On the Distribution of the Maximum

and Minimum of Ratios of Order Statistics*

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1. Introduction and Summary

Let $X_i (i=0,1,\ldots,p)$ be $(p+1)$ independent and identically distributed non-negative random variables each representing the jth order statistic in a random sample of size n from a continuous distribution $G(x)$ of a nonnegative random variable. Let $H_{j,n}(x)$ be the cumulative distribution function of $X_i (i=0,1,\ldots,p)$. Consider the ratios $Y_i = X_i/X_0$ $(i=1,2,\ldots,p)$. The random variables Y_i $(i=1,2,\ldots,p)$ are correlated and the distribution of the maximum and the minimum is of interest in problems of selection and ranking for restricted families of distribution. The distribution-free subset selection rules using the percentage points of these order statistics are investigated in a companion paper by Barlow and Gupta (1967). In the present paper, we discuss the distribution of these statistics, in general, for any $G(x)$ and then derive specific results for $G(x) = 1-e^{-x/\theta}$, $x > 0$, $\theta > 0$. Section 2 deals with the distribution of the maximum while Section 3 discusses the distribution of the minimum. Section 4 describes the tables of the percentage points of the two statistics.

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2. Distribution of Y_{max}

First we derive the joint distribution of $Y_i (i=1,2,\ldots,p)$. The joint density function for $X_0, X_1, \ldots, X_p$ is given by

$$(2.1) \qquad f(x_0,x_1,\ldots,x_p)=[j\binom{n}{j}]^{p+1} \prod_{t=0}^{p} G^{j-1}(x_t)\ [1-G(x_t)]^{n-j} g(x_t)$$

where $g(x) = dG(x)/dx$.

Making appropriate transformations the joint density of $Y_1, Y_2, \ldots, Y_p$ can be written as

$$(2.2) \quad f_1(y_1,y_2,\ldots,y_p)=[j\binom{n}{j}]^{p+1}\int_0^\infty y_0^p [G(y_0)\prod_1^p G(y_t y_0)]^{j-1}[(1-G(y_0)\prod_1^p(1-G(y_t y_0)]^{n-j} g(y_o)\prod_1^p g(y_t y_0)\, dy_0$$

For $G(x) = 1-e^{-x/\theta}$, (2.2) reduces to

$$(2.3) \quad f_2(y_1,y_2,\ldots,y_p)=[j\binom{n}{j}]^{p+1}\int_0^1 (-\log u)^p [(1-u)(1-u^{y_1})\ldots(1-u^{y_p})]^{j-1} u^{(n-j+1)(1+y_1+\ldots+y_p)-1}\, du$$

For $j = 1$, we obtain

$$(2.4) \qquad f_2(y_1,y_2,\ldots,y_p)= \frac{\Gamma(p+1)}{(1+y_1+\ldots+y_p)^{p+1}}\ ,\ 0 \le y_1,\ldots,y_p < \infty.$$

It should be noted that the distribution (2.4) is independent of n, as was also pointed out in the companion paper by Barlow and Gupta (1967). Again, (2.4) gives the joint density functions of several correlated F random variables each with (2,2) degrees of freedom. In this case, Y_{max} and Y_{min} are the largest and the smallest of several correlated F statistics with degrees of freedom (2,2). The distribution of the largest and the smallest of several correlated F statistics with different degrees of freedom have been discussed by Gupta (1963a) and Gupta and Sobel (1962) respectively.

The cumulative distribution function of Y_{max} can be obtained directly (without using (2,2)) as follows.

$$P\{Y_{max} \leq y\} \equiv H(y) = \int_0^\infty G^p_{j,n}(yx)\; g_{j,n}(x)\, dx \tag{2.5}$$

where

$$g_{j,n}(x) = j\binom{n}{j} G^{j-1}(x)[1-G(x)]^{n-j} g(x), \tag{2.6}$$

and

$$G_{j,n}(x) \equiv \int_0^x g_{j,n}(t)\, dt = I_{G(x)}(j,n-j+1) = \sum_{t=j}^{n} \binom{n}{t} G^t(x)(1-G(x))^{n-t}. \tag{2.7}$$

The density of Y_{max} is

$$h(y) = p \int_0^\infty x\, G^{p-1}_{j,n}(yx)\; g_{j,n}(yx)\; g_{j,n}(x) dx. \tag{2.8}$$

By expanding $G^p_{j,n}(yx)$ in powers of $1-G(yx)$, we can express (2.5) as

$$H_1(y) = j\binom{n}{j} \sum_{r=0}^{np} \int_0^\infty b(r,p;n,j)[1-G(xy)]^r G^{j-1}(x)[1-G(x)]^{n-j} g(x) dx \tag{2.9}$$

where $b(r,p;n,j)$ is the coefficient of y^r in $[\sum_{t=j}^{n} \binom{n}{t}(1-y)^t y^{n-t}]^p$ and is given by the following recursion relations.

$$(2.10)\qquad b(r,1;n,j) = \begin{cases} 1 \quad , & r = 0 \\ 0 \quad , & 1 \le r \le n-j \\ \binom{n}{r} \sum_{t=0}^{n-j} (-1)^{r-t} \binom{r}{t}, & n-j+1 \le r \le n \\ 0 \quad , & n < r < \infty \end{cases}$$

$$(2.11)\qquad b(r,p;n,j) = \begin{cases} 1 & r = 0 \\ 0 & 1 \le r \le n-j \\ b(r,p-1;n,j) + \sum_{t=n-j+1}^{r} b(t,1;n,j)b(r-t,p-1;n,j), & n-j+1 \le r \le n \\ b(r,p-1;n,j) + \sum_{t=n-j+1}^{n} b(t,1;n,j)b(r-t,p-1;n,j), & n+1 \le r \le np-n \\ \sum_{\max(n-j+1,r-np+n)}^{n} b(t,1;n,j)\, b(r-t,p-1;n,j), & np-n+1 \le r \le np \\ 0 \quad , & np < r < \infty \end{cases}$$

The density $h(y)$ can be written in a similar way. For the special case $G(x) = 1-e^{-x/\theta}$, we obtain

$$(2.12)\qquad H_1(y) = 1+ \sum_{r=n-j+1}^{np} \frac{b(r,p;n,j)}{(1+\frac{ry}{n})(1+\frac{ry}{n-1})\cdots(1+\frac{ry}{n-j+1})}$$

For j=1, the coefficients $b(r,p;n,1)=(-1)^{\ell}\binom{p}{\ell}$ if $r=n\ell$, $\ell=0,1,2,\ldots,p$ and zero otherwise. It follows that

$$(2.13)\qquad H_1(y)= \sum_{\ell=0}^{p} (-1)^{\ell} \binom{p}{\ell} \frac{1}{1+\ell y} ,$$

which is independent of n as it should be.

$$(2.14)\qquad h_1(y) = \sum_{\ell=0}^{p} (-1)^{\ell+1} \binom{p}{\ell} \frac{\ell}{(1+\ell y)^2}$$

Incidentally, one can obtain inequalities on the right hand sides of (2.12) and (2.13) by using the fact that $H_1(y)$ and $h_1(y)$ are the cdf and the density function.

3. Distribution of Y_{min}

The cdf $H_2(y)$ of Y_{min} is given by

$$(3.1)\qquad H_2(y) \equiv P\{Y_{min} \le y\} = 1-\int_0^\infty [1-G_{j,n}(yx)]^p g_{j,n}(x)\, dx,$$

where $g_{j,n}(x)$ and $G_{j,n}(x)$ are given by (2.6) and (2.7).

The density of Y_{min} is

$$(3.2)\qquad h_2(y) = p\int_0^\infty x[1-G_{j,n}(yx)]^{p-1} g_{j,n}(yx)\, g_{j,n}(x)\, dx.$$

Let $1-H_2(y) = F(y)$. By expanding $[1-G_{j,n}(yx)]^p$ in powers of $1-G(yx)$, we can write

$$(3.3)\quad F(y)=j\binom{n}{j} \sum_{r=0}^{np} \int_0^\infty b'(r,p;n,j)[1-G(xy)]^r\, G^{j-1}(x)[1-G(x)]^{n-j}\, g(x)\, dx$$

where $b'(r,p;n,j)$ is the coefficient of y^r in $[\sum_{t=0}^{j-1} \binom{n}{t}(1-y)^t y^{n-t}]^p$ and is given by the following recursion relations:

$$(3.4)\qquad b'(r,1;n,j) = \begin{cases} 0, & 0 \le r \le n-j \\ \sum_{k=0}^{j-1-n+r} (-1)^k \binom{n}{n-r+k}\binom{n-r+k}{k}, & n-j+1 \le r \le n \\ 0, & n < r < \infty \end{cases}$$

$$
(3.5)\quad b'(r,p;n,j) = \begin{cases} 0 & 0 \leq r \leq (n-j+1)p-1 \\ \displaystyle\sum_{m=\max(n-j+1,\,r-n(p-1))}^{\min(n,\,r-(p-1)(n-j+1)} b'(m,1;n,j)\,b'(r-m,p-1;n,j), & (n-j+1)p \leq r \leq np \\ 0 & np < r < \infty \end{cases}
$$

The density $h_2(y)$ can be written similarly. For the special case $G(x) = 1-e^{-x/\theta}$, we obtain

$$
(3.6)\qquad F(y) = \sum_{r=(n-j+1)p}^{np} \frac{b'(r,p;n,j)}{(1+\frac{ry}{n})(1+\frac{ry}{n-1})\ldots(1+\frac{ry}{n-j+1})}
$$

For $j = 1$, the coefficients $b(r,p;n,1) = 1$ if $r = np$ and zero otherwise. So (3.6) reduces to

$$
(3.7)\qquad F(y) = \frac{1}{1+yp},
$$

which is independent of n. In this case

$$
(3.8)\qquad h_2(y) = \frac{p}{(1+yp)^2}.
$$

4. Asymptotic Results

Let ξ_α denote the quantile of order α of the distribution $G(x)$, i.e. the root (assumed unique) of the equation $G(\xi) = \alpha$, where $0 < \alpha < 1$. We assume that, in some neighbourhood of $x = \xi_\alpha$, the density function $g(x)$ is continuous and has a continuous derivative $g'(x)$. In a sample of size n from the distribution $G(x)$, we take the jth smallest observation such that $j \leq n\alpha < j+1$. Then $X_{j,n}$ is asymptotically normal with mean ξ and standard deviation $\frac{11}{f(\xi_\alpha)}\sqrt{\frac{\alpha\bar{\alpha}}{n}}$, where $\bar{\alpha} = 1-\alpha$.

Thus we have, as $n \to \infty$ and $\frac{j}{n} \to \alpha$

$$(4.1) \qquad H_1(y) \approx \int_{-\infty}^{\infty} \Phi^p(xy + \frac{(y-1)\xi_\alpha f(\xi_\alpha)\sqrt{n}}{\sqrt{\alpha\bar{\alpha}}}) \, d\Phi(x)$$

and

$$(4.2) \qquad H_2(y) \approx 1 - \int_{-\infty}^{\infty} [1 - \Phi(xy + \frac{(y-1)\xi_\alpha f(\xi_\alpha)\sqrt{n}}{\sqrt{\alpha\bar{\alpha}}})]^p \, d\Phi(x)$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt \, .$$

For $p = 1$, we get

$$(4.3) \qquad H_1(y) = H_2(y) \approx \int_{-\infty}^{\infty} \Phi(xy + \frac{(y-1)\xi_\alpha f(\xi_\alpha)\sqrt{n}}{\sqrt{\alpha\bar{\alpha}}}) \, d\Phi(x)$$

Using the result

$$\int_{-\infty}^{\infty} \Phi(\alpha x + \beta) d\Phi(x) = \Phi(\frac{\beta}{\sqrt{1+\alpha^2}}) \, ,$$ this reduces to

$$(4.4) \qquad H_1(y) = H_2(y) \approx \Phi(\frac{(y-1)\xi_\alpha f(\xi_\alpha)\sqrt{n}}{\sqrt{1+y^2}\,\sqrt{\alpha\bar{\alpha}}})$$

So if $H_1(y) = P^*$

$$(4.5) \qquad \frac{(y-1)\xi_\alpha f(\xi_\alpha)\sqrt{n}}{\sqrt{1+y^2}\,\sqrt{\alpha\bar{\alpha}}} \quad \Phi^{-1}(P^*) \, ,$$

which can be written as

(4.6) $$y^2(1-B^2) - 2y + (1-B^2) = 0 ,$$

where $$B = \frac{\Phi^{-1}(P^*)\sqrt{\alpha\bar{\alpha}}}{\xi_\alpha f(\xi_\alpha)\sqrt{n}} .$$

Obviously, this quadratic equation in y, has two positive roots which are reciprocals of each other. The appropriate root can be determined using the fact that $H_1(y)$ is increasing in y and $H_1(1) = \frac{1}{2}$. So for $P^* > \frac{1}{2}$ (which will be the case for the selection procedures discussed in the companion paper), $y > 1$.

For the special case $G(x) = 1 - e^{-\frac{x}{\theta}}$,

$$e^{-\frac{\xi_\alpha}{\theta}} = 1 - \alpha \quad \text{and}$$

$$f(\xi_\alpha) = \frac{1}{\theta} e^{-\frac{\xi_\alpha}{\theta}} = \frac{1-\alpha}{\theta} .$$

So (4.1) and (4.2) reduce to

(4.7) $$H_1(y) \approx \int_{-\infty}^{\infty} \Phi^p(xy - \sqrt{\frac{n\bar{\alpha}}{\alpha}}\,(y-1)\log \bar{\alpha})\, d\Phi(x)$$

and

(4.8) $$H_2(y) \approx 1- \int_{-\infty}^{\infty} [1-\Phi(xy - \sqrt{\frac{n\bar{\alpha}}{\alpha}}\,(y-1)\log \bar{\alpha})]^p\, d\Phi(x) .$$

For a general p , to solve for y from $H_1(y) = P^*$ or $H_2(y) = P^*$ using (4.1) and (4.2), the table II of Gupta (1963b) can be used with interpolations if necessary.

5. Description of the tables

Table 1 provides for the case $j = 1$ the reciprocals of the percentage points of the distribution of Y_{max} corresponding to the probability levels $\alpha = P^* = .75$, .90 and .95 and the percentage points of the distribution of Y_{min} corresponding to the probability levels $\alpha = 1-P^* = .05$, .10 and .25 for $p = 1(1)9$. We note that when $j = 1$, the statistics Y_{max} and Y_{min} are the maximum and the minimum of several correlated F statistics with degrees of freedom (2,2) and hence the entries in Table 1 are the same as those for $\nu=2$ in Tables 1 A, B, C of Gupta (1963a) in the case of Y_{max} and same as those for $\nu=2$ in Tables 3 A, B, C of Gupta and Sobel (1962) in the case of Y_{min}, but are given for more places of decimals.

Tables 2 A through 2 E give the reciprocals of the percentage points of the distribution of Y_{max} corresponding to the probability levels $\alpha = P^* = .75$, .90, .95 for p=1 through 5 respectively. The ranges of N are: 5(1)15 in Tables 2 A,B,C, 5(1)12 in Table 2D and 5(1)10 in Table 2 E .

Tables 3A through 3E present the percentage points of the distribution of Y_{min} corresponding to the probability levels $\alpha = 1-P^* = .25$, .10, .05 for $p = 1$ through 5 respectively. The ranges of N: 5(1)15 in Tables 3A-D & 5(1)12 in Table 3E.

In all these tables the probability levels are chosen such that P^*, the infimum of the probability of correct selection in the companion paper by Barlow and Gupta is .75, .90 and .95 and the entries are either the percentage points or the reciprocals of the percentage points so that they will be the values of the constants d or c $(0 < c, d < 1)$ to be used in the selection procedures discussed in the companion paper.

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TABLE I

A. Reciprocals of 100α percentage points of $Y_{max} = \max_{1<i\leq p} X_i/X_0$ for $j=1$ and all n.

$\alpha = P^*$ / p	0.75	0.90	0.95
1	.33333	.11111	.05263
2	.20783	.07233	.03469
3	.16631	.05871	.02827
4	.14472	.05145	.02483
5	.13115	.04683	.02263
6	.12166	.04357	.02107
7	.11456	.04112	.01990
8	.10901	.03919	.01897
9	.10451	.03762	.01822

B. 100α percentage points of $Y_{min} = \min_{1<i\leq p} X_i/X_0$ for $j=1$ and all n.

$1-\alpha = P^*$ / p	0.75	0.90	0.95
1	.33333	.11111	.05263
2	.16667	.05556	.16667
3	.11111	.03704	.11111
4	.08333	.02778	.08333
5	.06667	.02222	.06667
6	.05556	.01852	.05556
7	.04762	.01587	.04762
8	.04167	.01389	.04167
9	.03704	.01235	.03704

For given p and P^*, the entries in Tables A and B are respectively the values of c and d (c and d are independent of n) for which

$$\int_0^\infty G_{1,n}^p\left(\frac{x}{c}\right) dG_{1,n}(x) = P^* \quad \text{and} \quad \int_0^\infty [1-G_{1,n}(xd)]^p \, dG_{1,n}(x) = P^*$$

where $G_{1,n}(\cdot)$ is the c.d.f. of the smallest order statistic in a sample of size n from the exponential distribution.

Table 2A

Reciprocals of the percentage points of $Y_{max} = \max_{1 \le i \le p} X_i/X_0$ for $p = 1$

N \ j	2	3	4	5	6	7	8	9	10	11	12	13	14
5	.48307	.55596	.59583	.60462									
	.24225	.32197	.37003	.38251									
	.15560	.22871	.27582	.28958									
6	.48353	.55788	.60192	.62654									
	.24265	.32397	.37699	.40851									
	.15591	.23045	.28228	.31441									
7	.48379	.55889	.60473	.63401	.64944								
	.24289	.32503	.38021	.41755	.43822								
	.15609	.23138	.28527	.32314	.34485								
8	.48396	.55949	.60626	.63757	.65802	.66737							
	.24303	.32566	.38198	.42189	.44902	.46206							
	.15620	.23193	.28692	.32735	.35558	.36965							
9	.48406	.55988	.60720	.63958	.66222	.67684	.68189						
	.24313	.32607	.38306	.42434	.45436	.47436	.48175						
	.15627	.23229	.28793	.32973	.36089	.38211	.39036						
10	.48414	.56014	.60782	.64082	.66463	.68159	.69211	.69396					
	.24320	.32635	.38377	.42587	.45743	.48058	.49534	.49837					
	.15633	.23253	.28860	.33121	.36396	.38843	.40435	.40801					
11	.48420	.56033	.60824	.64166	.66616	.68436	.69731	.70481	.70421				
	.24325	.32655	.38427	.42689	.45939	.48422	.50232	.51307	.51266				
	.15637	.23270	.28906	.33220	.36592	.39215	.41157	.42334	.42330				
12	.48424	.56047	.60855	.64224	.66720	.68614	.70040	.71041	.71558	.71305			
	.24329	.32669	.38462	.42761	.46071	.48657	.50648	.52074	.52833	.52513			
	.15639	.23283	.28939	.33290	.36724	.39455	.41588	.43136	.43979	.43671			
13	.48428	.56057	.60878	.64267	.66793	.68736	.70240	.71377	.72153	.72488	.72079		
	.24331	.32680	.38489	.42813	.46165	.48817	.50919	.52535	.53659	.54163	.53612		
	.15642	.23293	.28964	.33341	.36818	.39619	.41869	.43622	.44855	.45425	.44861		
14	.48430	.56066	.60896	.64299	.66847	.68823	.70379	.71597	.72513	.73112	.73300		
	.24334	.32689	.38509	.42853	.46234	.48933	.51106	.52839	.54163	.55044	.55338		
	.15643	.23300	.28982	.33380	.36888	.39737	.42064	.43941	.45389	.46365	.46707		
15	.48432	.56072	.60909	.64324	.66888	.68888	.70479	.71751	.7275	.73495			
	.24335	.32696	.38525	.42883	.46287	.49019	.51242	.53052	.54497	.55584			
	.15645	.23306	.28997	.33409	.36940	.39825	.42205	.44165	.45744	.46944			

For given p,n,j and $P^* = .75$(top), .90(middle), .95(bottom), the entries in this table are the values of c for which $\int_0^\infty G^p_{j,n}(\frac{x}{c})\, dG_{j,n}(x) = P^*$ where $G_{j,n}(\cdot)$ is the c.d.f. of the jth order statistic in a sample of size n from the exponential distribution.

TABLE 2B

Reciprocals of the percentage points of $Y_{max} = \max_{1 \le i \le p} X_i/X_0$ for p=2

N \ j	2	3	4	5	6	7	8	9	10	11	12	13	14
5	.34892	.42508	.46809	.47527									
	.18181	.25464	.29996	.31044									
	.11853	.18353	.22682	.23872									
6	.34945	.42738	.47564	.50261	.49813								
	.18220	.25665	.30712	.33734	.33528								
	.11882	.18521	.23315	.26322	.26304								
7	.34975	.42860	.47913	.51207	.52891								
	.18242	.25772	.31045	.34682	.36670								
	.11898	.18610	.23611	.27194	.29238								
8	.34994	.42934	.48104	.51659	.53993	.54982							
	.18256	.25835	.31228	.35139	.37819	.39056							
	.11908	.18663	.23774	.27616	.30324	.31642							
9	.35006	.42980	.48221	.51915	.54535	.56214	.56697						
	.18265	.25876	.31340	.35398	.38388	.40378	.41046						
	.11915	.18697	.23873	.27855	.30865	.32917	.33668						
10	.35015	.43012	.48298	.52074	.54847	.56833	.58038	.58137					
	.18272	.25904	.31414	.35559	.38718	.41049	.42519	.42740					
	.11920	.18720	.23939	.28005	.31178	.33567	.35111	.35408					
11	.35022	.43035	.48352	.52180	.55045	.57195	.58724	.59572	.59371				
	.18276	.25924	.31465	.35667	.38927	.41443	.43279	.44345	.44206				
	.11923	.18737	.23984	.28105	.31377	.33949	.35858	.36998	.36923				
12	.35026	.43051	.48390	.52254	.55179	.57428	.59131	.60316	.60885	.60442			
	.18280	.25939	.31502	.35743	.39069	.41697	.43732	.45184	.45926	.45492			
	.11926	.18749	.24017	.28175	.31512	.34196	.36305	.37834	.38644	.38259			
13	.35030	.43064	.48419	.52309	.55274	.57588	.59396	.60762	.61679	.62026	.61386		
	.18283	.25950	.31529	.35798	.39169	.41871	.44028	.45692	.46837	.47314	.46633		
	.11928	.18758	.24042	.28227	.31608	.34365	.36597	.38341	.39562	.40098	.39450		
14	.35033	.43074	.48441	.52350	.55344	.57702	.59579	.61056	.62162	.62865	.63030	.62224	
	.18285	.25958	.31550	.35840	.39244	.41996	.44233	.46026	.47393	.48289	.48544	.47654	
	.11930	.18766	.24060	.28265	.31679	.34487	.36800	.38675	.40123	.41088	.41394	.40520	
15	.35035	.43082	.48458	.52382	.55397	.57787	.59711	.61260	.62481	.63379	.63909	.63922	.62977
	.18287	.25965	.31567	.35872	.39300	.42089	.44381	.46259	.47763	.48889	.49578	.49647	.48575
	.11931	.18771	.24075	.28295	.31734	.34578	.36947	.38909	.40496	.41700	.42451	.42559	.41489

For given p,n,j and P* = .75(top), .90(middle), .95(bottom) the entries in this table are the values of c for which $\int_0^\infty G_{j,n}^p(\frac{x}{c})\, dG_{j,n}(x) = P^*$ where $G_{j,n}(\cdot)$ is the c.d.f. of the jth order statistic in a sample of size n from the exponential distribution.

Table 2C

Reciprocals of the percentage points of $Y_{max} = \max_{1 \le i \le p} X_i/X_0$ for $p = 3$

N \ j	2	3	4	5	6	7	8	9	10	11	12	13	14
5	.29806	.37258	.41523										
	.15749	.22607	.26937										
	.10324	.16388	.20487										
6	.29860	.37502	.42329	.45018									
	.15786	.22808	.27658	.30566									
	.10352	.16551	.21113	.23985									
7	.29891	.37631	.42703	.46038	.47708								
	.15808	.22914	.27994	.31528	.33440								
	.10367	.16639	.21406	.24854	.26810								
8	.29911	.37708	.42909	.46528	.48904	.49862							
	.15821	.22978	.28179	.31994	.34614	.35791							
	.10377	.16691	.21568	.25276	.27899	.29153							
9	.29924	.37758	.43035	.46804	.49495	.51207	.51640						
	.15830	.23018	.28292	.32257	.35198	.37149	.37762						
	.10384	.16724	.21667	.25515	.28442	.30437	.31137						
10	.29933	.37792	.43117	.46976	.49835	.51885	.53111	.53141					
	.15837	.23047	.28366	.32422	.35535	.37840	.39281	.39447					
	.10388	.16747	.21732	.25665	.28757	.31093	.32595	.32846					
11	.29939	.37816	.43174	.47091	.50051	.52283	.53865	.54720	.54431				
	.15841	.23066	.28418	.32532	.35750	.38246	.40067	.41107	.40911				
	.10392	.16763	.21777	.25765	.28958	.31479	.33353	.34459	.34341				
12	.29944	.37834	.43216	.47172	.50197	.52538	.54314	.55541	.56103	.55556			
	.15845	.23081	.28455	.32609	.35896	.38508	.40536	.41978	.42694	.42198			
	.10394	.16775	.21809	.25835	.29094	.31729	.33806	.35310	.36091	.35662			
13	.29948	.37834	.43246	.47231	.50301	.52713	.54605	.56034	.56983	.57310	.56550		
	.15848	.23092	.28483	.32666	.35999	.38688	.40842	.42504	.43642	.44091	.43342		
	.10396	.16784	.21834	.25887	.29191	.31901	.34103	.35825	.37026	.37536	.36842		
14	.29951	.37858	.43270	.47276	.50378	.52839	.54807	.56359	.57517	.58241	.58374	.57435	
	.15850	.23101	.28505	.32709	.36076	.38817	.41054	.42851	.44221	.45108	.45334	.44369	
	.10398	.16791	.21852	.25925	.29262	.32024	.34309	.36166	.37599	.38547	.38827	.37904	
15	.29954	.37866	.43289	.47310	.50436	.52953	.54953	.56585	.57871	.58812	.59352	.59323	.58232
	.15851	.23108	.28521	.32742	.36134	.38913	.41208	.43094	.44606	.45735	.46413	.46449	.45298
	.10399	.16797	.21867	.25956	.29317	.32116	.34458	.36404	.37980	.39173	.39908	.39990	.38869

For given p,n,j and P* = .75(top), .90(middle), .95(bottom), the entries in this table are the values of c for which $\int_0^\infty G_{j,n}^p(\frac{x}{c})\, dG_{j,n}(x) = P^*$ where $G_{j,n}(\cdot)$ is the c.d.f. of the jth order statistic in a sample of size n from the exponential distribution.

Table 2D

Reciprocals of the percentage points of $Y_{max} = \max_{1 \le i \le p} X_i/X_0$ for $p = 4$

N \ j	2	3	4	5	6	7	8	9	10	11
5	.26969 .14355 .09440	.34243 .20924 .15215	.38435 .25105 .19156							
6	.27023 .14392 .09466	.34493 .21123 .15377	.39269 .25827 .19778	.41919 .28647 .22552						
7	.27055 .14414 .09481	.34626 .21229 .15463	.39657 .26164 .20068	.42980 .29617 .23418	.44617 .31468 .25309					
8	.27075 .14427 .09491	.34705 .21293 .15514	.39870 .26351 .20229	.43490 .30086 .23839	.45867 .32656 .26399	.46788 .33786 .27605				
9	.27088 .14436 .09497	.34757 .21334 .15547	.40004 .26464 .20328	.43779 .30353 .24079	.46485 .33247 .26943	.48198 .35164 .28893	.48587 .35735 .29555			
10	.27097 .14442 .09502	.34791 .21361 .15570	.40086 .26539 .20392	.43958 .30519 .24228	.46841 .33590 .27259	.48910 .35865 .29552	.50133 .37280 .31021	.50110 .37405 .31240		
11	.27104 .14447 .09505	.34816 .21381 .15586	.40146 .26591 .20437	.44079 .30631 .24328	.47068 .33808 .27461	.49328 .36279 .29942	.50927 .38080 .31785	.51774 .39097 .32864	.51423 .38860 .32715	
12	.27109 .14450 .09508	.34835 .21396 .15598	.40188 .26629 .20470	.44163 .30709 .24399	.47220 .33956 .27597	.49596 .36545 .30193	.51399 .38559 .32243	.5264 .39987 .33723	.53188 .40681 .34481	.52572 .40142 .34022

For given p, n, j and $P^* = .75$(top), .90(middle), .95(bottom), the entries in this table are the values of c for which $\int_0^\infty G^p_{j,n}(\frac{x}{c})\, dG_{j,n}(x) = P^*$, where $G_{j,n}(\cdot)$ is the c.d.f. of the jth order statistic in a sample of size n from the exponential distribution.

TABLE 2E

Reciprocals of the percentage points of $Y_{max} = \max_{1 \leq i \leq p} X_i/X_0$ for $p = 5$

N \ j	2	3	4	5	6	7	8	9
5	.25101 .13424 .08844	.32219 .19776 .14410	.36340 .23842 .18232					
6	.25156 .13461 .08870	.32473 .19975 .14570	.37191 .24565 .18850	.39800 .27314 .21549				
7	.25188 .13482 .08885	.32609 .20081 .14655	.37587 .24903 .19139	.40887 .28288 .22413	.42490 .30091 .24253			
8	.25208 .13495 .08895	.32690 .20144 .14706	.37806 .25099 .19299	.41411 .28761 .22834	.43776 .31287 .23600	.44664 .32379 .26511		
9	.25221 .13504 .08901	.32742 .20185 .14739	.37940 .25204 .19397	.41707 .29029 .23073	.44412 .31884 .25888	.46117 .33770 .27802	.46469 .34308 .28433	
10	.25231 .13510 .08905	.32777 .20213 .14762	.38027 .25279 .19462	.41894 .29196 .23223	.4478 .3223 .26206	.468 .3448 .28463	.48065 .35870 .29905	.48001 .35965 .30097

For given p, n, j and $P^* = .75$(top), .90(middle), .95(bottom), the entries in this table are the values of c for which $\int_0^\infty G^p_{j,n}(\frac{x}{c})dG_{j,n}(x) = P^*$, where $G_{j,n}(\cdot)$ is the c.d.f. of the jth order statistic in a sample of size n from the exponential distribution.

TABLE 3A

Percentage points of the distribution of $Y_{min} = \min_{1 \le i \le p} X_i/X_0$ for p = 1

n \ j	2	3	4	5	6	7	8	9	10	11	12	13	14	15
5	.48307	.55596	.59583	.60462										
	.24225	.32197	.37004	.38251										
	.15560	.22871	.27582	.28958										
6	.48353	.55788	.60192	.62654	.62516									
	.24266	.32397	.37699	.40851	.40827									
	.15591	.23045	.28228	.31441	.31548									
7	.48379	.55889	.60473	.63401	.64944	.64125								
	.24289	.32503	.38021	.41755	.43822	.42890								
	.15609	.23138	.28527	.32314	.34485	.33648								
8	.48396	.55949	.60626	.63757	.65802	.66737	.65431							
	.24303	.32566	.38198	.42189	.44902	.46206	.44593							
	.15620	.23193	.28692	.32735	.35557	.36964	.35399							
9	.48407	.55988	.60720	.63958	.66222	.67685	.68189	.66520						
	.24313	.32607	.38306	.42434	.45436	.47436	.48175	.46032						
	.15627	.23229	.28793	.32972	.36089	.38211	.39036	.36890						
10	.48414	.56014	.60782	.64082	.66463	.68159	.69211	.69396	.67447					
	.24320	.32635	.38377	.42587	.45743	.48057	.49534	.49837	.47271					
	.15633	.23253	.28859	.33121	.36396	.38843	.40435	.40801	.38182					
11	.48420	.56033	.60824	.64166	.66616	.68436	.69731	.70481	.70421	.68250				
	.24325	.32654	.38427	.42689	.45939	.48422	.50232	.51307	.51266	.48353				
	.15637	.23270	.28905	.33220	.36592	.39215	.41157	.42333	.42330	.39316				
12	.48424	.56047	.60855	.64224	.66720	.68614	.70040	.71041	.71558	.71305	.68954			
	.24329	.32669	.38462	.42761	.46071	.48657	.50648	.52074	.52833	.52512	.49310			
	.15639	.23283	.28939	.33290	.36724	.39455	.41588	.43136	.43980	.43671	.40323			
13	.48427	.56057	.60878	.64267	.66793	.68736	.70240	.71377	.72153	.72488	.72078	.69578		
	.24331	.32680	.38489	.42813	.46165	.48817	.50919	.52535	.53659	.54163	.53612	.50166		
	.45642	.23293	.28964	.33341	.36818	.39619	.41869	.43621	.44855	.45425	.44860	.41226		
14	.48430	.56066	.60896	.64299	.66847	.68823	.70379	.71597	.72513	.73112	.73300	.72762	.70138	
	.24334	.32689	.38509	.42852	.46234	.48933	.51106	.52839	.54163	.55044	.55338	.54593	.50936	
	.15643	.23300	.28983	.33379	.36888	.39737	.42064	.43941	.45389	.46366	.46707	.45926	.42044	
15	.48432	.56072	.60909	.64324	.66888	.68888	.70479	.71751	.72751	.73494	.73951	.74018	.73373	.70643
	.24336	.32696	.38525	.42883	.46287	.49019	.51242	.53052	.54497	.55585	.56266	.56384	.55474	.51636
	.15645	.23306	.28997	.33409	.36940	.39825	.42205	.44165	.45745	.46945	.47707	.47856	.46887	.42787

For given p, n, j and P*= .75(top, .90(middle), .95(bottom), the entries in this table are the values of d for which $\int_0^\infty [1-G_{j,n}(xd)]^p \, dG_{j,n}(x) = P^*$ where $G_{j,n}(\cdot)$ is the c.d.f. of the jth order statistic in a sample of size n from the exponential distribution.

TABLE 3B

Percentage points of the distribution of $Y_{min} = \min_{1 \leq i \leq p} X_i/X_0$ for $p = 2$

n \ j	2	3	4	5	6	7	8	9	10	11	12	13	14	15
5	.31553	.39895	.44798	.46243										
	.16203	.23711	.28562	.30102										
	.10551	.17100	.21617	.23160										
6	.31594	.40084	.45438	.48616	.48824									
	.16234	.23881	.29187	.32510	.32778									
	.10574	.17244	.22180	.25390	.25725									
7	.31617	.40184	.45733	.49430	.51519	.50871								
	.16251	.23971	.29477	.33353	.35619	.34946								
	.10587	.17320	.22441	.26178	.28424	.27829								
8	.31631	.40243	.45894	.49820	.52479	.53822	.52548							
	.16262	.24024	.29636	.33758	.36652	.38147	.36752							
	.10595	.17365	.22585	.26559	.29416	.30929	.29598							
9	.31641	.40282	.45993	.50040	.52952	.54905	.55708	.53956						
	.16230	.24059	.29734	.33988	.37163	.39346	.40257	.38289						
	.10601	.17395	.22673	.26775	.29910	.32104	.33042	.31115						
10	.31648	.40308	.46058	.50177	.53225	.55450	.56894	.57289	.55161					
	.16275	.24082	.29798	.34131	.37459	.39954	.41600	.42053	.39620					
	.10604	.17415	.22731	.26909	.30196	.32701	.34379	.34857	.32436					
11	.31653	.40326	.46103	.50269	.53398	.55770	.57502	.58563	.58639	.56209				
	.16279	.24099	.29842	.34227	.37647	.40311	.42293	.43522	.43606	.40787				
	.10607	.17429	.22772	.27000	.30378	.33054	.35071	.36337	.36438	.33601				
12	.31656	.40340	.46135	.50333	.53515	.55976	.57864	.59226	.59990	.59811	.57132			
	.16281	.24111	.29875	.34294	.37775	.40542	.42708	.44292	.45187	.44968	.41824			
	.10609	.17439	.22800	.27063	.30501	.33281	.35486	.37116	.38046	.37833	.34639			
13	.31659	.40350	.46160	.50380	.53598	.56117	.58099	.59625	.60701	.61227	.60840	.57953		
	.16283	.24121	.29899	.34343	.37865	.40700	.42977	.44758	.46026	.46647	.46175	.42753		
	.10611	.17447	.22822	.27109	.30588	.33437	.35756	.37588	.38903	.39555	.39075	.35574		
14	.31662	.40358	.46178	.50415	.53659	.56218	.58262	.59888	.61134	.61981	.62314	.61754	.58691	
	.16285	.24128	.29917	.34380	.37932	.40813	.43165	.45065	.46538	.47548	.47941	.47255	.43592	
	.10612	.17453	.22839	.27144	.30653	.33549	.35944	.37899	.39428	.40483	.40900	.40191	.36421	
15	.31663	.40365	.46193	.50442	.53706	.56293	.58381	.60071	.61421	.62444	.63105	.63278	.62572	.59359
	.16287	.24134	.29931	.34409	.37983	.40898	.43300	.45279	.46879	.48103	.48898	.49100	.48229	.44356
	.10613	.17459	.22852	.27171	.30702	.33633	.36080	.38117	.39778	.41057	.41893	.42110	.41202	.37194

For given p, n, j and P*= .75(top), .90(middle), .95(bottom), the entries in this table are the values of d for which $\int_0^\infty [1-G_{j,n}(xd)]^p dG_{j,n}(x) = P^*$ where $G_{j,n}(\cdot)$ is the c.d.f. of the jth order statistic in a sample of size n from the exponential distribution.

TABLE 3C

Percentage points of the distribution of $Y_{min} = \min_{1 \le i \le p} X_i/X_0$ for $p = 3$

n \ j	2	3	4	5	6	7	8	9	10	11	12	13	14	15
5	.24830 .12895 .08451	.33214 .19983 .14516	.38352 .24751 .18868	.40058 .26412 .20473										
6	.24865 .12920 .08470	.33390 .20134 .14643	.38968 .25329 .19383	.42388 .28677 .22551	.42810 .29094 .22998									
7	.24885 .12935 .08481	.33483 .20214 .14710	.39253 .25597 .19623	.43191 .29472 .23287	.45493 .31803 .25546	.45008 .31284 .25084								
8	.24898 .12944 .08488	.33538 .20262 .14750	.39409 .25744 .19755	.43576 .29855 .23644	.46455 .32792 .26486	.47978 .34367 .28038	.46818 .33118 .26847							
9	.24906 .12950 .08492	.33574 .20293 .14776	.39505 .25834 .19835	.43794 .30072 .23845	.46930 .33283 .26955	.49075 .35527 .29163	.50024 .36519 .30154	.48344 .34685 .28365						
10	.24912 .12955 .08495	.33598 .20314 .14794	.39568 .25894 .19888	.43929 .30208 .23971	.47204 .33567 .27226	.49628 .36117 .29736	.51236 .37830 .31443	.51747 .38359 .31979	.49655 .36046 .29690					
11	.24916 .12958 .08498	.33616 .20329 .14806	.39611 .25935 .19925	.44020 .30298 .24056	.47378 .33748 .27399	.49954 .36464 .30075	.51859 .38508 .32113	.53059 .39802 .33415	.53225 .39957 .33574	.50798 .37244 .50798				
12	.24920 .12960 .08499	.33629 .20340 .14816	.39642 .25965 .19952	.44084 .30362 .24115	.47496 .33871 .27516	.50163 .36688 .30293	.52231 .38913 .32514	.53743 .40561 .34173	.54622 .41517 .35143	.54511 .41362 .34985	.51807 .38309 .31910			
13	.24922 .12962 .08501	.33638 .20348 .14823	.39666 .25987 .19972	.44130 .30409 .24159	.47580 .33958 .27600	.50307 .36841 .30443	.52473 .39178 .32766	.54156 .41020 .34633	.55361 .42348 .35981	.55984 .43027 .36672	.55644 .42610 .36245	.52707 .39266 .32853		
14	.24924 .12963 .08502	.33646 .20355 .14828	.39684 .26004 .19987	.44165 .30444 .24191	.47642 .34022 .27661	.50410 .36952 .30551	.52641 .39362 .32958	.54428 .41323 .34936	.55812 .42856 .36495	.56771 .43922 .37584	.57182 .44368 .38040	.56652 .43730 .37380	.53516 .40132 .33711	
15	.24926 .12964 .08503	.33652 .20360 .14833	.39698 .26017 .19999	.44193 .30471 .24217	.47689 .34071 .27708	.50487 .37034 .30631	.52762 .39495 .33090	.54619 .41535 .35149	.56112 .43195 .36838	.57255 .44476 .38149	.58013 .45323 .39019	.58249 .45572 .39272	.57556 .44741 .38409	.54250 .40921 .34494

For given p, n, j and $P^* = .75$(top), .90(middle), .95(bottom), the entries in this table are the values of d for which $\int_0^\infty [1-G_{j,n}(xd)]^p dG_{j,n}(x) = P^*$ where $G_{j,n}(\cdot)$ is the c.d.f. of the jth order statistic in a sample of size n from the exponential distribution.

Table 3D

Percentage points of the distribution of $Y_{min} = \min_{1 \le i \le p} X_i/X_0$ for $p = 4$

n \ j	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	.21024	.29285	.34497	.36357										
5	.10994	.17752	.22430	.24157										
	.07234	.12953	.17176	.18812										
	.21055	.29650	.35081	.38624	.39188									
6	.11017	.17891	.22974	.26316	.26830									
	.07251	.13060	.17658	.20783	.21301									
	.21073	.29537	.35364	.39407	.41825	.41462								
7	.11029	.17965	.23226	.27076	.29436	.29023								
	.07260	.13131	.17883	.21483	.23741	.23368								
	.21084	.29589	.35514	.39784	.42774	.44401	.43341							
8	.11037	.18009	.23365	.27442	.30391	.32010	.30866							
	.07266	.13168	.18007	.21822	.24644	.26216	.25121							
	.21092	.29622	.35606	.39996	.43243	.45491	.46532	.44930						
9	.11043	.18037	.23450	.27650	.30865	.33138	.34179	.32446						
	.07270	.13192	.18082	.22014	.25094	.27303	.28325	.26635						
	.21097	.29645	.35666	.40129	.43514	.46043	.47744	.48332	.46299					
10	.11046	.18056	.23506	.27779	.31140	.33711	.35460	.36040	.33821					
	.07273	.13208	.18132	.22134	.25355	.27858	.29579	.30151	.27960					
	.21101	.29661	.35708	.40218	.43686	.46367	.48368	.49650	.49881	.47494				
11	.11049	.18070	.23545	.27866	.31315	.34050	.36124	.37457	.37660	.35034				
	.07275	.13219	.18167	.22214	.25521	.28186	.30231	.31554	.31751	.29134				
	.21104	.29673	.35738	.40280	.43802	.46576	.48741	.50339	.51290	.51231	.48551			
12	.11051	.18080	.23573	.27927	.31433	.34268	.36522	.38203	.39198	.39087	.36114			
	.07276	.13228	.18192	.22271	.25634	.28398	.30622	.32295	.33288	.33169	.30184			
	.21106	.29682	.35761	.40326	.43885	.46719	.48984	.50756	.52037	.52721	.52422	.49495		
13	.11053	.18088	.23594	.27972	.31518	.34418	.36782	.38655	.40018	.40733	.40358	.37085		
	.07277	.13234	.18210	.22312	.25714	.28543	.30877	.32745	.34111	.34827	.34438	.31132		
	.21108	.29689	.35778	.40360	.43947	.46822	.49153	.51031	.52494	.53520	.53984	.53484	.50345	
14	.11054	.18094	.23610	.28005	.31580	.34526	.36962	.38954	.40521	.41621	.42101	.41499	.37966	
	.07278	.13239	.18224	.22343	.25773	.28648	.31055	.33042	.34616	.35725	.36206	.35582	.31994	
	.21109	.29695	.35792	.40387	.43993	.46899	.49275	.51223	.52798	.54012	.54829	.55109	.54438	.51116
15	.11055	.18099	.23623	.28031	.31627	.34606	.37092	.39163	.40856	.42170	.43050	.43329	.42532	.38769
	.07279	.13243	.18236	.22368	.25818	.28726	.31184	.33250	.34953	.36281	.37172	.37449	.36621	.32783

For given p, n, j and P*= .75(top), .90(middle), .95(bottom), the entries in this table are the values of d for which $\int_0^\infty [1-G_{j,n}(xd)]^p \, dG_{j,n}(x) = P^*$ where $G_{j,n}(\cdot)$ is the c.d.f. of the jth order statistic in a sample of size n from the exponential distribution.

TABLE 3E

Percentage points of the distribution of $Y_{min} = \min_{1 \le i \le p} X_i/X_0$ for $p = 5$

n \ j	2	3	4	5	6	7	8	9	10	11	12	13	14	15
5	.18512 .09728 .06419	.26616 .16219 .11872	.31847 .20805 .15989	.33808 .22583 .17643										
6	.18541 .09748 .06433	.26772 .16350 .11981	.32418 .21333 .16447	.36014 .24659 .19533	.36682 .25241 .20101									
7	.18557 .09760 .06442	.26855 .16419 .12038	.32682 .21573 .16660	.36779 .25391 .20205	.39268 .27766 .22457	.39000 .27431 .22150								
8	.18567 .09767 .06447	.26904 .16460 .12073	.32827 .21706 .16778	.37146 .25744 .20531	.40201 .28693 .23331	.41899 .30340 .24915	.40921 .29278 .23894							
9	.18574 .09772 .06451	.26935 .16486 .12095	.32916 .21787 .16850	.37354 .25944 .20716	.40663 .29154 .23767	.42977 .31441 .25974	.44082 .32517 .27017	.42550 .30864 .25402						
10	.18579 .09775 .06453	.26957 .16504 .12110	.32974 .21840 .16897	.37484 .26070 .20831	.40930 .29421 .24020	.43524 .32002 .26514	.45286 .33773 .28242	.45931 .34389 .28840	.43955 .32246 .26725					
11	.18582 .09777 .06455	.26972 .16517 .12120	.33015 .21877 .16930	.37570 .26153 .20908	.41100 .29591 .24181	.43845 .32333 .26833	.45908 .34425 .28880	.47245 .35783 .30215	.47525 .36022 .30442	.45184 .33468 .27899				
12	.18585 .09779 .06456	.26984 .16527 .12128	.33044 .21904 .16954	.37631 .26212 .20963	.41215 .29706 .24290	.44053 .32546 .27040	.46279 .34815 .29262	.47934 .36518 .30943	.48935 .37539 .31952	.48917 .37463 .31863	.46272 .34557 .28951			

For given p, n, j and P*= .75(top), .90(middle), .95(bottom), the entries in this table are the values of d for which $\int_0^\infty [1-G_{j,n}(xd)]^p \, dG_{j,n}(x) = P^*$ where $G_{j,n}(\cdot)$ is the c.d.f. of the jth order statistic in a sample of size n from the exponential distribution.

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	ROLE	WT	ROLE	WT	ROLE	WT
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