On the exact distribution of Wilks's criterion*

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1. Summary. In this paper, the exact distribution of Wilks's likelihood ratio criterion, Λ, is obtained, giving explicit expressions for the same for $p = 3, 4, 5$ and 6, where p is the number of variables (= the number of non-null characteristic roots of a matrix when $p \leq f_2$, the degree of freedom for hypothesis, see below). The distribution is expressed as finite series except where p and f_2 are both odd, in which case it is given in infinite series form. Lower percentage points are tabulated for selected values of $f_2 > 10$, extending the tabulations of Schatzoff (1966) for the above values of p.

2. Introduction. Let $\underset{\sim}{X}_1(p \times f_1)$ $(f_1 \geq p)$ and $\underset{\sim}{X}_2$ $(p \times f_2)$ be distributed in the form

$$(2.1) \qquad (2\Pi)^{-\frac{1}{2}p(f_1+f_2)}|\Sigma|^{-\frac{1}{2}(f_1+f_2)} \exp[-\tfrac{1}{2}\mathrm{tr}\Sigma^{-1}\{\underset{\sim}{X}_1\underset{\sim}{X}_1' + (\underset{\sim}{X}_2-\underset{\sim}{\mu})(\underset{\sim}{X}_2-\underset{\sim}{\mu})'\}] \ ,$$

and let the non-zero roots of the determinantal equation

$$(2.2) \qquad |\underset{\sim}{X}_2\underset{\sim}{X}_2' - \lambda\ \underset{\sim}{X}_1\underset{\sim}{X}_1'| = 0$$

be denoted by $0 < \lambda_1 \leq \ldots \leq \lambda_s < \infty$, where $s = \min(p, f_2)$. The likelihood

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ratio criterion for testing, $H_0: \underset{\sim}{\mu}(p \times f_2) = \underset{\sim}{0}$ against $\underset{\sim}{\mu} \neq \underset{\sim}{0}$ can be expressed in terms of the following criterion suggested by Wilks (1932), and Pearson and Wilks (1933):

$$(2.3) \qquad \Lambda = |\underset{\sim}{X}_1\underset{\sim}{X}_1'| \;/\; |\underset{\sim}{X}_1\underset{\sim}{X}_1' + \underset{\sim}{X}_2\underset{\sim}{X}_2'| = \prod_{i=1}^{s} \{1/(1+\lambda_i)\} \quad .$$

It may be noted that in the context of multivariate analysis of variance, $\underset{\sim}{X}_1\underset{\sim}{X}_1'$ and $\underset{\sim}{X}_2\underset{\sim}{X}_2'$, are the sums of product matrices for error and hypothesis respectively, and f_1 and f_2 are the corresponding degrees of freedom.

Wilks (1935) has obtained the exact null hypothesis distribution of Λ in the form of a $(p-1)$ - fold multiple integral, which he was able to evaluate for $p = 1,2$; $p = 3$ with $f_2 = 3,4$ and for $p = 4$ with $f_2 = 4$ only. A number of asymptotic approximations has been given for general p and f_2 . Bartlett (1938), observing the asymptotic behaviour of likelihood ratio statistics, obtained a chi-square approximation to $-f_2 \log_e \Lambda$, for testing independence of several groups of variates as an infinite series of chi-square distributions. Wilks's Λ criterion is a special case of the statistics considered by Wald and Brookner (1941), when the number of groups is equal to two. Rao (1948), using $-\{f_1 - \frac{1}{2}(p-f_2+1)\} \log_e \Lambda$ obtained the first three terms of a more rapidly convergent series. Finally, Rao's approximation was shown to be a special case of a more general result of Box (1949), who gave asymptotic approximations to functions of general likelihood ratio statistics.

Schatzoff (1966) has given a method for obtaining the exact distribution of Λ but has not given explicitly the density or the distribution function. In this paper, the density and the cdf are given in explicit form for values of p up to 6. Schatzoff (1966) tabulated the factors for converting $\chi^2_{pf_2}$

percentiles to exact percentiles of $-\{f_1-\frac{1}{2}(p-f_2+1)\}\log_e\Lambda$ for $p = 3(1)8$ and values of f_2 such that $pf_2 \leq 70$, using certain recurrence relations on IBM 7094. The method used in the paper is by far simple compared to that of Schatzoff so that the restriction $pf_2 \leq 70$ has been overcome easily. While Schatzoff's method is not suitable for handling the distribution problem for odd values of f_2, the method of this paper gives the distribution explicitly in all cases. Also, unlike Consul (1966) who gave the distribution for p up to 4, as infinite series, we are giving the distribution here in finite series form except when both p and f_2 are odd in which case alone the series is infinite. Further, the exact distributions of Λ given here for $p = 3, 4, 5$ and 6 are used to extend Schatzoff's tables in these cases for selected values of $f_2 \geq 11$. The results are presented in Tables 1 to 4.

3. Distributional properties of Λ. For purposes of notational ease, the symbol Λ will be replaced by U. Let us denote by $B[a,b;X]$ The density function

$$(3.1) \qquad [1/B(a,b)]X^{a-1}(1-X)^{b-1} \quad 0 \leq X \leq 1$$

of a Beta variable X. The following theorems, which we state without proof, appear in Anderson (1958, Chapter 8) and have been used in the next section.

Theorem 3.1. The distribution of U_{p,f_2,f_1} is the same as that of U_{f_2,p,f_1+f_2-p}.

This implies that without loss of generality we need consider only values $f_2 \geq p$.

Theorem 3.2. U_{p,f_2,f_1} is distributed like $X_1 \dots X_p$ where X_i are independently distributed as $B[\frac{1}{2}(f_1-i+1),\frac{1}{2}f_2;X_i]$.

Theorem 3.3. U_{2r,f_2,f_1} is distributed like $Y_1^2 \dots Y_r^2$, where Y_i are independently distributed as $B[f_1+1-2i,f_2;Y_i]$; U_{2s+1,f_2,f_1} is distributed as $Z_1^2 \dots Z_s^2 \cdot Z_{s+1}$, where $Z_i (i = 1,\dots,s)$ are independently distributed as $B[f_1+1-2i,f_2;Z_i]$ and Z_{s+1} is independently distributed as $B[\frac{1}{2}(f_1+1-p);\frac{f_2}{2};Z]$.

4. The method of derivation. An immediate consequence of theorem 3.2 is that, since $-\log U_{p,f_2,f_1} = \sum_{i=1}^{p} (-\log X_i) = \sum_{i=1}^{p} Y_i$ (say), the distribution problem in hand can be reduced to that of a sum of independently distributed random variables. The latter distribution can be handled by taking successive convolutions provided that the procedure yields expressions which can be easily integrated at each stage. Schatzoff (1966) has proved that this is in fact the case. But whereas he depends entirely on theorem 3.2 we make use of both theorems 3.2 and 3.3. And by doing so we get the exact distribution of U_{p,f_2,f_1} for $p = 3, 4, 5, 6$ in much simpler form than otherwise is possible. For example, for $p = 4$ we convolute once as against three times, as Schatzoff (1966) has done. Similarly for $p = 5$ and 6 we convolute only two times as against 4 and 5 times respectively .

Consider the beta random variable of theorem 3.2. The density of X_i is given by

$$(4.1)\qquad B[\tfrac{1}{2}(f_1-i+1), \tfrac{f_2}{2}; X_i] = K_i\, X_i^{\frac{f_1-i-1}{2}} (1-X_i)^{\frac{f_2-2}{2}}$$

$$0 < X_i < 1,\ f_1 \geq i$$

where

$$(4.2)\qquad K_i = [1/B(\tfrac{f_1-i+1}{2}, \tfrac{f_2}{2})] = \Gamma(\tfrac{f_1-i+1+f_2}{2}) \,/\, \Gamma(\tfrac{f_1-i+1}{2})\, \Gamma(\tfrac{f_2}{2}) \ .$$

When f_2 is even, $b = \frac{1}{2}(f_2-2)$ is an integer and using binomial theorem the right side of (4.1) can be written in the form

$$(4.3)\qquad B[\tfrac{f_1-i+1}{2}, \tfrac{f_2}{2}; X_i] = K_i \sum_{\ell=0}^{b} (-1)^{\ell} \binom{b}{\ell} X_i^{\frac{1}{2}(f_1-i-1+2\ell)} \ .$$

Now let us transform $Y_i = -\log_e X_i$, then the density of Y_i is given by

$$(4.4)\qquad K_i \sum_{\ell=0}^{b} (-1)^{\ell} \binom{b}{\ell} e^{-\frac{1}{2}Y_i(f_1-i+1+2\ell)}, \quad Y_i > 0, \quad i = 1,\dots, p \ .$$

Similarly in the light of theorem 3.3 we consider the random variable, $Z_i = X_{2i-1} X_{2i}$, then the density of Z_i is given by

$$(4.5)\qquad C_i\, Z_i^{\frac{1}{2}(f_1-2i-1)} (1-\sqrt{Z_i})^{f_2-1} \ ,$$

where

$$C_i = 1 \,/\, 2B(f_1-2i+1, f_2) \ .$$

Note that in this case an application of the binomial theorem gives a finite series unlike in (4.1). This is important for our method. Now make the transformation from $Y_i' = -\log_e Z_i$, and, as before, from (4.5) we get the density of Y_i' as

$$(4.6) \qquad C_i \sum_{\ell=0}^{f_2-1} (-1)^{\ell} \binom{f_2-1}{\ell} e^{-\frac{Y_i'}{2}(f_1+\ell-2i+1)}, \quad Y_i' > 0 \ .$$

Finally, following Schatzoff (1966), consider the density of $V = V_1 + V_2$, where the density of V_1 is given by

$$(4.7) \qquad \{a^{k+1} / \Gamma(k+1)\} \, v_1^k \, e^{a v_1} , \quad v > 0, \quad k = \text{non-negative integer} ,$$

and that of V_2 by

$$(4.8) \qquad b \, e^{b v_2} , \quad v_2 > 0 \ .$$

Schatzoff (1966) has shown that the density of V takes the form

$$(4.9) \qquad b^{k+2} \, e^{b v} \, v^{k+1} / \Gamma(k+2) , \quad a = b$$

and

$$(4.10) \qquad \{a^{k+1} b / \Gamma(k+1)\} \{[e^{a v} \sum_{r=1}^{k+1} (-1)^{r+1} \frac{k!}{(k-r+1)!} \frac{v^{k-r+1}}{(a-b)^r}] + e^{b v} (b-a)^{-(k+1)} k!\}, \quad a \neq b \ .$$

The above results can be readily applied to obtain the distribution of U_{p,f_2,f_1} in the following section.

5. Exact distribution of U_{p,f_2,f_1} for $p = 3,4,5,6$. In this section, we consider the density and cdf of U_{p,f_2,f_1} for $p = 3,4,5$ and 6. We will start with $p = 3$.

Case (i): $p = 3$. We have $U_{3,f_2,f_1} = X_1X_2X_3 = Z_1X_3$, and hence

$$(5.1) \qquad - \log U_{3,f_2,f_1} = Y_1' + Y_3 \quad .$$

Now use (4.4), (4.6), (4.9) and (4.10), we get the density of U_{3,f_2,f_1} in the following form:

$$(5.2) \qquad [1/2B(f_1-1,f_2)B(\tfrac{f_1-2}{2}, \tfrac{f_2}{2})] \Big[\sum_{m=1}^{\frac{f_2}{2}-1} (-1)^{m-1}\binom{f_2-1}{2m-1}\binom{\frac{f_2}{2}-1}{m}(-\log U)U^{\frac{f_1+2m-4}{2}}$$

$$+ 2 \sum_{\substack{\ell=0 \\ \ell\neq 2m-1}}^{f_2-1} \sum_{m=0}^{\frac{f_2}{2}-1} \frac{(-1)^{\ell+m}}{(2m-\ell-1)}\binom{f_2-1}{\ell}\binom{\frac{f_2}{2}-1}{m}\Big(U^{\frac{f_1+\ell-3}{2}} - U^{\frac{f_1+2m-4}{2}}\Big)\Big]$$

$$0 \leq U \leq 1 \quad ,$$

which is a finite series for f_2 even and infinite series for f_2 odd. For obtaining cdf of U_{3,f_2,f_1} we integrate (5.2) between $(0,u)$, $0 < u \leq 1$, obtaining

$$(5.3)\quad P[U_{3,f_2,f_1} \leq u] = [1/B(f_1-1,f_2)B(\frac{f_1-2}{2}, \frac{f_2}{2})][\sum_{m=0}^{\frac{f_2}{2}-1} \frac{(-1)^{m-1}\, u^{\frac{f_1+2m-2}{2}}}{(f_1+2m-2)^2} \binom{\frac{f_2}{2}-1}{m}\binom{f_2-1}{2m-1} \cdot (2-(f_1+2m-2)\log u)$$

$$+ 2 \sum_{\substack{\ell=0 \\ \ell \neq 2m-1}}^{f_2-1} \sum_{m=0}^{\frac{f_2}{2}-1} \frac{(-1)^{\ell+m}}{(2m-\ell-1)} \binom{f_2-1}{\ell}\binom{\frac{f_2}{2}-1}{m} \left(\frac{u^{\frac{f_1+\ell-1}{2}}}{f_1+\ell-1} - \frac{u^{\frac{f_1+2m-2}{2}}}{f_1+2m-2}\right)] .$$

Case (ii): p = 4. In this case, $-\log U_{4,f_2,f_1} = Y_1' + Y_2'$ and the density of U_{4,f_2,f_1} is obtained in the following finite series:

$$(5.4)\quad \prod_{i=1}^{2} \frac{1}{2B(f_1-2i+1,f_2)} [\sum_{\ell=0}^{f_2-3} \binom{f_2-1}{\ell}\binom{f_2-1}{\ell+2}(-\log U)\, U^{\frac{f_1+\ell-3}{2}}$$

$$+ 2 \sum_{\substack{\ell=0 \\ \ell \neq m-2}}^{f_2-1} \sum_{m=0}^{f_2-1} \frac{(-1)^{\ell+m}}{(m-\ell-2)} \binom{f_2-1}{\ell}\binom{f_2-1}{m} \left(U^{\frac{f_1+\ell-3}{2}} - U^{\frac{f_1+m-5}{2}}\right)] .$$

Further, the cdf of U_{4,f_2,f_1} is given by

$$(5.5)\quad P[U_{4,f_2,f_1} \leq u] = \tfrac{1}{2} \prod_{i=1}^{2} \frac{1}{B(f_1-2i+1,f_2)} [\sum_{\ell=0}^{f_2-3} \frac{u^{\frac{f_1+\ell-1}{2}}}{(f_1+\ell-1)^2} \binom{f_2-1}{\ell}\binom{f_2-1}{\ell+2} .$$

$$(2-(f_1+\ell-1)\log u)$$

$$+ 2 \sum_{\substack{\ell=0 \\ \ell \neq m-2}}^{f_2-1} \sum_{m=0}^{f_2-1} \frac{(-1)^{\ell+m}}{(m-\ell-2)} \binom{f_2-1}{\ell}\binom{f_2-1}{m} \left\{\frac{u^{\frac{f_1+\ell-1}{2}}}{f_1+\ell-1} - \frac{u^{\frac{f_1+m-3}{2}}}{f_1+m-3}\right\}] .$$

Case (iii): $p = 5$. For $p = 5$, $-\log U_{5,f_2,f_1} = Y_1' + Y_2' + Y_5$ and the density of U_{5,f_2,f_1} is given by

$$(5.6) \qquad K\, U^{\frac{f_1-6}{2}} \Big[\sum_{n=2}^{\frac{1}{2}f_2-1} (-1)^n f(2n-3,2n-1,n)\, U^n (\log U)^2$$

$$+ 4 \sum_{\substack{\ell=0 \\ \ell \neq 2n-3}}^{f_2-3} \sum_{n=0}^{\frac{1}{2}f_2-1} \frac{(-1)^n}{(2n-\ell-3)} f(\ell,\ell+2,n) \left\{ \frac{2}{(2n-\ell-3)} (U^n - U^{\ell+3}) - U^{\frac{\ell+3}{2}} \log U \right\}$$

$$- 4 \sum_{\substack{m=0 \\ m \neq 2n-1}}^{f_2-1} \sum_{n=2}^{\frac{1}{2}f_2-1} \frac{(-1)^{m+n-1}}{(m-2n+1)} f(2n-3,m,n)\, U^n \log U$$

$$- 8 \sum_{\substack{\ell=0 \\ \ell \neq m-2,2n-3}}^{f_2-1} \sum_{m=0}^{f_2-1} \sum_{n=0}^{\frac{1}{2}f_2-1} \frac{(-1)^{\ell+m+n}}{(m-\ell-2)(2n-\ell-3)} f(\ell,m,n) \left(U^n - U^{\frac{\ell+3}{2}}\right)$$

$$+ 4 \sum_{\substack{\ell=0 \\ \ell \neq 2n+3}}^{f_2-1} \sum_{n=1}^{\frac{1}{2}f_2-1} \frac{(-1)^{\ell+n+1}}{(2n-\ell-3)} f(\ell,2n-1,n)\, U^n \log U$$

$$+ 8 \sum_{\substack{\ell=0 \\ \ell \neq m-2,\, m \neq 2n-1}}^{f_2-1} \sum_{m=0}^{f_2-1} \sum_{n=0}^{\frac{1}{2}f_2-1} \frac{(-1)^{\ell+m+n}}{(m-\ell-2)(2n-m-1)} f(\ell,m,n) \left(U^n - U^{\frac{m+1}{2}}\right) \Big] ,$$

where

$$f(\ell,m,n) = \binom{f_2-1}{\ell} \binom{f_2-1}{m} \binom{\frac{f_2}{2}-1}{n}$$

and

$$K = \left[1/2B\left(\frac{f_1-4}{2}, \frac{f_2}{2}\right)\right] \prod_{i=1}^{2} [1/2B(f_1-2i+1, f_2)] \quad .$$

The series (5.6) is finite when f_2 is even but infinite for f_2 odd. The cdf can be obtained from (5.6) by integrating between (o,u) and is available in an unpublished report (by Arjun K. Gupta, Department of Statistics, Purdue University).

<u>Case iv</u>: <u>$p = 6$</u>. In this case, noting that $-\log U_{6,f_2,f_1} = Y_1' + Y_2' + Y_3'$, we get the density of U_{6,f_2,f_1} in the form

$$(5.7) \qquad K_1 U^{\frac{1}{2}(f_1-3)} \Bigg[\sum_{\ell=0}^{\frac{f_2-5}{2}} (-1)^\ell f_1(\ell,\ell+2,\ell+4)\, U^{\frac{\ell}{2}} (\log U)^2$$

$$+ 4 \sum_{\substack{\ell=0 \\ \ell \neq n-4}}^{\frac{f_2-3}{2}} \sum_{n=0}^{\frac{f_2-1}{2}} \frac{(-1)^n}{(n-\ell-4)^2} f_1(\ell,\ell+2,n)\left(2U^{\frac{n-4}{2}} - U^{\frac{\ell}{2}}(2+(n-\ell-4)\log U)\right)$$

$$+ 8 \sum_{\substack{\ell=0 \\ \ell \neq m-2,\, n-4}}^{\frac{f_2-1}{2}} \sum_{m=0}^{\frac{f_2-1}{2}} \sum_{n=0}^{\frac{f_2-1}{2}} \frac{(-1)^{\ell+m+n}}{(m-\ell-2)(n-\ell-4)} f_1(\ell,m,n)\left(U^{\ell/2} - U^{\frac{n-4}{2}}\right)$$

$$- 4 \sum_{\substack{\ell=0 \\ \ell \neq m-2}}^{\frac{f_2-5}{2}} \sum_{m=0}^{\frac{f_2-1}{2}} \frac{(-1)^m}{(m-\ell-2)} f_1(\ell,m,\ell+4)\, U^{\ell/2} \log U$$

$$- 8 \sum_{\substack{\ell=0 \\ \ell \neq m-2,\, m \neq n-2}}^{\frac{f_2-1}{2}} \sum_{m=0}^{\frac{f_2-1}{2}} \sum_{n=0}^{\frac{f_2-1}{2}} \frac{(-1)^{\ell+m+n}}{(m-\ell-2)(n-m-2)} f_1(\ell,m,n)\left(U^{\frac{m-2}{2}} - U^{\frac{n-4}{2}}\right)$$

$$+ 4 \sum_{\substack{\ell=0 \\ \ell \neq m-2}}^{\frac{f_2-1}{2}} \sum_{m=0}^{\frac{f_2-3}{2}} \frac{(-1)^\ell}{(m-\ell-2)} f_1(\ell,m,m+2)\, U^{\frac{m-2}{2}} \log U \Bigg] \quad ,$$

where

$$f_1(\ell,m,n) = \binom{f_2-1}{\ell}\binom{f_2-1}{m}\binom{f_2-1}{n}$$

and

$$K_1 = \tfrac{1}{2} \prod_{i=1}^{3} [1/2B(f_1-2i+1,f_2)] \quad .$$

6. Computation of percentage points. The expressions developed in the preceding section were used for the tabulation of percentage points of Λ. Values were first computed of U_{p,f_2,f_1} on CDC 6500 to a minimum accuracy of five significant digits based on four arguments (p,f_2,f_1,α), where α is the lower probability level. For larger values of $f_1 (> 30)$ Rao's approximation (Rao (1948)) was used. These values were then used to obtain correction factors for converting chi-square percentiles with pf_2 degrees of freedom to the exact percentiles of $-\{f_1-\frac{1}{2}(p-f_2+1)\} \log U_{p,f_2,f_1}$. Finally, tabulation of the correction factors, $C =$ [percentile of $-\{f_1-\frac{1}{2}(p-f_2+1)\} \log_e U_{p,f_2,f_1}$] / (percentile of $\chi^2_{pf_2}$), was made for each pair (p,f_2) with arguments $M = f_1-p+1$ and α. These factors are given to three decimal places although they were obtained generally to an accuracy of four decimals. The correction factors are presented in Tables 1 - 4 for $M = 1(1)10, 12(2)20, 24, 30, 40, 60, 120, \infty$; Table 1 gives the percentage points for $p = 3$, $f_2 = 12(2)22$ and $\alpha = .1, .05, .025, .01$ and $.005$; Table 2, for $p = 4$, $f_2 = 11(1)13(2)23$ and α as above; Table 3, for $p = 5$ $f_2 = 12(2)16$, $\alpha = .05, .01$, and Table 4, $f_2 = 11(1)13$ and $\alpha = .05$ and $.01$. In referring to these tables it may be pointed out that by theorem 3.1 the distribution of U_{p,f_2,f_1} is the same as that of U_{f_2,p,f_1+f_2-p} and hence interchanging the role of p and f_2 does

not affect the value of M.

7. Uses of the tabulations. There are at least three tests of multivariate hypotheses for which the tabulations in the paper are useful, namely, (Pillai, 1960)

I that of equality of the covariance matrices of two p-variate normal populations;

II that of equality of the p-dimensional mean vectors of ℓ p-variate normal populations having a common covariance matrix; and

III that of independence between a p-set and a q-set of variates in a (p+q)-variate normal population.

Test of hypothesis II is the one discussed so far in this paper using Wilks's Λ given in (2.3). In the context of II $f_2 = \ell-1$ and $f_1 = N-\ell$, where N is the total of the sizes of the ℓ samples. As in section 2, $\underset{\sim}{S} = \underset{\sim}{X}_1\underset{\sim}{X}_1'$ is the Within S.P. matrix and $\underset{\sim}{S}^* = \underset{\sim}{X}_2\underset{\sim}{X}_2'$ is the Between S.P. matrix and $\Lambda = \prod_{i=1}^{s} (1-\theta_i)$, where θ_i's are the non-zero characteristic roots of $\underset{\sim}{S}^*(\underset{\sim}{S}^*+\underset{\sim}{S})^{-1}$ and $\theta_i = \lambda_i/(1+\lambda_i)$, $i = 1,,,, s$.

Now consider the test of III, i.e. H_0: $\underset{\sim}{\Sigma}_{12} = 0$ against $\underset{\sim}{\Sigma}_{12} \neq 0$, where $\underset{\sim}{\Sigma}_{12}$ is the population covariance matrix between the p and q set of variables. Use as test criterion $\Lambda = \prod_{i=1}^{p} (1-\theta_i)$, where θ_i's are the characteristic roots of $\underset{\sim}{S}_{11}^{-1}\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\underset{\sim}{S}_{12}'$, where $\underset{\sim}{S}_{11}$ is the S.P. matrix of the sample of observations on the p-set of variates, $\underset{\sim}{S}_{22}$ that on the q-set, and $\underset{\sim}{S}_{12}$, the S.P. matrix between the observations on the p-set and the q-set, $p \leq q$ and $p+q < k$, the sample size. Here Λ is the (2/k)th power of the likelihood ratio criterion and equals $|\underset{\sim}{S}_{11} - \underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\underset{\sim}{S}_{12}'| \,/\, |\underset{\sim}{S}_{11}| = \prod_{i=1}^{p} (1-\theta_i)$. In this context $f_1 = k-q-1$ and $f_2 = q$.

For test of hypothesis I also the criterion $\Lambda = \prod_{i=1}^{p} (1-\theta_i)$ is useful, where θ_i's are now the characteristic roots of the matrix $\underset{\sim}{S}_1(\underset{\sim}{S}_1+\underset{\sim}{S}_2)^{-1}$, where $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ denote the usual S.P. matrices computed from two independent p-variate samples of sizes n_1 and n_2 respectively. Here if γ_i, $i = 1,\ldots, p$, are the characteristic roots of $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$, where $\underset{\sim}{\Sigma}_1$ and $\underset{\sim}{\Sigma}_2$ are the covariance matrices of the two p-variate normal populations, then the null hypothesis can be written as H_o: $\gamma_1 = \ldots = \gamma_p = 1$. Further, consider the one-sided alternate hypothesis H_1: $\gamma_i \geq 1$, $i = 1,\ldots, p$, $\sum_{i=1}^{p} \gamma_i > p$. The tabulations in the paper may be used for this test with $f_2 = n_1-1$ and $f_1 = n_2-1$. Unlike the tests of hypotheses II and III, tabulations for larger values of f_2 are also important in this test. Further, it should be pointed out that the Λ-test in this case is not related to the likelihood ratio criterion. However, it has been shown for $p = 2$, through power comparisons with respect to each γ_i, that this test compares favorably with other good tests for the purpose (Pillai and Jayachandran, 1968).

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Table 1. Chi-square adjustments to Wilks's criterion U.

Factor C for lower percentiles of U (upper percentiles of χ^2), p = 3

M \ α	$f_2 = 12$: .100	.050	.025	.010	.005	$f_2 = 14$: .100	.050	.025	.010	.005
1	1.718	1.791	1.860	1.949	2.013	1.780	1.857	1.931	2.026	2.095
2	1.382	1.410	1.437	1.470	1.495	1.427	1.458	1.486	1.523	1.549
3	1.256	1.272	1.287	1.306	1.319	1.292	1.309	1.326	1.346	1.361
4	1.188	1.199	1.209	1.221	1.230	1.217	1.229	1.240	1.254	1.264
5	1.146	1.154	1.161	1.170	1.176	1.171	1.179	1.188	1.198	1.205
6	1.117	1.123	1.129	1.136	1.141	1.138	1.145	1.152	1.159	1.165
7	1.097	1.101	1.106	1.111	1.115	1.115	1.121	1.126	1.132	1.136
8	1.081	1.085	1.089	1.093	1.097	1.097	1.102	1.106	1.111	1.115
9	1.069	1.073	1.076	1.080	1.082	1.084	1.088	1.091	1.095	1.099
10	1.060	1.063	1.066	1.069	1.071	1.073	1.076	1.079	1.082	1.085
12	1.046	1.048	1.050	1.053	1.054	1.057	1.059	1.061	1.064	1.066
14	1.037	1.039	1.040	1.042	1.043	1.046	1.048	1.049	1.052	1.053
16	1.030	1.032	1.033	1.034	1.035	1.037	1.039	1.041	1.042	1.044
18	1.025	1.026	1.027	1.029	1.029	1.031	1.033	1.034	1.035	1.036
20	1.021	1.022	1.023	1.024	1.025	1.027	1.028	1.029	1.030	1.031
24	1.016	1.017	1.017	1.018	1.019	1.020	1.021	1.022	1.023	1.023
30	1.011	1.011	1.012	1.012	1.013	1.014	1.015	1.015	1.016	1.016
40	1.007	1.007	1.007	1.008	1.008	1.009	1.009	1.009	1.010	1.010
60	1.003	1.003	1.004	1.004	1.004	1.004	1.004	1.005	1.005	1.005
120	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001
∞	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\chi^2_{pf_2}$	47.2122	50.9985	54.4373	58.6192	61.5812	54.0902	58.1240	61.7768	66.2062	69.3360

M \ α	$f_2 = 16$: .100	.050	.025	.010	.005	$f_2 = 18$: .100	.050	.025	.010	.005
1	1.835	1.916	1.995	2.095	2.169	1.886	1.971	2.053	2.158	2.235
2	1.469	1.501	1.532	1.571	1.599	1.508	1.542	1.575	1.616	1.646
3	1.325	1.344	1.362	1.384	1.400	1.357	1.377	1.396	1.420	1.437
4	1.245	1.258	1.271	1.285	1.296	1.272	1.286	1.299	1.315	1.327
5	1.195	1.204	1.213	1.224	1.232	1.218	1.228	1.238	1.249	1.258
6	1.159	1.167	1.174	1.182	1.188	1.179	1.188	1.195	1.204	1.211
7	1.133	1.139	1.145	1.152	1.157	1.151	1.158	1.164	1.171	1.177
8	1.114	1.119	1.123	1.129	1.133	1.129	1.135	1.140	1.146	1.151
9	1.098	1.102	1.106	1.111	1.115	1.112	1.117	1.121	1.127	1.130
10	1.085	1.089	1.092	1.097	1.099	1.099	1.103	1.107	1.111	1.114
12	1.067	1.070	1.073	1.076	1.078	1.078	1.081	1.084	1.087	1.090
14	1.054	1.057	1.059	1.061	1.063	1.064	1.066	1.068	1.071	1.073
16	1.045	1.047	1.049	1.051	1.052	1.053	1.055	1.057	1.059	1.061
18	1.038	1.039	1.041	1.043	1.044	1.045	1.046	1.048	1.050	1.051
20	1.032	1.034	1.035	1.036	1.037	1.038	1.040	1.041	1.043	1.044
24	1.025	1.026	1.026	1.027	1.028	1.029	1.030	1.031	1.032	1.033
30	1.017	1.018	1.018	1.019	1.020	1.021	1.021	1.022	1.023	1.023
40	1.011	1.011	1.011	1.012	1.012	1.013	1.013	1.014	1.014	1.015
60	1.005	1.006	1.006	1.006	1.006	1.006	1.007	1.007	1.007	1.007
120	1.002	1.002	1.002	1.002	1.002	1.002	1.002	1.002	1.002	1.002
∞	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\chi^2_{pf_2}$	60.9066	65.1708	69.0226	73.6826	76.9688	67.6728	72.1532	76.1920	81.0688	84.5019

p = number of variates; f_1 = error degrees of freedom; f_2 = hypothesis degrees of freedom; $M = f_1-p+1$; $C = [\text{percentile for } -\{f_1-\frac{1}{2}(p-f_2+1)\}\log_e U]/(\text{percentile for } \chi^2$ with pf_2 degrees of freedom).

Table 1 (Cont'd.)

M \ α	$f_2 = 20$ .100	.050	.025	.010	.005	$f_2 = 22$ .100	.050	.025	.010	.005
1	1.932	2.021	2.106	2.216	2.297	1.975	2.067	2.156	2.269	2.353
2	1.544	1.580	1.614	1.657	1.689	1.578	1.616	1.651	1.696	1.729
3	1.387	1.408	1.428	1.453	1.472	1.415	1.438	1.459	1.485	1.504
4	1.298	1.313	1.327	1.344	1.356	1.322	1.338	1.353	1.371	1.384
5	1.240	1.251	1.261	1.274	1.283	1.261	1.273	1.284	1.297	1.307
6	1.199	1.208	1.216	1.226	1.233	1.218	1.227	1.236	1.246	1.254
7	1.168	1.176	1.182	1.190	1.196	1.185	1.193	1.200	1.209	1.215
8	1.145	1.151	1.157	1.163	1.168	1.160	1.167	1.173	1.180	1.185
9	1.127	1.132	1.136	1.142	1.146	1.142	1.147	1.151	1.157	1.161
10	1.112	1.116	1.120	1.125	1.128	1.124	1.129	1.133	1.139	1.141
12	1.089	1.092	1.095	1.099	1.102	1.099	1.103	1.106	1.110	1.113
14	1.073	1.075	1.078	1.081	1.083	1.082	1.085	1.087	1.091	1.093
16	1.061	1.063	1.065	1.067	1.069	1.069	1.071	1.073	1.076	1.078
18	1.052	1.053	1.055	1.057	1.059	1.059	1.061	1.063	1.065	1.066
20	1.044	1.046	1.048	1.049	1.050	1.051	1.052	1.054	1.056	1.057
24	1.034	1.035	1.036	1.038	1.039	1.039	1.040	1.041	1.043	1.044
30	1.024	1.025	1.026	1.027	1.027	1.028	1.029	1.030	1.031	1.031
40	1.015	1.016	1.016	1.017	1.017	1.018	1.018	1.019	1.020	1.020
60	1.008	1.008	1.008	1.009	1.009	1.009	1.009	1.010	1.010	1.010
120	1.002	1.002	1.002	1.002	1.003	1.003	1.003	1.003	1.003	1.003
∞	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\chi^2_{pf_2}$	74.3970	79.0819	83.2976	88.3794	91.9517	81.0855	85.9649	90.3489	95.6257	99.3304

p = number of variates; f_1 = error degrees of freedom; f_2 = hypothesis degrees of freedom; $M = f_1-p+1$; C = [percentile for $-\{f_1-\frac{1}{2}(p-f_2+1) \log_e U\}$] / (percentile for χ^2 with pf_2 degrees of freedom).

Table 2. Chi-square adjustments to Wilks's criterion U .

Factor C for lower percentiles of U (upper percentiles of χ^2), p = 4

M \ α	$f_2 = 11$ .100	.050	.025	.010	.005	$f_2 = 12$ .100	.050	.025	.010	.005
1	2.248	2.256	2.277	2.315	2.347	2.288	2.299	2.322	2.362	2.396
2	1.330	1.353	1.374	1.402	1.422	1.350	1.373	1.396	1.424	1.446
3	1.221	1.235	1.247	1.262	1.274	1.238	1.252	1.264	1.280	1.292
4	1.164	1.173	1.181	1.191	1.198	1.177	1.186	1.195	1.205	1.213
5	1.127	1.134	1.140	1.147	1.152	1.139	1.145	1.152	1.159	1.165
6	1.103	1.108	1.112	1.118	1.122	1.112	1.118	1.122	1.128	1.132
7	1.085	1.089	1.092	1.097	1.100	1.093	1.097	1.101	1.106	1.109
8	1.071	1.075	1.078	1.081	1.084	1.079	1.082	1.085	1.089	1.092
9	1.061	1.064	1.066	1.069	1.071	1.068	1.070	1.073	1.076	1.079
10	1.053	1.055	1.057	1.060	1.062	1.059	1.061	1.063	1.066	1.068
12	1.041	1.043	1.044	1.046	1.048	1.046	1.047	1.049	1.051	1.053
14	1.033	1.034	1.035	1.037	1.038	1.037	1.038	1.039	1.041	1.042
16	1.027	1.028	1.029	1.030	1.031	1.030	1.031	1.032	1.033	1.034
18	1.022	1.023	1.024	1.025	1.026	1.025	1.026	1.027	1.028	1.029
20	1.019	1.020	1.020	1.021	1.022	1.021	1.022	1.023	1.024	1.024
24	1.014	1.015	1.015	1.016	1.016	1.016	1.017	1.017	1.018	1.018
30	1.010	1.010	1.010	1.011	1.011	1.011	1.011	1.012	1.012	1.013
40	1.006	1.006	1.006	1.007	1.007	1.007	1.007	1.007	1.008	1.008
60	1.003	1.003	1.003	1.003	1.003	1.003	1.003	1.004	1.004	1.004
120	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001
∞	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\chi^2_{pf_2}$	56.3685	60.4809	64.2014	68.7095	71.8925	60.9066	65.1708	69.0226	73.6826	76.9688

M \ α	$f_2 = 13$ .100	.050	.025	.010	.005	$f_2 = 15$ .100	.050	.025	.010	.005
1	2.327	2.340	2.364	2.406	2.442	2.400	2.416	2.444	2.490	2.529
2	1.369	1.393	1.405	1.446	1.468	1.406	1.432	1.456	1.488	1.511
3	1.254	1.268	1.281	1.298	1.310	1.284	1.299	1.313	1.331	1.344
4	1.190	1.200	1.209	1.220	1.228	1.216	1.226	1.236	1.248	1.256
5	1.150	1.157	1.163	1.171	1.177	1.172	1.180	1.187	1.195	1.202
6	1.122	1.127	1.132	1.139	1.143	1.141	1.147	1.153	1.159	1.164
7	1.102	1.106	1.110	1.115	1.119	1.118	1.123	1.128	1.133	1.137
8	1.086	1.090	1.093	1.097	1.100	1.101	1.105	1.109	1.113	1.118
9	1.074	1.077	1.080	1.084	1.086	1.088	1.091	1.094	1.098	1.101
10	1.065	1.067	1.070	1.073	1.075	1.077	1.080	1.082	1.086	1.088
12	1.050	1.052	1.054	1.057	1.058	1.060	1.063	1.065	1.067	1.069
14	1.041	1.042	1.044	1.045	1.047	1.049	1.050	1.052	1.054	1.055
16	1.033	1.035	1.036	1.037	1.038	1.040	1.042	1.043	1.045	1.046
18	1.028	1.029	1.030	1.031	1.032	1.034	1.035	1.036	1.038	1.039
20	1.024	1.025	1.025	1.026	1.027	1.029	1.030	1.031	1.032	1.033
24	1.018	1.019	1.019	1.020	1.020	1.022	1.023	1.023	1.024	1.025
30	1.012	1.013	1.013	1.014	1.014	1.015	1.016	1.016	1.017	1.017
40	1.008	1.008	1.008	1.008	1.009	1.010	1.010	1.010	1.011	1.011
60	1.004	1.004	1.004	1.004	1.004	1.005	1.005	1.005	1.005	1.005
120	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.002	1.002
∞	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\chi^2_{pf_2}$	65.4224	69.8321	73.8098	78.6157	82.0008	74.3970	79.0819	83.2976	88.3794	91.9517

p = number of variates; f_1 = error degrees of freedom; f_2 = hypothesis degrees of freedom; $M = f_1 - p + 1$; C = [percentile for $-\{f_1 - \frac{1}{2}(p - f_2 + 1) \log_e U\}$]/(percentile for χ^2 with pf_2 degrees of freedom).

Table 2 (Cont'd.)

M \ α			$f_2 = 17$					$f_2 = 19$		
	.100	.050	.025	.010	.005	.100	.050	.025	.010	.005
1	2.466	2.486	2.517	2.567	2.608	2.528	2.550	2.584	2.637	2.682
2	1.440	1.468	1.494	1.527	1.551	1.473	1.502	1.529	1.563	1.589
3	1.313	1.329	1.344	1.363	1.377	1.340	1.357	1.373	1.393	1.408
4	1.240	1.252	1.262	1.275	1.284	1.264	1.276	1.287	1.300	1.310
5	1.193	1.201	1.209	1.218	1.225	1.214	1.223	1.231	1.241	1.248
6	1.160	1.166	1.172	1.180	1.185	1.178	1.185	1.191	1.199	1.205
7	1.135	1.140	1.145	1.151	1.155	1.151	1.157	1.162	1.169	1.173
8	1.116	1.120	1.124	1.129	1.133	1.131	1.135	1.140	1.145	1.149
9	1.101	1.105	1.108	1.112	1.115	1.114	1.118	1.122	1.126	1.130
10	1.089	1.092	1.095	1.098	1.101	1.101	1.104	1.107	1.111	1.114
12	1.070	1.073	1.075	1.078	1.080	1.080	1.083	1.086	1.089	1.091
14	1.057	1.059	1.061	1.063	1.065	1.066	1.068	1.070	1.072	1.074
16	1.048	1.049	1.051	1.052	1.054	1.055	1.057	1.058	1.060	1.062
18	1.040	1.042	1.043	1.044	1.045	1.047	1.048	1.050	1.051	1.052
20	1.035	1.036	1.037	1.038	1.039	1.040	1.042	1.043	1.044	1.045
24	1.026	1.027	1.028	1.029	1.030	1.031	1.032	1.033	1.034	1.035
30	1.019	1.019	1.020	1.020	1.021	1.022	1.023	1.023	1.024	1.025
40	1.012	1.012	1.012	1.013	1.013	1.014	1.014	1.015	1.015	1.016
60	1.006	1.006	1.006	1.006	1.007	1.007	1.007	1.007	1.008	1.008
120	1.002	1.002	1.002	1.002	1.002	1.002	1.002	1.002	1.002	1.002
∞	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\chi^2_{pf_2}$	83.3079	88.2502	92.6885	98.0284	101.7760	92.1662	97.3510	101.9990	107.5820	111.4950

M \ α			$f_2 = 21$					$f_2 = 23$		
	.100	.050	.025	.010	.005	.100	.050	.025	.010	.005
1	2.585	2.610	2.646	2.703	2.750	2.639	2.666	2.705	2.764	2.813
2	1.504	1.534	1.562	1.598	1.624	1.533	1.564	1.593	1.630	1.657
3	1.367	1.384	1.401	1.422	1.437	1.391	1.410	1.428	1.449	1.465
4	1.287	1.299	1.311	1.325	1.335	1.309	1.322	1.334	1.349	1.359
5	1.234	1.243	1.252	1.262	1.270	1.253	1.263	1.272	1.283	1.291
6	1.196	1.203	1.210	1.218	1.224	1.213	1.221	1.228	1.237	1.243
7	1.167	1.173	1.179	1.186	1.191	1.183	1.189	1.195	1.202	1.208
8	1.145	1.150	1.155	1.160	1.164	1.159	1.165	1.170	1.176	1.180
9	1.127	1.132	1.136	1.140	1.144	1.140	1.145	1.149	1.154	1.158
10	1.113	1.117	1.120	1.124	1.127	1.125	1.129	1.132	1.137	1.140
12	1.091	1.094	1.096	1.100	1.102	1.100	1.103	1.106	1.109	1.112
14	1.074	1.077	1.079	1.081	1.083	1.083	1.086	1.088	1.091	1.093
16	1.063	1.064	1.066	1.068	1.070	1.070	1.072	1.074	1.076	1.078
18	1.053	1.055	1.057	1.058	1.060	1.060	1.062	1.063	1.065	1.067
20	1.046	1.048	1.049	1.050	1.051	1.052	1.054	1.055	1.057	1.058
24	1.036	1.037	1.038	1.039	1.040	1.040	1.042	1.043	1.044	1.045
30	1.026	1.026	1.027	1.028	1.028	1.029	1.030	1.031	1.032	1.032
40	1.016	1.017	1.017	1.018	1.018	1.019	1.019	1.020	1.020	1.021
60	1.008	1.009	1.009	1.009	1.009	1.010	1.010	1.010	1.010	1.011
120	1.002	1.003	1.003	1.003	1.003	1.003	1.003	1.003	1.003	1.003
∞	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\chi^2_{pf_2}$	100.9800	106.3950	111.2420	117.0570	121.1260	109.7560	115.3900	120.4270	126.4620	130.6810

p = number of variates; f_1 = error degrees of freedom; f_2 = hypothesis degrees of freedom; $M = f_1-p+1$; C = [percentile for $-\{f_1-\frac{1}{2}(p-f_2+1)\} \log_e U\}$]/(percentile for χ^2 with pf_2 degrees of freedom).

Table 3. Chi-square adjustments to Wilks's criterion $_2U$.
Factor C for lower percentiles of U (upper percentiles of χ^2), p = 5

M \ α	$f_2 = 12$ .050	$f_2 = 12$ .010	$f_2 = 14$ .050	$f_2 = 14$ .010	$f_2 = 16$ .050	$f_2 = 16$ .010
1	1.643	1.768	1.683	1.813	1.722	1.855
2	1.350	1.396	1.383	1.431	1.415	1.465
3	1.240	1.265	1.267	1.294	1.294	1.323
4	1.179	1.196	1.203	1.221	1.226	1.245
5	1.141	1.153	1.161	1.174	1.181	1.196
6	1.114	1.124	1.132	1.143	1.150	1.161
7	1.095	1.103	1.111	1.119	1.127	1.136
8	1.081	1.087	1.095	1.102	1.109	1.116
9	1.070	1.075	1.082	1.088	1.095	1.101
10	1.061	1.065	1.072	1.077	1.083	1.089
12	1.047	1.051	1.057	1.060	1.066	1.070
14	1.038	1.040	1.045	1.048	1.053	1.057
16	1.031	1.033	1.038	1.040	1.045	1.047
18	1.026	1.028	1.032	1.034	1.038	1.040
20	1.022	1.024	1.027	1.029	1.033	1.034
24	1.017	1.018	1.021	1.022	1.025	1.026
30	1.012	1.012	1.014	1.015	1.018	1.019
40	1.007	1.008	1.009	1.010	1.011	1.012
60	1.004	1.004	1.004	1.005	1.006	1.006
120	1.001	1.001	1.001	1.001	1.002	1.002
∞	1.000	1.000	1.000	1.000	1.000	1.000
$\chi^2_{pf_2}$	79.0819	88.3794	90.5312	100.4250	101.8790	112.3290

p = number of variates; f_1 = error degrees of freedom; f_2 = hypothesis degrees of freedom; $M = f_1-p+1$; C = [percentile for $-\{f_1-\frac{1}{2}(p-f_2+1)\log_e U\}$]/(percentile for χ^2 with pf_2 degrees of freedom).

Table 4. Chi-square adjustments to Wilks's criterion U.

Factor C for lower percentiles of U (upper percentiles of χ^2), p = 6

M \ α	$f_2 = 11$		$f_2 = 12$		$f_2 = 13$	
	.050	.010	.050	.010	.050	.010
1	1.589	1.704	1.605	1.722	1.621	1.739
2	1.321	1.363	1.335	1.378	1.349	1.393
3	1.220	1.243	1.232	1.255	1.244	1.268
4	1.164	1.180	1.175	1.191	1.185	1.201
5	1.129	1.140	1.138	1.150	1.148	1.159
6	1.105	1.114	1.113	1.122	1.121	1.130
7	1.088	1.094	1.095	1.102	1.102	1.109
8	1.074	1.080	1.081	1.086	1.088	1.093
9	1.064	1.069	1.696	1.075	1.075	1.080
10	1.055	1.060	1.061	1.065	1.066	1.070
12	1.044	1.047	1.048	1.051	1.052	1.055
14	1.035	1.037	1.038	1.040	1.042	1.044
16	1.029	1.030	1.032	1.033	1.035	1.037
18	1.024	1.026	1.027	1.028	1.029	1.031
20	1.020	1.022	1.023	1.024	1.025	1.026
24	1.015	1.016	1.017	1.018	1.019	1.020
30	1.011	1.011	1.012	1.013	1.013	1.014
40	1.007	1.007	1.007	1.008	1.008	1.009
60	1.003	1.003	1.004	1.004	1.004	1.004
120	1.001	1.001	1.001	1.001	1.001	1.001
∞	1.000	1.000	1.000	1.000	1.000	1.000
$\chi^2_{pf_2}$	85.9649	95.6257	92.8083	102.8160	99.6169	109.9580

p = number of variates; f_1 = error degrees of freedom; f_2 = hypothesis degrees of freedom; $M = f_1-p+1$; C = [percentile for $-\{f_1-\frac{1}{2}(p-f_2+1)\log_e U\}$]/(percentile for χ^2 with pf_2 degrees of freedom).

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