Exponential Ergodicity of the M|G|1 Queue

by

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1. Introduction

We consider a single server queue with Poisson input at rate $\lambda > o$, and independent service times, with distribution $H(t)$. The mean value of $H(t)$ is denoted by α and we assume that for some some $\eta > o$ and finite $K \geq 1$ and all $t \geq o$

(1) $$1 - H(t) \leq K e^{-\eta t} .$$

This condition is satisfied in many practical cases. For example, if $1 - H(t) \sim e^{-\zeta t} g(t)$ as $t \to \infty$ where $e^{-\epsilon t} g(t) \to o$ as $t \to \infty$ for all $\epsilon > o$, then (1) is satisfied by rating $\eta = \zeta - \epsilon$ for some $\epsilon > o$. If $h(s)$ is the Laplace-Stieltjes transform of $H(t)$, then (1) implies that $h(s)$ converges in the open half-plane

$$P(\eta) = \{s: \ \text{Re } s > - \eta\} .$$

Furthermore:

(2) $$h(s) \geq (s+\eta)^{-1} [s(1-K) + \eta], \quad s \geq o$$

$$h(s) \leq (s+\eta)^{-1} [s(1-K) + \eta], \quad - \eta < s \leq o .$$

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The first inequality implies that:

$$s^{-1}[h(s) - 1] \geq - K(s+\eta), \quad s \geq o$$

so that upon letting s tend to $o+$, we obtain:

$$\alpha\eta \geq K , \tag{3}$$

so that $\eta > \lambda K$ implies $1 > \lambda\alpha$. Therefore: $1 - \alpha\lambda < o$ implies $\eta < \lambda K$. But $1 < \alpha\lambda$ is the necessary and sufficient condition for a transient queue, so that for any such queue which satisfies (1), we always have $\eta < \lambda K$ satisfied. This also accounts for the asymmetry in the theorems given below.

Random times $T_o, T_1, \ldots$ are defined as follows

(i) $T_o = o$ a.s.

(ii) T_{n+1} is the time instant in which all customers, if any, present at time T_n complete service. If there are no customers at T_n, then T_{n+1} is the instant in which the the first customer to arrive after T_n completes service.

We assume throughout this paper that at $t = o$, there are $i \geq o$ customers in the queue and if $i \geq 1$, a service is just beginning. Using the results of [1] it is easy to modify our discussion for different initial conditions.

If we denote by $\xi(t)$ the queuelength at $t+$ and consider the bivariate sequence of random variables:

$$\{\xi(T_n), T_{n+1} - T_n; \quad n \geq o\}$$

we see that it is a semi-Markov process on the nonnegative integers, whose transition probability matrix is given by:

$$(4) \quad Q_{ij}(x) = \int_o^x e^{-\lambda y} \frac{(\lambda y)^j}{j!} \, d\, H^{(i)}(y), \qquad i > o, \quad j \geq o ,$$

$$= \int_o^x (1 - e^{-\lambda(x-y)}) \, d\, Q_{ij}(y), \qquad i = o, \; j \geq o$$

in which $H^{(i)}(\cdot)$ is the i-fold convolution of $H(\cdot)$.

2. The Busy Period

The mass distribution $G(t)$ of the length of the busy period is defined easily defined in terms of the $\{T_n\}$ sequence.

$$(5) \qquad G(t) = P[T_n \leq t, \; \xi(T_n) = o, \; \xi(T_m) \neq o, \quad m = 1, \ldots, n-1 \text{ for some } n \,|\, \xi(T_o) = 1] \quad .$$

Let $\gamma(s)$ be the Laplace-Stieltjes transform of $G(t)$ then $\gamma(s)$ is the unique solution in $P(o)$ to the equation:

$$(6) \qquad Z = h[s + \lambda - \lambda Z] \; ,$$

which lies in $|Z| < 1$ for all s in $P(o)$. If $(-\theta)$ is the abscissa of convergence of $\gamma(s)$, then we have [5].

Lemma 1 a) If $1 - \alpha\lambda > o$, then $\gamma(s)$ is the Laplace-Stieltjes transform of a probability distribution with mean $\lambda[1 - \alpha\lambda]^{-1}$. Moreover if $\eta > \lambda K$ then $\theta \geq [\sqrt{\eta} - \sqrt{\lambda K}]^2 > o$.

b) If $1 - \alpha\lambda = o$ then $\gamma(s)$ is the Laplace-Stieltjes transform of a probability distribution with infinite mean, so that $\theta = o$.

c) If $1 - \alpha\lambda < o$ then $\gamma(s)$ is the Laplace-Stieltjes transform of an improper probability distribution. Moreover if $o < \eta < \lambda K$ then $\theta > o$.

It then follows immediately from a classical theorem for Laplace-Stieltjes transforms, [6, p. 40], that:

<u>Theorem 1:</u>

If $\lambda K > \eta > o$, then for some $\theta > o$ and $M \geq o$,

$$G(\infty) - G(t) \leq M\, e^{-\theta t}$$

for all sufficiently large t.

If $\lambda K < \eta$, then for some $M \geq o$:

$$1 - G(t) \leq M\, e^{-[\sqrt{\eta} - \sqrt{\lambda K}]^2 t}$$

for all sufficiently large t .

<u>Remark</u>:

In the first (transient) case $G(\infty) < 1$, is the unique root of the equation:

$$Z = h(\lambda - \lambda Z)$$

in $(o,1)$.

We set:

$$\beta = \sqrt{\eta} - \sqrt{\lambda K} \tag{8}$$

3. The Markov Renewal Process.

Let $M_{ij}(t)$ be the expected number of visits to state j in $(o,t]$ given that $\xi(T_o) = i$. In [5] we proved the following theorem:

Theorem 2:

(a) If $1 - \alpha\lambda < o$, then the semi-Markov process, defined above, is transient and exponentially ergodic, i.e. for all i,j there exist constants, $o \leq K_{ij} < \infty$, $o \leq t_{ij} < \infty$ and $o < L_{ij} < \infty$ such that

$$(9) \qquad |M_{ij}(t) - L_{ij}| \leq K_{ij}\, e^{-\sigma t} \quad \text{for} \quad t \geq t_{ij}$$

where $\sigma = \min[\theta,\lambda]$ and θ is defined in lemma 1.c

(b) If $\eta > \lambda K$, then the semi-Markov process is positive recurrent and exponentially ergodic, i.e. for all i,j there exist constants $\lambda_j > o$, $o \leq K_{ij} < \infty$, $o \leq t_{ij} < \infty$ and $o \leq |L_{ij}| < \infty$ such that

$$(10) \qquad \left|M_{ij}(t) - \frac{t}{\mu_{jj}} - L_{ij}\right| \leq K_{ij}\, e^{-\lambda_j t} \quad \text{for} \quad t \geq t_{ij}$$

where μ_{jj} is the mean recurrence time to state j.

When $i = j = o$ we know slightly more than that. For if $m_{oo}(s)$ is the Laplace-Stieltjes transform of $M_{oo}(t)$, then

$$(11) \qquad m_{oo}(s) = \frac{\lambda}{\lambda+s}\, \gamma(s)\, [1 - \frac{\lambda}{\lambda+s}\, \gamma(s)]^{-1} \; .$$

The constant L_{oo} in (9) is then equal to $G(\infty)[1 - G(\infty)]^{-1}$.

In (10) the constant $\mu_{oo} = \lambda^{-1}(1-\lambda\alpha)^{-1}$ and L_{oo} is the usual intercept of the linear asymptote to the renewal function $M_{oo}(t)$.

A general characterization of the "rates" λ_j was given in [4] and [5], to which we refer for details.

Corollary 1. If $1-\alpha\lambda < o$ then for some $L > o$ and some $\tau > o$: $P_o(t) \leq L\, e^{-\tau t}$,

If $\eta > \lambda K$ then for some $L > o$ and some $\tau > o$:

(12) $$|P_o(t) - 1 + \alpha\lambda| \leq L\, e^{-\tau t} .$$

Proof: It is known [3] that $P_o(t)$ satisfies the renewal type equation:

$$P_o(t) = e^{-\lambda t} + \int_o^t e^{-\lambda(t-u)}\, d\, M_{1o}(u)$$

where the Laplace-Stieltjes transform of $M_{1o}(u)$ is given by:

(13) $$m_{1o}(s) = \frac{\lambda+s}{\lambda}\; m_{oo}(s) = \gamma(s)[1 - \frac{\lambda}{\lambda+s}\gamma(s)]^{-1} .$$

If $\eta > \lambda K$, the result follows by applying the key renewal theorem with remainder term, proved in [4]. If $1 - \lambda\alpha < o$, the limit of $P_o(t)$ as t tends to infinity is zero. The stated inequality follows from the fact that the function in (13) is analytic in some strip to the left of the imaginary axis.

4. The Number of Customers Served During a Busy Period.

Let f_n denote the probability that n customers are served during a busy period, and set

(14) $$F(s) = \sum_{n=o}^{\infty} f_n s^n \quad .$$

$F(s)$ is the unique solution of the equation

(15) $$Z = sh[\lambda - \lambda Z] \ ,$$

in the unit-disk $|Z| \leq 1$. [3]

Theorem 3. Let $\eta > \lambda K$, then $\sum_{n=o}^{\infty} f_n = 1$. Moreover

a. If $K > 1$, then

(16) $$f_n \leq \sqrt{\sigma}^{\,n-1} \left\{\frac{K-1}{\sqrt{\eta K} - \sqrt{\sigma}}\right\}^{2n-1} \quad \text{for all} \quad n \geq o$$

where $\sigma = \eta - (K-1)\lambda > o$

b. If $K = 1$, then

(17) $$f_n \leq \frac{\lambda+\eta}{2\lambda} \left[\frac{4\lambda\eta}{(\lambda+\eta)^2}\right]^n \quad \text{for all} \quad n \geq o \ .$$

Proof: Define

(18) $$f[s,F(s)] = \lambda F^2(s) + sF(s)\lambda(K-1) - F(s)(\eta+\lambda) + s[\eta-(K-1)\lambda]$$

It follows from (2) and (15) that for real values of s

$$f[s, F(s)] \leq o \qquad \text{if} \qquad F(s) \leq 1$$

$$f[s, F(s)] \geq o \qquad \text{if} \qquad 1 \leq F(s) < 1 + \frac{\eta}{\lambda} \ .$$

Assume first that $K > 1$, and consider the graph of the hyperbola $f[s, F(s)] = o$ with the bounds given above. It follows that $F(s)$ is well defined as long as $s \leq \frac{1}{\lambda[K-1]^2} \{\sqrt{\eta K} - \sqrt{\sigma}\}^2 \equiv s_o > 1$, and that for $s \leq s_o$

(19) $$|F(s)| \leq \frac{\sqrt{\sigma}}{\lambda(K-1)} \{\sqrt{\eta K} - \sqrt{\sigma}\} \quad .$$

From Cauchy's estimates on an analytic function we obtain that

(20) $$f_n \leq \max_{|s|=s_o} |F(s)| s_o^{-n} \quad .$$

Combining (19) and (20) we obtain (16).

In the case $K = 1$ (17) follows from (16) by letting $K \to 1$, or from a similar argument as above, using now the parabola $f[s, F(s)] = o$.

5. <u>The Waitingtime Distribution in the Stationary Case</u>.

Let $W^*(x)$ be the distribution of the virtual waitingtime of a customer arriving in the queue in its stationary phase. Let $w(s)$ be its Laplace-Stieltjes transform. Then [3]

(21) $$w(s) = \frac{1-\alpha\lambda}{1 - \lambda[\frac{1-h(s)}{s}]} \quad .$$

Since we assume that $\eta > o$ and $1 - \alpha\lambda > o$, we obtain that $1 - W^*(x) \leq L\, e^{-rt}$ for some $r > o$. Since $W^*(x)$ is a distribution function, $w(s)$ has its first singularity then at $-\eta$ or at the solution of

(22) $$\frac{1 - h(s)}{s} = \frac{1}{\lambda}$$

in $-\eta \leq s < o$. We denote this solution by σ if it exists.

<u>Lemma 2</u>. If $\eta > K\lambda$, then $\sigma > \eta - K\lambda$.

Proof: Let $-\eta + K\lambda \leq s < o$. We obtain from (2) that

$$\frac{1 - h(s)}{s} \leq \frac{K}{s + \eta}$$

so that

$$\frac{1}{\lambda} - \frac{1 - h(s)}{s} \geq \frac{s + \eta - \lambda K}{\lambda(s+\eta)} > o .$$

The lemma follows from the monotonicity of the Laplace-Stieltjes transform $\frac{1 - h(s)}{\alpha s}$.

Corollary: If $\eta > K\lambda$ then for some constant $C > o$ $1 - W^{*}(x) \leq C\, e^{-[\eta - K\lambda]x}$ for $x \geq x_o$. To improve the bound for σ im lemma 2, we prove:

Lemma 3: Let $-\eta < o$ be the abscissa of convergence of $h(s)$ and let $1 - H(t) = e^{-\eta t} g(t)$. Then (22) has a unique solution in $-\eta \leq s < o$ if and only if

$$\int_o^\infty g(t)dt \geq \frac{1}{\lambda} \tag{23}$$

Proof: Let $h_1(s) = \frac{1 - h(s)}{\alpha s}$ be the Laplace-Stieltjes transform of $\frac{1}{\alpha}\int_o^t [1 - H(x)]dx$. Then $h_1(o) = 1$, $h_1(s) > 1$ for $-\eta \leq s < o$ and $h_1(s)$ increasing as $s \downarrow -\eta$. Hence (22) has a unique solution in $-\eta \leq s < o$ if and only if

$$\lim_{s \downarrow -\eta} h_1(s) \geq \frac{1}{\alpha\lambda} . \tag{24}$$

Let $C = \lim_{s \downarrow -\eta} h(s) \leq \infty$, then (24) is satisfied if and only if $C \geq 1 + \eta\lambda^{-1}$. By the assumption on $1 - H(t)$ we obtain that

$$\frac{C - 1}{\eta} = \lim_{s \downarrow -\eta} h_1(s)\alpha = \int_0^\infty g(t)dt$$

which proves the desired result.

We remark that condition (23) essentially states that the mean arrival time cannot be too large.

If $g(t) \sim t^\alpha$ as $t \to \infty$ for $\alpha > -1$ as is the case with gamma distributions, then (23) is satisfied. In other cases (23) might fail. For example, let $1 - H(t) = e^{-\eta t} [\sqrt{t}\,(t + a)]^{-1}$ for $a > 0$. Then (23) is satisfied if and only if $a \leq \lambda^2\pi^2$, [7, p. 16, #3]. Similarly for $1 - H(t) = e^{-\eta t}t^{-2}\sin^2 at$, where (23) is satisfied if and only if $a \geq 2(\lambda\pi)^{-1}$, [7, p. 13, #2].

Collecting the above results we obtain

<u>Theorem 4</u>: Let $-\eta < 0$ be the abscissa of convergence of $h(s)$ and let $1 - H(t) = e^{-\eta t}g(t)$.

(i) If $\int_0^\infty g(t)dt < \frac{1}{\lambda}$ then for some $L > 0$: $1 - W^*(x) \leq L\, e^{-\eta x}$ for sufficiently large x.

(ii) If $\int_0^\infty g(t)dt \geq \frac{1}{\lambda}$ then for some $L > 0$ and all $\epsilon > 0$

$$1 - W^*(x) \leq L\, e^{-[\sigma-\epsilon]x} \quad \text{for sufficiently large } x .$$

6. The Virtual Waitingtime

Let $\eta(t)$ be the virtual waiting of a customer joining the queue at time t, and let

(25) $$W(k,x) = P[\eta(t) \leq x] .$$

In particular, if $\eta > \lambda K$, $W^*(x) = \lim_{t \to \infty} W(k,x)$ exists by Lindley's theorem [3]. We discuss now the exponential decay of $W(t,x)$ to its limit $W^*(x)$ as $t \to \infty$ under the initial condition $P[\eta(o) = o] = 1$. We put

(26) $$M(u,s) = \int_o^\infty \int_o^\infty e^{-ut-sx} d_t d_x W(t,x) .$$

From [3, p. 51] one easily proves:

Lemma 4: If $u - s + \lambda[1 - h(s)] > o$ then

(27) $$M(u,s) = \{u-s+\lambda[1-h(s)]\}^{-1}\{s-\lambda[1-h(s)] - \frac{us}{u+\lambda-\lambda\gamma(u)}\} .$$

Theorem 5: Let $\beta = \sqrt{\eta} - \sqrt{\lambda K} > o$. Then for both x and t sufficiently large there exists a function $M(x)$, independent of t such that

(28) $$|W(t,x) - W^*(x)| \leq M(x)\ e^{-\sqrt{\eta}\ \beta x\ -\ \beta^2 t}$$

Proof: Let $M(u,s) = \int_o^\infty e^{-ut} d_t \Omega(t,s)$ where [3, p. 51]

$$\Omega(t,s) = e^{st-[1-h(s)]\lambda t}\,[1-s\int_o^t e^{-su+[1-h(s)]\lambda u}P_o(u)du]$$

and

$$\int_o^\infty e^{-us}\,P_o(u)du = [s+\lambda-\lambda\gamma(s)]^{-1}\ .$$

We prove that $M(u,s)$ is convergent if $s \geq -\beta/\eta$ and $u > -\beta^2$. Referring to (27) we know that $s - \lambda[1 - h(s)]$ converges at $s = -\beta/\eta$, for $\beta < \sqrt{\eta}$ and hence $-\sqrt{\eta}\,\beta > -\eta$. By lemma 1.a $\gamma(u)$ converges for $u > -\beta^2$ and since $1 + \lambda[\frac{1-\gamma(u)}{u}] > o$ for $u > -\beta^2$ also $u[u + \lambda - \lambda\gamma(u)]^{-1}$ converges if $u > -\beta^2$.

The inequality (2) applied at $s = -\beta/\eta$ yields

$$u + \beta/\eta + \lambda[1-h(-\beta/\eta)] \geq u + \beta/\eta - \beta/\lambda K = u + \beta^2 > o$$

so that $M(u, -\beta/\eta)$ converges for $u > -\beta^2$. By a classical theorem on Laplace-Stieltjes transforms [6, p. 40] we obtain that

$$\Omega(t, -\beta/\eta) - w(-\beta/\eta) = o[e^{-\beta^2 t}] \text{ as } t \to \infty\ . \tag{29}$$

Let now

$$J(x) = e^{\beta^2 t}\ \{W(t,x) - W^*(x)\} \tag{30}$$

then

$$\int_o^\infty e^{-sx}d\,J(x) = e^{\beta^2 t}[\Omega(t,s) - w(s)]$$

and this converges for $s \geq -\beta/\eta$. By the same theorem used above we obtain that

(31) $$J(x) - J(\infty) = o[e^{-\beta\sqrt{\eta}\, x}] \quad \text{as} \quad x \to \infty .$$

Now $J(\infty) = o$. Hence for some x_o large enough there exists a constant $C(x_o)$ such that

$$J(x) \leq C(x_o)\, e^{-\beta\sqrt{\eta}\, x} .$$

Combining this expression in (29) and (30) finishes the proof of the theorem.

7. The Queuelength in Continuous Time.

Let $\xi(t)$ be the number of customers in the queue at time t and let

$$\pi_{iK}(t) = P[\xi(t) = k \mid \xi(o) = i] .$$

The connection between these functions and the renewal functions is: [2]

(32) $$\Pi_{ij}(t) = \int_o^t e^{-\lambda(t-u)} d\, M_{io}(u) \qquad \text{if} \quad j = o$$

$$\sum_{m=1}^{j} \int_o^t e^{-\lambda(t-u)} \frac{[\lambda(t-u)]^{j-m}}{(j-m)!} [1-H(t-u)] d\, M_{im}(u) \quad \text{if} \quad j \neq o.$$

Let us introduce $Q_m(t) = e^{-\lambda t} \frac{(\lambda t)^m}{m!} [1-H(t)] \quad m = 0,1,\ldots,j-1$

(33) $$A_m = \int_o^\infty Q_m(t)dt$$

We can assume $j \neq o$, since $j = o$ was considered in cor. 1.

Theorem 6: a. If $\eta > \lambda K$ then there exist constants $o \leq K_{ij} < \infty$ such that for sufficiently large t

(34) $$\left|\Pi_{ij}(t) - \sum_{m=1}^{j} \frac{A_{j-m}}{\mu_{mm}}\right| < K_{ij}\, e^{-r_j t}$$

where $r_j = \min_{1 \leq m \leq j} \lambda_m$ and λ_m and μ_{mm} are defined in Th. 2.

b. If $1 - \alpha\lambda > o$ then there exist constants $o \leq K_{ij} < \infty$ such that for sufficiently large t

(35) $$\Pi_{ij}(t) \leq K_{ij}\, e^{-\sigma t}$$

where σ is defined in Th. 2.

Proof: We obtain from (32) and (33) that

$$\left|\Pi_{ij}(t) - \sum_{m=1}^{j} \frac{A_{j-m}}{\mu_{mm}}\right| \leq \sum_{m=1}^{j} \left|\int_o^t Q_{j-m}(t-u)d\, M_{im}(u) - \frac{1}{\mu_{mm}}\int_o^\infty Q_{j-m}(t)dt\right| .$$

Applying the Key renewal theorem with remainder term [4] we obtain

(36) $$\left|\int_o^t Q_{j-m}(t-u)dM_{im}(u) - \frac{1}{\mu_{mm}}\int_o^\infty Q_{j-m}(u)du\right| \leq C_{i,j-m}\, e^{-\lambda_m t}$$

for some positive constants $C_{i,j-m}$ and sufficiently large t.
If we put $K_{ij} = \sum_{m=1}^{j} C_{i,j-m}$ and $v_j = \min_{1 \leq m \leq j} \lambda_m$ then (34) follows.

If $1 - \alpha\lambda < o$ the argument is similar to the proof of cor. 1.

Other queueing models may be studied by similar methods, for example GI|M|1 and bulk queues with Poisson input as discussed in [2].

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13. ABSTRACT

If an M|G|1 queue has a service time distribution which is exponentially bounded, then most all important quantities of the queue have distributions, which are exponentially bounded.

This is proved for the busy period, the number of customers served during a busy period, the waitingtime, the queuelength in continuous and in discrete time.

The method of proof is based on the exponential ergodicity theorems for semi-Markov processes.

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