The Transient Behavior of the Queue with Alternating Priorities, with special reference to the Waitingtimes

by

Marcel F. Neuts(*), Purdue University

and

Micha Yadin(**), Technion and Kansas State University

Department of Statistics

Division of Mathematical Sciences

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I. Introduction

The queue with alternating priorities is recognized to be of considerable practical importance. It may be visualized as consisting of two units I and II, with a single server alternating between them. In this paper we consider the case, where the server alternates between the units according to a "zero-switch" rule [1,2,10] i.e. he continues service in a unit until all customers there have been served and then he switches to the other unit, provided that there is at least one customer there.

Specifically, let the input processes to I and II be independent Poisson processes of rates λ_1 and λ_2. The service times in I and II form independent sequences of independent random variables with common distributions $H_1(\cdot)$ and $H_2(\cdot)$ respectively. The mean service times are α_1 and α_2 respectively.

Under the "zero-switch" rule, the server stays in a unit, until the queue in it becomes empty. Thereupon he switches to the other unit, unless the queue in front of the other unit is empty. In that case he waits for the first new

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arrival and begins service in the corresponding unit. The event that new arrivals appear simultaneously in both queues has, of course, probability zero.

Units I and II need not be distinct in reality. Physically they can consist of a single station with customers arriving at a rate $\lambda_1 + \lambda_2$ and being of Type I with probability $\lambda_1 / \lambda_1 + \lambda_2$ and of Type II with probability $\lambda_2 / \lambda_1 + \lambda_2$.

A more general class of switching rules will be discussed elsewhere.

II. The busy periods.

We will denote the distribution of the busy period for an $M|G|1$ queue with input rate λ_i and service time distribution $H_i(\cdot)$ by $G_i(\cdot)$, $i = 1,2$. It is known, from the classical theory of the $M|G|1$ queue - Takács [8] - that if the busy period starts off with $\nu \geq 1$ customers initially, one of which is beginning service, then the distribution of the duration of the busy period is given by the ν-fold convolution $G_i^{(\nu)}(\cdot)$ of $G_i(\cdot)$. Moreover the Laplace-Stieltjes transform $\gamma_i(s)$ of $G_i(\cdot)$ is the unique root in the unit disk of the equation:

$$(1) \qquad z = h_i(s + \lambda_i - \lambda_i z) , \quad \text{Re } s > 0 ,$$

where $h_i(s)$ is the Laplace-Stieltjes transform of $H_i(\cdot)$, $i = 1,2$.

Next we define the busy periods for the queue with alternating priorities. Suppose that at $t = 0$, there is one customer in unit I and none in unit II and the customer just begins service. The length of time until both units are simultaneously empty for the first time, is called a I-busy period. If there is only one customer, but he starts in unit II at $t = 0$, we call the corresponding timelength a II-busy period. We denote by $K_1(\cdot)$ and $K_2(\cdot)$ the distribution functions of busy periods of types I and II respectively and their Laplace-Stieltjes transforms by $\theta_1(s)$ and $\theta_2(s)$ respectively.

The transforms $\theta_1(s)$ and $\theta_2(s)$ may be obtained by application of the following theorem.

Theorem 1.

(a) The convex combination $(\lambda_1 + \lambda_2)^{-1}[\lambda_1\theta_1(s) + \lambda_2\theta_2(s)]$ is the unique root in the unit disk $|z| \leq 1$ of the equation:

$$(2) \qquad (\lambda_1 + \lambda_2)z = \lambda_1 h_1(s + \lambda_1 + \lambda_2 - \lambda_1 z - \lambda_2 z) + \lambda_2 h_2(s + \lambda_1 + \lambda_2 - \lambda_1 z - \lambda_2 z) \quad ,$$

for every s with $\mathrm{Re}\ s > 0$.

(b) For every s with $\mathrm{Re}\ s > 0$, the pair $\theta_1(s), \theta_2(s)$ is the unique solution to the system of equations:

$$(3) \qquad \begin{aligned} z_1 &= h_1(s + \lambda_1 + \lambda_2 - \lambda_1 z_1 - \lambda_2 z_2) \quad , \\ z_2 &= h_2(s + \lambda_1 + \lambda_2 - \lambda_1 z_1 - \lambda_2 z_2) \quad , \end{aligned}$$

in the region $|z_1| \leq 1$, $|z_2| \leq 1$.

(c) For every s with $\mathrm{Re}\ s > 0$, the pair $\theta_1(s), \theta_2(s)$ is the unique solution to the system of equations:

$$(4) \qquad z_1 = \gamma_1(s + \lambda_2 - \lambda_2 z_2) \quad ,$$

$$z_2 = \gamma_2(s + \lambda_1 - \lambda_1 z_1) \quad ,$$

in the region $|z_1| \leq 1, \ |z_2| \leq 1$.

(d) Either $\theta_1(o+) = \theta_2(o+) = 1$, or $\theta_1(o+) < 1$ and $\theta_2(o+) < 1$. The first alternative holds if and only if

$$(5) \qquad \lambda_1\alpha_1 + \lambda_2\alpha_2 \leq 1 \quad .$$

If $\lambda_1\alpha_1 + \lambda_2\alpha_2 > 1$, then the equation (2) has a real, positive root $\gamma^* < 1$. In terms of this root, we have:

$$(6) \qquad \theta_1(0+) = h_1(\lambda_1 + \lambda_2 - \lambda_1\gamma^* - \lambda_2\gamma^*) < 1 \quad ,$$

$$\theta_2(0+) = h_2(\lambda_1 + \lambda_2 - \lambda_1\gamma^* - \lambda_2\gamma^*) < 1 \quad .$$

(e) If $\lambda_1\alpha_1 + \lambda_2\alpha_2 < 1$, then the means of $K_1(\cdot)$ and $K_2(\cdot)$ are given by:

$$(7) \qquad -\theta_1'(o+) = \alpha_1(1 - \lambda_1\alpha_1 - \lambda_2\alpha_2)^{-1} \quad ,$$

$$-\theta_2'(o+) = \alpha_2(1 - \lambda_1\alpha_1 - \lambda_2\alpha_2)^{-1} \quad ,$$

and they are both infinite when $\lambda_1\alpha_1 + \lambda_2\alpha_2 = 1$.

Proof:

The mixture $\dfrac{\lambda_1 K_1(\cdot) + \lambda_2 K_2(\cdot)}{\lambda_1 + \lambda_2}$ does not depend on the switching rule or the order of service of the customers. It may be considered as the busy period of an $M|G|1$ queue with input rate $\lambda_1 + \lambda_2$ and service time distribution $(\lambda_1 + \lambda_2)^{-1} [\lambda_1 H_1(\cdot) + \lambda_2 H_2(\cdot)]$. Part (a) is therefore just an application of Takács' theorem [8] p. 58.

Next we verify that $\theta_1(s)$ and $\theta_2(s)$ satisfy the system (3). Suppose that during the first service in a I-busy period ν_1 customers of type I and ν_2 customers of type II arrive $(\nu_1 \geq 0, \nu_2 \geq 0)$. We modify the service discipline as follows. If $\nu_1 > 0$, we serve a first customer of type I and go on serving all new arrivals until the queue becomes empty, apart from the $\nu_1 - 1$ customers of type I and the ν_2 customers of type II. We successively do the same with the next customers of type I and also with the ν_2 customers of type II. Since we so generate ν_1 I-busy periods and ν_2 II-busy periods which are all mutually independent, we obtain:

$$K_1(x) = \sum_{\nu=0}^{\infty} \sum_{\nu_1=0}^{\nu} \int_0^x e^{-(\lambda_1+\lambda_2)y} \binom{\nu}{\nu_1} \lambda_1^{\nu_1} \lambda_2^{\nu-\nu_1} \frac{y^\nu}{\nu!} [K_1^{(\nu_1)} * K_2^{(\nu_2)}(x-y)]\, d\, H_1(y) \tag{8}$$

where $K_1^{(o)}(\cdot) = K_2^{(o)}(\cdot) = U_o(\cdot)$ is the degenerate distribution.

Upon taking transforms, (8) yields the first equation of (3) with $z_1 = \theta_1(s)$, $z_2 = \theta_2(s)$. The second equation is obtained by applying the same argument to a II-busy period.

The uniqueness of this solution to the system(3) follows by consideration of the equivalent system:

$$(9) \qquad z_1 = h_1(s + \lambda_1 + \lambda_2 - \lambda_1 z - \lambda_2 z) \quad ,$$

$$(\lambda_1+\lambda_2)z = \lambda_1 h_1(s+\lambda_1+\lambda_2-\lambda_1 z-\lambda_2 z) + \lambda_2 h_2(s + \lambda_1 + \lambda_2 - \lambda_1 z - \lambda_2 z) \quad ,$$

in which $z = (\lambda_1 + \lambda_2)^{-1} [\lambda_1 z_1 + \lambda_2 z_2]$.

A similar argument shows that $\theta_1(s)$ and $\theta_2(s)$ are the solutions to the system (4). Consider a I-busy period and let there be ν new arrivals in unit II during the first phase in unit I. Each one of them may be considered as the first customer of a II-busy period, in the same manner as before, yielding:

$$(10) \qquad K_1(x) = \sum_{\nu=0}^{\infty} \int_0^x e^{-\lambda_2 y} \frac{(\lambda_2 y)^\nu}{\nu!} K_2^{(\nu)}(x-y)\, d\, G_1(y) \quad ,$$

which yields the first equation in (4).

The uniqueness of this solution is proved by considering the equivalent expressions:

$$(11) \qquad z_1 = \gamma_1[s + \lambda_2 - \lambda_2\gamma_2(s + \lambda_1 - \lambda_1 z_1)] \quad ,$$

$$z_2 = \gamma_2(s + \lambda_1 - \lambda_1 z_1) \quad .$$

The standard argument, involving Rouché's theorem, shows that the first equation in (11) has a unique root in $|z_1| \leq 1$, which upon substitution gives a unique root in $|z_2| \leq 1$.

The remaining statements follow directly from the equations (2), (3) and (4).

Part (b) of this theorem was proved earlier by P.D. Welch [9].

II. An Imbedded semi-Markov process.

The queue under discussion has several interesting imbedded semi-Markov processes. Its time dependence may be studied in several different ways by relating the processes such as queue lengths or waitingtimes to those semi-Markov sequences. This method is now classical — see Neuts [3,4,5]— and we will not present it in detail here. Instead we will discuss a number of shortcuts to save on notation.

Macroscopically the queue goes through alternating busy and idle periods. The idle periods and busy periods are mutually independent and the former have a negative exponential distribution with mean $(\lambda_1 + \lambda_2)^{-1}$. The latter are either I-busy periods or II-busy periods, whose successive types form a sequence of Bernoulli trials, with probability $\lambda_1(\lambda_1 + \lambda_2)^{-1}$ of a I-period.

We suppose that at $t = 0$ there are i_1 customers in unit I and i_2 in unit II, with $i_1 \geq 1$, $i_2 \geq 0$. Furthermore a customer in unit I is just beginning service. In this case, there will be an initial I-busy period different from the other I-busy periods. It is easy to modify our discussion to cover other initial conditions for the queue.

The initial instants of the successive busy periods (and also their end-points) form a sequence of regeneration points for the entire queueing process. Therefore we may initially limit our discussion to the queue characteristics <u>within</u> a busy period. Later, it will be easy to "patch" all busy and idle periods together to obtain the complete time dependent equations.

We now consider the queue within a busy period.

Let $T_o = 0$ and let the queue lengths at T_o be (i_1, i_2). Further assume that a service is starting in unit I and let $T_1, T_2, \ldots, T_N$ denote the lengths of the successive time intervals spent in units I and II. The even intervals correspond to service in unit II and the odd ones to service in unit I. By the word "task" we understand the time interval spent, without interruption, in one unit. A I-task will be spent in unit I and a II-task in unit II. N is a random variable, corresponding to the number of tasks (in unit I or unit II), before the queues become empty. Note that N could be infinite. We denote by $\kappa_1, \kappa_2, \ldots$ the numbers of customers in the system at times $T_1, T_1 + T_2, \ldots$. A fortiori we have $\kappa_N = 0$ if N is finite.

If we define $\kappa_o = i_1$, then the bivariate sequence $\{T_n, \kappa_n, n \geq 0\}$ has the basic properties of a semi-Markov sequence, except for the fact that the even and odd-numbered transitions are governed by different transition probability matrices. This could, of course, be avoided by adding a third variable which indicates whether the server spends the corresponding phase in unit I or in jnit II. For simplicity, we chose not to do so.

Note that the odd kappa-variables $\kappa_1, \kappa_3, \ldots$ express numbers of customers in unit II, whereas the even ones describe numbers of customers in unit II.

The transition probabilities are given by:

$$(12) \qquad P\{T_1 \leq x, \quad \kappa_1 = j \mid \kappa_o = i_1\}$$

$$= \int_o^x e^{-\lambda_2 y} \frac{(\lambda_2 y)^{j-i_2}}{(j-i_2)!} \, d\, G_1^{(i_1)}(y), \qquad j \geq i_2 \ .$$

(13) For $n \geq 1$:

$$P\{T_{2n+1} \leq x \ , \ \kappa_{2n+1} = j \mid \kappa_{2n} = i \geq 1\} =$$

$$= \int_0^x e^{-\lambda_2 y} \frac{(\lambda_2 y)^j}{j!} \; d\, G_1^{(i)}(y) \quad ,$$

(14) For $n \geq 1$:

$$P\{T_{2n} \leq x, \ \kappa_{2n} = j \mid \kappa_{2n-1} = i \geq 1\}$$

$$= \int_0^x e^{-\lambda_1 y} \frac{(\lambda_1 y)^j}{j!} \; d\, G_2^{(i)}(y) \quad ,.$$

The following probabilities are of basic importance. Let $R_n(i_1,i_2;j;x)$ be the probability that a busy period starting at $t = 0$ with (i_1,i_2) customers, lasts for at least n tasks, that the n-th task ends not later than time x and that at the end of the n-th task, j customers are waiting.

By $r_n(i_1,i_2;j,s)$ we denote the Laplace-Stieltjes transform of $R_n(i_1,i_2;j,x)$ and write all formulae in the sequel directly in transforms. We have:

(15) $$r_1(i_1,i_2;j,s) = \int_0^\infty e^{-(s+\lambda_2)y} \frac{(\lambda_2 y)^{j-i_2}}{(j-i_2)!} \; d\, G_1^{(i_1)}(y) \quad ,$$

for $j \geq i_2$.

(16) $$r_{2n+1}(i_1,i_2;j,s) = \sum_{\nu=1}^{\infty} r_{2n}(i_1,i_2;\nu,s) \int_0^\infty e^{-(s+\lambda_2)y} \frac{(\lambda_2 y)^j}{j!} \; d\, G_1^{(\nu)}(y) \ ,$$

for $j \geq 0, \ n \geq 1$.

(17) $$r_{2n}(i_1,i_2;j,s) =$$

$$\sum_{\nu=1}^{\infty} r_{2n-1}\,(i_1,i_2;\nu;s) \int_0^{\infty} e^{-(s+\lambda_1)y} \frac{(\lambda_1 y)^j}{j!}\, d\, G_2^{(\nu)}(y) \quad ,$$

for $j \geq 0$, $n \geq 1$.

Taking generating functions on j, we obtain for $|z| \leq 1$:

(18) $$r_1(i_1,i_2;z,s) = z^{i_2}\, \gamma_1^{i_1}(s + \lambda_2 - \lambda_2 z) \quad ,$$

(19) $$r_{2n+1}(i_1,i_2;z,s) =$$

$$\sum_{j=0}^{\infty} z^j \sum_{\nu=1}^{\infty} r_{2n}(i_1,i_2;\nu;s) \int_0^{\infty} e^{-(s+\lambda_2)y} \frac{(\lambda_2 y)^j}{j!}\, d\, G_1^{(\nu)}(y) =$$

$$= r_{2n}\,[i_1,i_2;\gamma_1(s+\lambda_2 - \lambda_2 z),s] - r_{2n}(i_1,i_2;o,s), \quad n \geq 1 \quad .$$

(20) $$r_{2n}(i_1,i_2;z,s) =$$

$$r_{2n-1}[i_1,i_2;\gamma_2(s + \lambda_1 - \lambda_1 z),s] - r_{2n-1}(i_1,i_2;o,s), \quad n \geq 1 \quad .$$

Formulae (18) - (20) may be simplified by the introduction of the following iterative sequences of functions:

(21) $$\varphi_o(z,s) = z \quad ,$$

$$\varphi_{n+1}(z,s) = \gamma_1[s + \lambda_2 - \lambda_2 \psi_n(z,s)] \quad , \quad n \geq 0 \quad .$$

$$\psi_o(z,s) = z \quad ,$$

$$\psi_{n+1}(z,s) = \gamma_2[s + \lambda_1 - \lambda_1 \psi_n(z,s)] \quad , \quad n \geq 0 \quad .$$

It is easy to verify that:

(22) $$\varphi_{2n}[z,s] = \varphi_{2n-1}[\gamma_2(s + \lambda_1 - \lambda_1 z),s] \quad , \quad n \geq 1 \quad ,$$

$$\varphi_{2n+1}[z,s] = \varphi_{2n}[\gamma_1(s + \lambda_2 - \lambda_2 z),s] \quad , \quad n \geq 0 \quad ,$$

$$\psi_{2n}[z,s] = \psi_{2n-1}[\gamma_1(s + \lambda_2 - \lambda_2 z),s] \quad , \quad n \geq 1 \quad ,$$

$$\psi_{2n+1}[z,s] = \psi_{2n}[\gamma_2(s + \lambda_1 - \lambda_1 z),s] \quad , \quad n \geq 0 \quad .$$

In terms of these iterates, defined in (21), it is possible to write (18) - (20) as a single formula:

(23) $$r_n(i_1,i_2;z,s) =$$

$$\varphi_n^{i_1}(z,s)\, \psi_{n-1}^{i_2}(z,s) - \varphi_{n-1}^{i_1}(o,s)\, \psi_{n-2}^{i_2}(o,s) \quad ,$$

for $n \geq 1$. We set $\psi_{-1}(z,s) = \varphi_{-1}(z,s) = 0$

Functional iterates, as defined in (21), are reminiscent of the theory of branching processes. In fact, the imbedded semi-Markov process, discussed here, is related to that appearing in a discussion of the $M|G|1$ queue by Neuts [3] . This suggests another approach to the distribution of the busy period, summarized below.

The asymmetry of formula (23) stems from the fact that we assumed that the server starts off with a I-task.

Let us define for $i_1 \geq 0$, $i_2 \geq 1$, the transform $\tilde{r}_n(i_1,i_2;z,s)$ which has the same definition as $r_n(i_1,i_2;z,s)$, except that the server starts off with a II-task, then:

$$(24) \qquad \tilde{r}_n(i_1,i_2;z,s) = \psi_n^{i_2}(z,s)\,\varphi_{n-1}^{i_1}(z,s) - \psi_{n-1}^{i_2}(o,s)\,\varphi_{n-2}^{i_1}(o,s)$$

for $n \geq 1$.

<u>Another Derivation of the distribution of the Busy Periods.</u>

For $i_1 = 1$, $i_2 = 0$, we obtain:

$$(25) \qquad r_n(1,0;z,s) = \varphi_n(z,s) - \varphi_{n-1}(o,s), \quad n \geq 1 \quad ,$$

which for $z = 0$, is the L.S.-transform of the probability $R_n(1,0;0,x)$ that a I-busy period lasts for exactly n tasks and ends no later than x.

The probability that the I-busy period lasts for <u>at most</u> n tasks and ends no later than x is given by $\sum_{\nu=1}^{n} R_\nu(1,0;\,0,x)$ and its transform is simply $\varphi_n(o,s)$. Repeating verbatim the argument given in [3], we prove the following theorem.

Theorem 2

For $s \geq 0$, the functional iterates $\varphi_n(o,s)$ are monotone increasing in n for every s and converge to $\theta_1(s)$. Their analytic continuations in $\mathrm{Re}\, s \geq 0$ converge to $\theta_1(s)$ for all $\mathrm{Re}\, s \geq 0$.

An analogous result holds for $\psi_n(o,s)$ and $\theta_2(s)$.

Theorem 3

If the server starts in unit I (or in unit II) and if the initial queue-lengths are i_1 and i_2, with $i_1 \geq 1$, $i_2 \geq 0$ (or $i_1 \geq 0$, $i_2 \geq 1$), then the duration of the initial busy period has a probability distribution with L.S. transform $\theta_1^{i_1}(s)\, \theta_2^{i_2}(s)$.

Proof:

The probability, that the initial busy period consists of at most N tasks and has a length at most x, is given by:

$$\sum_{\nu=1}^{N} R_\nu(i_1,i_2;o,x) \quad ,$$

which, upon transformation yields:

$$(25) \qquad \sum_{\nu=1}^{N} r_\nu(i_1,i_2;o,s) = \varphi_N^{i_1}(o,s) \quad \psi_{N-1}^{i_2}(o,s) \quad ,$$

which tends to $\theta_1^{i_1}(s)\, \theta_2^{i_2}(s)$ as N tends to infinity.

The number of tasks during a busy period.

Formula (25) gives the transform of the joint probability distribution of the duration of the (initial) busy period and the number of tasks in it. If we denote the number of tasks during a I-busy period, respectively a II-busy period, by B_1, respectively B_2, then:

$$(26) \qquad P\{B_1 \le n\} = \varphi_n(o,o) \ ,$$

$$P\{B_2 \le n\} = \psi_n(o,o) \ , \qquad n \ge 1 \ ,$$

yielding proper distributions if and only if $\lambda_1\alpha_1 + \lambda_2\alpha_2 \le 1$. When $\lambda_1\alpha_1 + \lambda_2\alpha_2 < 1$, we can find an easy upper bound for the probabilities (26). To this effect, we note, as was done in [3], that $\varphi_n(z,o)$ and $\psi_n(z,o)$ are strictly increasing, strictly convex functions in $[0,1]$ for $n \ge 1$. Since their graphs lie entirely above their tangents at $z = 1$, we obtain:

$$(27) \qquad \varphi_{2n}(o,o) < 1 - \left(\frac{\lambda_1\alpha_2}{1-\lambda_2\alpha_2}\right)^n \left(\frac{\lambda_2\alpha_1}{1-\lambda_1\alpha_1}\right)^n \ ,$$

$$\varphi_{2n+1}(o,o) < 1 - \left(\frac{\lambda_2\alpha_1}{1-\lambda_1\alpha_1}\right)^{n+1} \left(\frac{\lambda_1\alpha_2}{1-\lambda_2\alpha_2}\right)^n \ ,$$

$$\psi_{2n}(o,o) < 1 - \left(\frac{\lambda_1\alpha_2}{1-\lambda_2\alpha_2}\right)^n \left(\frac{\lambda_2\alpha_1}{1-\lambda_1\alpha_1}\right)^n \ ,$$

$$\psi_{2n+1}(o,o) < 1 - \left(\frac{\lambda_1\alpha_2}{1-\lambda_2\alpha_2}\right)^{n+1} \left(\frac{\lambda_2\alpha_1}{1-\lambda_1\alpha_1}\right)^n \ ,$$

by straightforward differentiation in (22) after setting $s = 0$.

The distributions in (26) were studied by random walk methods by M. Yadin [10], in the case where the service times are negative exponential.

IV. The Virtual Waitingtime Process

In contrast with the $M|G|1$ queue, three different concepts of virtual waitingtime can be given here.

a. Suppose a (virtual) customer arrives at time t. If he arrives in unit I, his waitingtime in the system will be denoted by $\eta_1(t)$ and if he arrives in unit II, by $\eta_2(t)$. For fixed t, $\eta_1(t)$ and $\eta_2(t)$ will, in general , be dependent random variables. We will study their joint distribution below.

b. If we fail to distinguish between the units in which the customer arrives, we may define the virtual waitingtime $\eta(t)$ as:

$$\eta(t) = \frac{\lambda_1\eta_1(t) + \lambda_2\eta_2(t)}{\lambda_1 + \lambda_2} \quad . \tag{28}$$

It is easy to obtain the distribution of $\eta(t)$ from the joint distribution of $\eta_1(t)$ and $\eta_2(t)$, but it is difficult to obtain the distribution of $\eta(t)$ directly

c. Suppose that, at time t, both Poisson arrival processes are interrupted, then we denote by $\tilde{\eta}(t)$ the additional time required to serve all the customers of either type present at time t. The distribution of $\tilde{\eta}(t)$ may be obtained easily along with the joint distribution of the queue-lengths in both units at time t. We will postpone its discussion to a later section.

We assume that $t = 0$ is the beginning of a I-task and that $i_1 \geq 1$, $i_2 \geq 0$ customers are then present.

1. The Probability that the server is idle at time t.

Clearly the events $\{\eta_1(t) = 0\}$, $\{\eta_2(t) = 0\}$, and $\{\tilde{\eta}(t) = 0\}$ are equivalent.

The endpoints of busy periods, regardless of their types form a renewal process, with renewal function $M(t)$ given by:

(29) $$m(\xi) = \int_0^\infty e^{-\xi t} \, d\,M(t) =$$

$$= \theta_1^{i_1}(\xi)\theta_2^{i_2}(\xi) \{1-(\xi+\lambda_1+\lambda_2)^{-1}[\lambda_1\theta_1(\xi)+\lambda_2\theta_2(\xi)]\}^{-1} \quad ,$$

for $\operatorname{Re} \xi > 0$.

By an elementary renewal argument, we get:

(30) $$P\{\eta_1(t) = 0 | i_1, i_2\} = P\{\eta_2(t) = 0 | i_1, i_2\}$$

$$= P\{\tilde{\eta}(t) = 0 | i_1, i_2\} = \int_0^t e^{-(\lambda_1+\lambda_2)(t-u)} \, d\,M(u) \quad ,$$

so that:

(31) $$\int_0^\infty e^{-\xi t} P\{\tilde{\eta}(t) = 0\} dt = (\xi + \lambda_1 + \lambda_2)^{-1} m(\xi) =$$

$$= \theta_1^{i_1}(\xi)\, \theta_2^{i_2}(\xi)[\xi + \lambda_1 - \lambda_1\theta_1(\xi) + \lambda_2 - \lambda_2\theta_2(\xi)]^{-1} \quad .$$

Using the Key renewal theorem and (31), we obtain as $t \to \infty$,

(32) $$\lim_{t \to \infty} P\{\eta(t) = 0\} = 1 - \lambda_1\alpha_1 - \lambda_2\alpha_2 \quad ,$$

when $\lambda_1\alpha_1 + \lambda_2\alpha_2 < 1$ and zero if and only if the reverse inequality holds.

2. <u>The joint distribution of $\eta_1(t)$ and $\eta_2(t)$</u> .

We will calculate the transform:

$$(33) \qquad \Omega(\xi,s_1,s_2) = \int_0^\infty e^{-\xi t} E\{e^{-s_1\eta_1(t) - s_2\eta_2(t)}\} dt$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty e^{-\xi t - s_1 x - s_2 y} dt \, d\, P\{\eta_1(t) \leq x, \eta_2(t) \leq y\} \quad .$$

<u>Remark</u>: All the probability distributions and expectation, which we consider here, are conditional upon the given initial conditions. For simplicity of notation, we do not write them every time.

We first "remove" the mass at (o,o) in the joint distribution by writing:

$$(34) \qquad \Omega(\xi,s_1,s_2) - \int_0^\infty e^{-\xi t} P\{\eta_1(t) = \eta_2(t) = 0\} dt =$$

$$\int_0^\infty \int_0^\infty \int_0^\infty e^{-\xi t - s_1 x - s_2 y} dt \, d\, P\{0 < \eta_1(t) \leq x, 0 < \eta_2(t) \leq y\} \quad .$$

The term, subtracted on the left is given in (31). In the term on the right, we need consider only those pathfunctions of the process for which t belongs to a busy period.

We proceed by giving a careful decomposition of the event

$$(35) \qquad B(x,y,t) = \{0 < \eta_1(t) \leq x, \quad 0 < \eta_2(t) \leq y\}$$

into mutually disjoint simpler events.

Step 1: Let L_ν and L'_ν denote the epochs at which the ν-th busy period begins and ends. Let I_ν be the type (1 or 2) of the ν-th busy period, then:

$$(36) \qquad B(x,y,t) = \bigcup_{k=1}^{2} \bigcup_{\nu=1}^{\infty} \{0 < \eta_1(t) \leq x, \quad 0 < \eta_2(t) \leq y \;, \quad L_\nu \leq t < L'_\nu \;, \; I_\nu = k\} \quad .$$

For $\nu = 1$, we have:

$$(37) \qquad L_1 = 0, \quad I_1 = 1 \qquad \text{a.s.}$$

$$\{0 < \eta_1(t) \leq x, \; 0 < \eta_2(t) \leq y, \; L'_1 > t, \; I_1 = 2\} = \emptyset \;,$$

because of the given initial conditions.

Step 2: Consider any of the events in (36) for $\nu \geq 2$. During the interval $[L_\nu,t)$ some tasks may have been completed. Let L''_ν be the epoch at which the last task-completion before t occurs. If there is no task-completion in $[L_\nu,t)$, set $L''_\nu = L_\nu$. Let $N_\nu \geq 0$ be the number of task completions in $[L_\nu,t)$. With this notation the timepoint t belongs to the $(N_\nu + 1)$st task of the ν-th busy period. If $N_\nu + 1$ is odd, then this task has the type I_ν, whereas if $N_\nu + 1$ is even, the task containing t has the type $(I_\nu + 1)$ mod 2. We may write:

$$(38) \qquad \{0 < \eta_1(t) \le x,\ \ 0 < \eta_2(t) \le y,\ \ L_\nu \le t < L'_\nu\ ,\ \ I_\nu = k\}$$

$$= \bigcup_{\rho=0}^{\infty} \{0 < \eta_1(t) \le x,\ 0 < \eta_2(t) \le y,\ L''_\nu \le t < L'_\nu\ ,\ I_\nu = k,\ N_\nu = 2\rho+1\}$$

$$.\cup.\ \bigcup_{\rho=0}^{\infty} \{0 < \eta_1(t) \le x,\ 0 < \eta_2(t) \le y,\ L''_\nu \le t < L'_\nu,\ I_\nu = k,\ N_\nu = 2\rho\}\ .$$

Step 3. We consider the path-function of the process between L''_ν and t. Suppose that the task, containing t is of type I and that it began at time $L''_\nu = u < t$. Suppose that at time u, there were $\beta \ge 1$ customers of type I in the system (and, of course, none of type II). Conditional upon all this information, the task containing t may be considered as a busy period in an $M|G|1$ queue with input rate λ_1 and service time distribution $H_1(\cdot)$ and with β customers initially. This "busy period" must go on for a length of time, at least $t-u$. The virtual waitingtime $\eta_1(t)$ is then equal to the time required to serve all customers of type I, who are present at time t. These customers may either be leftovers from the first β or may be new arrivals during the interval $[u,t)$. The virtual waitingtime $\eta_2(t)$ is more complicated. The server must first complete the I-task begun at time u and must then serve all II-customers, who have arrived in the interval $[u,t)$, if any. The time required to do all this is the virtual waitingtime $\eta_2(t)$.

Should the task, containing t, be a II-task, then the preceding description applies, with the role of I and II-customers reversed.

In order to express the integral in (34) as simply as possible, we first derive a number of auxiliary results, working our way backwards through the verbal description given above.

(A). Auxiliary problem 1.

Suppose we have an $M|G|1$ queue with β customers at $t = 0$, one of which is beginning service. ($\beta \geq 1$). The input rate is λ_1 and the service time distribution is $H_1(\circ)$. There is also an independent secondary Poisson arrival process with rate λ_2 of customers of type II. These have a service time distribution $H_2(\cdot)$ and may be served only after the busy period of ordinary (type I) customers has ended. We consider this queue at time t and ask for the probability that:

(i) At time t, the server is still serving customers of type I.

(ii) The total service time $\eta^*(t)$ of all I-customers present at time t is less than or equal to x.

(iii) The last customer of type II to arrive before time t leaves the system not later than $t + y$. Denote this probability by $\chi_t^{(\beta)}(x,y)$. It is clearly zero for $x > y$.

Now, let $\psi_t^{(\beta)}(x,v)$ be the probability that, at time t, the initial busy period in the $M|G|1$ queue has not yet ended, that $\eta^*(t)$ does not exceed x and that the initial busy period ends before $t + v$. This probability was calculated by Neuts [6]. We obtain

$$(39) \qquad \chi_t^{(\beta)}(x,y) = \sum_{\delta=0}^{\infty} \int_0^x \int_x^y \int_x^{y'} e^{-\lambda_2 t} \frac{(\lambda_2 t)^\delta}{\delta!} \, d\, H_2^{(\delta)}(y'-v) \, d_{x',v} \, \psi_t^{(\beta)}(x',v) \, ,$$

whence:

$$(40) \qquad \int_0^\infty e^{-\xi t}\, dt \int_0^\infty \int_x^\infty e^{-s_1 x - s_2 y}\, d_{x,y}\, \chi_t^{(\beta)}(x,y) =$$

$$\sum_{\delta=0}^{\infty} \int_0^\infty e^{-\xi t - \lambda_2 t} \frac{(\lambda_2 t)^\delta}{\delta!}\, dt\, h_2^\delta(s_2) \int_0^\infty \int_x^\infty e^{-s_1 x - s_2 y}\, d_{x,y} \psi_t^{(\beta)}(x,y)$$

$$= \int_0^\infty e^{-[\xi + \lambda_2 - \lambda_2 h_2(s_2)]t}\, dt \int_0^\infty \int_x^\infty e^{-s_1 x - s_2 y}\, d_{x,y}\, \psi_t^{(\beta)}(x,y) \quad ,$$

but this is just the transform of $\psi_t^{(\beta)}(x,y)$ which was obtained in Neuts [6], formula (16), evaluated at $\xi + \lambda_2 - \lambda_2 h_2(s_2)$. Substituting we obtain:

$$(41) \qquad \int_0^\infty e^{-\xi t}\, dt \int_0^\infty \int_x^\infty e^{-(s_1 x + s_2 y)}\, d_{x,y}\, \chi_t^{(\beta)}(x,y) =$$

$$\{s_1 + s_2 - \xi - \lambda_2 + \lambda_2 h_2(s_2) + \lambda_1 h_1[s_1 + s_2 + \lambda_1 - \lambda_1 \gamma_1(s_2)] - \lambda_1 \gamma_1(s_2)\}^{-1}$$

$$\{\gamma_1^\beta[\xi + \lambda_2 - \lambda_2 h_2(s_2)] - h_1^\beta[s_1 + s_2 + \lambda_1 - \lambda_1 \gamma_1(s_2)]\} \quad .$$

(B) <u>Auxiliary problem 2</u>.

Considering the queue with alternating priorities, let there be $i_1 \geq 1$ I-customers and $i_2 \geq 0$ II-customers at $t = 0$, which is the beginning of a I-task.

What is the probability that at time t:

(i) the server has never been idle in $[0,t)$.

(ii) the virtual waitingtimes $\eta_1(t)$ and $\eta_2(t)$ satisfy:

$$(42) \qquad 0 < \eta_1(t) \leq x \;, \qquad 0 < \eta_2(t) \leq y \quad .$$

Denote this probability by ${}_1\Lambda_{i_1,i_2}(x,y,t)$, then we have:

$$(43)\qquad {}_1\Lambda_{i_1,i_2}(x,y,t) = {}_1\tilde{\chi}_t^{(i_1,i_2)}(x,y) +$$

$$\sum_{\beta=1}^{\infty}\sum_{n=1}^{\infty}\int_0^x\int_0^y\int_0^t d\, R_{2n}(i_1,i_2;\beta,u)\, d_{x',y'}\,{}_1\chi_{t-u}^{(\beta)}(x',y')$$

$$+\sum_{\beta=1}^{\infty}\sum_{n=0}^{\infty}\int_0^x\int_0^y\int_0^t d\, R_{2n+1}(i_1,i_2;\beta,u)\, d_{x',y'}\,{}_2\chi_{t-u}^{(\beta)}(x',y') \quad ,$$

in which ${}_1\tilde{\chi}_t^{(i_1,i_2)}(x,y)$ is the same probability as in part A except that at the beginning of time already $i_2 \geq 0$ customers of type II are present. The transform of this probability is easily expressible, analogous to (41). The prefix 1 or 2 to the χ-probabilities in the formula (43) indicates whether formula (41) should be applied with the parameters of unit I or II playing the role of the principal queue in part A.

Upon taking transforms in (43) and applying (41) we obtain:

$$(44) \quad \int_0^\infty e^{-\xi t}\, dt \int_0^\infty \int_0^\infty e^{-(s_1 x + s_2 y)} d_{x,y}\; {}_1\Lambda_{i_1,i_2}(x,y,t) =$$

$$\{s_1+s_2-\xi-\lambda_2+\lambda_2 h_2(s_2) + \lambda_1 h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)] - \lambda_1\gamma_1(s_2)\}^{-1}$$

$$\{\gamma_1^{i_1}[\xi+\lambda_2-\lambda_2 h_2(s_2)] - h_1^{i_1}[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)]\}\; h_2^{i_2}(s_2)$$

$$+ \{s_1+s_2-\xi-\lambda_2+\lambda_2 h_2(s_2) + \lambda_1 h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)] - \lambda_1\gamma_1(s_2)\}^{-1}$$

$$\sum_{\beta=1}^{\infty} \sum_{n=1}^{\infty} r_{2n}(i_1,i_2;\beta,\xi)\; \{\gamma_1^{\beta}[\xi+\lambda_2-\lambda_2 h_2(s_2)] - h_1^{\beta}[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)]\}$$

$$+ \{s_1+s_2-\xi-\lambda_1+\lambda_1 h_1(s_1) + \lambda_2 h_2[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)] - \lambda_2\gamma_2(s_1)\}^{-1}$$

$$\sum_{\beta=1}^{\infty} \sum_{n=0}^{\infty} r_{2n+1}(i_1,i_2;\beta,\xi)\{\gamma_2^{\beta}[\xi+\lambda_1-\lambda_1 h_1(s_1)] - h_2^{\beta}[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)]\} .$$

If $i_1 \geq 0$, $i_2 \geq 1$ and if at $t = 0$, the server starts off with a II-task, then an analogous argument leads to:

$$(45) \qquad \int_0^\infty e^{-\xi t}\, dt \int_0^\infty \int_0^\infty e^{-s_1 x - s_2 y}\, d_{x,y}\; {}_2\Lambda_{i_1,i_2}(x,y,t) =$$

$$\{s_1+s_2-\xi-\lambda_1+\lambda_1 h_1(s_1) + \lambda_2 h_2[s_1+s_2+\lambda_2\gamma_2(s_1)] - \lambda_2\gamma_2(s_1)\}^{-1}$$

$$\{\gamma_2^{i_2}[\xi+\lambda_1-\lambda_1 h_1(s_1)] - h_2^{i_2}[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)]\}\, h_1^{i_1}(s_2)$$

$$+ \{s_1+s_2-\xi-\lambda_1+\lambda_1 h_1(s_1) + \lambda_2 h_2[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)] - \lambda_2\gamma_2(s_1)\}^{-1}$$

$$\sum_{\beta=1}^{\infty} \sum_{n=1}^{\infty} \tilde{r}_{2n}(i_1,i_2;\beta,\xi)\, \{\gamma_2^{\beta}[\xi+\lambda_1-\lambda_1 h_1(s_1)] - h_2^{\beta}[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)]\}$$

$$+ \{s_1+s_2-\xi-\lambda_2+\lambda_2 h_2(s_2) + \lambda_1 h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)] - \lambda_1\gamma_1(s_2)\}^{-1}$$

$$\sum_{\beta=1}^{\infty} \sum_{n=0}^{\infty} \tilde{r}_{2n+1}(i_1,i_2;\beta,\xi)\, \{\gamma_1^{\beta}[\xi+\lambda_2-\lambda_2 h_2(s_2)] - h_1^{\beta}[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)]\}.$$

In terms of the above probabilities, the probability required in formula (34) may be easily expressed. We relate the virtual waitingtimes at t to the last beginning of a busy period before t. We obtain:

$$(46) \qquad P\{0 < \eta_1(t) \le x\ ,\quad 0 < \eta_2(t) \le y\} = {}_1\Lambda_{i_1,i_2}(x,y,t) +$$

$$+ \int_0^t \int_0^u e^{-(\lambda_1+\lambda_2)(u-v)}\; {}_1\Lambda_{1,0}(x,y,t-u)\, \lambda_1\, du\, d\, M(v)$$

$$+ \int_0^t \int_0^u e^{-(\lambda_1+\lambda_2)(u-v)}\; {}_2\Lambda_{0,1}(x,y,t-u)\, \lambda_2\, du\, d\, M(v) \quad ,$$

whence:

$$(47) \qquad \int_0^\infty e^{-\xi t}\, dt \int_0^\infty \int_0^\infty e^{-s_1 x - s_2 y}\, d_{x,y}\, P\{0 < \eta_1(t) \leq x,\ 0 < \eta_2(t) \leq y\}$$

$$= \int_0^\infty e^{-\xi t}\, dt \int_0^\infty \int_0^\infty e^{-s_1 x - s_2 y}\, d_{x,y}\ {}_1\Lambda_{i_1, i_2}(x,y,t)$$

$$+ \frac{\lambda_1 m(\xi)}{\xi+\lambda_1+\lambda_2} \int_0^\infty e^{-\xi t}\, dt \int_0^\infty \int_0^\infty e^{-s_1 x - s_2 y}\, d_{x,y}\ {}_1\Lambda_{1,0}(x,y,t)$$

$$+ \frac{\lambda_2 m(\xi)}{\xi+\lambda_1+\lambda_2} \int_0^\infty e^{-\xi t}\, dt \int_0^\infty \int_0^\infty e^{-s_1 x - s_2 y}\, d_{x,y}\ {}_2\Lambda_{0,1}(x,y,t)$$

$$= S_1(s_1,s_2)\, \{\gamma_1^{i_1}[\xi+\lambda_2-\lambda_2 h_2(s_2)] - h_1^{i_1}[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)]\}\, h_2^{i_2}(s_1)$$

$$+ S_1(s_1,s_2) \sum_{\beta=1}^{\infty} \sum_{n=1}^{\infty} r_{2n}(i_1,i_2;\beta,\xi)\, \{\gamma_1^{\beta}[\xi+\lambda_2-\lambda_2 h_2(s_2)]$$

$$- h_1^{\beta}[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)]\}$$

$$+ S_2(s_1,s_2) \sum_{\beta=1}^{\infty} \sum_{n=0}^{\infty} r_{2n+1}(i_1,i_2;\beta,\xi)\, \{\gamma_2^{\beta}[\xi+\lambda_1-\lambda_1 h_1(s_1)]$$

$$- h_2^{\beta}[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)]\}$$

$$+ \frac{\lambda_1 \theta_1^{i_1}(\xi)\, \theta_2^{i_2}(\xi)}{\xi+\lambda_1-\lambda_1\theta_1(\xi)+\lambda_2-\lambda_2\theta_2(\xi)}\, \{S_1(s_1,s_2)\, \{\gamma_1[\xi+\lambda_2-\lambda_2 h_2(s_2)]$$

$$- h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)]\}$$

$$+ S_1(s_1,s_2) \sum_{\beta=1}^{\infty} \sum_{n=1}^{\infty} r_{2n}(1,0;\beta,\xi) \{\gamma_1^{\beta} [\xi+\lambda_2-\lambda_2 h_2(s_2)]$$

$$- h_1^{\beta}[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)]\}$$

$$+ S_2(s_1,s_2) \sum_{\beta=1}^{\infty} \sum_{n=0}^{\infty} r_{2n+1}(1,0;\beta,\xi)\{\gamma_2^{\beta}[\xi+\lambda_1-\lambda_1 h_1(s_1)]$$

$$- h_2^{\beta}[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)]\}\Big\}$$

$$+ \frac{\lambda_2 \theta_1^{i_1}(\xi)\ \theta_2^{i_2}(\xi)}{\xi+\lambda_1-\lambda_1\theta_1(\xi)+\lambda_2-\lambda_2\theta_2(\xi)} \Big\{S_2(s_1,s_2)\{\gamma_2[\xi+\lambda_1-\lambda_1 h_1(s_1)]$$

$$- h_2[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)]\}$$

$$+ S_2(s_1,s_2) \sum_{\beta=1}^{\infty} \sum_{n=1}^{\infty} \tilde{r}_{2n}(0,1;\beta,\xi)\{\gamma_2^{\beta}[\xi+\lambda_1-\lambda_1 h_1(s_1)]$$

$$-h_2^{\beta}[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)]\}$$

$$+ S_1(s_1,s_2) \sum_{\beta=1}^{\infty} \sum_{n=0}^{\infty} \tilde{r}_{2n+1}(0,1;\beta,\xi)\{\gamma_1^{\beta}[\xi+\lambda_2-\lambda_2 h_2(s_2)]$$

$$- h_1^{\beta} [s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)]\}\Big\} \quad ,$$

where:

$$(48) \qquad S_1^{-1}(s_1,s_2) = s_1+s_2-\xi-\lambda_2+\lambda_2 h_2(s_2) + \lambda_1 h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)] - \lambda_1\gamma_1(s_2) \ ,$$

and

$$S_2^{-1}(s_1,s_2) = s_1+s_2-\xi-\lambda_1+\lambda_1h_1(s_1)+\lambda_2h_2[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)] - \lambda_2\gamma_2(s_1) \quad .$$

This formula may be somewhat simplified, using the following observations:

$$(49) \qquad \sum_{\beta=1}^{\infty} \sum_{n=1}^{\infty} r_{2n}(i_1,i_2;\beta,\xi) \left\{\gamma_1^{\beta}[\xi+\lambda_2-\lambda_2h_2(s_2)] - h_1^{\beta}[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)]\right\}$$

$$= \sum_{n=1}^{\infty} \Big\{r_{2n}\Big[i_1,i_2;\gamma_1[\xi+\lambda_2-\lambda_2h_2(s_2)],\xi\Big]$$

$$- r_{2n}\Big[i_1,i_2;h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)],\xi\Big]\Big\}$$

$$= \sum_{n=1}^{\infty} \Big\{\varphi_{2n}^{i_1}\Big[\gamma_1[\xi+\lambda_2-\lambda_2h_2(s_2)],\xi\Big] \psi_{2n-1}^{i_2}\Big[\gamma_1[\xi+\lambda_2-\lambda_2h_2(s_2)],\xi\Big]$$

$$- \varphi_{2n}^{i_1}\Big[h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)],\xi\Big]\psi_{2n-1}^{i_2}\Big[h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)],\xi\Big]\Big\}$$

$$= \sum_{n=1}^{\infty} \Big\{\varphi_{2n+1}^{i_1}\Big[h_2(s_2),\xi\Big] \psi_{2n}^{i_2} \Big[h_2(s_2),\xi\Big]$$

$$- \varphi_{2n}^{i_1}\Big[h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)],\xi\Big] \psi_{2n-1}^{i_2}\Big[h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)],\xi\Big]\Big\}.$$

Similarly, we have:

$$(50)\qquad \sum_{\beta=1}^{\infty}\sum_{n=0}^{\infty} r_{2n+1}(i_1,i_2;\beta,\xi)\Big\{\gamma_2^{\beta}[\xi+\lambda_1-\lambda_1 h_1(s_1)] - h_2^{\beta}[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)]\Big\} =$$

$$= \sum_{n=0}^{\infty}\Big\{\varphi_{2n+2}^{i_1}\big[h_1(s_1),\xi\big]\,\psi_{2n+1}^{i_2}\big[h_1(s_1),\xi\big] - \varphi_{2n+1}^{i_1}\big[h_2[s_1+s_2+\lambda_1-\lambda_1\gamma_2(s_1)],\xi\big]\,\psi_{2n}^{i_2}\big[h_2[s_1+s_2+\lambda_1-\lambda_1\gamma_2(s_1)],\xi\big]\Big\}$$

and

$$(51)\qquad \sum_{\beta=1}^{\infty}\sum_{n=1}^{\infty} \tilde{r}_{2n}(0,1;\beta,\xi)\,\Big\{\gamma_2^{\beta}[\xi+\lambda_1-\lambda_1 h_1(s_1)]-h_2^{\beta}[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)]\Big\}$$

$$= \sum_{n=1}^{\infty}\Big\{\psi_{2n+1}\big[h_1(s_1),\xi\big] - \psi_{2n}\big[h_2[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)],\xi\big]\Big\}\quad ,$$

and

$$(52)\qquad \sum_{\beta=1}^{\infty}\sum_{n=0}^{\infty} \tilde{r}_{2n+1}(0,1;\beta,\xi)\Big\{\gamma_1^{\beta}[\xi+\lambda_2-\lambda_2 h_2(s_2)] - h_1^{\beta}[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)]\Big\}$$

$$= \sum_{n=0}^{\infty}\Big\{\psi_{2n+2}\big[h_2(s_2),\xi\big] - \psi_{2n+1}\big[h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)],\xi\big]\Big\}\quad ;$$

by use of the formulae (22), (23) and (24). Formula (47) may now be rewritten as follows:

$$(53) \quad \int_0^\infty e^{-\xi t}\, dt \int_0^\infty \int_0^\infty e^{-(s_1 x + s_2 y)} d_{x,y}\, P\{0 < \eta_1(t) \leq x,\ 0 < \eta_2(t) \leq y\}$$

$$\begin{aligned}
&= S_1(s_1,s_2) \sum_{n=0}^{\infty} \Big\{ \varphi_{2n+1}^{i_1}\big[h_2(s_2),\xi\big]\, \psi_{2n}^{i_2}\big[h_2(s_2),\xi\big] \\
&\qquad - \varphi_{2n}^{i_1}\big[h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)],\xi\big]\, \psi_{2n-1}^{i_2}\big[h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)],\xi\big]\Big\} \\
&+ S_2(s_1,s_2) \sum_{n=0}^{\infty} \Big\{ \varphi_{2n+2}^{i_1}\big[h_1(s_1),\xi\big]\, \psi_{2n+1}^{i_2}\big[h_1(s_1),\xi\big] \\
&\qquad - \varphi_{2n+1}^{i_1}\big[h_2[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)],\xi\big]\, \psi_{2n}^{i_1}\big[h_2[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)],\xi\big]\Big\} \\
&+ \frac{\theta_1^{i_1}(\xi)\, \theta_2^{i_2}(\xi)}{\xi+\lambda_1-\lambda_1\theta_1(\xi)+\lambda_2-\lambda_2\theta_2(\xi)} \Big\{ \lambda_1 S_1(s_1,s_2) \sum_{n=0}^{\infty} \Big\{ \varphi_{2n+1}\big[h_2(s_2),\xi\big] \\
&\qquad\qquad - \varphi_{2n}\big[h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)],\xi\big]\Big\} \\
&+ \lambda_1 S_2(s_1,s_2) \sum_{n=0}^{\infty} \Big\{ \varphi_{2n+2}\big[h_1(s_1),\xi\big] - \varphi_{2n+1}\big[h_2[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)],\xi\big]\Big\} \\
&+ \lambda_2 S_2(s_1,s_2) \sum_{n=0}^{\infty} \Big\{ \psi_{2n+1}\big[h_1(s_1),\xi\big] - \psi_{2n}\big[h_2[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)],\xi\big]\Big\} \\
&+ \lambda_2 S_1(s_1,s_2) \sum_{n=0}^{\infty} \Big\{ \psi_{2n+2}\big[h_2(s_2),\xi\big] - \psi_{2n+1}\big[h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_2(s_1)],\xi\big]\Big\} \Big\} .
\end{aligned}$$

Finally, using formulae (33), (34) and (53) we obtain the following theorem.

<u>Theorem 4</u>

The transform $\Omega(\xi,s_1,s_2)$ of the joint probability distribution of $\eta_1(t)$ and $\eta_2(t)$ is given by:

$$(54)\qquad \Omega(\xi,s_1,s_2) = \theta_1^{i_1}(\xi)\,\theta_2^{i_2}(\xi)\Big[\xi+\lambda_1-\lambda_1\theta_1(\xi)+\lambda_2-\lambda_2\theta_2(\xi)\Big]^{-1}\Big\{1+$$

$$\lambda_1 S_1(s_1,s_2)\sum_{n=0}^{\infty}\Big\{\varphi_{2n+1}[h_2(s_2),\xi]-\varphi_{2n}\Big[h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)],\xi\Big]\Big\}$$

$$+\lambda_1 S_2(s_1,s_2)\sum_{n=0}^{\infty}\Big\{\varphi_{2n+2}\Big[h_1(s_1),\xi\Big]-\varphi_{2n+1}\Big[h_2[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1),\xi\Big]\Big\}$$

$$+\lambda_2 S_2(s_1,s_2)\sum_{n=0}^{\infty}\Big\{\psi_{2n+1}\Big[h_1(s_1),\xi\Big]-\psi_{2n}\Big[h_2[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)],\xi\Big]\Big\}$$

$$+\lambda_2 S_1(s_1,s_2)\sum_{n=0}^{\infty}\Big\{\psi_{2n+2}\Big[h_2(s_2),\xi\Big]-\psi_{2n+1}\Big[h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)],\xi\Big]\Big\}\Big\}$$

$$+S_1(s_1,s_2)\sum_{n=0}^{\infty}\Big\{\varphi_{2n+1}^{i_1}\Big[h_2(s_2),\xi\Big]\,\psi_{2n}^{i_2}\Big[h_2(s_2),\xi\Big]$$

$$-\varphi_{2n}^{i_1}\Big[h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2)],\xi\Big]\,\psi_{2n-1}^{i_2}\Big[h_1[s_1+s_2+\lambda_1-\lambda_1\gamma_1(s_2),\xi\Big]\Big\}$$

$$+S_1(s_1,s_2)\sum_{n=0}^{\infty}\Big\{\varphi_{2n+2}^{i_1}\Big[h_1(s_1),\xi\Big]\psi_{2n+1}^{i_2}\Big[h_1(s_1),\xi\Big]$$

$$-\varphi_{2n+1}^{i_1}\Big[h_2[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)],\xi\Big]\psi_{2n}^{i_2}\Big[h_2[s_1+s_2+\lambda_2-\lambda_2\gamma_2(s_1)],\xi\Big]\Big\}.$$

Corollary 1.

By setting $s_1 = 0$ or $s_2 = 0$, we obtain the marginal distributions of $\eta_1(t)$ and $\eta_2(t)$. Provided $\gamma_1(o+) = \gamma_2(o+) = 1$, a condition which is always satisfied in the equilibrium queue, the formula (54) then simplifies somewhat since:

$$(55) \qquad S_1^{-1}(s_1,o) = S_2^{-1}(s_1,o) = s_1 + \lambda_1 h_1(s_1) - \lambda_1 - \xi \quad ,$$

$$S_1^{-1}(o,s_2) = S_2^{-1}(o,s_2) = s_2 + \lambda_2 h_2(s_2) - \lambda_2 - \xi \quad ,$$

$$h_1[s_2 + \lambda_1 - \lambda_1\gamma_1(s_2)] = \gamma_1(s_2) \quad ,$$

$$h_2[s_1 + \lambda_2 - \lambda_2\gamma_2(s_1)] = \gamma_2(s_1) \quad .$$

Corollary 2.

The distribution of $\eta(t)$, the virtual waitingtime of an arbitrary customer arriving at time t, as defined in (23) is found by setting $s_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2} s$ and $s_2 = \frac{\lambda_2}{\lambda_1 + \lambda_2} s$ in formula (54). Unfortunately the resulting formula does not simplify.

Corollary 3.

The limiting joint distribution of $\eta_1(t)$ and $\eta_2(t)$, if it exists, is found by taking the limit $\lim\limits_{\xi \to o+} \xi\, \Omega(\xi,s_1,s_2)$ in (54). It is easy to see that this limit is zero, when $1 \leq \lambda_1\alpha_1 + \lambda_2\alpha_2$. If $1 > \lambda_1\alpha_1 + \lambda_2\alpha_2$, we obtain the limiting joint distribution, which is then a proper distribution.

The transforms $\Omega_1^*(s_1)$ and $\Omega_2^*(s_2)$ are given by:

$$(56) \qquad \Omega_1^*(s_1) = \lim_{\xi \to o+} \xi\, \Omega(\xi,s_1,o) = (1-\lambda_1\alpha_1-\lambda_2\alpha_2)\Bigg\{\frac{s_1}{s_1+\lambda_1h_1(s_1)-\lambda_1}$$

$$+ \frac{\lambda_1}{s_1+\lambda_1h_1(s_1)-\lambda_1} \sum_{n=0}^{\infty} \Big[1-\varphi_{2n+1}[\gamma_2(s_1),o]\Big]$$

$$+ \frac{\lambda_2}{s_1+\lambda_1h_1(s_1)-\lambda_1} \sum_{n=0}^{\infty} \Big[1-\psi_{2n}[\gamma_2(s_1),o]\Big]\Bigg\} ,$$

and

$$(57) \qquad \Omega_2^*(s_2) = \lim_{\xi \to o+} \xi\, \Omega(\xi,o,s_2) = (1-\lambda_1\alpha_1-\lambda_2\alpha_2)\Bigg\{\frac{s_2}{s_2+\lambda_2h_2(s_2)-\lambda_2}$$

$$+ \frac{\lambda_1}{s_2+\lambda_2h_2(s_2)-\lambda_2} \sum_{n=0}^{\infty} \Big[1-\varphi_{2n}[\gamma_1(s_2),o]\Big]$$

$$+ \frac{\lambda_2}{s_2+\lambda_2h_2(s_2)-\lambda_2} \sum_{n=0}^{\infty} \Big[1-\psi_{2n+1}[\gamma_1(s_2),o]\Big]\Bigg\} .$$

These formulae are obtained from (54) by passage to the limit and cancelling out the common terms. It is not possible in general to find closed form expressions for the sums appearing on the right.

As a check on our calculations, we may verify that $\Omega_1^*(o+) = \Omega_2^*(o+) = 1$. However, this requires de l' Hôpital's rule and the following formulae, which are also needed in the calculation of the moments.

$$(58)\qquad \gamma_1'(o+) = -\alpha_1(1-\lambda_1\alpha_1)^{-1} ,$$

$$\gamma_2'(o+) = -\alpha_2(1-\lambda_2\alpha_2)^{-1} ,$$

$$\gamma_1''(o+) = \mu_1''(1-\lambda_1\alpha_1)^{-3} , \qquad \mu_1'' = \int_o^\infty x^2 d\, H_1(x) ,$$

$$\gamma_2''(o+) = \mu_2''(1-\lambda_2\alpha_2)^{-3} , \qquad \mu_2'' = \int_o^\infty x^2 d\, H_2(x) ,$$

and differentiating with respect to z, with the second variable equal to zero:

$$\varphi_{2n+1}'(1,0) = \left(\frac{\lambda_2\alpha_1}{1-\lambda_1\alpha_1}\right)^{n+1} \left(\frac{\lambda_1\alpha_2}{1-\lambda_2\alpha_2}\right)^{n}$$

$$\varphi_{2n}'(1,0) = \psi_{2n}'(1,0) = \left(\frac{\lambda_2\alpha_1}{1-\lambda_1\alpha_1}\right)^{n} \left(\frac{\lambda_1\alpha_2}{1-\lambda_2\alpha_2}\right)^{n} ,$$

$$\psi_{2n+1}'(1,0) = \left(\frac{\lambda_2\alpha_1}{1-\lambda_1\alpha_1}\right)^{n} \left(\frac{\lambda_1\alpha_2}{1-\lambda_2\alpha_2}\right)^{n+1} ,$$

$$\varphi_{2n+1}''(1,0) = \frac{\lambda_2^2\mu_1''}{(1-\lambda_1\alpha_1)^3} \left(\frac{\lambda_2\alpha_1}{1-\lambda_1\alpha_1}\right)^{2n} \left(\frac{\lambda_1\alpha_2}{1-\lambda_2\alpha_2}\right)^{2n}$$

$$+ \left(\frac{\lambda_2\alpha_1}{1-\lambda_1\alpha_1}\right)^{2} \psi_{2n}''(1,0) , \qquad n \geq 0$$

$$\varphi_{2n}''(1,0) = \frac{\lambda_2^2\mu_1''}{(1-\lambda_1\alpha_1)^3} \left(\frac{\lambda_2\alpha_1}{1-\lambda_1\alpha_1}\right)^{2n-2} \left(\frac{\lambda_1\alpha_2}{1-\lambda_2\alpha_2}\right)^{2n}$$

$$+ \left(\frac{\lambda_2\alpha_1}{1-\lambda_1\alpha_1}\right)^{2} \psi_{2n-1}''(1,0), \qquad n \geq 1 ,$$

and analogous formulae for $\psi_{2n}''(1,0)$ and $\psi_{2n+1}''(1,0)$. If we define:

$$A_1 = \sum_{n=0}^{\infty} \varphi''_{2n+1}(1,0) \qquad B_1 = \sum_{n=0}^{\infty} \psi''_{2n+1}(1,0) \quad ,$$

$$A_2 = \sum_{n=0}^{\infty} \varphi''_{2n}(1,0) \qquad B_2 = \sum_{n=0}^{\infty} \psi''_{2n}(1,0) \quad ,$$

then

$$A_1 = \lambda_2^2\, \gamma_1''(o) \left[1 - \frac{\lambda_1^2\, \lambda_2^2\, \alpha_1^2\, \alpha_2^2}{(1-\lambda_1\alpha_1)^2\, (1-\lambda_2\alpha_2)^2}\right]^{-1} + \lambda_2^2\, \gamma_1'^2(o)\, B_2 \quad ,$$

$$A_2 = \frac{\lambda_2^2\, \mu_1''}{(1-\lambda_1\alpha_1)^3}\; \frac{\lambda_1^2\, \alpha_2^2}{(1-\lambda_2\alpha_2)^2} \left[1 - \frac{\lambda_1^2\, \lambda_2^2\, \alpha_1^2\, \alpha_2^2}{(1-\lambda_1\alpha_1)^2\, (1-\lambda_2\alpha_2)^2}\right]^{-1} + \frac{\lambda_2^2\, \alpha_1^2}{(1-\lambda_1\alpha_1)^2}\; B_1 \quad ,$$

and two analogous equations, with A,B and $1,2$ interchanged. Solving for A_1, B_1, A_2 and B_2, we obtain:

$$A_1 = \left[1 - \frac{\lambda_1^2\, \lambda_2^2\, \alpha_1^2\, \alpha_2^2}{(1-\lambda_1\alpha_1)^2\, (1-\lambda_2\alpha_2)^2}\right]^{-1} \frac{(1-\lambda_2\alpha_2)\, \lambda_2^2}{(1-\lambda_1\alpha_1)^2\, (1-\lambda_1\alpha_1-\lambda_2\alpha_2)} \left\{\mu_1'' + \frac{\lambda_1^2\lambda_2\alpha_1^3\mu_2''}{(1-\lambda_2\alpha_2)^3}\right\} \tag{59}$$

$$B_2 = \left[1 - \frac{\lambda_1^2\, \lambda_2^2\, \alpha_1^2\, \alpha_2^2}{(1-\lambda_1\alpha_1)^2\, (1-\lambda_2\alpha_2)^2}\right]^{-1} \frac{\lambda_1\, \lambda_2^2}{(1-\lambda_1\alpha_1-\lambda_2\alpha_2)(1-\lambda_1\alpha_1)} \left\{\frac{\alpha_2\, \mu_1''}{1-\lambda_1\alpha_1} + \frac{\lambda_1\, \alpha_1\, \mu_2''}{(1-\lambda_2\alpha_2)^2}\right\}$$

and corresponding expressions for A_2 and B_1 .

These expressions are obtained by differentiating once or twice in the functional equations (1) and (21). We assume throughout that the service time distributions have finite second moments.

Taking the limits as $s_1 \to o+$ and $s_2 \to o+$ in (56) and (57), we find that $\Omega_1^*(o+) = \Omega_2^*(o+) = 1$, showing that the limiting distributions are proper for a stable queue. In order to find the limiting expected waitingtime, we first differentiate with respect to s_1 or s_2 in (56) or (57) and take the limits as $s_1 \to o$ or $s_2 \to o$.

Denoting these expectations by $\hat{EW}_1$ and EW_2 respectively, we find after repeated applications of de l'Hopital's rule that:

$$(60) \qquad E(W_1) = \frac{1}{2}\frac{1}{(1-\lambda_1\alpha_1)[(1-\lambda_1\alpha_1)^2(1-\lambda_2\alpha_2)^2-\lambda_1^2\lambda_2^2\alpha_1^2\alpha_2^2]}\Big\{\lambda_1\mu_1''\Big[(1-\lambda_1\alpha_1)^2(1-\lambda_2\alpha_2)^2$$

$$+\lambda_2^2\alpha_2^2(1+\lambda_2\alpha_2)-\lambda_1\alpha_1\lambda_2^2\alpha_2^2(1+\lambda_1\alpha_1)\Big]$$

$$+\lambda_2\mu_2''\Big[(1-\lambda_1\alpha_1)^2\ (1-\lambda_2\alpha_2)^2+\lambda_1^2\lambda_2^2\alpha_1^2\alpha_2^2\Big]\Big\} ,$$

and an analogous expression, with the indices 1 and 2 interchanged for $E(W_2)$.

V. <u>The Queuelength Process.</u>

We denote by $\xi_1(t)$ and $\xi_2(t)$ the numbers of customers in units I and II at time t+. In this section we will study the joint distribution of $\xi_1(t)$ and $\xi_2(t)$. Clearly $\xi_1(t) = \xi_2(t) = 0$, if and only if the server is idle, so,

$$(61) \qquad P\{\xi_1(t) = \xi_2(t) = 0 \mid \xi_1(0) = i_1, \ \ \xi_2(0) = i_2\}$$

$$= P\{\eta_1(t) = \eta_2(t) = 0 \mid \xi_1(0) = i_1, \ \ \xi_2(0) = i_2\}$$

and its transform is given by formula (31).

Next, we define:

(62) $$\Pi_1(j_1,j_2,t) = P_1\{\xi_1(t) = j_1,\ \xi_2(t) = j_2 \mid \xi_1(0) = i_1,\ \xi_2(0) = i_2\}$$

and

(63) $$\Pi_2(j_1,j_2,t) = P_2\{\xi_1(t) = j_1,\ \xi_2(t) = j_2 \mid \xi_1(0) = i_1,\ \xi_2(0) = i_2\} ,$$

where the subscript 1 or 2 denotes, that at time t, the server is in unit I, respectively II. We recall, that in our choice of initial conditions, the server starts a service in unit I at time $t = 0$.

As in the case of the virtual waitingtime process, we need to perform a few auxiliary calculations, which relate to the $M|G|1$ queue.

(A) Auxiliary problem 1.

Consider an $M|G|1$ queueing process with input rate λ_1 and service time distribution $H_1(\cdot)$ and $\beta \geq 1$ customers at $t = 0$. Further there is a secondary, independent Poisson arrival process of rate λ_2. We wish to calculate the probability that:

1) The queue is never empty in $[o,t)$
2) At time t, there are j_1 customers of type I, $j_1 \geq 1$
3) At time t, there are j_2 customers of type II present, given that there were i_2 at $t = 0$. $j_2 \geq i_2 \geq 0$.

We denote this probability by ${}_1\Xi^{(i_1,i_2)}(j_1,j_2,t)$. We have:

$$(64)\qquad {}_1\Xi^{(i_1,i_2)}(j_1,j_2,t) =$$

$$e^{-\lambda_2 t}\,\frac{(\lambda_2 t)^{j_2-i_2}}{(j_2-i_2)!}\sum_{n=0}^{\infty}\sum_{\nu=1}^{j_1}\int_0^t e^{-\lambda_1(t-u)}\,\frac{[\lambda_1(t-u)]^{j_1-\nu}}{(j_1-\nu)!}$$

$$[1-H_1(t-u)]\;d\,G_{\beta\nu}^{(n)}(u)$$

where the probability mass-functions $G_{\beta\nu}^{(n)}(u)$ are defined as follows. $G_{\beta\nu}^{(o)}(u) = \delta_{\beta\nu}\,U(u)$, where $U(\cdot)$ is the distribution degenerate at zero, and for $n \geq 1$, $G_{\beta\nu}^{(n)}(u)$ is the probability that, in an $M|G|1$ queue of input rate λ_1 and service time distribution $H_1(\cdot)$, with β customers initially, the initial busy period involves at least n services, that the n-th service is completed before time u and that at the end of the n-th service, there are ν customers waiting. The quantities $G_{\beta\nu}^{(n)}(u)$ were studied first by Takács [7]. We recall here, that if $g_{\beta\nu}^{(n)}(\zeta)$ is the Laplace-Stieltjes transform of $G_{\beta\nu}^{(n)}(u)$, then:

$$(65)\qquad \sum_{n=0}^{\infty}\sum_{\nu=1}^{\infty} g_{\beta\nu}^{(n)}(\zeta)\,Z^{\nu} = \frac{Z[Z^{\beta}-\gamma_1^{\beta}(\zeta)]}{Z-h_1(\zeta+\lambda_1-\lambda_1 Z)}\,.$$

See also Neuts [6], formula (6).

Using formulae (64) and (65), we find easily:

$$(66)\qquad \sum_{j_1=1}^{\infty}\sum_{j_2=0}^{\infty} z_1^{j_1} z_2^{j_2}\int_0^{\infty} e^{-\zeta t}\,{}_1\Xi^{(i_1,i_2)}(j_1,j_2,t)\,dt$$

$$= \frac{1-h_1(\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2)}{\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2}\;\frac{z_1}{z_1-h_1(\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2)}\;z_2^{i_2}[z_1^{\beta}-\gamma_1^{\beta}(\zeta+\lambda_2-\lambda_2 z_2)]\,.$$

(B) Auxiliary problem 2.

Consider an $M|G|1$ queueing process with input rate λ_1, servicetime distribution $H_1(\cdot)$ and β customers initially. Further, there is a secondary, independent Poisson process of rate λ_2, which is the arrival process for customers of type II, with servicetime distribution $H_2(\cdot)$. There are i_2 customers of type II initially. We are interested in the probability of an event, which is the intersection of the following two. Firstly, we require that in $[o,t)$, the $M|G|1$ queue of customers of type I is never idle. Secondly, we suppose that at time t, both arrival processes are interrupted. The server must first attend to all customers of type I then present and when their service is completed, all those of type II, present at t, must be served. We denote the time required until there are no more I-customers by $\omega_1(t)$ and the additional time until there are no more II-customers by $\omega_2(t)$. We ask, in addition to the above, that $\omega_1(t) \leq x$, $\omega_2(t) \leq y$.

We denote the required probability by ${}_1Q^{(\beta,i_2)}(x,y,t)$. It must satisfy the relation:

$$(67)\qquad {}_1Q^{(\beta,i_2)}(x,y,t) =$$

$$\sum_{n=0}^{\infty} \sum_{j_1=1}^{\infty} \sum_{\nu=1}^{j_1} \sum_{j_2=i_2}^{\infty} \int_{o}^{x}_{(x_1)} \int_{o}^{y}_{(y_1)} \int_{o}^{t}_{(u)} \int_{t}^{t+x_1}_{(v)} e^{-\lambda_2 t} \frac{(\lambda_2 t)^{j_2-i_2}}{(j_2-i_2)!}$$

$$e^{-\lambda_1(t-u)} \frac{[\lambda_1(t-u)]^{j_1-\nu}}{(j_1-\nu)!} \, d\, G_{\beta\nu}^{(n)}(u) \, d\, H_1(v-u) \, d\, H_1^{(j_1-1)}(t+x_1-v)$$

$$d\, H_2^{(j_2)}(y_1) \quad ,$$

for all $x \geq 0$, $y \geq 0$. The probabilistic argument yielding (65) is fairly easy. We denote by u the time of the last service completion (of type I) before t, by v the time of the first service completion after t and we denote the numbers of customers at time t by j_1 and j_2. Formula (67) is then obtained by summing over all allowable values of u, v, j_1 and j_2. Upon taking transforms, we find:

$$(68) \qquad \int_0^\infty e^{-\xi t}\, dt \int_0^\infty \int_0^\infty e^{-(s_1x+s_2y)}\, d_{x,y}\, {}_1Q^{(\beta,i_2)}(x,y,t) =$$

$$h_2^{i_2}(s_2)\, \frac{h_1^\beta(s_1) - \gamma_1^\beta[\xi+\lambda_2-\lambda_2h_2(s_2)]}{s_1-\xi-\lambda_1+\lambda_1h_1(s_1)-\lambda_2+\lambda_2h_2(s_2)} \quad .$$

The calculations leading from (67) to (68) are rather lengthy and involve formula (65) crucially.

C. Auxiliary problem 3.

Consider the queue with alternating priorities, which starts in unit I with initial queuelengths $i_1 \geq 1$ and $i_2 \geq 0$. By $\theta_{11}^{(i_1,i_2)}(j_1,j_2,t)$ we denote the probability that at time t:

1. The server has never been idle.
2. He is serving in unit I.
3. $\xi_1(t) = j_1 \geq 1$, $\xi_2(t) = j_2 \geq 0$.

We also define $\theta_{12}^{(i_1,i_2)}(j_1,j_2,t)$ and for $i_1 \geq 0$, $i_2 \geq 1$, $\theta_{21}^{(i_1,i_2)}(j_1,j_2,t)$ and $\theta_{22}^{(i_1,i_2)}(j_1,j_2,t)$. The definitions are completely analogous, except for the initial unit served and the unit being served at time t. These are indicated in an obvious way by the modified indices. We will give details of the calculations only for the probabilities $\theta_{11}^{(i_1,i_2)}(j_1,j_2,t)$ and state the results for the others. The proofs are completely analogous. We have:

$$\text{(69)} \qquad \theta_{11}^{(i_1,i_2)}(j_1,j_2,t) = {}_1\Xi^{(i_1,i_2)}(j_1,j_2,t) +$$

$$+\sum_{n=1}^{\infty}\sum_{\beta=1}^{\infty}\int_0^t {}_1\Xi^{(\beta,o)}(j_1,j_2,t-u)\, d\, R_{2n}(i_1,i_2;\beta,u) \quad ,$$

by consideration of all possible switches of unit, which may occur in $[o,t)$.

From formulae (22), (23), (64) and (66) we obtain:

$$\text{(70)} \qquad \sum_{j_1=1}^{\infty}\sum_{j_2=0}^{\infty}\int_0^{\infty} e^{-\zeta t}\,\theta_{11}^{(i_1,i_2)}(j_1,j_2,t)\,dt\, z_1^{j_1} z_2^{j_2} =$$

$$\frac{1-h_1(\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2)}{\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2} \cdot \frac{z_1}{z_1-h_1(\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2)} \cdot$$

$$\Big\{z_1^{i_1} z_2^{i_2} - z_2^{i_2}\,\gamma_1^{i_1}(\zeta+\lambda_2-\lambda_2 z_2) + \sum_{\beta=1}^{\infty}\sum_{n=1}^{\infty} r_{2n}(i_1,i_2;\beta,\zeta) \cdot$$

$$[z_1^{\beta} - \gamma_1^{\beta}(\zeta+\lambda_2-\lambda_2 z_2)]\Big\} =$$

$$\frac{1-h_1(\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2)}{\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2} \cdot \frac{z_1}{z_1-h_1(\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2)} \cdot$$

$$\Big\{z_1^{i_1} z_2^{i_2} + \sum_{n=0}^{\infty}\Big[\varphi_{2n}^{i_1}(z_1,\zeta)\psi_{2n-1}^{i_2}(z_1,\zeta)-\varphi_{2n+1}^{i_1}(z_2,\zeta)\psi_{2n}^{i_2}(z_2,\zeta)\Big]\Big\} \quad ,$$

Likewise:

$$\text{(71)} \quad \sum_{j_1=1}^{\infty} \sum_{j_2=0}^{\infty} \int_0^{\infty} e^{-\zeta t}\, \theta_{12}^{(i_1,i_2)}(j_1,j_2,t)\, dt =$$

$$\frac{1-h_2(\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2)}{\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2} \cdot \frac{z_2}{z_2-h_2(\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2)} \cdot$$

$$\sum_{n=0}^{\infty} \left[\varphi_{2n+1}^{i_1}(z_2,\zeta)\psi_{2n}^{i_2}(z_2,\zeta)-\varphi_{2n+2}^{i_1}(z_1,\zeta)\psi_{2n+1}^{i_2}(z_1,\zeta)\right] ,$$

$$\text{(72)} \quad \sum_{j_1=0}^{\infty} \sum_{j_2=1}^{\infty} \int_0^{\infty} e^{-\zeta t}\, \theta_{21}^{(i_1,i_2)}(j_1,j_2,t)\, dt =$$

$$\frac{1-h_1(\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2)}{\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2} \cdot \frac{z_1}{z_1-h_1(\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2)} \cdot$$

$$\sum_{n=0}^{\infty} \left[\psi_{2n+1}^{i_2}(z_1,\zeta)\varphi_{2n}^{i_1}(z_1,\zeta)-\psi_{2n+2}^{i_2}(z_2,\zeta)\varphi_{2n+1}^{i_1}(z_2,\zeta)\right] ,$$

and:

$$\text{(73)} \quad \sum_{j_1=0}^{\infty} \sum_{j_2=1}^{\infty} \int_0^{\infty} e^{-\zeta t}\, \theta_{22}^{(i_1,i_2)}(j_1,j_2,t)\, dt =$$

$$\frac{1-h_2(\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2)}{\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2} \cdot \frac{z_2}{z_2-h_2(\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2)} \cdot$$

$$\left\{z_1^{i_1} z_2^{i_2} + \sum_{n=0}^{\infty} \left[\psi_{2n}^{i_2}(z_2,\zeta)\varphi_{2n-1}^{i_1}(z_2,\zeta)-\psi_{2n+1}^{i_2}(z_1,\zeta)\varphi_{2n}^{i_1}(z_1,\zeta)\right]\right\} .$$

Formulae (70)- (73) lead to transforms for the $\Pi_1(j_1,j_2,t)$ and $\Pi_2(j_1,j_2,t)$, defined in (62) and (63). We summarize these results in the following theorem.

Theorem 5

The joint distribution of the queuelengths $\xi_1(t)$ and $\xi_2(t)$ and the type of the unit served at time t is given by the transforms:

$$(74) \qquad \sum_{j_1=1}^{\infty} \sum_{j_0=0}^{\infty} z_1^{j_1} z_2^{j_2} \int_0^{\infty} e^{-\zeta t}\, \Pi_1(j_1,j_2,t)\, dt =$$

$$\frac{1-h_1(\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2)}{\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2} \cdot \frac{z_1}{z_1-h_1(\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2)} \cdot$$

$$\Big\{ z_1^{i_1} z_2^{i_2} + \sum_{n=0}^{\infty} \Big[\varphi_{2n}^{i_1}(z_1,\zeta)\psi_{2n-1}^{i_2}(z_1,\zeta) - \varphi_{2n+1}^{i_1}(z_2,\zeta)\psi_{2n}^{i_2}(z_2,\zeta) \Big]$$

$$+ \frac{\theta_1^{i_1}(\zeta)\,\theta_2^{i_2}(\zeta)}{\zeta+\lambda_1-\lambda_1\theta_1(\zeta)+\lambda_2-\lambda_2\theta_2(\zeta)} \sum_{n=0}^{\infty} \Big\{ \lambda_1 \Big[\varphi_{2n}(z_1,\zeta) - \varphi_{2n+1}(z_2,\zeta) \Big]$$

$$+ \lambda_2 \Big[\psi_{2n+1}(z_1,\zeta) - \psi_{2n+2}(z_2,\zeta) \Big] \Big\} \Big\} ,$$

and

$$(75) \qquad \sum_{j_1=0}^{\infty} \sum_{j_2=1}^{\infty} z_1^{j_1} z_2^{j_2} \int_0^{\infty} e^{-\zeta t}\, \Pi_2(j_1,j_2,t)\, dt =$$

$$\frac{1-h_2(\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2)}{\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2} \cdot \frac{z_2}{z_2-h_2(\zeta+\lambda_1-\lambda_1 z_1+\lambda_2-\lambda_2 z_2)} \cdot$$

$$\left\{ \sum_{n=0}^{\infty} \left[\varphi_{2n+1}^{i_1}(z_2,\zeta)\psi_{2n}^{i_2}(z_2,\zeta) - \varphi_{2n+2}^{i_1}(z_1,\zeta)\psi_{2n+1}^{i_2}(z_1,\zeta) \right] \right.$$

$$+ \frac{\theta_1^{i_1}(\zeta)\ \theta_2^{i_2}(\zeta)}{\zeta+\lambda_1-\lambda_1\theta_1(\zeta)+\lambda_2-\lambda_2\theta_2(\zeta)} \left\{ \lambda_1 \sum_{n=0}^{\infty} \left[\varphi_{2n+1}(z_2,\zeta) - \varphi_{2n+2}(z_1,\zeta) \right] \right.$$

$$\left. \left. + \lambda_2 \sum_{n=0}^{\infty} \left[\psi_{2n}(z_2,\zeta) - \psi_{2n+1}(z_1,\zeta) \right] \right\} \right\} ,$$

for $\operatorname{Re} \zeta > 0$, $|z_1| \le 1$, $|z_2| \le 1$.

Proof:

The usual argument yields the results immediately. Either the server has never been idle by time t or he has. The second term, relates the queuelength at time t to the time of the beginning of the busy period to which time t belongs.

Corollary 1

If $\lambda_1\alpha_1+\lambda_2\alpha_2 < 1$, then $\lim_{t\to\infty} \Pi_1(j_1,j_2,t) = \Pi_1(j_1,j_2)$ and $\lim_{t\to\infty} \Pi_2(j_1,j_2,t) = \Pi_2(j_1,j_2)$ exist and are positive. If $\lambda_1\alpha_1+\lambda_2\alpha_2 \ge 1$, these limits are zero.

Proof:

The existence of the limits follows from the key renewal theorem for semi-Markov processes. This is immediate from the time-domain versions of the equations (74) and (75). We do not give the details here, but refer to Neuts [4] for a detailed discussion of a simpler case.

Corollary 2.

If $\lambda_1\alpha_1+\lambda_2\alpha_2 < 1$, then :

$$(76)\qquad \sum_{j_1=1}^{\infty}\sum_{j_2=0}^{\infty} \Pi_1(j_1,j_2)z_1^{j_1}z_2^{j_2} = \frac{1-h_1(\lambda_1-\lambda_1z_1+\lambda_2-\lambda_2z_2)}{\lambda_1-\lambda_1z_1+\lambda_2-\lambda_2z_2}\ .$$

$$\frac{z_1}{z_1-h_1(\lambda_1-\lambda_1z_1+\lambda_2-\lambda_2z_2)}\ (1-\lambda_1\alpha_1-\lambda_2\alpha_2)\sum_{n=0}^{\infty}\Big\{\lambda_1\big[\varphi_{2n}(z_1,0)$$

$$-\varphi_{2n+1}(z_2,0)\big] + \lambda_2\big[\psi_{2n+1}(z_1,0)-\psi_{2n+2}(z_2,0)\big]\Big\}\ ,$$

$$(77)\qquad \sum_{j_1=0}^{\infty}\sum_{j_1=1}^{\infty} \Pi_2(j_1,j_2)z_1^{j_1}z_2^{j_2} = \frac{1-h_2(\lambda_1-\lambda_1z_1+\lambda_2-\lambda_2z_2)}{\lambda_1-\lambda_1z_1+\lambda_2-\lambda_2z_2}\ .$$

$$\frac{z_2}{z_2-h_2(\lambda_1-\lambda_1z_1+\lambda_2-\lambda_2z_2)}\ (1-\lambda_1\alpha_1-\lambda_2\alpha_2)\sum_{n=0}^{\infty}\Big\{\lambda_1\big[\varphi_{2n+1}(z_2,0)$$

$$-\varphi_{2n+2}(z_1,0)\big] + \lambda_2\big[\psi_{2n}(z_2,0)-\psi_{2n+1}(z_1,0)\big]\Big\}\ .$$

Proof:

Since the limits exist, their generating functions are given by multiplying by ζ in (74) and (75) and taking the limit as $\zeta \to 0+$.

Corollary 3

Upon setting $z_1 = z_2 = 1$ in (76) and (77), we obtain the limiting probability that the server is in unit I, respectively unit II. Denoting these probabilities by Π_1^* and Π_2^*, we obtain:

$$\Pi_1^* = \lambda_1\alpha_1 \;, \qquad \Pi_2^* = \lambda_2\alpha_2 \;, \tag{78}$$

by using formulae (58) .

Remark: From (76) and (77), one may obtain several other interesting results in a routine fashion. Asymptotic moments of the queuelengths follow by straightforward differentiation and by using formulae (58) to obtain the limiting values as $z_1 \nearrow 1$, $z_2 \nearrow 1$.

VI. The Waitingtime $\tilde{\eta}(t)$.

If at time t, both arrival processes are interrupted, $\tilde{\eta}(t)$ denotes the additional time until the queue becomes empty. By $\Omega_{11}(x,t)$ we denote the probability that at t, unit I is being served and $\tilde{\eta}(t) \leq x$. Likewise, $\Omega_{12}(x,t)$ denotes the probability that at t, unit II is being served and $\tilde{\eta}(t) \leq x$, conditional upon the usual initial conditions.

The Laplace-Stieltjes transforms:

$$\tilde{\Omega}_{11}(s,\zeta) = \int_0^\infty e^{-\zeta t}\, dt \int_0^\infty e^{-sx}\, d_x\, \Omega_{11}(x,t) \quad , \tag{79}$$

and

$$\tilde{\Omega}_{12}(x,\zeta) = \int_0^\infty e^{-\zeta t}\, dt \int_0^\infty e^{-sx}\, d_x\, \Omega_{12}(x,t) \quad ,$$

are found by exactly the same argument as that leading up to (74) and (75), except that the result of the auxiliary problem 2 instead of auxiliary problem 1 intervenes.

We list only the major steps and the final results.

$$(80) \qquad \tilde{\Omega}_{11}(s,\zeta) =$$

$$\{\zeta-s+\lambda_1-\lambda_1 h_1(s)+\lambda_2-\lambda_2 h_2(s)\}^{-1} \cdot$$

$$\Big\{h_2^{i_2}(s)h^{i_1}(s)-h_2^{i_2}(s)\gamma^{i_1}[\zeta+\lambda_2-\lambda_2 h_2(s)]$$

$$+\sum_{n=1}^{\infty}\{r_{2n}[i_1,i_2;h_1(s),\zeta]-r_{2n}[i_1,i_2;\gamma_1(\zeta+\lambda_2-\lambda_2 h_2(s)),\zeta]\}$$

$$+[\zeta+\lambda_1-\lambda_1\theta_1(\zeta)+\lambda_2-\lambda_2\theta_2(\zeta)]^{-1}\,\theta_1^{i_1}(\zeta)\,\theta_2^{i_2}(\zeta) \cdot$$

$$\Big\{\lambda_1\sum_{n=0}^{\infty}\{r_{2n}[1,0;h_1(s),\zeta]-r_{2n}[1,0;\gamma_1(\zeta+\lambda_2-\lambda_2 h_2(s)),\zeta]\Big\}$$

$$+\lambda_2\sum_{n=0}^{\infty}\{\tilde{r}_{2n}[0,1;h_2(s),\zeta]-\tilde{r}_{2n+1}[0,1;\gamma_2(\zeta+\lambda_1-\lambda_1 h_1(s)),\zeta]\}\Big\}\Big\} \ .$$

Upon simplification, using (22) - (24), we obtain:

$$(81) \qquad [\zeta - s + \lambda_1 - \lambda_1 h_1(s) + \lambda_2 - \lambda_2 h_2(s)]\, \tilde{\Omega}_{11}(s,\zeta) =$$

$$h_1^{i_1}(s)\, h_2^{i_2}(s) - \gamma_1^{i_1}[\zeta + \lambda_2 - \lambda_2 h_2(s)]\, h_2^{i_2}(s)$$

$$+ \sum_{n=1}^{\infty} \left[\varphi_{2n}^{i_1}[h_1(s),\zeta]\psi_{2n-1}^{i_2}[h_1(s),\zeta] - \varphi_{2n+1}^{i_1}[h_2(s),\zeta]\psi_{2n}^{i_2}[h_2(s),\zeta]\right]$$

$$+ [\zeta + \lambda_1 - \lambda_1\theta_1(\zeta) + \lambda_2 - \lambda_2\theta_2(\zeta)]^{-1}\, \theta_1^{i_1}(\zeta)\, \theta_2^{i_2}(\zeta).$$

$$\left\{\lambda_1 \sum_{n=0}^{\infty} \left[\varphi_{2n}[h_1(s),\zeta] - \varphi_{2n+1}[h_2(s),\zeta]\right] + \right.$$

$$\left. + \lambda_2 \sum_{n=0}^{\infty} \left[\psi_{2n+1}[h_1(s),\zeta] - \psi_{2n+2}[h_2(s),\zeta]\right]\right\} \quad ,$$

and an analogous expression for $\tilde{\Omega}_{12}(s,\zeta)$. When $\lambda_1\alpha_1 + \lambda_2\alpha_2 < 1$, the probabilities $\Omega_{11}(x,t)$ and $\Omega_{12}(x,t)$ have non-trivial limits as $t \to \infty$. In the classical way we obtain the transforms of these limits:

$$(82) \qquad \tilde{\Omega}_1(s) = \lim_{\zeta \to 0} \zeta\, \tilde{\Omega}_{11}(s,\zeta) = (1 - \lambda_1\alpha_1 - \lambda_2\alpha_2)[s - \lambda_1 + \lambda_1 h_1(s) - \lambda_2 + \lambda_2 h_2(s)]^{-1}$$

$$\left\{\lambda_1 \sum_{n=0}^{\infty} \left[\varphi_{2n+1}[h_2(s),0] - \varphi_{2n}[h_1(s),0]\right]\right.$$

$$\left. + \lambda_2 \sum_{n=0}^{\infty} \left[\psi_{2n+2}[h_2(s),0] - \psi_{2n+1}[h_1(s),0]\right]\right\} \quad ,$$

and:

$$(83) \qquad \tilde{\Omega}_2(s) = \lim_{\zeta \to 0} \zeta\, \tilde{\Omega}_{12}(s,\zeta) = (1-\lambda_1\alpha_1-\lambda_2\alpha_2)[s-\lambda_1+\lambda_1 h_1(s)-\lambda_2+\lambda_2 h_2(s)]^{-1}$$

$$\Big\{\lambda_1 \sum_{n=0}^{\infty} \Big[\varphi_{2n+2}[h_1(s),0] - \varphi_{2n+1}[h_2(s),0]\Big]$$

$$+ \lambda_2 \sum_{n=0}^{\infty} \Big[\psi_{2n+1}[h_1(s),0] - \psi_{2n}[h_2(s),0]\Big]\Big\} \quad .$$

Letting s tend to zero, a single application of de l'Hôpital's rule yields that:

$$(84) \qquad \tilde{\Omega}_1(0) = \lambda_1\alpha_1 \quad , \qquad \tilde{\Omega}_2(0) = \lambda_2\alpha_2$$

which are the limits as t tends to infinity of the probabilities, that the server is, at time t, in unit I, respectively unit II.

We immediately obtain the limiting distribution of $\tilde{\eta}(t)$. Its Laplace-Stieltjes transform $\tilde{\Omega}(s)$ is given by:

$$(85) \qquad \tilde{\Omega}(s) = 1-\lambda_1\alpha_1-\lambda_2\alpha_2 + \tilde{\Omega}_1(s) + \tilde{\Omega}_2(s) = (1-\lambda_1\alpha_1-\lambda_2\alpha_2)$$

$$\Big\{1+[s-\lambda_1+\lambda_1 h_1(s)-\lambda_2+\lambda_2 h_2(s)]^{-1}[\lambda_1 \lim_{n\to\infty} \varphi_{2n}[h_1(s),0]-\lambda_1 h_1(s)$$

$$+ \lambda_2 \lim_{n\to\infty} \psi_{2n}[h_2(s),0] - \lambda_2 h_2(s)]\Big\}$$

but it is easy to show that in a stable queue, both limits appearing in (85) are equal to one.

Simplifying, we obtain:

$$(86) \qquad \tilde{\Omega}(s) = \frac{(1-\lambda_1\alpha_1 - \lambda_2\alpha_2)s}{s-\lambda_1+\lambda_1h_1(s)-\lambda_2+\lambda_2h_2(s)} \quad .$$

This is a remarkable result, since $\tilde{\Omega}(s)$ is also the transform of the limiting distribution for the virtual waitingtime in an $M|G|1$ queue with input rate $\lambda_1 + \lambda_2$ and service time distribution

$$\frac{\lambda_1H_1(\cdot) + \lambda_2H_2(\cdot)}{\lambda_1 + \lambda_2}$$

for which $\lambda_1\alpha_1 + \lambda_2\alpha_2 < 1$. [8] .

Concluding remarks.

It is not surprising, in view of the complex nature of the queueing process studied here, that few simple results are obtained. This paper again shows the power of the imbedded semi-Markov process in the study of queues with Poisson input. The inequalities (27) show that for very stable queues, i.e. $\lambda_1\alpha_1 + \lambda_2\alpha_2$ not close to one, the infinite series of transforms actually converge rapidly, so that the first few terms will approximate the distributions accurately. The computation of higher moments of the limiting distributions is also routine, but tedious.

We further remark that exactly the same method will yield corresponding results for the case in which the server must switch from one unit to another according to an irreducible finite state Markov chain, when there are two or more units. We have limited our discussion to two units, mainly because of the profuse notation already required there.

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13. ABSTRACT

A server operates two units with independent Poisson arrival processes and with independent, general servicetimes. He serves each unit until its queue becomes empty and then switches to the other unit of customers are waiting there. If both units are empty, he operates the unit in which the first customer arrives and stays in it until it again becomes empty.

We study the time dependent and asymptotic distributions for the queuelength processes and for several virtual waitingtime processes. The analysis is in terms of imbedded semi-Markov processes.

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