geq 1)$ is a martingale difference sequence then X_n converges a.s. to a random variable as $n \to \infty$ on the set where $\sup |X_n| < \infty$ a.s. and

$$(2) \qquad \Sigma_{n=1}^{\infty} E(D_n \; I(|D_n| > K)|\mathcal{F}_{n-1})$$

converges a.s. for all integers $K \geq N$ for some integer N. This is a martingale version of the aforementioned result with condition (2) playing the role of symmetry.

Given a stochastic sequence $(D_n, \mathcal{F}_n, n \geq 1)$ and a sequence of positive constants $(a_n, n \geq 1)$ we may form the truncated random variables $D_n' = D_n \; I(|D_n| \leq a_n)$. Then the sequence $(D_n' - E(D_n'| \mathcal{F}_{n-1}), \mathcal{F}_n, n \geq 1)$ is a martingale difference sequence. Let $X_m = \Sigma_{n=1}^m D_n$ and $X_m' = \Sigma_{n=1}^m (D_n' - E(D_n'|\mathcal{F}_{n-1}))$. Then we may investigate the convergence properties of the martingale $(X_n', \mathcal{F}_n, n \geq 1)$ and thereby deduce convergence properties for $(X_n, \mathcal{F}_n, n \geq 1)$. This simple technique was found to be very useful in developing proofs for the results stated in this chapter.

2. The Relationship of Conditions of the Form $\Sigma_{n=1}^{\infty} E(D_n^2 / (b_n^2 + D_n^2)) < \infty$ to a.s. Convergence.

In a recent note by Szynal [25] transmitted with remarks by Levy [20], two results pertaining to the strong law of large numbers for independent random variables are stated:

(3) Let $(D_n, n \geq 1)$ be a sequence of independent random variables with mean zero, $X_n \equiv \Sigma_{m=1}^{n} D_m$ for $n \geq 1$, $(b_n, n \geq 1)$ be a sequence of positive constants with $b_n \uparrow \infty$ and $\Sigma_{n=1}^{\infty} E[D_n^2 / (b_n^2 + D_n^2)] < \infty$. Then X_n / b_n converges a.s. to zero as $n \to \infty$.

(4) Let $(D_n, n \geq 1)$ be a sequence of independent random variables with mean zero, $X_n \equiv \Sigma_{m=1}^{n} D_m$ for $n \geq 1$, and $\Sigma_{n=1}^{\infty} E[|D_n|^{2r} / (n^{r+1} + |D_n|^{2r})] < \infty$ for some $r \geq 1$. Then X_n/n converges a.s. to zero as $n \to \infty$.

The following example shows that the above two results are incorrect as stated.

EXAMPLE 3.1. Let $(D_n, n \geq 3)$ be a sequence of independent random variables such that $P[D_n = n^{9/4}] = 1/n^2$ and $P[D_n = -n^{9/4} / (n^2 - 1)] = (n^2 - 1) / n^2$. Note that $ED_n = 0$ and that $\Sigma_{n=3}^{\infty} E[|D_n|^{2r}/(n^{r+1} + |D_n|^{2r})] = \Sigma_{n=3}^{\infty} E[(|D_n|^{2r} / (n^{r+1} + |D_n|^{2r}))(I(|D_n| > n^2) + I(|D_n| \leq n^2))] \leq \Sigma_{n=3}^{\infty} (P[|D_n| > n^2] + 1 / (1 + n^{1+r/2}((n^2 - 1)/n^2)^{2r})) < \infty$ for $r \geq 1$. Since $\Sigma_{n=3}^{\infty} P[|D_n| > n^2] < \infty$, it follows that there exists an integer valued random variable N such that $n \geq N$ implies that $D_n = -n^{9/4} / (n^2 - 1)$ a.s. Hence D_n and therefore X_n / n diverges to $-\infty$. Setting $r = 1$, it is clear that the hypotheses of (3) are satisfied for the case $b_n = n$. The hypotheses of (4) are satisfied for all $r \geq 1$. Thus the example is established.

The following easily established result for martingales allows us to give corrected versions of (3) and (4) as well as state some other related results.

THEOREM 3.1.

i) Given constants $b_n > 0$, $f_n > 0$, $r \geq 1/2$, and $s > 0$, suppose it is known for any martingale difference sequence $(D_n, \mathscr{F}_n, n \geq 1)$ that $\sum_{n=1}^{\infty} E(|D_n|^{2r} / (f_n b_n^s)) < \infty$ implies $\sum_{n=1}^{\infty} D_n / b_n$ converges a.s. to a random variable. Then it follows for any stochastic sequence $(D_n, \mathscr{F}_n, \mathscr{F} \geq 1)$ that $\sum_{n=1}^{\infty} E(|D_n|^{2r} / (|D_n|^{2r} + f_n b_n^s)) < \infty$ implies that $\sum_{n=1}^{\infty} (D_n - a_n) / b_n$ converges a.s. to a random variable, where $a_n = E(D_n I(|D_n|^{2r} \leq f_n b_n^s)|\mathscr{F}_{n-1})$.

ii) Given constants $b_n > 0$, $f_n > 0$, $r \geq 1/2$ and $s > 0$, suppose it is known for any martingale difference sequence $(D_n, \mathscr{F}_n, n \geq 1)$ that $\sum_{n=1}^{\infty} E(|D_n|^{2r} / (f_n b_n^s)|\mathscr{F}_{n-1}) < \infty$ a.s. on a set B implies that $\sum_{n=1}^{\infty} D_n / b_n$ converges a.s. to a random variable on B. Then it follows for any stochastic sequence $(D_n, \mathscr{F}_n, n \geq 1)$ that $\sum_{n=1}^{\infty} E(|D_n|^{2r} / (|D_n|^{2r} + f_n b_n^s)|\mathscr{F}_{n-1}) < \infty$ a.s. on a set A implies that $\sum_{n=1}^{\infty} (D_n - a_n)/b_n$ converges a.s. to a random variable on A, where $a_n = E(D_n I(|D_n|^{2r} \leq f_n b_n^s)|\mathscr{F}_{n-1})$.

iii) Given constants $0 < b_n \to \infty$, $f_n > 0$, $r \geq 1/2$, and $s > 0$, suppose it is known for any martingale difference sequence $(D_n, \mathscr{F}_n, n \geq 1)$ that $\sum_{n=1}^{\infty} E[|D_n|^{2r} / (f_n b_n^s)] < \infty$ implies that $\sum_{n=1}^{m} D_n / b_m$ converges a.s. to zero as $m \to \infty$. Then it follows for any stochastic sequence $(D_n, \mathscr{F}_n, n \geq 1)$ that $\sum_{n=1}^{\infty} E[|D_n|^{2r} / (|D_n|^{2r} + f_n b_n^s)] < \infty$

implies that $\Sigma_{n=1}^{m} (D_n - a_n) / b_m$ converges a.s. to zero as $m \to \infty$ where $a_n = E(D_n I(|D_n|^{2r} \le f_n b_n^s)|\mathcal{F}_{n-1})$.

iv) Given constants $0 < b_n \to \infty$, $f_n > 0$, $r \ge 1/2$, and $s > 0$, suppose it is known for any martingale difference sequence $(D_n, \mathcal{F}_n, n \ge 1)$ that $\Sigma_{n=1}^{\infty} E(|D_n|^{2r} / (f_n b_n^s)|\mathcal{F}_{n-1}) < \infty$ a.s. on a set B implies that $\Sigma_{n=1}^{m} D_n / b_m$ converges a.s. to zero on B as $m \to \infty$. Then it follows for any stochastic sequence $(D_n, \mathcal{F}_n, n \ge 1)$ that

$\Sigma_{n=1}^{\infty} E(|D_n|^{2r} / (|D_n|^{2r} + f_n b_n^s)|\mathcal{F}_{n-1}) < \infty$ a.s. on a set A implies that $\Sigma_{n=1}^{m} (D_n - a_n) / b_m$ converges a.s. to zero as $m \to \infty$ on A, where $a_n = E(D_n I(|D_n|^{2r} \le f_n b_n^s)|\mathcal{F}_{n-1})$.

PROOF. We establish (ii) only since the proofs of (i) and (iii) and (iv) are very similar to that of (ii). Assume

$\Sigma_{n=1}^{\infty} E(|D_n|^{2r} / (|D_n|^{2r} + f_n b_n^s)|\mathcal{F}_{n-1}) < \infty$ on a set A. Let

$D_n' = D_n I(|D_n|^{2r} \le f_n b_n^s)$. $\Sigma_{n=1}^{\infty} P(D_n \ne D_n'|\mathcal{F}_{n-1}) = \Sigma_{n=1}^{\infty} P(|D_n|^{2r} > f_n b_n^s$ $|\mathcal{F}_{n-1}) = \Sigma_{n=1}^{\infty} E([|D_n|^{2r} / (|D_n|^{2r} + f_n b_n^s)][(|D_n|^{2r} + f_n b_n^s) / |D_n|^{2r}]$ $\cdot I(|D_n|^{2r} > f_n b_n^s)|\mathcal{F}_{n-1}) \le 2 \Sigma_{n=1}^{\infty} E(|D_n|^{2r}/(|D_n|^{2r} + f_n b_n^s)|\mathcal{F}_{n-1}) < \infty$ on A. Now $(I(D_n \ne D_n') - P(D_n \ne D_n'|\mathcal{F}_{n-1}), \mathcal{F}_n, n \ge 1)$ is a sequence of uniformly bounded martingale differences. Thus by a result of Doob ([8], p. 320) it follows that $P[D_n \ne D_n' \text{ i.o.}, A] = 0$.

$(\Sigma_{n=1}^{m}(D_n' - E(D_n'|\mathcal{F}_{n-1})), \mathcal{F}_m, m \ge 1)$ is a martingale.

$\Sigma_{n=1}^{\infty} E(|D_n' - E(D_n'|\mathcal{F}_{n-1})|^{2r} / (f_n b_n^s)|\mathcal{F}_{n-1}) \le 2^{2r} \Sigma_{n=1}^{\infty} E(|D_n'|^{2r}/(f_n b_n^s)|\mathcal{F}_{n-1})$ $= 2^{2r} \Sigma_{n=1}^{\infty} E([|D_n|^{2r}/(f_n b_n^s + |D_n|^{2r})][(f_n b_n^s + |D_n|^{2r})/(f_n b_n^s)]$ $\cdot I(|D_n|^{2r} \le f_n b_n^s)|\mathcal{F}_{n-1}) \le 2^{2r+1} \Sigma_{n=1}^{\infty} E(|D_n|^{2r}/(|D_n|^{2r} + f_n b_n^s)|\mathcal{F}_{n-1}) < \infty$

a.s. on A using the c_r inequality ([22], p. 155) and the fact that $E(|D_n'|^{2r}|\mathcal{F}_{n-1}) \geq |E(D_n'|\mathcal{F}_{n-1})|^{2r}$ a.s. If the hypotheses of (ii) hold, it follows that $\Sigma_{n=1}^{\infty}(D_n' - E(D_n'|\mathcal{F}_{n-1})) / b_n$ converges a.s. to a random variable on A. Since $P[D_n' \neq D_n$ i.o., $A] = 0$, (ii) is established. This completes the proof.

We now apply Theorem 3.1 to state several a.s. convergence results, each result corresponding to a known convergence result.

COROLLARY 3.1. Let $(D_n, \mathcal{F}_n, n \geq 1)$ be a stochastic sequence, $b_n > 0$, and $\Sigma_{n=1}^{\infty} E(|D_n|^{2r} / (|D_n|^{2r} + b_n^{2r})|\mathcal{F}_{n-1}) < \infty$ on a set A for some $1/2 \leq r \leq 1$. Then $\Sigma_{n=1}^{\infty} (D_n - a_n) / b_n$ converges a.s. to a random variable on A where $a_n = E(D_n I(|D_n| \leq b_n)|\mathcal{F}_{n-1})$.

PROOF. If $(D_n/b_n, \mathcal{F}_n, n \geq 1)$ is a martingale difference sequence, it follows by results of Levy for the case $r = 1$ ([19], Theorem 68) and Chow for the case $1/2 \leq r < 1$ ([3], p. 554) that $\Sigma_{n=1}^{\infty} E(|D_n|^{2r} / b_n^{2r}|\mathcal{F}_{n-1}) < \infty$ a.s. on a set B implies that $\Sigma_{n=1}^{\infty} D_n/b_n$ converges a.s. to a random variable on B. Thus Theorem 3.1(ii) applies with $1/2 \leq r \leq 1$, $s = 2r$, and $f_n = 1$.

REMARK. In the special case where $(D_n, n \geq 1)$ is an independent sequence of random variables, Corollary 3.1 also follows from a theorem of Loeve ([21], p. 286).

COROLLARY 3.2. Let $(D_n, \mathcal{F}_n, n \geq 1)$ be a stochastic sequence with $\Sigma_{n=1}^{\infty} E(|D_n|^{2r} / (n^{r+1} + |D_n|^{2r})) < \infty$ for some $r \geq 1$. Then $\Sigma_{n=1}^{m} (D_n - a_n) / m$ converges a.s. to zero as $m \to \infty$, where $a_n = E(D_n I(|D_n|^{2r} \leq n^{r+1}|\mathcal{F}_{n-1})$.

PROOF. If $(D_n, \mathcal{F}_n, n \geq 1)$ is a martingale difference sequence with $\Sigma_{n=1}^{\infty} E[|D_n|^{2r} / n^{r+1}] < \infty$ for some $r \geq 1$, then $\Sigma_{n=1}^{m} D_n / m$ converges a.s. to zero as $m \to \infty$, by a result of Chow ([4],[2]). Thus Theorem 3.1 (iii) applies with $s = r+1$, $f_n = 1$, $r \geq 1$, and $b_n = n$.

COROLLARY 3.3. Let $(D_n, \mathcal{F}_n, n \geq 1)$ be a stochastic sequence with $\Sigma_{n=1}^{\infty} E(|D_n|^{2r} / (c_n^{1-r} + |D_n|^{2r})) < \infty$ for some $r > 1$ and positive constants c_n such that $\Sigma_{n=1}^{\infty} 1/c_n < \infty$. Then $\Sigma_{n=1}^{\infty}(D_n - a_n)$ converges a.s. to a random variable where $a_n = E(D_n I(|D_n|^{2r} \leq c_n^{1-r})|\mathcal{F}_{n-1})$.

PROOF. If $(D_n, \mathcal{F}_n, n \geq 1)$ is a martingale difference sequence with $\Sigma_{n=1}^{\infty} E|D_n|^{2r} / c_n^{1-r} < \infty$, for some $r > 1$, and positive constants c_n such that $\Sigma_{n=1}^{\infty} 1/c_n < \infty$ then $\Sigma_{n=1}^{\infty} D_n$ converges a.s. to a random variable by a result of Chow ([3], p. 555). Thus Theorem 3.1 (i) applies with $b_n = 1$, $r > 1$ and $f_n = c_n^{1-r}$.

COROLLARY 3.4. Let $(D_n, \mathcal{F}_n, n \geq 1)$ be a martingale difference sequence, $b_n > 0$, $E \sup_{n \geq 1} (|D_n| / b_n) < \infty$ and $\Sigma_{n=1}^{\infty} E[|D_n|^{2r}/(|D_n|^{2r} + b_n^{2r})] < \infty$ for some $1/2 \leq r \leq 1$. Then $\Sigma_{n=1}^{\infty} E((D_n/b_n)I(|D_n| \leq b_n)|\mathcal{F}_{n-1}) \equiv \Sigma_{n=1}^{\infty} a_n/b_n$ converges a.s.

PROOF. By Corollary 3.1, $\Sigma_{n=1}^{\infty} (D_n - a_n)/ b_n$ converges a.s. $\Sigma_{n=1}^{\infty} E[|D_n|^{2r}/(|D_n|^{2r} + b_n^{2r})] < \infty$ implies that $\Sigma_{n=1}^{\infty} D_n^2 / b_n^2 < \infty$ a.s.. According to a result of Burkholder ([1], p. 1498), $E \sup_{n \geq 1} (|D_n|/b_n) < \infty$ and $\Sigma_{n=1}^{\infty} D_n^2 / b_n^2 < \infty$ a.s. together imply that $\Sigma_{n=1}^{\infty} D_n / b_n$ converges a.s.. Hence $\Sigma_{n=1}^{\infty} a_n / b_n$ converges a.s.. The result is established.

If $b_n \uparrow \infty$, we can deduce stability results from Corollary 3.1 by applying the Kronecker lemma in the well-known manner. ([12], pp. 238 - 9). The stability result corresponding to Corollary 3.1

$(\Sigma_{n=1}^{m} (D_n - a_n) / b_m$ converges a.s. to zero as $m \to \infty$ if the hypotheses of Corollary 3.1 are satisfied and $b_n \uparrow \infty)$ applied in the special case where the random variable D_n are independent and $r = 1$ gives a corrected version of (3).[1] Corollary 3.2 applied in the special case where the random variables D_n are independent gives a corrected version of (4). If the hypotheses of Corollary 3.1 hold, it is clear that $\Sigma_{n=1}^{\infty} D_n / b_n$ converges a.s. to a random variable on the set A if and only if $\Sigma_{n=1}^{\infty} a_n / b_n$ converges a.s. to a random variable on A. Similar statements can of course be made about Corollaries 3.2 and 3.3 and are omitted. If in the statement of the theorem or in the statement of any of the corollaries, the random variables D_n are assumed to be independent, it should be noted that the a_n in each case do indeed become constants as the notation suggests. If further the random variables are assumed to be symmetrical as well as independent, then $a_n = 0$. Thus under the additional assumption of symmetry, (3) and (4) are valid as stated. In the independent, symmetric case a converse to Corollary 3.1 can be stated.

PROPOSITION 3.1. Let $(D_n, n \geq 1)$ be a sequence of independent symmetric random variables and $(b_n, n \geq 1)$ a sequence of positive constants. If $\Sigma_{n=1}^{\infty} D_n / b_n$ converges a.s. to a random variable then $\Sigma_{n=1}^{\infty} E[D_n^2 / (b_n^2 + D_n^2)] < \infty$.

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[1] In a research report done concurrently with the present work, Heyde [13] states in a theorem that if $(D_n, n \geq 1)$ is a sequence of independent random variables with $0 < b_n \uparrow \infty$, then $\Sigma_{n=1}^{m}(D_n - a_n)/b_m$ converges a.s. to zero as $m \to \infty$ where $a_n = E[D_n I(|D_n| \leq b_n)]$. This is also the result we refer to here.

PROOF. We may assume $b_n = 1$ without loss of generality. $\Sigma_{n=1}^{\infty} E[(D_n^2 / (1 + D_n^2))I(D_n^2 \geq 1)] \leq \Sigma_{n=1}^{\infty} P[D_n^2 \geq 1] < \infty$ by the three series theorem, ([22], p. 237). $\Sigma_{n=1}^{\infty} E[(D_n^2/(1+ D_n^2))I(D_n^2 < 1)] \leq \Sigma_{n=1}^{\infty} ED_n^2 \; I(|D_n|^2 \leq 1) < \infty$ by the three series theorem.

COROLLARY 3.5.[2] Let $(D_n, n \geq 1)$ be a sequence of independent symmetric random variables and $(b_n, n \geq 1)$ a sequence of positive constants. Then $\Sigma_{n=1}^{\infty} D_n / b_n$ converges a.s. to a random variable if and only if $\Sigma_{n=1}^{\infty} E[D_n^2 / (b_n^2 + D_n^2)] < \infty$.

PROOF. Immediate from Proposition 3.1 and Corollary 3.1.

3. Relationship of the a.s. Convergence of $\Sigma_{n=1}^{\infty} D_n$ and $\Sigma_{n=1}^{\infty} D_n^2$ in the Independent Case

PROPOSITION 3.2. Let $(D_n, n \geq 1)$ be a sequence of independent random variables with $(b_n, n \geq 1)$ a sequence of positive constants. Then $\Sigma_{n=1}^{\infty} E[D_n^2/(b_n^2 + D_n^2)] < \infty$ if and only if $\Sigma_{n=1}^{\infty} D_n^2 / b_n^2 < \infty$.

PROOF. The sufficiency is obvious since $\Sigma_{n=1}^{\infty} D_n^2 / (b_n^2 + D_n^2) < \infty$ implies that $\Sigma_{n=1}^{\infty} D_n^2 / b_n^2 < \infty$. To establish the necessity assume $\Sigma_{n=1}^{\infty} D_n^2 / b_n^2 < \infty$. By the three series theorem $\Sigma_{n=1}^{\infty} E[(D_n^2/b_n^2)I(D_n^2 \leq b_n^2)] < \infty$ and $\Sigma_{n=1}^{\infty} P[D_n^2 > b_n^2] < \infty$. But $\Sigma_{n=1}^{\infty} E[D_n^2 / (b_n^2 + D_n^2)] = \Sigma_{n=1}^{\infty} E[(D_n^2 / (b_n^2 + D_n^2))(I(D_n^2 \leq b_n^2) + I(D_n^2 > b_n^2))] \leq \Sigma_{n=1}^{\infty} (E[(D_n^2 / b_n^2) I(D_n^2 \leq b_n^2)] + P[D_n^2 > b_n^2]) < \infty$.

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[2] Szynal [26] has also recently made the same observation.

COROLLARY 3.6. Let $(D_n, n \geq 1)$ be a sequence of independent random variables and $(b_n, n \geq 1)$ a sequence of positive constants with $\Sigma_{n=1}^{\infty} D_n^2 / b_n^2 < \infty$ a.s. Then $\Sigma_{n=1}^{\infty} E[(D_n / b_n) I(|D_n| \leq b_n)]$ converges if and only if $\Sigma_{n=1}^{\infty} D_n / b_n$ converges a.s. to a random variable.

PROOF. We may assume $b_n = 1$ for all $n \geq 1$ without loss of generality. The necessity is well-known. To establish the sufficiency, assume $\Sigma_{n=1}^{\infty} D_n^2 < \infty$ a.s. and $\Sigma_{n=1}^{\infty} E[D_n I(|D_n| \leq 1)]$ converges. By Proposition 3.2, $\Sigma_{n=1}^{\infty} E[D_n^2 / (1 + D_n^2)] < \infty$. By Corollary 3.1, $\Sigma_{n=1}^{\infty} (D_n - E([D_n I(|D_n| \leq 1)]))$ converges a.s. to a random variable. Hence $\Sigma_{n=1}^{\infty} D_n$ converges a.s. to a random variable.

PROPOSITION 3.3. Let $(D_n, n \geq 1)$ be a sequence of independent random variables such that $\Sigma_{n=1}^{\infty} D_n$ converges a.s. to a random variable and $(R_n, n \geq 1)$ be an independent sequence of random variables independent of the D_n also with $P[R_n = 1] = P[R_n = -1] = 1/2$. Then (i) $\Sigma_{n=1}^{\infty} D_n^2 < \infty$ a.s. if and only if (ii) $\Sigma_{n=1}^{\infty} E^2[|D_n| I(|D_n| \leq c)] < \infty$ for some $c > 0$ if and only if (iii) $\Sigma_{n=1}^{\infty} E^2[D_n I(|D_n| \leq c] < \infty$ for some $c > 0$ if and only if (iv) $\Sigma_{n=1}^{\infty} D_n R_n$ converges a.s. to a random variable.

PROOF. We show that (iii) implies (i) implies (ii) implies (iii) implies (iv). (iii) implies (i): By the three series theorem, $\Sigma_{n=1}^{\infty} D_n R_n$ converging a.s. implies that $\Sigma_{n=1}^{\infty} E\, D_n^2\, I(D_n^2 \leq 1) < \infty$ and that $\Sigma_{n=1}^{\infty} P[|D_n| \geq 1] < \infty$. Hence $\Sigma_{n=1}^{\infty} D_n^2 < \infty$ a.s.

i) implies (ii): $\Sigma_{n=1}^{\infty} D_n^2 < \infty$ implies that $\Sigma_{n=1}^{\infty} E[D_n^2\, I(|D_n| \leq c)] < \infty$ for all $c > 0$ by the three series theorem. But $E^2[|D_n| I(|D_n| \leq c)] \leq E[D_n^2\, I(|D_n| \leq c)]$ and hence $\Sigma_{n=1}^{\infty} E^2[D_n\, I(|D_n| \leq c)] < \infty$.

ii) implies (iii): Trivial.

iii) implies (iv): We establish the a.s. convergence of $\sum_{n=1}^{\infty} D_n R_n$ by verifying the conditions of the three series theorem.

$\sum_{n=1}^{\infty} P[|D_n R_n| > c] = \sum_{n=1}^{\infty} P[|D_n| > c] < \infty$ since $\sum_{n=1}^{\infty} D_n$ converges a.s. $D_n R_n$ are symmetric and hence $\sum_{n=1}^{\infty} E[D_n R_n I(|D_n R_n| \le c)] = 0$.

$\sum_{n=1}^{\infty} E[(D_n R_n I(|D_n R_n| \le c))^2] = \sum_{n=1}^{\infty} E(D_n^2 I(|D_n| \le c))$. $\sum_{n=1}^{\infty} D_n$ converges a.s. implies that $\sum_{n=1}^{\infty} E[D_n^2 I(|D_n| \le c)] - E^2[D_n I(|D_n| \le c)] < \infty$.

By hypothesis, $\sum_{n=1}^{\infty} E^2[D_n I(|D_n| \le c)] < \infty$ and hence $\sum_{n=1}^{\infty} E[D_n^2 I(|D_n| \le c)] < \infty$. Thus by the three series theorem, $\sum_{n=1}^{\infty} D_n R_n$ converges a.s..

REMARK. Consider the special case where $D_n = w_n Y_n$ where $(w_n, n \ge 1)$ is a sequence of positive constants such that $\sum_{n=1}^{\infty} w_n = \infty$ and $(Y_n, n \ge 1)$ is a sequence of independent identically distributed random variables with distribution function F. Define, for each $x > 0$, $N(x)$ as the number of integers for which $b_n / w_n \le x$ for a given sequence of positive constants $(b_n, n \ge 1)$ with $b_n \uparrow \infty$. Jamison, Orey, and Pruitt [16] introduce the condition

(5) $\int_{-\infty}^{\infty} x^2 \int_{|x|}^{\infty} \frac{N(y)}{y^3} \, dy \, dF(x) < \infty$ in order to study the a.s. convergence of $\sum_{n=1}^{N} D_n / b_N$ to zero in the case that $b_N = \sum_{n=1}^{N} w_n$. Heyde [13] generalizes their work to the following: If $0 \le b_n \uparrow \infty$ and $\int_{-\infty}^{\infty} x^2 \int_{|x|}^{\infty} \frac{N(y)}{y^3} \, dy \, d\,F(x) < \infty$ then $\sum_{n=1}^{m} (D_n - E[D_n I(|D_n| \le b_n)])/b_m$ converges a.s. to zero as $m \to \infty$. This result is established by showing

that (5) is equivalent to the condition that $\Sigma_{n=1}^{\infty} E(D_n^2/(b_n^2 + D_n^2)) < \infty$. Hence by Proposition 3.2, it follows that $\Sigma_{n=1}^{\infty} D_n^2 / b_n^2 < \infty$ is equivalent to the condition (5), i.e. that the condition (5) of Jamison, Orey, and Pruitt is equivalent to the a.s. convergence of the appropriate sum of squares.

4. $E(\Sigma_{n=1}^{\infty} |D_n|^{2r} / f_n)^{1/(2r)} < \infty$ Implies a.s. Convergence of $\Sigma_{n=1}^{\infty} D_n$ and Related Results

Let $f_n(r) \equiv f_n = \begin{cases} 1 & \text{if } 0 < r \leq 1 \\ c_n^{1-r} & \text{if } r > 1 \end{cases}$ where $(c_n, n \geq 1)$ is a sequence of positive constants such that $\Sigma_{n=1}^{\infty} c_n^{-1} < \infty$ and let $(D_n, \mathcal{F}_n, n \geq 1)$ be a martingale difference sequence. Without loss of generality in the sequel, we assume $c_n \geq 1$ for all $n \geq 1$.

THEOREM 3.2. If $E(\Sigma_{n=1}^{\infty} |D_n|^{2r} / f_n)^{1/2r} < \infty$ for some $r > 0$, then $\Sigma_{n=1}^{\infty} D_n$ converges a.s..

PROOF. As a first step we establish a numerical inequality which is stated as a lemma.

LEMMA. Let $(c_n, n \geq 1)$ be a sequence of non-negative constants and $r > 1$. Then there exists a constant $M > 0$ independent of the values of the c_n such that $c_m/((\Sigma_{n=1}^{m} c_n)^{1-1/(2r)} + (\Sigma_{n=1}^{m-1} c_n)^{1-1/(2r)})$ $\leq M((\Sigma_{n=1}^{m} c_n)^{1/(2r)} - (\Sigma_{n=1}^{m-1} c_n)^{1/(2r)})$ for all $m \geq 2$.

PROOF OF LEMMA. The above inequality holds if and only if

$c_m \leq M(c_m + (\Sigma_{n=1}^{m-1} c_n)^{1-1/(2r)}(\Sigma_{n=1}^{m} c_n)^{1/(2r)} - (\Sigma_{n=1}^{m} c_n)^{1-1/(2r)}$

$(\Sigma_{n=1}^{m-1} c_n)^{1/(2r)})$ for all $m \geq 2$ which holds if and only if $(1 - M)$

$c_m \leq M(\Sigma_{n=1}^{m} c_n)^{1/(2r)}(\Sigma_{n=1}^{m-1} c_n)^{1/(2r)}[(\Sigma_{n=1}^{m-1} c_n)^{1-1/r} - (\Sigma_{n=1}^{m} c_n)^{1-1/r}]$

for all $m \geq 2$. Thus it suffices to show the existence of an $M > 0$ such that for all $a \geq 0$ and $b > 0$, $(M-1)M^{-1} a \geq (a+b)^{1/(2r)}$ $(b)^{1/(2r)}((a+b)^{1-1/r} - b^{1-1/r})$ which holds if and only if $(M-1)M^{-1}(a/b) \geq (a/b+1)^{1/(2r)}((a/b+1)^{1-1/r} - 1) = (a/b+1)^{1-1/(2r)}$ $- (a/b+1)^{1/(2r)}$. Let $\beta = (M-1)M^{-1}$ and $x = a/b+1$. Then it suffices to show the existence of a $\beta < 1$ such that $-\beta + \beta x \geq x^{1-1/(2r)} - x^{1/(2r)}$ for all $x \geq 1$. Let $f(x) = \beta x - x^{1-1/(2r)} + x^{1/(2r)} - \beta$. $f(1) = 0$. Hence it suffices to show that $f'(x) \geq 0$ for all $x \geq 1$ and some $\beta < 1$. $f'(x) = \beta + (2r)^{-1} x^{1/(2r)-1} - (1-(2r)^{-1})x^{-1/(2r)}$ $\geq \beta - 1 + (2r)^{-1}$. Thus it suffices to choose $\beta = 1 - (2r)^{-1}$ to establish the lemma.

Without loss of generality, we assume $D_1 = 1$ a.s. . By hypotheses $\Sigma_{n=1}^{\infty} |D_n|^{2r} < \infty$ for $0 < r \leq 1/2$ and hence $\Sigma_{n=1}^{\infty} D_n$ converges a.s.. For $r = 1$, the result is an exact restatement of a result of Burkholder ([1], p. 1497). Since $(\Sigma_{n=1}^{\infty} |D_n|^{2r})^{1/(2r)} \geq (\Sigma_{n=1}^{\infty} D_n^2)^{1/2}$ for $1/2 < r \leq 1$, the result for $1/2 < r \leq 1$ follows immediately from Burkholder's result.

For an integer $K \geq 1$ and $r > 1$ let

$$t(K) \equiv t = \inf \{m : K(\Sigma_{n=1}^{m} |D_n|^{2r}/ f_n)^{1/(2r)} < \Sigma_{n=1}^{m} |D_n|^{2r}/(f_n + |D_n|^{2r})\} .$$

Let $X_m = \Sigma_{n=1}^{m} D_n$ and $X'_m = \begin{cases} X_m & \text{if } t \geq m \\ X_t & \text{if } t < m \end{cases}$, i.e.

$X'_m = \Sigma_{n=1}^{m} D_n \, I(t \geq n) \equiv \Sigma_{n=1}^{m} D'_n$. Then

$$\Sigma_{n=1}^{\infty} |D'_n|^{2r}/(|D'_n|^{2r} + f_n) = \begin{cases} \Sigma_{n=1}^{t-1} |D_n|^{2r}/(|D_n|^{2r} + f_n) + |D_t|^{2r}/(|D_t|^{2r} + f_t) & \text{if } t < \infty \\ \Sigma_{n=1}^{\infty} |D_n|^{2r}/(|D_n|^{2r} + f_n) & \text{if } t = \infty \end{cases} .$$

Noting that $|D_t|^{2r}/(|D_t|^{2r} + f_t) \leq (|D_t|^{2r}/f_t)^{1/2r}$ it follows that

$E\,\Sigma_{n=1}^{\infty}|D_n'|^{2r}/(|D_n'|^{2r} + f_n) \leq (K+1)\,E(\Sigma_{n=1}^{\infty}\,|D_n|^{2r}/f_n)^{1/2r} < \infty$. Then by Corollary 3.3, it follows that $\Sigma_{n=1}^{\infty}(D_n' - a_n)$ converges a.s. with $a_n = E[D_n'\,I(|D_n'|^{2r} \leq f_n)|\mathcal{F}_{n-1})$. We now establish that $\Sigma_{n=1}^{\infty}\,E(D_n'\,I(|D_n'|^{2r} \leq f_n)|\mathcal{F}_{n-1}) \equiv \Sigma_{n=1}^{\infty}\,a_n$ converges a.s. on the set $[t = \infty]$. For $n \geq 2$, on the set $[t = \infty]$, $|E(D_n' I(|D_n'|^{2r} \leq f_n)|\mathcal{F}_{n-1})|$

$= |E(D_n\,I(t \geq n)\,I(|D_n|^{2r}\,I(t \geq n) \leq f_n)\,|\,\mathcal{F}_{n-1})|$

$= |E(D_n\,I(|D_n|^{2r}\,I(t \geq n) \leq f_n)\,|\mathcal{F}_{n-1})|$

$= |E(D_n\,I(|D_n|^{2r}\,I(t \geq n) > f_n\,|\,\mathcal{F}_{n-1})| \leq E(|D_n|I(|D_n|^{2r} > f_n)|\mathcal{F}_{n-1})$

using the fact that $E(D_n\,|\,\mathcal{F}_{n-1}) = 0$ a.s. Let A_n be the set $[|D_n|^{2r-1} < (1/(f_n^{-1} - f_n^{(1-2r)/(2r)}))(\Sigma_{m=1}^{n-1}|D_m|^{2r}/f_m)^{(2r-1)/(2r)}]$.

$E(|D_n|\,I\,(|D_n|^{2r} > f_n)\,|\,\mathcal{F}_{n-1}) = E(|D_n|I(A_n)I(|D_n|^{2r} > f_n)\,|\,\mathcal{F}_{n-1})$ $+ E(|D_n|\,I(A_n')\,I(|D_n|^{2r} > f_n)\,|\,\mathcal{F}_{n-1})$. We estimate each term separately.

$E(|D_n|I(A_n)I(|D_n|^{2r} > f_n)\,|\,\mathcal{F}_{n-1}) \leq E((1/(f_n^{-1} - f_n^{(1-2r)/(2r)}))^{1/(2r-1)}$ $(\Sigma_{m=1}^{n-1}|D_m|^{2r}/f_m)^{1/2r}I(|D_n|^{2r} > f_n)\,|\,\mathcal{F}_{n-1}) \leq c(\Sigma_{m=1}^{\infty}|D_m|^{2r}/f_m)^{1/(2r)}$ $P(|D_n|^{2r} > f_n\,|\,\mathcal{F}_{n-1})$ for some constant $c > 0$, noting that $1/(f_n^{-1} - f_n^{(1-2r)/(2r)}) \to 0$ since $f_n \to 0$ as $n \to \infty$. Thus

$\Sigma_{n=2}^{\infty}E(|D_n|\,I(A_n)\,I(|D_n|^{2r} > f_n)|\mathcal{F}_{n-1}) \leq c(\Sigma_{m=1}^{\infty}|D_m|^{2r}/f_m)^{1/2r}\,\Sigma_{n=2}^{\infty}$ $P(|D_n|^{2r} > f_n|\mathcal{F}_{n-1})$. Note that $(\Sigma_{n=1}^{m}\,I(|D_n|^{2r} > f_n)$ $- P(|D_n|^{2r} > f_n|\,\mathcal{F}_{n-1}),\ \mathcal{F}_m,\ m \geq 1)$ is a martingale with uniformly bounded martingale differences. It then follows by a result of Doob ([8], p. 320) that $\Sigma_{n=1}^{\infty} I(|D_n|^{2r} > f_n) < \infty$ a.s. if and only if $\Sigma_{n=1}^{\infty} P(|D_n|^{2r} > f_n|\mathcal{F}_{n-1}) < \infty$ a.s.. But we know that $\Sigma_{n=1}^{\infty} I(|D_n|^{2r} > f_n) < \infty$ a.s. since $\Sigma_{n=1}^{\infty}|D_n|^{2r}/f_n < \infty$ a.s. by hypothesis. Thus $\Sigma_{n=1}^{\infty} P(|D_n|^{2r} > f_n|\mathcal{F}_{n-1}) < \infty$ a.s. and therefore $\Sigma_{n=2}^{\infty} E(|D_n|\ I(A_n)\ I(|D_n|^{2r} > f_n)|\mathcal{F}_{n-1}) < \infty$ a.s..

By definition of A_m', on A_m' we have

$$|D_m|^{2r-1}/f_m - |D_m|^{2r-1}f_m^{(1-2r)/(2r)} \geq (\Sigma_{n=1}^{m-1} |D_n|^{2r}/f_n)^{(2r-1)/(2r)}$$

which implies that

$$2|D_m|^{2r-1}/f_m \geq (\Sigma_{n-1}^{m}|D_n|^{2r}/f_n)^{(2r-1)/(2r)} + (\Sigma_{n=1}^{m-1} |D_n|^{2r}/f_n)^{(2r-1)/(2r)}$$

using the inequality $|a+b|^s \leq |a|^s + |b|^s$ for $0 < s \leq 1$. Hence on A_m', $1/2 = |D_m|^{2r-1} f_m^{-1}/(|D_m|^{2r-1} f_m^{-1} + |D_m|^{2r-1} f_m^{-1}) \leq |D_m|^{2r-1} f_m^{-1}/ ((\Sigma_{n=1}^{m} |D_n|^{2r}/f_n)^{1-1/(2r)} + (\Sigma_{n=1}^{m-1} |D_n|^{2r}/f_n)^{1-1/(2r)})$ for $m \geq 2$.

Thus, $\Sigma_{n=2}^{\infty}(1/2)\ E(|D_n|\ I(A_n')\ I(|D_n|^{2r} > f_n)|\mathcal{F}_{n-1})$

$$\leq \Sigma_{m=2}^{\infty} E(|D_m|^{2r} f_m^{-1}/((\Sigma_{n=1}^{m} |D_n|^{2r} / f_n)^{1-1/(2r)} + (\Sigma_{n=1}^{m-1} |D_n|^{2r}/f_n)^{1-1/(2r)}\ |\mathcal{F}_{m-1})$$

a.s. Now

$$\Sigma_{m=2}^{\infty}E(|D_m|^{2r} f_m^{-1}/((\Sigma_{n=1}^{m} |D_n|^{2r}/f_n)^{1-1/(2r)} + (\Sigma_{n=1}^{m-1} |D_n|^{2r}/f_n)^{1-1/(2r)})$$

$$\leq E\ \Sigma_{m=2}^{\infty} M[(\Sigma_{n=1}^{m} |D_n|^{2r}/f_n)^{1/(2r)} - (\Sigma_{n=1}^{m-1} |D_n|^{2r}/f_n)^{1/(2r)}]$$

$$\leq E\ M(\Sigma_{m=2}^{\infty} |D_m|^{2r}/f_m)^{1/2r} < \infty ,$$

using the lemma with $c_m = |D_m|^{2r} f_m^{-1}$. $E|D_1| < \infty$. Thus $\Sigma_{n=1}^{\infty}E(|D_n|I(A_n')I(|D_n|^{2r} > f_n)|\mathcal{F}_{n-1}) < \infty$ a.s. Combining, it follows that $\Sigma_{n=1}^{\infty}E(D_n'\ I(|D_n'|^{2r} \leq f_n)|\ \mathcal{F}_{n-1})$ converges a.s. on the set $[t = \infty]$. Since the a.s. convergence of

$\Sigma_{n=1}^{\infty} (D_n' - E(D_n' I(|D_n'|^{2r} \leq f_n)|\mathcal{F}_{n-1}))$ is already established, it follows that $\Sigma_{n=1}^{\infty} D_n$ converges a.s. on the set $[t = \infty]$, where we use the fact that $D_n = D_n'$ for all $n \geq 1$ on the set $[t = \infty]$. But $\Omega = \bigcup_{K=1}^{\infty} [t(K) = \infty]$ since $\Sigma_{n=1}^{\infty} |D_n|^{2r} / (f_n + |D_n|^{2r}) < \infty$ a.s. Thus $\Sigma_{n=1}^{\infty} D_n$ converges a.s..

COROLLARY 3.7. Let $f_n = (n(\log n)^{1+\epsilon})^{1-r}$ for $n \geq 3$ and some $r > 1$ and $\epsilon > 0$. If $E(\Sigma_{n=3}^{\infty} |D_n|^{2r} / f_n)^{1/2r} < \infty$ for some $r > 1$, then $\Sigma_{n=3}^{\infty} D_n$ converges a.s..

PROOF. Immediate from Theorem 3.2 since $\Sigma_{n=3}^{\infty} (n(\log n)^{1+\epsilon})^{-1} < \infty$.

REMARK. Corollary 3.7 sharpens a result of Chow ([3], p. 555) which states that $\Sigma_{n=1}^{\infty} E|D_n|^{2r} (n(\log n)^{1+\epsilon})^{r-1} < \infty$ for some $r > 1$ and $\epsilon > 0$ implies that $\Sigma_{n=1}^{\infty} D_n$ converges a.s..

Corollary 3.7 is sharp in the sense that $\epsilon > 0$ may not be dropped. The following example shows this even in the case that $(D_n, n \geq 1)$ is a sequence of independent random variables with mean zero.

EXAMPLE 3.2. Let $(D_n, n \geq 3)$ be a sequence of independent symmetric random variables with $|D_n| = (n \log n \log \log n)^{-1/2}$ a.s. and $g_n = (n \log n)^{1-r}$ for some $r > 1$.

$\Sigma_{n=3}^{\infty} |D_n|^{2r}/g_n = \Sigma_{n=3}^{\infty} (n \log n(\log \log n)^r)^{-1} < \infty$. Thus $E(\Sigma_{n=3}^{\infty} |D_n|^{2r} / g_n)^{1/(2r)} < \infty$. But $\Sigma_{n=3}^{\infty} ED_n^2 I(|D_n| \leq 1)$ $= \Sigma_{n=3}^{\infty} (n \log n \log \log n)^{-1} = \infty$ and hence $\Sigma_{n=1}^{\infty} D_n$ diverges a.s. by the three series theorem.

COROLLARY 3.8. If (6) $E|D_t| \; I(t < \infty) f_t^{-1/(2r)} < \infty$ for some $r > 1$ and for all stopping variables t, then $\Sigma_{n=1}^{\infty} D_n$ converges a.s. on the set $[\Sigma_{n=1}^{\infty}|D_n|^{2r}/f_n < \infty]$.

PROOF. For an integer $K \geq 1$, let $t(M) \equiv t = \inf \{m: (\Sigma_{n=1}^{m} |D_n|^{2r} f_n^{-1})^{1/(2r)} > M\}$. Let $X_m = \Sigma_{n=1}^{m} D_n$ and

$$X'_m = \begin{cases} X_m & \text{if } t \geq m \\ X_t & \text{if } t < m \end{cases}, \text{ i.e. } X'_m = \Sigma_{n=1}^{m} D_n \, I(t \geq n) \equiv \Sigma_{n=1}^{m} D'_n .$$

$$E(\Sigma_{n=1}^{\infty} |D'_n|^{2r}/f_n)^{1/(2r)} \leq M + E|D_t| \, I(t < \infty) \, f_t^{-1/(2r)} < \infty .$$

$(X'_n, \mathcal{F}_n, n \geq 1)$ is a martingale according to a well known result of Doob ([8], p. 300). Thus $\Sigma_{n=1}^{\infty} D'_n$ converges a.s. by Theorem 3.2.

Thus on the set $\bigcup_{M=1}^{\infty} [t(M) = \infty] = [\Sigma_{n=1}^{\infty} |D_n|^{2r} / f_n < \infty]$, it follows that $\Sigma_{n=1}^{\infty} D_n$ converges a.s..

REMARKS. Without the validity of Theorem 3.2, (6) would have to be replaced by the much stronger condition that $E[|D_t|^{2r} I(t < \infty) f_t^{-1}] < \infty$ for some $r > 1$ and all stopping times t in order that Corollary 3.8 remain valid.

Burkholder proves that if $E \sup_{n\geq 1} |D_n| < \infty$, then the set $[\Sigma_{n=1}^{\infty} D_n^2 < \infty] = [\Sigma_{n=1}^{\infty} D_n$ converges $]$ a.s.. In view of Corollary 3.8 and the fact that $E \sup_{n\geq 1} |D_n| < \infty$ implies that $E|D_t| I(t < \infty) < \infty$ for all stopping times t one might expect some sort of converse to Corollary 3.8 to hold, for example that if $E \sup_{n\geq 1} (|D_n|/(g_n)^{1/(2r)}) < \infty$ then the set $[\Sigma_{n=1}^{\infty} D_n$ converges a.s.$] \subset [\Sigma_{n=1}^{\infty}|D_n|^{2r} g_n^{-1} < \infty]$ where $g_n = (n \log n)^{1-r}$ for some $r > 1$ or some similar choice for g_n. However, the following example shows that this statement fails, even for $g_n = n^{1-r}$.

EXAMPLE 3.3. Let $f_n = (n(\log n)^{1+\epsilon})^{1-r}$ for some $\epsilon > 0$, $r > 1$ and $n \geq 2$.

$$\text{Let } D_n = \begin{cases} f_n^{1/(2r)} & \text{with probability } p_n \\ -f_n^{1/(2r)} & \text{with probability } p_n \\ 0 & \text{with probability } 1-2p_n \end{cases}$$

where $p_n = 1/(n(\log n)^{1+\epsilon})^{1/r}$ and $(D_n, n \geq 1)$ is assumed to be a sequence of independent random variables. Computing, $ED_n^2 = (n(\log n)^{1+\epsilon})^{-1}$. Hence $\Sigma_{n=2}^{\infty} ED_n^2 < \infty$. Since D_n is symmetric and D_n converges to zero a.s. as $n \to \infty$ it follows by the three series theorem that $\Sigma_{n=1}^{\infty} D_n$ converges a.s. $|D_n|/f_n^{1/(2r)} \leq 1$ and hence $E \sup_{n\geq 2} (|D_n|/(f_n)^{1/(2r)}) < \infty$ holds and thus $E \sup_{n\geq 2} (|D_n|/(g_n)^{1/(2r)}) < \infty$ holds. Since $|D_n|^{2r} n^{r-1} \leq 1$ and $\Sigma_{n=2}^{\infty} \text{Var} (|D_n|^{2r} n^{r-1}) = \infty$, it follows by a result of Kolmogorov ([22], p. 236) that $\Sigma_{n=2}^{\infty} |D_n|^{2r} n^{r-1} = \infty$ a.s.. Thus the example is established.

Based upon Chow's result that $E \Sigma_{n=1}^{\infty} |D_n|^{2r} / n^{r+1} < \infty$ for some $r > 1$ implies that $\Sigma_{n=1}^{m} D_n/m$ converges a.s. to zero, we use the approach of Theorem 3.2 to state the following theorem.

THEOREM 3.3. If $E(\Sigma_{n=1}^{\infty} |D_n|^{2r} / n^{r+1})^{1/2r} < \infty$ for some $r > 1$ then $\Sigma_{n=1}^{m} (D_n - a_n)/m$ converges a.s. to zero where $a_n = E(D_n I(|D_n|^{2r} \leq n^{r+1})|\mathcal{F}_{n-1})$.

PROOF. The details are omitted. One defines a stopping rule analogous to that defined in the proof of Theorem 3.2 and uses Corollary 3.2 to complete the proof.

COROLLARY 3.9. If $E(|D_t|\ I(t < \infty)t^{-(r+1)/(2r)}) < \infty$ for all stopping variables t, then $\Sigma_{n=1}^{m} (D_n - a_n) / m$ converges a.s. to zero on the set where $\Sigma_{n=1}^{\infty}|D_n|^{2r}/n^{r+1} < \infty$ for some $r > 1$ where $a_n = E(D_n\ I(|D_n|^{2r} \le n^{r+1})|\mathcal{F}_{n-1})$.

PROOF. For an integer $K \ge 1$, let

$t(M) \equiv t = \inf_m \{m: (\Sigma_{n=1}^{m} |D_n|^{2r} / n^{r+1})^{1/(2r)} > M\}$. Let $X_m = \Sigma_{n=1}^{m} D_n$

and $X_m' = \begin{cases} X_m & \text{if } t \ge m \\ X_t & \text{if } t < m \end{cases}$, i.e. $X_m' = \Sigma_{n=1}^{m} D_n I(t \ge n) \equiv \Sigma_{n=1}^{m} D_n'$.

$$E(\Sigma_{n=1}^{\infty} |D_n'|^{2r} / n^{r+1})^{1/(2r)} \le M + E(|D_t|\ I(t < \infty)\ t^{-(r+1)/(2r)}) < \infty$$

$(X_n', \mathcal{F}_n, n \ge 1)$ is a martingale according to a well known result of Doob. Thus $\Sigma_{n=1}^{m}(D_n' - E(D_n'\ I(|D_n'|^{2r} \le n^{r+1})|\mathcal{F}_{n-1})) / m$ converges a.s. to zero as $m \to \infty$ by Theorem 3.3. On the set $[t = \infty], D_n = D_n'$ and

$$E(D_n'\ I(|D_n'|^{2r} \le n^{r+1})|\mathcal{F}_{n-1}) = -E(D_n'\ I(|D_n'|^{2r} > n^{r+1})|\mathcal{F}_{n-1})$$

$$= -E(D_n\ I(|D_n|^{2r} > n^{r+1})\ I(t \ge n)|\mathcal{F}_{n-1}) = E(D_n I(|D_n|^{2r} \le n^{r+1}|\mathcal{F}_{n-1}) .$$

Thus on the set $\bigcup_{M=1}^{\infty} [t(M) = \infty] = [\Sigma_{n=1}^{\infty} |D_n|^{2r} / n^{r+1} < \infty]$, the desired result follows.

REMARK. Obviously if $|D_n| \le K\, n^{(r+1)/(2r)}$ for some constant K or if $(D_n, n \ge 1)$ is an independent symmetric sequence of random variables, we may conclude that $\Sigma_{n=1}^{m} D_n/m$ converges a.s. to zero if the hypotheses of Corollary 3.9 hold.

5. $E|D_n|^{2r}\,(\log^+(|D_n|)^{1+\epsilon} \leq M$

<u>implies a.s. convergence of</u> $\Sigma_{n=1}^m D_n/m^{1/(2r)}$

<u>to zero in the martingale case for</u> $1/2 \leq r \leq 1$

Let $(D_n, \mathcal{F}_n, n \geq 1)$ be a stochastic sequence and $(a_n, n \geq 1)$ be a sequence of positive constants.

THEOREM 3.4. Let $\psi(x)$ be a positive even function which is non-decreasing for $x > 0$. Suppose either

i) $\psi(x) / x$ is non-increasing for $x > 0$ or

ii) $\psi(x) / x$ is non-decreasing and $\psi(x) / x^2$ is non-increasing for $x > 0$ and $(D_n, \mathcal{F}_n, n \geq 1)$ is a martingale difference sequence. Then on the set where $\Sigma_{n=1}^\infty E(\psi(D_n)/\psi(a_n)|\mathcal{F}_{n-1}) < \infty$ it follows that $\Sigma_{n=1}^\infty D_n / a_n$ converges a.s..

PROOF. Let $D_n' = D_n\, I(|D_n| < a_n)$. We establish the result under (ii) first. According to a result of Chow ([3], p. 554), it suffices to show that $\Sigma_{n=2}^\infty E((D_n^2/a_n^2)I(|D_n| \leq a_n) + (|D_n|/a_n)I(|D_n| > a_n)|\mathcal{F}_{n-1}) < \infty$ a.s. on the set where $\Sigma_{n=1}^\infty E(\psi(D_n)/\psi(a_n)|\mathcal{F}_{n-1}) < \infty$.

$\Sigma_{n=2}^\infty E((D_n^2/a_n^2)\; I(|D_n| \leq a_n)|\mathcal{F}_{n-1}) \leq \Sigma_{n=2}^\infty E(\psi(D_n)/\psi(a_n)|\mathcal{F}_{n-1})$ since $\psi(x) / x^2 \leq \psi(y) / y^2$ for $x \geq y > 0$ by (ii).

$\Sigma_{n=2}^\infty E((|D_n|/(a_n)\; I(|D_n| > a_n)|\mathcal{F}_{n-1}) \leq \Sigma_{n=2}^\infty E(\psi(D_n)/\; \psi(a_n)|\mathcal{F}_{n-1})$ since $\psi(x) / x \geq \psi(y) / y$ for $x \geq y > 0$.

Thus the result is established for case (ii). To prove the result under (i), we must make only minor modifications in the proof of Chow's result in [3] referred to above in the proof.

$|E(D'_n / a_n|\mathscr{F}_{n-1})| = |E(D_n / a_n)I(|D_n| < a_n)|\mathscr{F}_{n-1})|$

$\leq E((|D_n| / a_n)\ I(|D_n| < a_n)|\mathscr{F}_{n-1}) \leq E(\psi(D_n) / \psi(a_n)|\mathscr{F}_{n-1})$ since

$\psi(x) / x \leq \psi(y) / y$ for $x \geq y > 0$. $\Sigma_{n=2}^{\infty} P(D_n \neq D'_n \mid \mathscr{F}_{n-1})$

$= \Sigma_{n=2}^{\infty} P(|D_n| > a_n \mid \mathscr{F}_{n-1}) \leq \Sigma_{n=2}^{\infty} E(\psi(D_n)/\psi(a_n)|\mathscr{F}_{n-1})$ since $\psi(x)$ is non decreasing in x. Thus, on the set where

$\Sigma_{n=2}^{\infty} E(\psi(D_n)/\psi(a_n)|\mathscr{F}_{n-1}) < \infty$, it follows that $\Sigma_{n=2}^{\infty} P(|D_n| > a_n|\mathscr{F}_{n-1}) < \infty$.

Note that $(\Sigma_{n=1}^{m} (I(D_n \neq D'_n) - P(D_n \neq D'_n |\mathscr{F}_{n-1})), \mathscr{F}_m, m \geq 1)$ is a martingale with uniformly bounded martingale differences. It then follows by a result of Doob ([8], p. 320) that $\Sigma_{n=2}^{\infty} I(|D_n| \neq D'_n) < \infty$ a.s. on the set where $\Sigma_{n=2}^{\infty} E(\psi(D_n)/\psi(a_n)|\mathscr{F}_{n-1}) < \infty$.

Let $X_n = D'_1 / a_1 + D'_2 / a_2 + \ldots + D'_n / a_n - E(D'_2 / a_2|\mathscr{F}_1)$

$- E(D'_3 / a_3|\mathscr{F}_2) - \ldots - E(D'_n / a_n|\mathscr{F}_{n-1})$ and $X_n - X_{n-1} = \widetilde{D}_n$ for $n \geq 2$.

$E(X_n^2|\mathscr{F}_{n-1}) - X_{n-1}^2 \leq E((D'_n)^2/a_n^2|\mathscr{F}_{n-1}) = E((D_n^2/a_n^2)\ I(|D_n| \leq a_n)|\mathscr{F}_{n-1})$

$\leq E(\psi(D_n)/\psi(a_n)|\mathscr{F}_{n-1})$ since $\psi(x)/x^2 \geq \psi(y) / y^2$ for $0 < x \leq y$ by (i). Thus $(X_n, \mathscr{F}_n, n \geq 1)$ is a martingale with

$\Sigma_{n=2}^{\infty} E((X_n^2|\mathscr{F}_{n-1}) - X_{n-1}^2) = \Sigma_{n=2}^{\infty} E((\widetilde{D}_n)^2|\mathscr{F}_{n-1}) < \infty$ on the set where

$\Sigma_{n=2}^{\infty} E(\psi(D_n) / \psi(a_n)|\mathscr{F}_{n-1}) < \infty$. Hence by a martingale convergence theorem of Doob ([8], p. 320) it follows that X_n converges a.s. to a random variable as $n \to \infty$ on the set where $\Sigma_{n=2}^{\infty} E(\psi(D_n)/\psi(a_n)|\mathscr{F}_{n-1}) < \infty$.

Thus on the set where $\Sigma_{n=2}^{\infty} E(\psi(D_n)/\psi(a_n)|\mathscr{F}_{n-1}) < \infty$, it follows that $\Sigma_{n=1}^{\infty} D_n / a_n$ converges a.s..

REMARK. In the special case where $(D_n, n \geq 1)$ is a sequence of independent random variables, Theorem 3.4 reduces to a result of Chung [7]. The same remark applies to Corollary 3.10 given below.

COROLLARY 3.10. Let $(D_n, \mathcal{F}_n, n \geq 1)$ be a martingale difference sequence with $E|D_n|^{2r} (\log^+|D_n|)^{1+\epsilon} \leq M$ for constants $\epsilon > 0, 1/2 \leq r \leq 1$, and $M < \infty$. Then $\Sigma_{n=1}^m D_n / m^{1/2r}$ converges a.s. to zero as $m \to \infty$.

PROOF. Let $\psi(x) = |x|^{2r} (\log^+ |x|)^{1+\epsilon}$ and $a_n = n^{1/(2r)}$ in Theorem 3.4 and note that $\psi(x)$ satisfies the hypotheses of Theorem 3.4 under condition (ii).

$$\Sigma_{n=2}^{\infty} E(\psi(D_n) / \psi(a_n)) \leq \Sigma_{n=2}^{\infty} M/(n(\log^+ n^{1/2r})^{1+\epsilon}) < \infty \quad .$$

Hence $\Sigma_{n=1}^{\infty} D_n / n^{1/(2r)}$ converges a.s. and the result follows by an application of the Kronecker Lemma.

6. A Martingale Convergence Theorem Related to the Concept of Symmetry

Let $(X_n, \mathcal{F}_n, n \geq 1)$ be a stochastic sequence and $(D_n, \mathcal{F}_n, n \geq 1)$ be the difference sequence associated with it, i.e. $D_n = X_n - X_{n-1}$ for $n \geq 2$ and $D_1 = X_1$.

THEOREM 3.5. Let $A = [\sup X_n < \infty,\ \sup |D_n| < \infty,\ \Sigma_{n=1}^{\infty} E(D_n I(|D_n| > K)|\mathcal{F}_{n-1})$ converges for all integers $K \geq N$ for some integer $N]$. Then X_n converges a.s. on A.

PROOF. Fix an integer $K \geq N$. Let $X'_m = \Sigma_{n=1}^m (D_n I(|D_n| \leq K) - E(D_n I(|D_n| \leq K)|\mathcal{F}_{n-1})$. Then $(X'_n, \mathcal{F}_n, n \geq 1)$ is a martingale with its martingale differences uniformly bounded by 2K. Thus X'_n converges a.s. to a random variable on the set $\sup X'_n < \infty$ by a result of Doob ([8], p. 320). Let $A(K) = [|D_n| \leq K$ for all $n \geq 1]$. On

$A(K)$, $X'_m = \Sigma_{n=1}^m (D_n - E(D_n I(|D_n| \le K)|\mathcal{F}_{n-1})) = X_m - \Sigma_{n=1}^m E(D_n I(|D_n| \le K)|\mathcal{F}_{n-1})$. Thus on $AA(K)$ it follows that $\sup X'_n < \infty$. Hence X'_n converges a.s. on the set $AA(K)$. But $\Sigma_{n=1}^\infty E(D_n I(|D_n| \le K)|\mathcal{F}_{n-1})$ converges a.s. on $AA(K)$ by hypotheses. Hence X_n converges on $\bigcup_{K=N}^\infty AA(K)$.

$\sup |D_n| < \infty$ a.s. on A implies that $\bigcup_{K=N}^\infty AA(K) = A$. The result is established.

COROLLARY 3.11. Let $A = [\sup |X_n| < \infty, \Sigma_{n=1}^\infty E(D_n I(|D_n| > K)|\mathcal{F}_{n-1})$ converges a.s. for all integers $K \ge N$ for some integer $N]$. Then X_n converges on A.

PROOF. $\sup |X_n| < \infty$ implies $\sup |D_n| < \infty$. Hence Theorem 3.5 applies.

REMARK. We cannot drop $\sup |D_n| < \infty$ as a restrictive condition on the set A in the statement of Theorem 3.5 as the following example shows.

EXAMPLE 3.4. Let $(D_n, n \ge 1)$ be a sequence of independent random variables defined as follows:

$$D_n = \begin{cases} -\log n & \text{with probability } 1/n \\ n & \text{with probability } (\log n)/n^2 \\ 0 & \text{otherwise} \end{cases} .$$

We note that $\sup X_n < \infty$ a.s. and that $\Sigma_{n=1}^\infty E(D_n I(|D_n| > K))$ converges for all integers $K \ge 1$. But $\Sigma_{n=1}^\infty D_n$ clearly diverges a.s. to $-\infty$.

REMARK. Originally Theorem 3.1 and Corollaries 3.1 - 3.3 were stated for independent random variables. The author wishes to thank Y.S. Chow for pointing out that these results could be stated in the martingale case without essential modification of their proofs and also for pointing out Corollary 3.4.

CHAPTER IV

AN EXTENSION OF THE KOLMOGOROV LAW OF THE ITERATED LOGARITHM TO THE MARTINGALE CASE

1. Introduction

Throughout Chapter IV, we let $(X_n, \mathcal{F}_n, n \geq 1)$ be a martingale defined on a probability space $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}_n$ is the σ-field generated by $(X_1, \ldots, X_n)$. Throughout, we limit consideration to σ-fields which are generated by countable families of random variables. Let $D_n = X_n - X_{n-1}$ for $n \geq 1$ where $X_o \equiv 0$. We call $(D_n, \mathcal{F}_n, n \geq 1)$ a martingale difference sequence. Given a set A, let $I(A)$ be the indicator function of A and A' be the compliment of A. By $a_n \approx b_n$, we mean $\lim_{n\to\infty} a_n/b_n = 1$. Logarithms are always to the base ℓ. Let $\infty \uparrow s_n^2 = \Sigma_{i=1}^n ED_i^2$, $t_n = \sqrt{2 \log \log s_n^2}$, and $(K_n, n \geq 1)$ be a sequence of constants converging to zero. If $(D_n, n \geq 1)$ is in particular a sequence of independent random variables with $ED_n = 0$ and $|D_n|/s_n \leq K_n t_n^{-1}$ a.s., then $\limsup_{n\to\infty} X_n/(s_n t_n) = 1$ a.s. according to the well known Kolmogorov law of the iterated logarithm ([22], pp. 260-3). Theorems 4.1 and 4.2 below generalize this result to the martingale case. Assuming that $(D_n, \mathcal{F}_n, n \geq 1)$ is a martingale difference sequence, $|D_n|/s_n \leq K_n\, t_n^{-1}$ a.s., and $E(D_n^2|\mathcal{F}_{n-1}) \leq b^2 ED_n^2$ a.s. for $n \geq 1$ and some constant $b > 0$. Theorem 4.1 states that $\limsup_{n\to\infty} X_n/(s_n t_n) \leq b$, a.s. If in addition $|E(D_n^2|\mathcal{F}_{n-1})/ED_n^2 - 1| \leq r_n$ a.s. where $(r_n, n \geq 1)$ is a sequence of constants converging to zero, Theorem 4.2 states that the conclusion of the iterated logarithm holds in full strength; i.e., that $\operatorname*{Lim\,sup}_{n\to\infty} X_n/(s_n\, t_n) = 1$ a.s. As in the independent case, exponential inequalities provide the computational backbone of the proofs of these two theorems.

2. Generalization of the Iterated Logarithm to the Martingale Case

Throughout, since we deal only with a countable family of random variables and σ-fields generated by countable families of random variables, we may without loss of generality assume that the conditional probabilities considered are regular. (See [22], pp. 358-65 for a treatment of Doob's work on the existence of regular conditional probabilities.)

LEMMA 4.1. Let $(D_k, \mathcal{F}_k, 1 \leq k \leq n)$ be a sequence of martingale differences with $\mathcal{F}_o \subset \mathcal{F}_1$ a σ-field, not necessarily trivial. Let $c \geq \max_{1 \leq k \leq n} |D_k|/s_n$ a.s. where c is a constant. Assume there exists a constant $b > 0$ such that $E(D_k^2|\mathcal{F}_{k-1}) \leq b^2\, ED_k^2$ a.s. for $1 \leq k \leq n$. Fix $\epsilon > 0$. Then a.s.,

$$(1) \qquad P(X_n/(bs_n) > \epsilon|\mathcal{F}_o) \leq \begin{cases} \exp(-(\epsilon^2/2)(1-\epsilon c/(2b))) & \text{if } \epsilon\, c/b \leq 1 \\ \exp(-\epsilon b/(4c)) & \text{if } \epsilon\, c/b \geq 1 \end{cases} .$$

REMARK. Except for martingale considerations, the proof of Lemma 4.1 follows closely the proof of the corresponding result for independent random variables given in [22], pp. 254-7.

PROOF. Consider a random variable D and a σ-field $\mathcal{F}$ such that $|D| \leq c'$, $E(D|\mathcal{F}) = 0$, and $E(D^2|\mathcal{F}) \leq b^2\, ED^2$ a.s. for constants c' and $b > 0$. Choose $t > 0$ such that $t\, c' \leq 1$. Computing,

$E(\exp(t\,D)|\mathscr{F}) = 1 + t^2/2\; E(D^2|\mathscr{F}) + t^3/3!\; E(D^3|\mathscr{F}) + \ldots + \; \leq 1 +$
$t^2/2\; b^2\; ED^2 + t^3/3!\; c'b^2\; ED^2 + t^4/4!\; c'^2 b^2\; ED^2 + \ldots + = 1 +$
$t^2\; b^2\; ED^2/2\; (1 + t\; c'/3 + t^2\; c'^2/(3\cdot 4) + \ldots \leq 1 + t^2\; b^2\; ED^2/2$
$(1 + t\; c'/2) \leq \exp\; (t^2\; b^2\; ED^2/2\; (1 + t\; c'/2))$. Hence

(2) $$E(\exp(tD)|\mathscr{F}) \leq \exp(t^2 b^2\; ED^2/2\; (1 + t\; c'/2)) \quad \text{a.s. if} \quad t\; c' \leq 1.$$

$$E\; \exp(tX_n/(s_n b)|\mathscr{F}_{n-1}) = \exp(tX_{n-1}/(s_n b)) \cdot E(\exp(tD_n/(s_n b)|\mathscr{F}_{n-1})$$
$$\leq \exp(tX_{n-1}/(s_n b))\; \exp\; ((t^2/2)(1 + tc/(2b))\; ED_n^2/s_n^2) \text{ a.s.}$$
$$\text{for} \quad t\; c/b \leq 1$$

by (2) with $D = D_n/(s_n b)$ and $c' = c/b$. Taking conditional expectations on both sides of the above inequality with respect to $\mathscr{F}_{n-2}$ yields $E(\exp(tX_n/(s_n b)|\mathscr{F}_{n-2}) \leq \exp(tX_{n-2}/(s_n b))\; \exp((t^2/2)(1 + tc/(2b))$ $(ED_{n-1}^2 + ED_n^2)\; /\; s_n^2)$ a.s. Induction yields

(3) $$E(\exp(tX_n/(s_n b))|\mathscr{F}_o) \leq \exp((t^2/2)(1 + tc/(2b))) \text{ a.s. for } t\; c/b \leq 1.$$

$E(\exp(tX_n/(s_n b)|\mathscr{F}_o) = \int \exp(t\; x) dP(X_n/(s_n b) \leq x|\mathscr{F}_o)$ a.s. ([22], p. 360) using the fact that $P(X_n/(s_n b) \leq x|\mathscr{F}_o)$ is a regular conditional probability. Hence it follows by (3) that $P(X_n/(s_n b) > \epsilon|\mathscr{F}_o)$ $\leq \exp(-\epsilon t) E(\exp(tX_n/(s_n b))|\mathscr{F}_o) \leq \exp(-\epsilon t)\; \exp((t^2/2)(1 + tc/(2b)))$ a.s. if $t\; c/b \leq 1$. Thus

$$P(X_n/(s_n b) > \epsilon|\mathscr{F}_o) \leq \begin{cases} \exp\; (-(\epsilon^2/2)(1 - \epsilon c/(2b))) \text{ a.s. if } \; \epsilon\; c/b \leq 1 \\ \\ \exp\; (-\epsilon\; b/(4c)) \quad \text{a.s. if } \; \epsilon\; c/b > 1 \end{cases}$$

follows by setting $t = \epsilon$, $t = b/c$ respectively.

THEOREM 4.1. Let $(D_k, \mathscr{F}_k, k \geq 1)$ be a martingale difference sequence where $\mathscr{F}_k$ is the σ-field generated by $(D_1, D_2, \ldots, D_k)$ and such that $D_n/s_n \leq K_n t_n^{-1}$ a.s. and $s_n^2 \uparrow \infty$. Let $E(D_n^2 | \mathscr{F}_{n-1}) \leq b^2 ED_n^2$ a.s. for all $n \geq 1$ and some constant $b > 0$. Then $\lim\sup_{n\to\infty} X_n/(s_n t_n) \leq b$ a.s.

PROOF. We proceed as in the proof of the Kolmogorov iterated logarithm given in [22]. $s_n^2 \uparrow \infty$, $s_{n+1}^2 / s_n^2 \to 1$ implies for every $p > 1$, there exists a sequence of integers $\infty \uparrow n_k(p) \equiv n_k$ such that $s_{n_k} \approx p^k$. Choose $\delta > 0$. We show that $P[X_n > (1+\delta) s_n t_n b \text{ i.o.}] = 0$. Let $X^*_{n_k} = \operatorname{Max}_{n \leq n_k} X_n$. $P[X_n > (1+\delta) s_n t_n b \text{ i.o.}] \leq P[X^*_{n_k} > (1+\delta) s_{n_{k-1}} t_{n_{k-1}} b \text{ i.o.}]$. Noting that $(1+\delta) s_{n_{k-1}} t_{n_{k-1}} \approx ((1+\delta)/p) s_{n_k} t_{n_k}$, we choose $0 < \delta' < \delta$ and $p > 1$ such that $(1+\delta)/p > 1 + \delta'$. Then for k sufficiently large, $P[X^*_{n_k} > (1+\delta) b\, s_{n_{k-1}} t_{n_{k-1}}] \leq P[X^*_{n_k} > (1+\delta') b\, s_{n_k} t_{n_k}]$. Clearly it suffices to prove that $\Sigma_{k=1}^{\infty} P[X^*_{n_k} > (1+\delta') b\, s_{n_k} t_{n_k}]$.

We now derive a Levy type inequality. Let $B_n = [X_{n_k} - X_n + \sqrt{2b^2 E(X_{n_k} - X_n)^2} \geq 0]$ for $n \leq n_k$, $X'_n = \operatorname{Max}_{j \leq n} (X_j - \sqrt{2b^2 E(X_j - X_{n_k})^2})$ for $n \leq n_k$, and $A_n = [X'_{n-1} < \epsilon,\ X_n - \sqrt{2b^2 E(X_n - X_{n_k})^2} \geq c]$ for $n \leq n_k$ and some $\epsilon > 0$ to be chosen later.

$$P[B'_n | \mathscr{F}_n] \leq \frac{E((X_{n_k} - X_n)^2 | \mathscr{F}_k)}{2b^2 E(X_{n_k} - X_n)^2} \leq \frac{1}{2}$$

since $E((X_{n_k} - X_n)^2 \mid \mathcal{F}_n) = E(\sum_{j=n+1}^{n_k} D_j^2 \mid \mathcal{F}_n) = E(E(D_{n_k}^2 \mid \mathcal{F}_{n_{k-1}}) \mid \mathcal{F}_n) +$
$E(E(D_{n_{k-1}}^2 \mid \mathcal{F}_{n_{k-2}}) \mid \mathcal{F}_n) + \ldots + E(D_{n+1}^2 \mid \mathcal{F}_n) \leq b^2 \sum_{j=n+1}^{n_k} ED_j^2 = b^2\, E(X_{n_k} - X_n)^2$.
Note that $[X_{n_k} \geq \epsilon] \supset \bigcup_{n=1}^{n_k} A_n B_n$. Hence $P[X_{n_k} \geq \epsilon] \geq$
$E\, \Sigma_{n=1}^{n_k} E(I(A_n)\, I(B_n) \mid \mathcal{F}_n) \geq \frac{1}{2} E\, \Sigma_{n=1}^{n_k} I(A_n) = \frac{1}{2} P[X'_{n_k} \geq \epsilon]$. Let $\epsilon =$
$b(1 + \delta')\, s_{n_k} t_{n_k} - \sqrt{2}\, b\, s_{n_k}$. Then $2\, P[X_{n_k} \geq b(1 + \delta')\, s_{n_k} t_{n_k} -$
$\sqrt{2}b\, s_{n_k}] \geq P[\max_{n \leq n_k} (X_n - \sqrt{2b^2\, E(X_n - X_{n_k})^2}) \geq b(1 + \delta')\, s_{n_k} t_{n_k} - \sqrt{2}b\, s_{n_k}]$
$\geq P[\max_{n \leq n_k} X_n \geq b(1 + \delta')\, s_{n_k} t_{n_k}]$. Also $P[X_{n_k} \geq b(1 + \delta'')\, s_{n_k} t_{n_k}] \geq$
$P[X_{n_k} \geq b(1 + \delta')\, s_{n_k} t_{n_k} - \sqrt{2}\, b\, s_{n_k}]$ for $0 < \delta'' < \delta'$ and k sufficiently large. Thus $2P[X_{n_k} \geq b(1 + \delta'')\, s_{n_k} t_{n_k}] \geq P[X^*_{n_k} \geq$
$b(1 + \delta')\, s_{n_k} t_{n_k}]$. Now, according to Lemma 4.1, $2P[X_{n_k} \geq b(1+\delta'')$
$s_{n_k} t_{n_k}] \leq 2 \exp[-(1 + \delta'')^2\, t_{n_k}^2 / 2\, (1 - (1 + \delta'')\, t_{n_k}\, c_k/(2b))]$ where c_k
is minimum constant such that $c_k \geq \max_{j \leq n_k} |D_j| / s_{n_k}$ a.s. and $\mathcal{F}_0 = (\Omega, \phi)$.
Here we have used the fact that $c_k\, t_{n_k} \to 0$ which is a consequence of the hypothesis that $|D_n| / s_n \leq K\, t_n^{-1}$ a.s. Given any $\epsilon > 0$,
$\exp[-(1+\delta'')^2 (t_{n_k}^2 / 2)(1-(1+\delta'')\, t_{n_k}\, c_k / (2b))] \leq \exp(-(1+\delta'')^2\, t_{n_k}^2 (1-\epsilon)/2)$
for k sufficiently large.

Choose $\epsilon > 0$ such that $\eta \equiv (1 + \delta'')^2 (1-\epsilon) > 1$. Then
$\exp(-\eta\, t_{n_k}^2/2) \approx 1/(2k \log p)^{\eta}$. Thus $\Sigma_{k=1}^{\infty} P[X_{n_k} > b(1 + \delta'')\, s_{n_k} t_{n_k}]$
$< \infty$ and the desired result follows.

LEMMA 4.2. Let $(D_k, \mathcal{F}_k, 1 \leq k \leq n)$ be a sequence of martingale differences with $\mathcal{F}_o \subset \mathcal{F}_1$ a σ-field, not necessarily trivial. Let $c \geq \underset{1 \leq k \leq n}{\text{Max}} |D_k|/s_n$ a.s. where c is a constant. Assume there exists constants $a > 0$ and $b > 0$ such that $a^2\, ED_k^2 \leq E(D_k^2|\mathcal{F}_{k-1}) \leq b^2\, ED_k^2$ a.s. Then for a constant $\gamma > 0$, there exists a function $g(\gamma) < 1$ with $\lim_{\gamma \to 0} g(\gamma) = 1$ [3] such that $a/b > g(\gamma)$, $c = c(\gamma)$ sufficiently small and $\epsilon = \epsilon(\gamma)$ sufficiently large implies that $P[X_n/(s_n a) > \epsilon|\mathcal{F}_o] > \exp(-\epsilon^2/2\ (1+\gamma))$ a.s.

REMARK. Except for martingale considerations similar to those used in the proof of Lemma 4.1, the proof of Lemma 4.2 follows closely the proof of the corresponding result for independent random variables given in [22], pp. 254-7.

PROOF. Consider an arbitrary $t > 0$, random variable D, and a σ-field $\mathcal{F}$ such that $|D| \leq c'$, $E(D|\mathcal{F}) = 0$, and $E(D^2|\mathcal{F}) \geq a^2\, ED^2$ a.s. for positive constants a and c'. Computation shows that

$$(4)\qquad E(\exp(tD)|\mathcal{F}) \geq 1 + t^2a^2ED^2/2\ (1 - t\,c'/2)$$
$$\geq \exp(t^2a^2ED^2(1-tc')/2) \text{ a.s. for } 0 < t\,c' \leq 1.$$

$E(\exp(tX_n/(s_na))|\mathcal{F}_{n-1}) = \exp(tX_{n-1}/(s_na))\, E(\exp(tD_n/(s_na)|\mathcal{F}_{n-1}) \geq \exp(tX_{n-1}/(s_na))\, \exp((t^2/2)ED_n^2(1-t\,c/a)/s_n^2)$ a.s. for $t\,c/a \leq 1$ by (4) with $D = D_j/(a\, s_n)$ and $c' = c/a$. Induction yields $E \exp(t\, X_n/(s_na)|\mathcal{F}_o) \geq \exp((t^2/2)(1-t\,c/a))$ a.s. for $t\,c/a \leq 1$.

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[3] The actual form of g is somewhat complicated and unimportant. Choose $1 > \beta > 0$ such that $(1+\beta^2/2+2\beta)/(1-\beta)^2 \leq 1+\gamma$. Then $1 > g(\gamma) > 0$ and $g^2(\gamma) > \text{Max}((1+2\beta+\beta^2/1.5)/(1+\beta)^2,\ (1-2\beta+\beta^2/1.5)/(1-\beta)^2)$ are a pair of conditions sufficient to determine an acceptable g.

Thus given any t and $\alpha > 0$ we can choose c sufficiently small so that

(5) $$E(\exp(t X_n/(s_n a)|\mathcal{F}_o) \geq \exp((t^2/2)(1-\alpha)) \quad \text{a.s.}$$

Using the regularity of the conditional probabilities involved, we may write

$$E(\exp(tX_n/(s_n a))|\mathcal{F}_o) = -\int_{-\infty}^{\infty} \exp(t\ x)\ d\ P(X_n/(a\ s_n) \geq x|\mathcal{F}_o) \quad \text{a.s.}$$

$$= t\int_{-\infty}^{\infty} P(X_n/(a\ s_n) \geq x|\mathcal{F}_o) \exp(t\ x)\ dx \quad \text{a.s.}$$

$$= t\left[\int_{-\infty}^{o} + \int_{o}^{t(1-\beta)} + \int_{t(1-\beta)}^{t(1+\beta)} + \int_{t(1+\beta)}^{8tb^2/a^2} + \int_{8tb^2/a^2}^{\infty}\right]$$

$$P(X_n/(a\ s_n) \geq x|\mathcal{F}_o) \exp\ (t\ x)\ dx \equiv J_1 + \dots + J_5 \ .$$

We estimate the five integrals separately.

J_1: $t\int_{-\infty}^{o} P(X_n/(a\ s_n) \geq x|\mathcal{F}_o) \exp(t\ x)dx \leq t\int_{-\infty}^{o} \exp(t\ x)\ dx = 1$ a.s.

J_5: By Lemma 4.1, $P(X_n/(b\ s_n) \geq x\ a/b|\mathcal{F}_o) \leq \exp(-x\ a/(4c)) \leq \exp(-2t\ x)$ a.s. if $x\ a\ c/b^2 \geq 1$ where c is chosen such that $c \leq a/(8t)$. By Lemma 4.1, $P(X_n/(b\ s_n) \geq x\ a/b|\mathcal{F}_o) \leq \exp((-x^2a^2/(2b^2))(1-x\ a\ c/(2b^2))) \leq \exp(-x^2\ a^2/(4b^2))$ a.s. if $x\ a\ c/b^2 < 1$. Since $x > 8t\ b^2/a^2$ for J_5, it follows that $P(X_n/(b\ s_n) \geq x\ a/b|\mathcal{F}_o) \leq \exp(-2t\ x)$ a.s. when $x\ a\ c/b^2 < 1$. Thus $J_5 \leq \int_{8t\ b^2/a^2}^{\infty} e^{-tx}\ dx \leq 1$ a.s.

J_2 and J_4: We choose c sufficiently small such that $(8t\ b^2/a^2)$ $(a\ c/b^2) \leq 1$. Since $x \leq 8t\ b^2/a^2$ for J_2 and J_4, it follows that

$x\ a\ c/b^2 \leq 1$. Thus by Lemma 4.1, $\exp(t\ x)\ P(X_n/(b\ s_n) > a\ x/b|\mathcal{F}_o) \leq \exp(tx-a^2x^2/(2b^2)\ (1-4c\ t/a))$ a.s. Let $h(x) = t\ x\ -(a^2/(2b^2))(1-4c\ t/a)\ x^2$. $h(x)$ is maximized at $x' = t/((a^2/b^2)(1-4ct/a))$. Now we choose c sufficiently small so that x' lies in the interval $[t(1-\beta),\ t(1+\beta)]$ of J_3. This is possible since $b^2/a^2 < 1 + \beta$ by definition of g. Then $h(x) \leq h(t(1-\beta))$ for x lying in interval $[0,t(1-\beta)]$ of J_2 and $h(x) \leq h(t(1+\beta))$ for x lying in interval $[t(1+\beta),\ 8tb^2/a^2]$ of J_4. $h(t(1-\beta))=(t^2/2)(1-\beta)[2-(a^2/b^2)(1-\beta) + 4c\ t(a/b^2)(1-\beta)]$. By definition of g, $a^2/b^2 > (1-2\beta+\beta^2/1.5)/(1-\beta^2)$ and hence $2 - a^2/b^2(1-\beta) \leq (1-\beta^2/1.5)\ /\ (1-\beta)$. Now we choose c sufficiently small so that $h(t(1-\beta)) \leq (t^2/2)(1-\beta^2/2)$. Combining,

$$J_2 \leq t\int_o^{t(1-\beta)} \exp(h(x))dx \leq t^2\ \exp((t^2/2)(1-\beta^2/2))\quad \text{a.s.}$$

Similarly, for J_4, $h(t(1+\beta)) = (t^2/2)(1+\beta)[2-(a^2/b^2)(1+\beta) + (4c\ t\ a/b^2)(1+\beta)]$ .. By definition of g, $a^2/b^2 > (1+2\beta+\beta^2/1.5)\ /\ (1+\beta)^2$ and hence $2-(a^2/b^2)(1+\beta) < (1-\beta^2/1.5)\ /\ (1+\beta)$. Now we choose c sufficiently small so that $h(t(1+\beta)) \leq (t^2/2)(1-\beta^2/2)$. Combining,

$$J_4 \leq t\int_{t(1+\beta)}^{8tb^2/a^2} \exp(h(x))dx \leq 16t^2\ \exp((t^2/2)(1-\beta^2/2)) \quad .$$

Here we use the fact that $b^2/a^2 \leq 2$ which follows by the definition of g. We now choose c sufficiently small so that (5) holds with $\alpha = \beta^2/4$. Then, it follows that $J_2 + J_4 \leq 17t^2\ \exp((t^2/2)(1-\alpha))\ \exp(-t^2\ \alpha/2) \leq 17t^2\ E(\exp(t\ X_n/(s_na))|\mathcal{F}_o)\ \exp(-t^2\ \alpha/2)$ a.s.

$$J_3: \quad t\int_{t(1-\beta)}^{t(1+\beta)} \exp(tx)\ P[X_n/(s_n\ a) \geq x|\mathcal{F}_o]dx$$

$$\leq t^2(2\beta)\ \exp(t^2(1+\beta))\ P(X_n/(s_na) \geq t(1-\beta)|\mathcal{F}_o) \quad .$$

Let $\epsilon = t(1-\beta)$. Combining the above, it follows for t chosen sufficiently large and c chosen sufficiently small that

$$J_2 + J_4 \leq 1/4\ E(\exp(t\ X_n / (s_n\ a))\ |\mathcal{F}_o) \quad \text{a.s.} \quad \text{and}$$

$$J_1 + J_5 \leq 2 \leq 1/4\ E(\exp(t\ X_n / (s_n\ a))\ |\mathcal{F}_o) \quad \text{a.s.}\ .$$

Thus, for c sufficiently small and ϵ sufficiently large,

$$1/2 \exp((t^2 / 2)(1 - \alpha)) \leq 1/2\ E(\exp(t\ X_n / (s_n\ a))|\mathcal{F}_o)$$

$$\leq J_3 \leq t^2\ 2\ \beta \exp(t^2(1+\beta))\ P(X_n / (s_n\ a) > \epsilon|\mathcal{F}_o) \quad \text{a.s.}\ ;$$

i.e., $P(X_n/(s_n\ a) > \epsilon|\mathcal{F}_o) \geq 1/(4\beta^2 t^2) \exp(-(t^2/2)(1+\beta^2/4+2\beta))$

$$\geq \exp(-\ \epsilon^2/(2(1-\beta)^2)(1+\beta^2 / 2+2\beta)) \quad \text{a.s.}$$

But $(1+\beta^2/2+2\beta) / (1-\beta)^2 \leq 1 + \gamma$ by hypothesis and hence $P[X_n / (s_n\ a) > \epsilon|\mathcal{F}_o) > \exp(-\ \epsilon^2 / 2\ (1+\gamma))$ a.s. for c sufficiently small and ϵ sufficiently large. The result is established.

THEOREM 4.2. Let $(D_k, \mathcal{F}_k, k \geq 1)$ be a martingale difference sequence where $\mathcal{F}_k$ is the σ - field generated by $(D_1, D_2, \ldots, D_k)$ and such that $D_n / s_n \leq K_n t_n^{-1}$ a.s. and $s_n^2 \uparrow \infty$. Assume there exists a sequence of constants $(r_n, n \geq 1)$ converging to 0 such that $|(E[D_n^2|\mathcal{F}_{n-1}] / ED_n^2) - 1| \leq r_n$ a.s.. Then $\operatorname{Lim\,sup}_{n \to \infty} X_n / (s_n\ t_n) = 1$ a.s.

PROOF. We again proceed as in the proof of the Kolmogorov iterated logarithm given in [22]. By hypothesis, there exists constants $\{a_k, k \geq 1\}$ and $\{b_k, k \geq 1\}$ with $1 > a_k \to 1$ and $1 < b_k \to 1$ such that $a_k^2\ ED_k^2 \leq E[D_k^2\ |\ \mathcal{F}_{k-1}] \leq b_k^2\ ED_k^2$ a.s. . Fix an integer N.

By Theorem 4.1, $\operatorname{Lim\,sup}_{n \to \infty}((X_n - X_N)/\sqrt{2(s_n^2 - s_N^2) \log\log (s_n^2 - s_N^2)}) \leq \sup_{k \geq N} b_k \equiv b_N^*$.

But b_N^* converges to 1 as N approaches ∞ by hypothesis.

Since $X_N / \sqrt{2(s_n^2 - s_N^2) \log\log (s_n^2 - s_N^2)}$ converges to zero as n approaches ∞ with N fixed, it follows that $\operatorname*{Lim\,sup}_{n \to \infty} x_n/(s_n t_n) \leq 1$ a.s..

To complete the proof, it is sufficient to show for $\delta > 0$ that $(1-\delta)s_n t_n$ belongs to the lower class of X_n, i.e. that $P[X_n > (1-\delta)s_n t_n \text{ i.o.}] = 1$. We choose a sequence of integers $\infty \uparrow n_k(p) \equiv n_k$ such that $s_{n_k} \approx p^k$ for $p > 1$ to be chosen later.

It is sufficient to show $P[X_{n_k} > (1-\delta) s_{n_k} t_{n_k} \text{ i.o.}]$. Let $u_k^2 = s_{n_k}^2 - s_{n_{k-1}}^2 \approx s_{n_k}^2 (1-\frac{1}{p^2})$ and $v_k = (2 \log\log u_k^2)^{1/2} \approx t_{n_k}$. Let the event $A_k(\delta'') = [X_{n_k} - X_{n_{k-1}} > (1-\delta'') u_k v_k a_k^*]$ where $0 < \delta'' < \delta$ and $a_k^* = \operatorname*{Min}_{n_{k-1} < n \leq n_k} a_n$. We claim that $P[A_k(\delta'') \text{ i.o.}] = 1$. Since $(\Sigma_{j=1}^k (I(A_j(\delta'')) - P(A_j(\delta'')|\mathscr{F}_{n_{j-1}})), \mathscr{F}_{n_k}, k \geq 1)$ is a martingale, it follows easily by a result of Doob ([8], p. 320), that $\Sigma_{k=1}^\infty P(A_k(\delta'')|\mathscr{F}_{n_{k-1}}) = \infty$ a.s. implies that $P[A_k(\delta'') \text{ i.o.}] = 1$. Let $\epsilon_k = (1-\delta)v_k$, c_k be the minimum constant such that $c_k \geq \operatorname*{Max}_{n_{k-1} < n \leq n_k} |D_n| / u_k$, and $1 + \gamma = 1/(1-\delta'')$ define γ. Since $\epsilon_k \to \infty$, $c_k \to 0$, and $1 > a_k / b_k \to 1$ as $k \to \infty$, we may apply Lemma 4.2 to obtain

$$P((X_{n_k} - X_{n_{k-1}}) / (u_k a_k^*) > (1-\delta'') v_k | \mathscr{F}_{n_{k-1}}) \geq \exp(-(1-\delta'')^2 v_k^2 (1+\gamma)/2) \text{ a.s.}$$

for k sufficiently large. Since $\exp(-(1-\delta'')^2 v_k^2 (1+\delta)/2) \approx 1/(2k \log p)^{1-\delta''}$, it follows that $\Sigma_{k=1}^\infty P(A_k(\delta'')|\mathscr{F}_{n_{k-1}}) = \infty$ a.s. and hence that $P[A_k(\delta'') \text{ i.o.}] = 1$.

Choosing $\delta' \ni \delta > \delta' > \delta''$ and noting that $a_k^* \to 1$ as $k \to \infty$, it is clear that $P[X_{n_k} - X_{n_{k-1}} > (1-\delta')u_k v_k \text{ i.o.}] = 1$. Let the event $B_k = \{|X_{n_{k-1}}| \leq 2s_{n_{k-1}} t_{n_{k-1}}\}$. Without loss of generality we can and do assume $b \equiv \sup b_k \leq 3/2$. Then, applying Theorem 4.1, $P(B_k' \text{ i.o.}) = 0$. Thus $P[\{X_{n_k} - X_{n_{k-1}} > (1-\delta')u_k v_k\} B_k \text{ i.o.}] = 1$.

$\{X_{n_k} - X_{n_{k-1}} > (1-\delta') u_k v_k\} B_k \subset \{X_{n_k} > (1-\delta') v_k u_k - 2s_{n_{k-1}} t_{n_{k-1}}\}$

and $(1-\delta')u_k v_k - 2s_{n_{k-1}} t_{n_{k-1}} \approx ((1-\delta')(1-1/p^2)^{1/2} - 2/p)s_{n_k} t_{n_k}$. We now choose p sufficiently large so that $(1-\delta')(1-1/p^2)^{1/2} - 2/p > 1-\delta$. Then $[\{X_{n_k} - X_{n_{k-1}} > (1-\delta')u_k v_k)\} B_k \text{ i.o.}] \subset [X_{n_k} > (1-\delta)s_{n_k} t_{n_k} \text{ i.o.}]$ and hence the result is established.

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13. ABSTRACT Let $(\Omega,\mathcal{F},P)$ be a probability space, $(D_n, n \geq 1)$ be a sequence of independent random variables, a_{nk} be a matrix of real numbers, $T_{nm} = \sum_{k=1}^{m} a_{nk}D_k$, and T_n be the almost sure limit as $m \to \infty$ when it exists. T_n is said to converge completely to zero [15] if $\sum_{n=1}^{\infty} P[|T_n| > \epsilon] < \infty$ for all $\epsilon > 0$. Various conditions are given for the complete or almost sure converence of T_n to zero, extending or improving results given by Hsu and Robbins [15], Erdos [10], Pruitt [24], and Chow [6]. According to Chow [6] a random variable D is generalized Gaussian if there exists an $\alpha \geq 0$ such that for every real t, $E \exp(tD) \leq \exp(t^2\alpha^2/2)$. In Chapter II, we extend to the martingale case a result of Chow [6] concerning the complete convergence of T_n to zero where the D_n's are generalized Gaussian. Other almost sure convergence results for generalized Gaussian random variables are also given. In Chapter III a number of almost sure convergence results are established in the martingale case. In particular a result of Szynal [25] about independent random variables is corrected and extended to the martingale case. A strong law of large numbers for independent random variables due to Chung [7] is also extended to the martingale case. Ideas of Burkholder [1] and Chow [3] are combined to yield a new martingale convergence theorem. In Chapter IV an extension of the Kolmogorov law of the iterated logarithm to the martingale case is made.

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