On the exact distribution of Pillai's $V^{(s)}$ criterion*

by

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1. Introduction and Summary. The joint distribution of s non-null characteristic roots of a matrix in multivariate analysis given by Fisher [1], Girshick [2], Hsu [3] and Roy [12] can be expressed in the form

$$f(\theta_1,\dots,\theta_s) = C(s,m,n) \prod_{i=1}^{s} \theta_i^m (1-\theta_i)^n \prod_{i>j} (\theta_i - \theta_j) \quad , \tag{1.1}$$

$$0 < \theta_1 \le \cdots \le \theta_s < 1 \quad ,$$

where

$$C(s,m,n) = \{\Pi^{\frac{1}{2}s} \prod_{i=1}^{s} \Gamma[\tfrac{1}{2}(2m+2n+s+i+2)]\} / \tag{1.2}$$

$$\{\prod_{i=1}^{s} \Gamma[\tfrac{1}{2}(2m+i+1)]\Gamma[\tfrac{1}{2}(2n+i+1)\Gamma(\tfrac{1}{2}i)\} \quad ,$$

and the parameters m and n are defined differently for various situations as described by Pillai [7], [9]. Now Pillai's $V^{(s)}$ criterion may be defined as $\sum_{i=1}^{s} \theta_i$. In this paper an attempt is made to obtain the exact cdf of $V^{(s)}$ extending the work of Nanda [5] who derived its cdf in the special cases of $m = 0$ and $s = 2$ and 3. Explicit expressions for the cdf are obtained in this paper for $s = 3$ and integral values of $m \le 3$, and $s = 4$ and $m = 0$

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and 1. Further, exact upper percentage points of $V^{(s)}$ are presented for values of s and m given above and selected values of n. It may be pointed out that the exact cdf for $s = 2$ was obtained by Mikhail [4] and the exact non-central cdf for $s = 2$ for different tests by Pillai and Jayachandran [11].

2. Nanda's method for $m = 0$. In this section, the method of approach of Nanda [5] will be given briefly in order to describe the extension of the method in the next section. In (1.1) transform $g_i = n\theta_i$, $i = 1,2,\ldots,s$, and let $n \to \infty$, then

$$(2.1) \qquad f_1(g_1,g_2,\ldots,g_s) = \kappa(s,m) \prod_{i=1}^{s} (e^{-g_i} g_i^m) \prod_{i>j} (g_i - g_j) \quad ,$$

$$0 < g_1 \leq \cdots \leq g_s < \infty \quad ,$$

where

$$\kappa(s,m) = \pi^{s/2} \Big/ \Big\{ \prod_{i=1}^{s} \Gamma[\tfrac{1}{2}(2m+i+1)]\Gamma(\tfrac{1}{2}i) \Big\} \quad .$$

Now consider the cdf of the largest root, g_s, and transform $xy_i = g_i$, we get

$$(2.2) \qquad \int_{0<g_1\leq\cdots\leq g_s<x} \cdots \int \prod_{i=1}^{s} (e^{-g_i} g_i^m) \prod_{i>j} (g_i - g_j) \prod_{i=1}^{s} dg_i$$

$$= x^{ms+\frac{1}{2}s(s+1)} \int_{0<y_1\leq\cdots\leq y_s<1} \cdots \int e^{-x\sum_{i=1}^{s} y_i} \prod_{i=1}^{s} y_i^m \prod_{i>j} (y_i - y_j) \prod_{i=1}^{s} dy_i \quad .$$

Now replace y_i by $1 - y_i$ $(i = 1,2,\ldots,s)$ on the right side of (2.2) and then change m to n on both sides, we get

$$\text{(2.3)} \qquad C(s,0,n)\, V(x;s-1,s-2,\ldots,1,0;-1) = e^{-sx}\, x^{sn+\frac{1}{2}s(s+1)}\, M(x,0,n,s) \quad ,$$

where $M(x,0,n,s)$ denotes the moment generating function of $V^{(s)}$ when $m = 0$ and $V(x;\, q_p,\ldots,q_1;t)$ is given by

$$\text{(2.4)} \qquad V(x;\, q_s,\ldots,\, q_1;t) = \begin{vmatrix} \int_o^x y_s^{n+q_s}\, e^{ty_s}\, dy_s & \cdots & \int_o^x y_s^{n+q_1}\, e^{ty_s}\, dy_s \\ \int_o^{y_2} y_1^{n+q_s}\, e^{ty_1}\, dy_1 & \cdots & \int_o^{y_2} y_1^{n+q_1}\, e^{ty_1}\, dy_1 \end{vmatrix} \quad .$$

Now for illustrating the method further let us consider $s = 3$ which is as far as Nanda has proceeded. Thus

$$\text{(2.5)} \qquad M(x,0,n,3) = C(3,0,n)\, e^{3x}\, x^{-(3n+6)}\, V(x;2,1,0;-1) \quad .$$

Now it can be shown that

$$\text{(2.6)} \qquad V(x;2,1,0;-1) = 2\, I(x;2n+3,-2)\, I(x;n,-1) - 2\, I(x;2n+2,-2)\, I(x;n+1,-1)$$
$$+ (n+1)^{-1}[-2I_o(x;n+2,-1)\, I(x;2n+2,-2) + I_o(x;2n+3,-2)\, I(x;n+1,-1)],$$

where

$$I(x;q,t) = \int_o^x e^{ty}\, y^q\, dy$$

and

$$I_o(x;q,t) = e^{ty}\, y^q \Big|_o^x \quad .$$

Further, put $y = xu$ and we get from (2.6)

$$(2.7)\qquad V(x;2,1,0;-1) = x^{3n+5}\,[2I(1;2n+3,-2x)\,I(1;n,-x)$$

$$-\,2I(1;2n+2,-2x)\,I(1;n+1,-x)$$

$$-\,(n+1)^{-1}\,\{-2e^{-x}\,I(1;2n+2,-2x) + e^{-2x}\,I(1;n+1,-x)\}]\ .$$

By integrating by parts we can get x^{3n+6} as a common factor on the right side of the above equation. We use the value of $V(x;2,1,0;-1)$ thus obtained in (2.5) to get

$$(2.8)\qquad M(x,0,n,3) = C(3,0,n)\,[I(n+2,x) / \{(n+2)(2n+3)\} + 2I(2n+4,2x) / (n+2)$$

$$-\,4I(2n+3,2x) / (2n+3) + 2I(2n+3,2x)\,I(n+1,x) /$$

$$(2n+3)] / (n+1)\ ,$$

where

$$I(q,\alpha x) = \int_0^1 e^{\alpha x(1-u)}\,u^q\,du = \int_0^1 e^{\alpha ux}\,(1-u)^q\,du\ .$$

As the mgf is known, the cdf can be obtained, once we know the contribution of each integral on the right side of (2.8). We get corresponding terms for $I(n+\ell,x)$ and $I(2n+k,2x)$ by integrating the density $(1-u)^{n+\ell}$ of u and $(1-u)^{2n+k}$ of $2u$ respectively. Hence

$$(2.9)\qquad I(n+\ell,x) \longrightarrow \begin{cases} \{1-(1-z)^{n+\ell+1}\}/(n+\ell+1) & 0 \le z \le 1 \\ (n+\ell+1)^{-1} & 0 \le z \le 2 \end{cases}$$

and

(2.10) $$I(2n+k,2x) \longrightarrow \begin{cases} \{1-(1-z/2)^{2n+k+1}\} \;/\; (2n+k+1) & 0 \le z \le 2 \\ (2n+k+1)^{-1} & 2 \le z \le 3 \end{cases} .$$

However, to get the contribution of $I(n+\ell,x)\ I(2n+k,2x)$ we consider y_1 and y_2, two random variables, distributed in $(0,1)$ with densities $(1-y_1)^{n+\ell}$ and $(1-y_2)^{2n+k}$. Then

(2.11) $$I(n+\ell,x)\ I(2n+k,2x) \to \int\int_{y_1+2y_2 \le z} (1-y_1)^{n+\ell}(1-y_2)^{2n+k}\, dy_1\, dy_2 \;, \qquad 0 \le z \le 3 \;.$$

The integral in (2.11) is obtained through geometrical consideration of the region of integration (See Section 6 for a similar case) in intervals $0 \le z \le 1$, $1 \le z \le 2$ and $2 \le z \le 3$ separately (See Nanda [5]). Thus, for $m = 0$

(2.12) $$\Pr(V^{(3)} \le z) \;=\; 1 - C(3,0,n)\, W(3,0)$$

where

$$W(3,0) = \begin{cases} p_{0\,3}(1-z)^{n+3} + \sum_{j=4}^{5} q_{0\,j}\,(1-\tfrac{1}{2}z)^{2n+j} + \omega_{0\,3\,2}\, B(z_1,z_2;\ 2n+4,n+3) & 0 \le z \le 1 \\ \Big[\ \sum_{j=4}^{5} q_{0\,j}\,(1-\tfrac{1}{2}z)^{2n+j} + \omega_{0\,3\,2}\, B(z_1,1;2n+4,n+3) & 0 \le z \le 2 \\ \omega_{0\,3\,2}\, B(0,1;2n+4,n+3) & 2 \le z \le 3 \end{cases} ,$$

where

$$\omega_{m\,k\,\ell} = 2^{n+\ell} z_2^{-(3n+k+\ell+1)}\, h_{m\,k\,\ell}, \quad z_1 = (2-z)/(3-z), \quad z_2 = 2(3-z)^{-1} \quad ,$$

$$p_{0\,3} = \{(n+1)(n+2)(n+3)(2n+3)\}^{-1}, \qquad q_{0\,4} = -\,\{(n+1)(n+2)^2\}^{-1} \quad ,$$

$$q_{0\,5} = 2\{(n+1)(n+2)(2n+5)\}^{-1}, \quad h_{m\,3\,2} = 2\{(n+1)(n+2)(2n+3)\}^{-1}, \quad m = 0,1,2,3,$$

and

$$B(a,b,c,d) = \int_a^b x^{c-1}\,(1-x)^{d-1}\,dx \quad .$$

3. Extension of Nanda's method. In order to obtain the mgf of $V^{(s)}$ for larger values of m, we proceed as follows: Multiply the right side of (2.1) by the factor $\prod_{i=1}^{s} (1-g_i)^n$ and replace $\kappa(s,m)$ by $C(s,m,n)$. Now, as in (2.3), interchanging m and n, let us consider the following integral thus obtained:

$$(3.1) \qquad \int\!\cdots\!\int_{0<g_1\le\cdots\le g_s<x} \prod_{i=1}^{s} (e^{-g_i} g_i^n) \prod_{i>j} (g_i - g_j) \sum_{i_0+\cdots+i_s = m} (m!/i_0!\cdots i_s!)(-a_1)^{i_1}\cdots ((-1)^s a_s)^{i_s} \prod_{i=1}^{s} dg_i \quad ,$$

where $a_i\,(i = 1,\ldots,s)$ is the ith elementary symmetric function (esf) in s g's. Now using Pillai's lemma [10] on the multiplication of a basic Vandermonde determinant by powers of esf's, transforming $x\,y_i = g_i$ first and then y_i to $1 - y_i$, we can express the mgf in the form

$$(3.2) \qquad M(x,m,n,s) = e^{sx}\, x^{-sn-\frac{1}{2}s(s+1)} \sum_{\delta} C_\delta V(x; q_s, q_{s-1}, \ldots, q_1; -1)\;\; x^{-\alpha} \quad ,$$

where $\sum_\delta$ denotes summation over all possible determinants obtained by using Pillai's Lemma [10], $\alpha = \sum_{j=1}^{s} (q_j - j + 1)$, and C_δ are constants independent of x.

Next step is to evaluate the determinants under $\sum_\delta$. For that we apply Pillai's reduction formula [8] which in the present context may be stated as follows:

Lemma 1. The determinant $V(x;q_s,q_{s-1},\ldots,q_1,t) = A^{(s)} + B^{(s)} + (n+q_s)\, C^{(s)}$, where

$$A^{(s)} = -I_0(x;q_s,t)\, V(x;q_{s-1},q_{s-2},\ldots,q_1;t) \quad ,$$

$$B^{(s)} = 2 \sum_{j=s-1}^{1} (-1)^{s-j-1}\, I(x;q_s+q_j,2t)\, V(x;q_{s-1},\ldots,q_{j+1},q_{j-1},\ldots,q_1;t) \; ,$$

and

$$C^{(s)} = V(x;q_s-1,\; q_{s-1},\ldots,q_1;t) \quad .$$

We may illustrate the method by considering $m = 1$ and $s = 3$. In this case one of the determinants occuring in (3.2) is $V(x;3,1,0;-1)$ which is evaluated as follows.

$$V(x;3,1,0;-1) = A^{(3)} + B^{(3)} + (n+3)\, C^{(3)} \quad , \tag{3.3}$$

where

$$A^{(3)} = -\, I_0(x;n+3,-1)\, V(x;1,0;-1) \quad ,$$

$$B^{(3)} = 2I(x;2n+4,-2)\ I(x;n,-1) - 2I(x;2n+3,-2)\ I(x;n+1,-1) \quad ,$$

and

$$C^{(3)} = V(x;2,1,0;-1) \quad .$$

Now, note that the multiplying factor for $V(x;3,1,0;-1)$ is $x^{-(3n+7)}$. But $I_0(x;n+3,-1)\ V(x;1,0;-1)$ gives only $x,^{3n+5}$ hence $A^{(3)}$ has to be integrated by parts twice. Similarly $B^{(3)}$ and $C^{(3)}$ have to be integrated by parts to take care of the power of x. In general this is true for all $V(x;q_3,q_2,q_1;-1)$. Thus we get

$$(3.4) \qquad V(x;3,1,0;-1)\ x^{-(3n+7)}\ e^{3x}$$

$$=\{(n+1)(n+2)(2n+3)\}^{-1}[I_0(2n+3,2x)\ I(n+3,x)-4(n+2)I_0(n+3,x)I(2n+3,2x)$$

$$+ 6(n+2)\ I(n+1,x)\ I(2n+4,2x) + 3I_0(2n+4,2x)\ \ I(n+2,x)$$

$$+ 4(2n+3)\ I_0(n+1,x)\ I(2n+5,2x) - 4(n+3)\ I_0(n+2,x)\ I(2n+4,2x)],$$

where

$$I_0(q,\alpha x) = e^{\alpha x(1-u)}\ u^q\big|_o^1 \quad .$$

Similarly other determinants involved in (3.2) with $m = 1$ and $s = 3$ can be evaluated to obtain the mgf in this case. The cdf can be derived from the mgf as indicated in the previous section. Further, the above method can be applied in the general case i.e. for $M(x,m,n,s)$ using (3.2).

4. <u>The exact cdf of $V^{(3)}$ for $m = 1,2,3$</u>. Using methods outlined in sections 2 and 3, the cdf of $V^{(3)}$ has been derived for $m = 1,2$ and 3. The mgf's and cdf's in the corresponding cases are given below:

(i) $\underline{m = 1}$. The mgf is given by

$$(4.1) \quad M(x,1,n,3) = e^{3x}\, x^{-(3n+6)} C(3,1,n)[V(x;2,1,0;-1) - x^{-1}\cdot V(x;3,1,0;-1) + x^{-2}\, V(x;3,2,0;-1) - x^{-3}\, V(x;3,2,1;-1)] \; .$$

Then

$$(4.2) \quad \Pr(V^{(3)} \le z) = 1 - C(3,1,n)\, W(3,1)$$

where

$$W(3,1) = \begin{cases} \Big\{ \sum_{j=3}^{4} p_{1j}(1-z)^{n+j} + \sum_{j=4}^{7} q_{1j}(1-z/2)^{2n+j} + \sum_{k=3}^{5} \omega_{1k2}\, B(z_1,z_2;2n+k+1,n+3) \\ \quad + \omega_{153}\; B(z_1,z_2;2n+6,n+4) + \omega_{134}\; B(z_1,z_2;2n+4,n+5) \Big\} & 0 \le z \le 1 \\ \Big\{ \sum_{j=4}^{7} q_{1j}(1-z/2)^{2n+j} + \sum_{k=3}^{5} \omega_{1k2}\, B(z_1,1;2n+k+1,n+3) \\ \quad + \omega_{153}\; B(z_1,1;2n+6,n+4) + \omega_{134}\; B(z_1,1;2n+4,n+5) \Big\} & 1 \le z \le 2 \\ \Big\{ \sum_{k=3}^{5} \omega_{1k2}\, B(0,1;2n+k+1,n+3) + \omega_{153}\; B(0,1;2n+6,n+4) \\ \quad + \omega_{134}\; B(0,1;2n+4,n+5) \Big\} & 2 \le z \le 3 \end{cases} ,$$

where

$$p_{13} = (2n+1)p_{03},\; p_{14} = \{(4n^3 + 20n^2 + 31n + 12)\, p_{03}\}/\{(n+4)(2n+5)\} ,$$

$$q_{14} = \{2(n^3 + 4n^2 + 3n - 3)\, q_{04}\}/\{(n+3)(n+4)(2n+3)\} ,$$

$q_{15} = -2/\{(n+2)(n+3)(2n+5)\}$, $q_{16} = -2/\{(n+1)(n+2)(n+3)^2\}$,

$q_{17} = 2/\{(n+1)(n+2)(n+3)(2n+7)\}$, $h_{132} = h_{032}$, $h_{142} = -3h_{132}$,

$h_{152} = 2/\{(n+1)(n+2)\}$, $h_{153} = -2/\{(n+2)(n+3)(2n+5)\}$ and

$h_{134} = -2/\{(n+3)(n+4)\}$.

(ii) m = 2. The mgf is given by

$$\begin{aligned}(4.3)\qquad M(x,2,n,3) = e^{3x}\, x^{-(3n+6)}\, C(3,2,n)[&V(x;2,1,0;-1) -2x^{-1} V(x;3,1,0;-1) \\ &+ x^{-2}\{V(x;4,1,0;-1) + 3V(x;3,2,0;-1)\} \\ &- 2x^{-3}\{V(x;4;2,0;-1) + 2V(x;3,2,1;-1)\} \\ &+ x^{-4}\{V(x;4,3,0;-1) + 3V(x;4,2,1;-1)\} \\ &- 2x^{-5} V(x;4,3,1;-1) + x^{-6} V(x;4,3,2;-1)] \;.\end{aligned}$$

Therefore

$$(4.4)\qquad \Pr(W^{(3)} \le z) = 1 - C(3,2,n)\, W(3,2)$$

where

$$W(3,2)=\begin{cases}\left\{\sum_{j=3}^{5} p_{2j}(1-z)^{n+j}+\sum_{j=4}^{9} q_{2j}(1-z/2)^{2n+j}+\sum_{\ell=2}^{4}\sum_{k=\ell+1}^{7} \omega_{2k\ell}B(z_1,z_2;2n+k+1,n+\ell+1)\right\} & 0\le z\le 1\\ \{\text{same as above except } p_{2j}=0\ (j=3,4,5) \text{ and replace } z_2 \text{ by } 1\} & 1\le z\le 2\\ \{\text{same as above except } p_{2j}=0,\ j=3,4,5; q_{2j}=0,\ j=4,\ldots,9 \text{ and replace } z_1 \text{ and } z_2 \text{ by } 0 \text{ and } 1 \text{ respectively}\} & 2\le z\le 3\end{cases}$$

where

$$p_{23} = -\{6(n^2+3n+1)\ p_{03}\}/\{(n+3)(2n+5)\},\quad p_{24} = -\{(4n^3+12n^2+17n+3)\ p_{03}\}/\{(n+4)(2n+5)\}\ ,$$

$$p_{25} = 30p_{03}/\{(n+4)(2n+5)(2n+7)\},\quad q_{24} = -q_{04}/(2n+3),\quad q_{25} = -4(3n^2+10n+12)/r(n),$$

$$q_{26} = 2(3n^3+29n^2+90n+94)(2n+3)/\{(n+3)(n+4)r(n)\},\quad q_{27}= 2n(2n+5)/r(n+1)\ ,$$

$$q_{28}= -6/\{(n+1)_4(n+4)\},\quad q_{29} = 4p_{03}(2n+3)/\{(n+4)(2n+9)\}\ ,$$

$$h_{232} = h_{032},\quad h_{242} = 2h_{142},\quad h_{252} = 6(2n^2+10n+13)\ h_{152}/\{(2n+3)(2n+5)\}\ ,$$

$$h_{262} = -2h_{152}(2n+7)/(n+3),\quad h_{272} = h_{152},\quad h_{243} = -6/\{(n+2)(n+3)\}\ ,$$

$$h_{253} = 8/\{(n+2)(n+3)\},\quad h_{263} = -9h_{153},\quad h_{273} = -4/\{(n+2)(n+3)\}\ ,$$

$$h_{254} = -4/\{(n+4)(2n+5)\},\quad h_{264} = -2h_{134},\quad h_{274} = -h_{134}/(2n+7)\ ,$$

and where

$$r(n) = (n+1)(n+2)(n+3)(2n+3)(2n+5),\quad \text{and}\quad (a)_i = a(a+1)\ \ldots\ (a+i-1)\ .$$

(iii) m = 3. The mgf is given by

$$(4.5)\qquad M(x,3,n,3) = e^{3x}\, x^{-(3n+6)}\, C(3,3,n)$$

$$[V(2,1,0)-3x^{-1}V(3,1,0)+3x^{-2}\{V(4,1,0)+2V(3,2,0)\}$$

$$-x^{-3}\{V(5,1,0)+8V(4,2,0)+10V(3,2,1)\}+3x^{-4}\{V(5,2,0)+2V(4,3,0)+3V(4,2,1)\}$$

$$-3x^{-5}\{V(5,3,0)+2V(5,2,1)+3V(4,3,1)\}+x^{-6}\{V(5,4,0)+8V(5,3,1)+10V(4,3,2)\}$$

$$-3x^{-7}\{V(5,3,2)+V(5,4,1)\} + 3x^{-8}V(5,4,2) - x^{-9}V(5,4,3)] \;.$$

For brevity in notation $V(x;q_s,q_{s-1},\ldots,q_1;-1)$ will be written $V(q_s,q_{s-1},\ldots,q_1)$ if there is no room for confusion as in (4.5) above. Now from (4.5) we get the cdf

$$(4.6)\qquad \Pr(V^{(3)} \le z) = 1 - C(3,3,n)\, W(3,3)$$

where

$$W(3,3)=\begin{cases} \{\sum_{j=3}^{6} p_{3j}(1-z)^{n+j} + \sum_{j=4}^{11} q_{3j}(1-z/2)^{2n+j} + \sum_{\ell=2}^{5}\sum_{k=\ell+1}^{9} \omega_{3k\ell} \\ \qquad\qquad B(z_1,z_2;2n+k+1,n+\ell+1)\} & 0 \le z \le 1 \\ \sum_{j=4}^{11} q_{3j}(1-z/2)^{2n+j} + \sum_{\ell=2}^{5}\sum_{k=\ell+1}^{9} \omega_{3k\ell}\, B(z_1,1;2n+k+1,n+\ell+1) & 1 \le z \le 2 \\ \sum_{\ell=2}^{5}\sum_{k=\ell+1}^{9} \omega_{3k\ell}\, B(0,1;2n+k+1,n+\ell+1) & 2 \le z \le 3 \;. \end{cases}$$

where

$$p_{33} = (4n^3+ 38n^2+ 85n+33) \;/\; \{(n+3)(n+4)r(n)\} \quad ,$$

$$p_{34} = -(8n^5+64n^4+216n^3+395n^2+316n+216) \;/\; \{(n+4)^2(2n+7)r(n)\}$$

$$p_{35} = (8n^5+68n^4+250n^3+439n^2+417n+78) \;/\; \{(n+4)(n+5)(2n+7)r(n)\} \quad ,$$

$$p_{36} = -630 \;/\; \{(n+4)_3(2n+7)(2n+9)r(n)\},\; q_{34}= q_{24},$$

$$q_{35} = -4(6n^2+19n+21)p_{03}/(2n+5),\;\; q_{36}= (40n^4+464n^3+1885n^2+3327n+2244) \;/ \; \{(n+3)(n+4)r(n)\} \quad ,$$

$$q_{37}= 2(10n^5+63n^4-308n^3-3429n-9668n-9120)/\{(n+1)_5(n+3)(2n+5)(2n+7)\} \quad ,$$

$$q_{38}= (8n^4+95n^3+339n^2+250n-422)\; q_{28}/\{3(n+5)(2n+7)\} \quad ,$$

$$q_{39}= -2(n^2+3n+8)/\{(n+2)_4(2n+9)\},\;\; q_{3,10}= -24/(n+1)_5(n+5),$$

$$q_{3,11} = 12/\{(n+1)_5(2n+11)\} \quad ,$$

$$h_{332}= h_{132},\;\; h_{342}= 3h_{142},\;\; h_{352}= 3(8n^2+44n+63)h_{152}/\{(2n+3)(2n+5)\} \quad ,$$

$$h_{362}= -2(64n^3+500n^2+1261n+1035)/r(n),\;\; h_{372}= 6(5n+19)/\{(n+1)_3\} \quad ,$$

$$h_{382}= -6(2n+9)/\{(n+1)(n+2)(n+4)\},\;\; h_{392}= h_{152},\;\; h_{343}= 2h_{243} \quad ,$$

$$h_{353}= 4(16n+43)/\{(n+2)(n+3)(2n+5)\},\;\; h_{363}= 3(6n^2+26n+25)h_{153}/(n+3) \quad ,$$

$$h_{373}= 3(20n^2+144n+265)h_{153}/(2n+7),\;\; h_{383}= 16(2n+9)/(n+2)_3 \quad ,$$

$$h_{393}= -6/(n+2)_2,\;\; h_{354}=6h_{254},\;\; h_{364}= 3(7n+20)h_{264}/(2n+5) \quad ,$$

$$h_{374}= -2(16n+59)h_{274},\;\; h_{384}= -18h_{274},\;\; h_{394}= 3h_{264}/2 \quad ,$$

$h_{365} = -2/\{(n+3)(n+5)\}$, $h_{375} = 12/\{(n+5)(2n+7)\}$,

$h_{385} = -6/\{(n+4)(n+5)\}$ and $h_{395} = -2/\{(n+4)(n+5)(2n+9)\}$.

5. Percentage Points of $V^{(3)}$. In this section the expressions for the cdf derived are made use of in computing exact upper percentage points of $V^{(3)}$ for $m = 1,2,$ and 3 and selected values of n. Approximate percentage points obtained by Pillai [9], using the two moment quotients, were taken and used to compute on IBM 7094 the probability corresponding to these approximate percentage points. Further, exact percentage points were computed starting with the approximate ones. Table 1 gives the results thus obtained. It may be pointed out that Pillai's beta approximation to the distribution of $V^{(s)}$ [6], [9] is good enough for values of n beyond those selected here.

Table 1. Approximate and Exact Percentage Points of $V^{(3)}$

n	Approximate Upper percentage points (Pillai's tables)	Probability	Exact Upper percentage points
		m = 1	
		5%	
5	1.288	.95032	1.28722
10	0.892	.95008	0.89184
15	0.682	.95006	0.68191
20	0.552	.95012	0.55185
25	0.464	.95053	0.46340
		1%	
5	1.459	.99004	1.45858
10	1.028	.98988	1.02894
15	0.794	.98991	0.79456
20	0.647	.98999	0.64707
25	0.545	.98984	0.54574
		m = 2	
		5%	
5	1.476	.94951	1.47715
10	1.053	.94996	1.05308
15	0.818	.95026	0.81758
20	0.668	.95001	0.66799
25	0.565	.95025	0.56471
		1%	
5	1.639	.98962	1.64237
10	1.191	.99002	1.19083
15	0.933	.98996	0.93324
20	0.767	.98999	0.76707
25	0.651	.98998	0.65107
		m = 3	
		5%	
10	1.190	.95008	1.18985
15	0.937	.95029	0.93653
20	0.772	.95008	0.77189
25	0.657	.95050	0.65638
		1%	
10	1.326	.98999	1.32610
15	1.053	.98996	1.05325
20	0.873	.98997	0.87317
25	0.745	.99550	0.74556

6. <u>Exact cdf of $V^{(4)}$</u>. The mgf in this case can be obtained by putting $s = 4$ in (3.2), use of lemma 1 and further integration by parts as indicated in section 3. In the final form the mgf involves the integrals $I(n+\ell,x)$, $I(2n+k,2x)$, $I(n+\ell,x)\ I(2n+k,2x)$ and $I(2n+\ell,2x)\ I(2n+k,2x)$. In order to obtain the cdf of $V^{(4)}$ it is necessary to know the contribution of each of the four types of integral terms above. Of these, the first three can be immediately written down similar to the results of $s = 3$. We get

$$(6.1)\qquad I(n+\ell,x) \to \begin{cases} \{1-(1-z)^{n+\ell+1}\}/(n+\ell+1) & 0 \le z \le 1 \\ 1/(n+\ell+1) & 1 \le z \le 4 \end{cases},$$

$$(6.2)\qquad I(2n+k,2x) \to \begin{cases} A = \{1-(1-\tfrac{1}{2}z)^{2n+k+1}\}/(2n+k+1) & 0 \le z \le 2 \\ 1/(2n+k+1) & 2 \le z \le 4 \end{cases}.$$

$$(6.3)\qquad I(n+\ell,x)\ I(2n+k,2x) \begin{cases} B = \{A/(n+\ell+1) \\ \quad -aB(z_1,z_2;\ 2n+k+1,n+\ell+2)\} & 0 \le z \le 1 \\ \text{Change } z_2 \text{ to } 1 \text{ in } B & 1 \le z \le 2 \\ \text{Put } z = 2\text{, change } (z_1,z_2) \text{ to } (0,1) \text{ in } B & 2 \le z \le 3 \\ 1 / \{(n+\ell+1)(2n+k+1)\} & 3 \le z \le 4 \end{cases},$$

where

$$a = 2^{n+\ell+1} / (n+\ell+1)z_2^{3n+k+\ell+2} .$$

Now, to obtain the contribution $F(k,\ell)$ (say) of $I(2n+\ell,2x)\ I(2n+k),2x)$, proceed as follows:

Let y_1 and y_2 be two r.v. distributed in [0,1] with densities $(1-y_1)^{2n+\ell}$ and $(1-y_2)^{2n+k}$ respectively, then

$$(6.4) \qquad F(k,\ell) = \iint_S (1-y_1)^{2n+\ell}(1-y_2)^{2n+k}\, dy_1\, dy_2$$

where $S = \{(y_1,y_2);\ 0 \le y_i \le 1,\ i = 1,2;\ 2y_1 + 2y_2 \le z\}$.

It is obvious that the value of $F(k,\ell)$ depends on that of z and let

$$(6.5) \qquad F(k,\ell) = F_i(k,\ell) \qquad z \in S_i$$

where $S_i = [2(i-1),2i] \cap S$ and $i = 1,2.$

Consider the unit square, OPQR, in Fig. 1 and let AB be the line $y_1 + y_2 = \frac{z}{2}$ $(0 \le z < 2)$ and CD the line $y_1 + y_2 = z/2$ $(2 \le z \le 4)$. Then $F_1(k,\ell)$ and $F_2(k,\ell)$ are obtained by integrating over the areas OAB and OPCDR respectively.

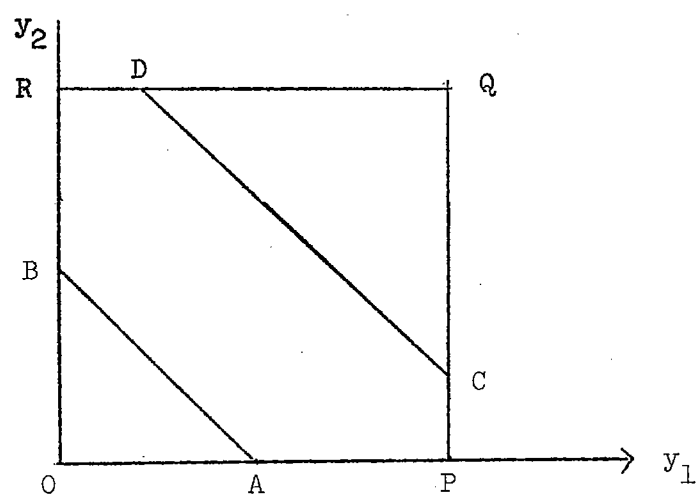

Figure 1. Region of Integration of $F(k,\ell)$

$$(6.6) \qquad F_1(k,\ell) = \int_0^{\frac{1}{2}z} \int_0^{\frac{1}{2}z-y_1} (1-y_1)^{2n+\ell} (1-y_2)^{2n+k} \, dy_1 \, dy_2$$

$$= \int_0^{\frac{1}{2}z} (1-y_1)^{2n+\ell} \frac{[1-(1-\frac{1}{2}z+y_1)^{2n+k+1}]}{2n+k+1} \, dy_1$$

$$= \frac{1-(1-\frac{1}{2}z)^{2n+\ell+1}}{(2n+\ell+1)(2n+k+1)} - \frac{B(z_3,z_4;\ 2n+k+2,2n+\ell+1)}{(2n+k+1)z_4^{(4n+k+\ell+2)}}$$

and

$$(6.7) \qquad F_2(k,\ell) = \int_{\frac{1}{2}z-1}^{1} \int_{\frac{1}{2}z-y_1}^{1} (1-y_1)^{2n+\ell} (1-y_2)^{2n+k} \, dy_1 \, dy_2$$

$$= \frac{1}{(2n+\ell+1)(2n+k+1)} - \frac{B(0,1;2n+k+2,2n+\ell+1)}{(2n+k+1)z_4^{(4n+k+\ell+2)}} ,$$

where $z_3 = (2-z)/(4-z)$ and $z_4 = 2/(4-z)$. Therefore

$$(6.8) \qquad I(2n+\ell,2x)\ I(2n+k,2x) \rightarrow \begin{cases} F_1(k,\ell) & 0 \leq z \leq 2 \\ F_2(k,\ell) & 2 \leq z \leq 4 \end{cases} .$$

7. <u>The exact cdf of $V^{(4)}$ for $m = 0$.</u> Using the method outlined above the mgf and cdf of $V^{(4)}$ for $m = 0$ can be obtained. First, for $m = 0$ and $s = 4$ we get from (3.2)

$$(7.1) \qquad M(x,0,n,4) = e^{4x} x^{-(4n+10)} C(4,0,n)\ V(x;3,2,1,0;-1) .$$

Now by the method of the previous section we get

$$(7.2)\quad M(x,0,n,4) = C(4,0,n)$$

$$[1/\{(n+2)r(n)\}\ \{-(4n+9)\ I(n+2,x) + I(n+1,x)\}$$

$$+ \{4(n+3)/((n+2)r(n))\}\ I(2n+4,2x) - \{4/((n+1)_2(2n+5))\}\ I(2n+5,2x)$$

$$+ \{2/((n+1)_2(2n+3))\}\ \{2I(2n+6,2x) - I(n+1,x)\ I(2n+4,2x)\}$$

$$+ \{8(n+3)/r(n)\}\ I(2n+3,2x)\ I(2n+5,2x)$$

$$- \{2/((n+1)(n+3))\}\ I(n+1,x)\ I(2n+6,2x)$$

$$+ \{4/((n+1)(2n+5))\}\ \{I(n+1,x)\ I(2n+5,2x) + I(n+2,x)\ I(2n+5,2x)\}$$

$$- \{2/(n+2)^2\}\ I(n+2,x)\ I(2n+4,2x)] \quad .$$

Now, the cdf is obtained by using results (6.1) to (6.3) and (6.8) in (7.2), which gives

$$(7.3)\quad \Pr(V^{(4)} \leq z) = 1 - C(4,0,n)\ W(4,0) \quad ,$$

where

$$(7.4)\quad W(4,0) = \{-(4n+9)/((n+2)(n+3)r(n))\}(1-z)^{n+3}-\{1/((n+2)(n+4)r(n)\}(1-z)^{n+4}$$

$$- \{4/((n+1)(n+3)(2n+5)^2)\}(1-\tfrac{1}{2}z)^{2n+5}+\{2(n^2+3n+3)/$$

$$((n+1)_3(n+2)_2(2n+3))\}(1-\tfrac{1}{2}z)^{2n+6}$$

$$+\{2(2n+5)/((2n+7)r(n)\}(1-\tfrac{1}{2}z)^{2n+7}+\{4(n+3)/$$

$$((n+2)r(n))\}z_4^{-(4n+10)}B(z_3,z_4;2n+5,2n+6)$$

$$+\{2^{n+3}/((n+1)z_2^{3n+9})\}[-((n+2)_2)^{-1}B(z_1,z_2;2n+7,n+3)$$

$$+\{2z_2/((n+2)(2n+5)B(z_1,z_2;2n+6,n+3)$$

$$-\{z_2^2/((n+2)^2(2n+3))\}B(z_1,z_2;2n+5,n+3)$$

$$+\{4/(n+3)(2n+5))\}B(z_1,z_2;2n+6,n+4)$$

$$-\{2z_2(n+1)/((n+2)^2(n+3))\}B(z_1,z_2;2n+5,n+4)]$$

when $0 \leq z \leq 1$.

For obtaining $W(4,0)$ in the other three intervals, we may make the following changes on the right side of (7.4).

a) $1 \leq z \leq 2$

(1) Drop all terms involving $1 - z$, and

(2) Change z_2 to 1;

b) $2 \leq z \leq 3$

(1) Drop all terms involving $(1-z)$,

(2) Drop all terms involving $(1 - \frac{1}{2}z)$,

(3) Replace z_1 and z_2 by 0 and 1 respectively, and

(4) Replace z_3 and z_4 by 0 and 1 respectively;

c) $3 \leq z \leq 4$

(1) Drop all terms involving z except those with

$$B(z_3,z_4;\ 2n+\ell+2, 2n+k+2),$$ and

(2) Replace z_3 and z_4 by 0 and 1 respectively.

8. Exact cdf of $V^{(4)}$ for $m = 1$. In this case, the mgf of $V^{(4)}$ can be written as

$$(8.1) \qquad M(x,1,n,4) = C(4,m,n)e^{4x}\, x^{-(4n+10)}[V(x;3,2,1,0;-1)-x^{-1}\, V(x;4,2,1,0;-1)$$
$$+x^{-2}V(x;4,3,1,0;-1)-x^{-3}V(x;4,3,2,0;-1)+x^{-4}V(x;4,3,2,1;-1)]\,.$$

On simplifying (8.1) and using results (6.1) to (6.3) and (6.8), the cdf of $V^{(4)}$ is obtained in the form:

$$(8.2) \qquad \Pr(V^{(4)} \leq z) = 1 - C(4,1,n)\, W(4,1) \quad ,$$

where

(8.3) $$W(4,1) = [\{-(4n^4+38n^3+136n^2+213n+114)/(t(n))\}(1-z)^{n+3}$$

$$-\{2(8n^6+124n^5+790n^4+2647n^3+4923n^2+4816n+1917)/((n+4)(2n+7)t(n))\}(1-z)^{n+4}$$

$$+\{6/((n+5)(2n+7)t(n))\}(1-z)^{n+5}+\{2(10n^2+37n+31)(n+4)/((n+2)(2n+5)s(n))\}(1-\tfrac{1}{2}z)^{2n+5}$$

$$+\{(16n^5+200n^4+984n^3+2374n^2+2837n+1374)/((n+2)(n+3)(2n+7)s(n))\}(1-\tfrac{1}{2}z)^{2n+6}$$

$$-\{4(4n^5+56n^4+325n^3+959n^2+1426n+860)/((n+2)(2n+7)^2s(n))\}(1-\tfrac{1}{2}z)^{2n+7}$$

$$+\{4(4n^4+40n^3+145n^2+225n+131)/((n+4)(2n+7)s(n))\}(1-\tfrac{1}{2}z)^{2n+8}+\{6/((2n+9)s(n))\}(1-\tfrac{1}{2}z)^{2n+9}$$

$$+\{2^{n+3}z_2^{-(3n+8)}/(n+1)_3\}[\{2(2n^2+5n+4)(n+3)/((n+2)(2n+3)(2n+5)\}B(z_1,z_2;2n+6,n+3)$$

$$-z_2^{-1}B(z_1,z_2;2n+7,n+3)-\{(n+3)/(n+4)\}z_2^{-3}B(z_1,z_2;2n+9,n+3)$$

$$-\{2(n+1)/(n+2)\}B(z_1,z_2;2n+5,n+4)+\{2(2n^2+9n+6)/((n+2)(2n+5))\}z_2^{-1}B(z_1,z_2;2n+6,n+4)$$

$$-\{2(2n+7)(n+1)/(n+3)(2n+5)\}z_2^{-2}B(z_1,z_2;2n+7,n+4)+\{2(2n+3)/(2n+7)\}z_2^{-3}B(z_1,z_2;2n+8,n+4)$$

$$-\{2(n+1)/(n+4)\}z_2^{-4}B(z_1,z_2;2n+9,n+4)+\{4(2n^2+9n+13)/((n+4)(2n+5))\}z_2^{-2}B(z_1,z_2;2n+6,n+5)$$

$$+\{8(n+2)/(n+3)_2\}z_2^{-3}B(z_1,z_2;2n+7,n+5)+\{4(2n^2+9n+8)/((n+4)(2n+7))\}z_2^{-4}B(z_1,z_2;2n+8,n+5)]$$

$$+\{4z_4^{-(4n+10)}/(n+1)_3\}[\{(n+3)/((n+2)(2n+3)(2n+5))\}B(z_3,z_4;2n+5,2n+6)$$

$$+\{n/(2n+3)\}z_4^{-1}B(z_3,z_4;2n+5,2n+7)$$

$$+\{(n+3)/(2n+3)\}z_4^{-2}B(z_3,z_4;2n+5,2n+8)-\{(n^2+2n+2)/((n+2)(2n+3))\}z_4^{-1}B(z_3,z_4;2n+7,2n+5)$$

$$-\{n+5)/(2n+5)\}z_4^{-2}B(z_3,z_4;2n+6,2n+7)-\{4/((2n+5)(2n+7))\}z_4^{-3}B(z_3,z_4;2n+6,2n+8)$$

$$+\{(n+1)/((n+4)(2n+5)(2n+7))\}z_4^{-4}B(z_3,z_4;2n+9,2n+6)], \quad 0 \le z \le 1 \quad ,$$

where $s(n) = (n+1)_4(2n+3)(2n+5)$ and $t(n) = (n+2)_2 s(n)$. For obtaining the cdf of $V^{(4)}$ when $z \geq 1$, make changes a) to c) as in the previous section.

9. Percentage points for $V^{(4)}$. As in section 5, upper percentage points are computed from expressions given in the above sections for the exact cdf of $V^{(4)}$ for $m = 0$ and 1, but only for selected values of n. These are presented in Table 2 along with approximate percentage points from Pillai [9] and the corresponding probabilities.

Table 2. Approximate and Exact Percentage Points of $V^{(4)}$

n	Approximate Upper percentage points (Pillai's tables)	Probability	Exact Upper percentage points
		m = 0	
		5%	
5	1.411	.95048	1.40976
10	0.974	.94999	0.97401
15	0.744	.95008	0.74386
20	0.602	.95028	0.60160
25	0.505	.95001	0.50499
		1%	
5	1.594	.99009	1.59305
10	1.118	.98994	1.11847
15	0.861	.98986	0.86191
20	0.701	.98998	0.70111
25	0.590	.98982	0.59088
		m = 1	
		5%	
5	1.693	.94983	1.69343
10	1.203	.95022	1.20253
15	0.932	.94992	0.93212
		1%	
5	1.875	.98985	1.87653
10	1.352	.99002	1.35179
15	1.056	.98996	1.05628
20	0.866	.98984	0.86690

Finally, it may be pointed out that the values of m for which exact cdf of $V^{(s)}$ has been obtained in the paper are very useful especially for the test of (i) independence between a p-set and a q-set of variates from a normal population in which case $m = \frac{1}{2}(q-p-1)$, $p \leq q$, and (ii) the equality of mean vectors of ℓ p-variate normal populations having a common covariance matrix in which case $m = \frac{1}{2}(|\ell-p-1|-1)$.

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