# On the moment generating function of Pillai's $V^{(s)}$ criterion*

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1. Introduction and Summary. The moment generating function (mgf) of Pillai's $V^{(s)}$ criterion in the central case has been considered by Pillai [8], [9] and James [3]. In the present paper, the mgf's of $V^{(s)}$ are considered in the following non-central cases: (i) MANOVA and (ii) canonical correlation. The lower order moments of $V^{(s)}$ for (i) and (ii) in the non-central case were obtained earlier by Pillai [10] for $s = 2$ in the linear case i.e. when there is only one non-zero population root. These results for (i) were extended by Khatri and Pillai [5], [6], [7] to general $s$ in the linear case and in the planar case i.e. when there are two non-zero population roots. Only the first two moments were considered in the planar case while the first four moments were obtained in the linear case. The results of this paper further facilitates the derivation of the general moments.

2. Preliminaries. The following lemmas will be used in the sequal for the derivation of the mgf's:

Lemma 1: Let $S(p \times p)$ be a positive definite (p.d.) symmetric matrix, $\underset{\sim}{T}(p \times p)$, a complex matrix whose real part is p.d. symmetric, and $\underset{\sim}{U}(p \times p)$, a symmetric matrix , then

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$$(2.1)\qquad \frac{2^{\frac{1}{2}p(p-1)}\Gamma_p(b)}{(2\Pi i)^{\frac{1}{2}p(p+1)}} \int_{R(\underset{\sim}{T})>\underset{\sim}{0}} e^{tr\, \underset{\sim}{T}} |\underset{\sim}{T}|^{-b} {}_qF_r(a_1,\ldots,a_q;\, b_1,\ldots,b_r;\, \underset{\sim}{T}^{-1}\underset{\sim}{S},\underset{\sim}{U})d\underset{\sim}{T}$$

$$= {}_qF_{r+1}\,(a_1,\ldots,a_q;\, b_1,\ldots,b_r;\, \underset{\sim}{S},\, \underset{\sim}{U}) \quad ,$$

(James [3], Constantine [1]) ,

where the hypergeometric function of matrix argument is defined by James [3] as

$$(2.2)\qquad {}_qF_r(a_1,\ldots,a_q;\, b_1,\ldots,b_r;\underset{\sim}{S},\underset{\sim}{T}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa\cdots(a_q)_\kappa\, C_\kappa(\underset{\sim}{S})\, C_\kappa(\underset{\sim}{T})}{(b_1)_\kappa\cdots(b_r)_\kappa\, C_\kappa(I_p)\, k!} \quad ,$$

where $a_1,\ldots,\, a_q,\, b_1,\ldots,\, b_r$ are real or complex constants and the multivariate coefficient $(a_\kappa)$ is given by

$$(a)_\kappa = \prod_{i=1}^{p} (a - \tfrac{1}{2}(i-1))_{\kappa_i} \quad ,$$

where $(a)_k = a(a+1)\,\ldots\,(a+k-1)$. The partition $\kappa$ of $k$ is such that

$$\kappa = (k_1,\, k_2,\ldots,\, k_p) \quad (k_1 \geq k_2 \geq \ldots \geq k_p \geq 0) \quad ,$$

$k_1 + \ldots + k_p = k$ , and the zonal polynomials, $C_\kappa(\underset{\sim}{S})$, are expressible in terms of elementary symmetric functions of the latent roots of $\underset{\sim}{S}$ [3] .

Lemma 2. If $\underset{\sim}{S}$, $\underset{\sim}{T}$ and $\underset{\sim}{U}$ are as in Lemma 1, then

$$(2.3)\qquad \frac{1}{\Gamma_p(a)} \int_{\underset{\sim}{S}>0} e^{-tr\, \underset{\sim}{S}} |\underset{\sim}{S}|^{a-\frac{1}{2}(p+1)} {}_qF_r(a_1,\ldots,a_q;b_1,\ldots,b_r;\underset{\sim}{S}\,\underset{\sim}{T},\underset{\sim}{U})d\underset{\sim}{S}$$

$$= {}_{q+1}F_r\,(a_1,\ldots,a_q,a;\, b_1,\ldots,b_r;\, \underset{\sim}{T},\, \underset{\sim}{U}) \quad ,$$

(James [3], Constantine [1]) .

Lemma 3. If $S(p \times p)$ is p.d., then

$$(2.4) \qquad {}_1F_1(a,b;S) = \frac{\Gamma_p(b)}{\Gamma_p(a)\,\Gamma_p(b-a)} \int_0^I e^{\mathrm{tr}\, S\, T} |T|^{a-\frac{1}{2}(p+1)} |I-T|^{b-a-\frac{1}{2}(p+1)} dT \quad ,$$

(James [3]) .

Lemma 4. Let $Z$ be a complex symmetric matrix whose real part is p.d., and let $T$ be an arbitrary complex symmetric matrix. Then

$$(2.5) \qquad \int_{S>0} e^{-\mathrm{tr}\, Z\, S} |S|^{t-\frac{1}{2}(p+1)} C_\kappa(S\, T) dS = (t)_\kappa\, \Gamma_p(t) |Z|^{-t}\, C_\kappa(T\, Z^{-1}) \quad ,$$

where $R(t) > \frac{1}{2}(p-1)$ . (Constantine [1]) .

It is easy to see that lemma 2 follows from lemma 4. A lemma similar to lemma 4 has been used to prove lemma 1 which can be referred to [1]. Also note that ${}_0F_0(S) = e^{\mathrm{tr}\, S}$ and

$$(2.6) \qquad C_\kappa(I + A) \,/\, C_\kappa(I) = \sum_{n=0}^{k} \sum_{\eta} a_{\kappa,\eta}\, C_\eta(A) \,/\, C_\eta(I) \quad ,$$

where $a_{\kappa,\eta}$ are constants (Constantine [2]) .

3. Cdf of $V^{(s)}$ for MANOVA. Let $X$ be a $p \times f_2$ matrix variate $(p \le f_2)$ and $Y$ a $p \times f_1$ matrix variate $(p \le f_1)$ and let the columns be all independently normally distributed with covariance matrix $\Sigma$, $E(X) = M$ and $E(Y) = 0$. Let $\ell_1, \ldots, \ell_p$ be the characteristic roots of

$$|X\, X' - \ell(Y\, Y' + X\, X')| = 0 \quad ,$$

and $\omega_1, \ldots, \omega_p$ those of $|\underset{\sim}{M}\,\underset{\sim}{M}' - \omega\,\underset{\sim}{\Sigma}| = 0$, then the joint density function of $\ell_1, \ldots, \ell_p$ is given by Constantine [1] and by James [3] in the form

$$(3.1)\qquad e^{-\frac{1}{2}\mathrm{tr}\underset{\sim}{\Omega}}\, C(p,f_1,f_2)\,{}_1F_1(\tfrac{1}{2}\nu;\tfrac{1}{2}f_2;\tfrac{1}{2}\underset{\sim}{\Omega},\underset{\sim}{L})\,|\underset{\sim}{L}|^{\frac{1}{2}(f_2-p-1)}\,|\underset{\sim}{I}-\underset{\sim}{L}|^{\frac{1}{2}(f_1-p-1)} \prod_{i>j} (\ell_i - \ell_j)\,,$$

$$0 < \ell_1 \leq \ldots \leq \ell_p < 1\,,$$

where $\underset{\sim}{L} = \underset{\sim}{X}'(\underset{\sim}{Y}\,\underset{\sim}{Y}' + \underset{\sim}{X}\,\underset{\sim}{X}')^{-1}\underset{\sim}{X}$, $\underset{\sim}{\Omega} = \underset{\sim}{M}'\,\underset{\sim}{\Sigma}^{-1}\,\underset{\sim}{M}$, $\nu = f_1 + f_2$,

$$(3.2)\qquad C(p,f_1,f_2) = \{\Pi^{\frac{1}{2}p^2}\,\Gamma_p(\tfrac{1}{2}\nu)\} \,/\, \{\Gamma_p(\tfrac{1}{2}f_1)\,\Gamma_p(\tfrac{1}{2}f_2)\,\Gamma_p(\tfrac{1}{2}p)\}\,,$$

and the determinants in (3.1) are expressed as products of the characteristic roots of their respective matrices. In the context of (i), $f_2 = \ell-1$ and $f_1 = N-\ell$, $N$ being the pooled sample size of the samples from $\ell$ populations. Here $V^{(s)} = V^{(p)} = \sum_{i=1}^{p} \ell_i$. Now, by an application of lemmas 1 and 2, we get

$$(3.3)\qquad E(e^{t\,\mathrm{tr}\underset{\sim}{L}}) = \frac{e^{-\frac{1}{2}\mathrm{tr}\underset{\sim}{\Omega}}\,2^{\frac{1}{2}p(p-1)}}{\Gamma_p(\frac{1}{2}f_1)(2\Pi i)^{\frac{1}{2}p(p+1)}} \int_{\underset{\sim}{S}>\underset{\sim}{0}} e^{-\mathrm{tr}\underset{\sim}{S}}|\underset{\sim}{S}|^{\frac{1}{2}(\nu-p-1)} \int_{\underset{\sim}{R}(\underset{\sim}{T})>\underset{\sim}{0}} e^{\mathrm{tr}\underset{\sim}{T}}|\underset{\sim}{T}|^{-\frac{1}{2}f_2}$$

$$\int_{\underset{\sim}{L}>\underset{\sim}{0}}^{\underset{\sim}{I}} e^{\mathrm{tr}(\underset{\sim}{I}t + \underset{\sim}{S}\,\underset{\sim}{T}^{-1}\frac{1}{2}\underset{\sim}{\Omega})\underset{\sim}{L}}|\underset{\sim}{L}|^{\frac{1}{2}(f_2-p-1)}|\underset{\sim}{I}-\underset{\sim}{L}|^{\frac{1}{2}(f_1-p-1)}\,d\underset{\sim}{L}\,d\underset{\sim}{T}\,d\underset{\sim}{S}\,.$$

Further, using lemma 4 and integrating with respect to $\underset{\sim}{L}$,

$$(3.4)\quad E(e^{t\,\mathrm{tr}\underset{\sim}{L}}) = \frac{e^{-\frac{1}{2}\mathrm{tr}\underset{\sim}{\Omega}}\Gamma_p(\tfrac{1}{2}f_2)2^{\frac{1}{2}p(p-1)}}{\Gamma_p(\tfrac{1}{2}\nu)(2\Pi i)^{\frac{1}{2}p(p+1)}} \int_{R(\underset{\sim}{T})>0} e^{\mathrm{tr}\,\underset{\sim}{T}}|\underset{\sim}{T}|^{-\frac{1}{2}f_2} \int_{\underset{\sim}{S}>0} e^{-\mathrm{tr}\underset{\sim}{S}}|\underset{\sim}{S}|^{\frac{1}{2}(\nu-p-1)}$$

$$ {}_1F_1(\tfrac{1}{2}f_2;\tfrac{1}{2}\nu;\underset{\sim}{I}t + \underset{\sim}{S}\underset{\sim}{T}^{-1}\tfrac{1}{2}\underset{\sim}{\Omega})\,d\underset{\sim}{S}\,d\underset{\sim}{T} \ .$$

Again, with the help of (2.6), lemma 4 and a similar lemma, we get

$$(3.5)\quad E(e^{t\,\mathrm{tr}\underset{\sim}{L}}) = e^{-\frac{1}{2}\mathrm{tr}\underset{\sim}{\Omega}} \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{n=0}^{k}\sum_{\eta} \frac{(\frac{1}{2}f_2)_\kappa\,(\frac{1}{2}\nu)_\eta\, a_{\kappa,\eta}\, t^{k-n} C_\kappa(\underset{\sim}{I})\, C_\eta(\frac{1}{2}\underset{\sim}{\Omega})}{(\frac{1}{2}\nu)_\kappa\,(\frac{1}{2}f_2)_\eta\, k!\, C_\eta(\underset{\sim}{I})} \ .$$

4. <u>Cdf of $V^{(s)}$ for canonical correlation.</u> Let the columns of $\begin{pmatrix}\underset{\sim}{X}_1\\ \underset{\sim}{X}_2\end{pmatrix}$ be $\nu$ independent normal $(p+q)$-variates, $(p \le q,\ p + q \le \nu,\ \nu + 1 = n,$ the sample size) with zero means and covariance matrix

$$\Sigma = \begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{12}' & \Sigma_{22}\end{pmatrix} \ .$$

Let $\underset{\sim}{R} = \mathrm{diag}(r_i)$, where $r_1^2,\ldots,r_p^2$, are the characteristic roots of the equation

$$|\underset{\sim}{X}_1\underset{\sim}{X}_2'(\underset{\sim}{X}_2\underset{\sim}{X}_2')^{-1}\underset{\sim}{X}_2\underset{\sim}{X}_1' - r^2\underset{\sim}{X}_1\underset{\sim}{X}_1'| = 0$$

and $P = \mathrm{diag}(\rho_1^2,\ldots,\rho_p^2)$ are characteristic roots of the equation

$$|\underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}\underset{\sim}{\Sigma}_{12}' - \rho^2\underset{\sim}{\Sigma}_{11}| = 0 \ .$$

Then, the distribution of $r_1^2,\ldots,r_p^2$ is given by Constantine [1], in the following form [3]

$$(4.1) \qquad |I-P^2|^{\frac{1}{2}\nu}\; C(p,f_1,f_2)\; {}_2F_1(\tfrac{1}{2}\nu,\tfrac{1}{2}\nu;\tfrac{1}{2}f_2;P^2,R^2)|R^2|^{\frac{1}{2}(f_2-p-1)}|I-R^2|^{\frac{1}{2}(f_1-p-1)} \cdot$$

$$\prod_{i>j}(r_i^2 - r_j^2)\prod_{i=1}^{p} dr_i^2 \,, \qquad 0 < r_1^2 \le \ldots \le r_p^2 < 1 \,,$$

where $f_2 = q$ and $f_1 = \nu-q$ . Again, in this case $V^{(s)} = V^{(p)} = \sum_{i=1}^{p} r_i^2$ .

Now using lemma 1 once and lemma 2 twice and integrating with respect to $R^2$ with the help of lemma 3 we get the mgf of $V^{(p)} = \mathrm{tr}\, R^2$ in the form

$$(4.2) \qquad E(e^{t\,\mathrm{tr}R^2}) = \frac{2^{\frac{1}{2}p(p-1)}\Gamma_p(\frac{1}{2}f_2)\,|I-P^2|^{\frac{1}{2}\nu}}{\Gamma_p^2(\frac{1}{2}\nu)(2\Pi i)^{\frac{1}{2}p(p+1)}} \int_{R(T)>0} e^{\mathrm{tr}\,T}|T|^{-\frac{1}{2}f_2} \cdot$$

$$\int_{S_2>0} e^{-\mathrm{tr}S_2}|S_2|^{\frac{1}{2}(\nu-p-1)} \int_{S_1>0} e^{-\mathrm{tr}S_1}|S_1|^{\frac{1}{2}(\nu-p-1)} \cdot$$

$${}_1F_1(\tfrac{1}{2}f_2;\; \tfrac{1}{2}\nu;\; It + S_1S_2T^{-1}P^2)\; dS_1\; dS_2\; dT \,.$$

Again, using (2.6), lemma 4 and a similar lemma we have

$$(4.3) \qquad E(e^{t\,\mathrm{tr}R^2}) = |I-P^2|^{\frac{\nu}{2}} \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{n=0}^{k}\sum_{\eta} \frac{(\frac{1}{2}f_2)_\kappa((\frac{1}{2}\nu)_\eta)^2 a_{\kappa,\eta} t^{k-n} C_\kappa(I) C_\eta(P^2)}{(\frac{1}{2}\nu)_\kappa\;(\frac{1}{2}f_2)_\eta\; k!\;\; C_\eta(I)} \,,$$

5. Remarks. Khatri [4] has obtained the non-central mgf of $V^{(p)}$ associated with the test $\lambda\Sigma_1 = \Sigma_2$, $\lambda > 0$, where $\Sigma_1$ and $\Sigma_2$ are the covariance matrices of two p-variate normal populations. However, a factor $(\frac{1}{2}\nu)_\eta$ has been omitted in the expression for the mgf and hence the correct expression is given by

$$E(e^{tV^{(p)}}) = |\lambda\Lambda|^{-\frac{1}{2}f_1} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{n=0}^{k} \sum_{\eta} \frac{(\frac{1}{2}f_1)_\kappa \, (\frac{1}{2}\nu)_\eta \, a_{\kappa,\eta} t^{k-n} C_\kappa(I) C_\eta(I-(\lambda\Lambda)^{-1})}{(\frac{1}{2}\nu)_\kappa \; k! \; C_\eta(I)},$$

where $\Lambda = \Sigma_1 \Sigma_2^{-1}$, $f_1 = n_1 - 1$ and $f_2 = n_2 - 1$, where $n_1$ and $n_2$ are the respective sizes of the samples from each of the two populations.

Further it should be pointed out that the moments of $V^{(p)}$ obtained by Pillai [10] for (i) and (ii) for $p = 2$ in the linear case and those by Khatri and Pillai [5], [6], [7] for (i) for general linear and planar cases were verified to follow from (3.5) and (4.3) to the extent $a_{\kappa,\eta}$ coefficients are available in Constantine [2] and further tabulations carried out by us.

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