SOME CENTRAL AND NONCENTRAL DISTRIBUTION PROBLEMS IN MULTIVARIATE ANALYSIS*

by

Arjun Kumar Gupta

Purdue University

Department of Statistics

Division of Mathematical Sciences

Mimeograph Series No. 139

January 1968

* This research was supported by the National Science Foundation, Grant. No. GP-7663.

CHAPTER I

BACKGROUND AND PRELIMINARIES

1.1 General Background

In multivariate analysis, we generally wish to test three hypotheses, namely,

(I) that of equality of the dispersion matrices of two p-variate normal populations,

(II) that of equality of the p-dimensional mean vectors for ℓ p-variate normal populations (which is mathematically identical with the general problem of multivariate analysis of variance of means); and

(III) that of independence between a p-set and a q-set of variates in a (p+q) - variate normal population, with $p \leq q$.

All tests proposed so far for these hypotheses have been shown to depend, when the hypotheses to be tested are true, only on the characteristic roots of matrices based on sample observations. For example, in case (I), all the tests proposed so far are based on the characteristic roots of the matrix $S_1(S_1+S_2)^{-1}$, where S_1 and S_2 denote the sum of product (S.P.) matrices and where both are almost everywhere positive definite (a.e.p.d.). Thus $S_1(S_1+S_2)^{-1}$ is a.e.p.d., whence it follows that all the p characteristic roots are greater than zero and less than unity. In case (II), the matrix is $S^*(S^*+S)^{-1}$, where S^* denotes the "between" S.P. matrix of means weighted by the sample sizes and S denotes the "within" S.P. matrix (pooled from the S.P. matrices of ℓ samples). Then S is a.e.p.d., and S* is at least positive semidenfinite of rank $s = \min(p, \ell-1)$. Thus, a.e., s of the characteristic roots are greater than zero and less than unity and the p-s remaining roots are zero. In case(III), the matrix is $S_{11}^{-1}S_{12}S_{22}^{-1}S_{12}'$, where S_{11} is the S. P. matrix of the sample of observations on the p-set of variates, S_{22} that on the q-set, and S_{12}, the S.P. matrix between the observations on the p-set and those on the q-set. If $p \leq q$ and $p + q < k$, where k is the sample size, then a.e. the p characteristic roots of this matrix are greater than zero and less than unity.

In other words, these characteristic roots, form the sample functions, using which all tests in multivariate analysis, are constructed. So, their discussion for introductory purposes is perhaps mandatory.

Consider the following determinantal equations,

$$|A - \theta (A + B)| = 0 \tag{1.1.1}$$

and

$$|A - \lambda B| = 0 \tag{1.1.2}$$

where A and B are independent S.P. matrices, based on n_1 and n_2 sample sizes respectively and can be defined differently for different hypotheses (I), (II) and (III), as mentioned above. In each case, if the hypothesis to be tested is true, the $s \leq p$ nonzero roots θ_i, where $0 < \theta_1 \leq \theta_2 \leq \cdots \leq \theta_s < 1$, have the same joint distribution, the form of which was given by Roy (1939), Hsu (1939) and Fisher (1939). The distribution can be written in the form

$$(1.1.3) \qquad C(s,m,n) \prod_{i=1}^{s} \theta_i^m (1-\theta_i)^n \prod_{i>j} (\theta_i-\theta_j) \prod_{i=1}^{s} d\theta_i$$

$$0 < \theta_1 \leq \cdots \leq \theta_s < 1 ,$$

where

$$(1.1.4) \qquad C(s,m,n) = \frac{\Pi^{s/2} \prod_{i=1}^{s} \Gamma\left(\frac{2m+2n+s+i+2}{2}\right)}{\prod_{i=1}^{s} \Gamma\left(\frac{2m+i+1}{2}\right) \Gamma\left(\frac{2n+i+1}{2}\right) \Gamma\left(\frac{i}{2}\right)}$$

Here m and n are to be interpreted differently for the different situations. For example, in case (I), with n_1 and n_2 as the sample sizes,

$$(1.1.5) \qquad m = \tfrac{1}{2}(n_1-p-2) , \qquad n = \tfrac{1}{2}(n_2-p-2) .$$

In case (II), with N the total of the sizes of ℓ samples,

$$(1.1.6) \qquad m = \tfrac{1}{2}(|\ell-p-1|-1) , \qquad n = \tfrac{1}{2}(N-\ell-p-1) .$$

In case (III),

$$(1.1.7) \qquad m = \tfrac{1}{2}(q-p-1) , \qquad n = \tfrac{1}{2}(k-p-q-2) .$$

The corresponding joint distribution of λ_i's$(\lambda_i = \frac{\theta_i}{1-\theta_i}$, $i=1,\ldots,s)$ under the respective hypotheses, is given by

$$(1.1.8)\quad C(s,m,n) \prod_{i=1}^{s} \lambda_i^m (1+\lambda_i)^{-(m+n+s+1)} \prod_{i>j} (\lambda_i-\lambda_j) \prod_{i=1}^{s} d\lambda_i$$

$$0 < \lambda_1 \leq \cdots \leq \lambda_s < \infty .$$

where m,n and $C(s,m,n)$ are defined as above.

Nanda (1948 a) has shown that if $\xi_i = n\,\theta_i (i=1,\ldots,s)$, then the limiting distribution of ξ_i's as n tends to infinity is given by

$$(1.1.9)\qquad k(s,m) \prod_{i=1}^{s} \xi_i^m e^{-\xi_i} \prod_{i>1} (\xi_i-\xi_j) \prod_{i=1}^{s} d\xi_i$$

$$0 < \xi_1 \leq \cdots \leq \xi_s < \infty$$

where

$$(1.1.10)\qquad k(s,m) = \Pi^{s/2} / \left[\prod_{i=1}^{s} \Gamma\left(\frac{2m+i+1}{2}\right) \Gamma\left(\frac{i}{2}\right)\right] .$$

The distribution (1.1.9) can also be arrived at as that of $\xi_i = \frac{1}{2}\gamma_i$ $(i=1,\ldots,s)$ where γ_i's are the roots of the equation $|S-\gamma\,\Sigma| = 0$ where S is the variance - covariance matrix computed from a sample taken from an s-variate normal population with dispersion matrix Σ .

1.2 Statistic Proposed for Tests of Hypotheses (I), (II) and (III)

We list below the statistics based on the characteristic roots θ_i, λ_i or ξ_i, which can be used to test the hypotheses (I), (II) and (III) with the suitable choice of the independent S.P. matrices A and B.

(i) Roy's (1945, 1953) criteria of the largest root $\theta_s(\lambda_s)$ and the smallest root $\theta_1(\lambda_1)$.

(ii) Hotelling's (1951) T_o^2 statistic defined as

$$T_o^2 = (n_2-1)\,\mathrm{tr}B^{-1}A = (n_2-1)\sum_{i=1}^{s}\lambda_i = (n_2-1)\sum_{i=1}^{s}\left(\frac{\theta_i}{1-\theta_i}\right) \tag{1.2.1}$$

(iii) Wilks' (1932) Λ-criterion defined as

$$\Lambda = \frac{|B|}{|A+B|} = \prod_{i=1}^{s}(1-\theta_i) = \prod_{i=1}^{s}(1+\lambda_i)^{-1} \tag{1.2.2}$$

(iv) Pillai's (1954) V-statistic defined as

$$V = \mathrm{tr}\,[(A+B)^{-1}A] = \sum_{i=1}^{s}\theta_i = \sum_{i=1}^{s}\left(\frac{\lambda_i}{1+\lambda_i}\right) \tag{1.2.3}$$

(v) Finally, we propose the statistic $W_2^{(s)}$ defined as

$$W_2^{(s)} = \sum_{i<j}\xi_i\,\xi_j \quad . \tag{1.2.4}$$

Arguments for the use of these test criteria are that the characteristic roots (i) are invariant under all linear transformations of the variates, (ii) are unaffected by a change in the unit of measurement, and (iii) are independent of the magnitudes of the population variances and covariances.

Nanda (1948 a) gave the joint limiting form of (1.1.3) which we have listed in (1.1.9). Following him, the joint limiting form of (1.1.8) is easily proved also to be the same as (1.1.9) by setting $\xi_i = n\lambda_i$ in (1.1.8) and then letting $n \to \infty$. This fact that the joint limiting forms of both (1.1.3) and (1.1.8) are the same enables us to conclude that the limiting distributions of the statistics T_o^2 and V will be the same except for the constant multiplier. The same can be said in the case of Roy's statistic.

No great headway has been made so far in finding the distribution of the various statistics defined above. The classical T_o^2 is known (Rao, 1952) to be distributed, under the null hypotheses, as central chi-square with $(n_1-1)p$ degrees of freedom (d.f.). In the case of non-centrality parameter $\delta^2 \neq 0$, the classical T_o^2 is a non-central chi-square distributed with $(n_1-1)p$ d.f.. The exact distribution of studentized T_o^2 for both central and noncentral cases is not known in compact standard form. Ito (1956) has given, under the null hypotheses, its approximate distribution as an asymptotic expression of chi-square each with $(n_1-1)p$ d.f..

Wilks (1932) has given the exact distribution of Λ for $n = 1,2$ and any p, and for $p = 1,2$ and any n, by comparing the moments of Λ with those of F-ratio. Bartlett (1938), Rao (1948), Box (1949) have suggested different approximations but the exact distribution and its tabular values are not yet completely available.

Roy (1942) obtained the distribution of the largest, smallest and any intermediate one of the roots of the determinantal equation (1.1.1). Nanda (1948 a) gave a different method for deriving these distributions for $s = 2(1)5$. The limiting forms of these distributions were given by Nanda (1948 b). Pillai (1954) gave exact expressions for the distribution of the largest root up to $s = 10$ and also an approximation which he generalized (1965, 1967). He (1960) published tables for $s = 2(1)5$, $m = 0(1)4$ and $n = 5$ to 1,000 which he later (1967) extended up to $s = 20$.

Pillai (1954, 1955, 1960) has given an approximation to his statistic V and has tabulated it for $s = 2(1)5$, $m = -.5(.5)5(5)80$ and $n = 5(5)80$. Nanda (1950) has also given the exact distribution for the special case when $m = 0$.

We shall be concerned here with the distributions of Λ and $W_2^{(s)}$ in the null and the non null case.

1.3 Recent Advances

The theory of multivariate analysis took a new turn when matrix variates were introduced as arguments. A.T. James (1954, 1955, 1960, 1961) used the theory of averaging over orthogonal groups, zonal polynomials, Bessel functions and hypergeometric functions with matrix argument and derived (a) non-central Wishart distribution (1955),
(b) distribution of the latent roots of the covariance matrix (1960),
(c) distribution of non-central means with known covariances (1961).

Constantine (1963) discovered the power series representation of hypergeometric functions with matrix argument and using the Laplace transform gave an alternate derivation of (a), (b) and (c). He also derived the distribution of canonical correlation coefficients in the general case. More recently, Constantine (1966) obtained the non-central distribution of Hotelling's generalized T_0^2 but it converges only in $[0,1)$.

Schotzoff (1966) gave a form for the distribution of Wilks' Λ using convolution operation. But he did not give the exact distribution explicitly for $p > 2$ and his tables of percentage points were restricted such that $pf_2 \leq 70$. Consul (1966), using inverse Mellin transform; gave exact distribution of Λ pr p=3 and 4 which involve hypergeometric functions. Present investigations have extended these results giving explicit distribution for $p = 3(1)6$ in finite series form (except when p and f_2 are both odd, in which case the series is infinite) and enabling computations of percentiles overcoming the barrier $pf_2 \leq 70$.

1.4 Mathematical Glossary

Some known results, which are used frequently in this thesis, are given in this section. These results, stated without proof, are given in brief in order to ease and make clear subsequent use of them and checking of conditions for their use.

1.4.1 Vandermonde's Determinant

Let us consider a type of determinant (due to Vandermonde) which plays an important role in the development of Chapter II. Denote by V_o the Vandermonde's determinant of the form

$$(1.4.1.1)\quad V_o = \begin{vmatrix} x_p^{p-1} & x_p^{p-2} & \cdots & x_p & 1 \\ x_{p-1}^{p-1} & x_{p-1}^{p-2} & \cdots & x_{p-1} & 1 \\ \cdots & \cdots & \cdots & \cdots & . \\ x_2^{p-1} & x_2^{p-2} & \cdots & x_2 & 1 \\ x_1^{p-1} & x_1^{p-2} & \cdots & x_1 & 1 \end{vmatrix}$$

where $x_1,\ldots,x_p$ are p variables. The determinant can be shown to be equal to the expression

$$(1.4.1.2)\qquad V_o = \prod_{i>j}^{p} (x_i - x_j) \quad .$$

The determinant V_o has several interesting properties. In Chapter II we are interested in the product of a Vandermonde determinant and the powers of elementary symmetric functions. To this effect we have a very useful Lemma due to Pillai (1964), which we state below.

Lemma 1. Let $D(g_s, g_{s-1},\ldots, g_1)$, $(g_j \geq 0,\ j=1,2,\ldots,s)$, denote the determinant

$$(1.4.1.3)\quad D(g_s, g_{s-1}, \ldots, g_1) = \begin{vmatrix} x_s^{g_s} & x_s^{g_{s-1}} & \cdots & x_s^{g_1} \\ \vdots & & & \\ x_1^{g_s} & x_1^{g_{s-1}} & \cdots & x_1^{g_1} \end{vmatrix} .$$

If $a_r (r \leq s)$ denotes the rth elementary symmetric function in s x's , then

(i)

$$(1.4.1.4)\quad a_r\, D(g_s, g_{s-1}, \ldots, g_1) = \sum{}' \; D(g_s', g_{s-1}', \ldots, g_1') ,$$

where $g_j' = g_j + \delta$, $j = 1,2,\ldots,s$, $\delta = 0,1$ and Σ' denotes the sum over the $\binom{s}{r}$ combinations of s g's taken r at a time for which r indices $g_j' = g_j + 1$ such that $\delta = 1$ while for other indices $g_j' = g_j$ such that $\delta = 0$.

(ii)

$$(1.4.1.5)\quad a_r\, a_h\, D(g_s, g_{s-1}, \ldots, g_1) = \sum{}'' \; D(g_s'', g_{s-1}'', \ldots, g_1'') ,$$

where $h \leq s$, $g_j'' = g_j' + \delta$, $j = 1,2,\ldots,s$, $\delta = 0, 1$ and Σ'' denotes summation over the $\binom{s}{r}\binom{s}{h}$ terms obtained by taking h at a time of the s g's in each D in Σ' in (1.4.1.4) for which h indices $g_j'' = g_j' + 1$ while for other indices $g_j'' = g_j'$.

(iii) $(a_r)^k (a_h)^{\ell}\, D(g_s, g_{s-1}, \ldots, g_1)$, $(k, \ell \geq 0)$ can be expressed as a sum of $\binom{s}{r}^k \binom{s}{h}^\ell$ determinants obtained by performing on $D(g_s, g_{s-1}, \ldots, g_1)$ in any order (i) k times and (i) ℓ times with $r = h$.

However, if at least two of the indices in any determinant are equal, the corresponding term in the summation vanishes.

1.4.2 Generalized Hypergeometric Functions

The generalized hypergeometric function ${}_pF_q$ is Pachhammer's notation for the series

$$(1.4.2.1) \quad {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!}$$

where the hypergeometric coefficient $(a)_k$ is given by

$$(a)_k = a(a+1) \ldots (a+k-1) \quad .$$

${}_pF_q$ is a function of the real or complex numbers $a_1 \ldots a_p$, $b_1 \ldots b_q$.

The multivariate distributions involve a generalization of this function to the case in which the variable x is replaced by a symmetric matrix $S(m \times m)$ and ${}_pF_q$ is real or complex valued symmetric function of the latent roots of S.

We take our definition of the general system of hypergeometric functions of matrix argument as the power series representation discovered by Constantine (1963). As we are dealing with symmetric functions of m variables, the power series can be expanded in terms of one of the types of symmetric polynomials. For any such type of basis of the symmetric polynomials, the individual homogeneous polynomials of degree k are usually indexed by partitions $\kappa = (k_1, k_2, \ldots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, $k_1 + \cdots + k_m = k$, of k into not more than m parts. Hence whereas in the case of a single variable, we sum over all integers k, in the case of a matrix variable, we sum over all partitions κ of all integers. While in theory any basis of the symmetric polynomials would do, in practice a colossal simplification of the coefficients is achieved if certain homogeneous symmetric polynomials, $C_\kappa(S)$, called zonal polynomials, described in the next section, are used.

The hypergeometric functions which appear in the distributions of the matrix variates are given by Constantine (1963).

Definition 1.

$$(1.4.2.2) \quad {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; S) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa(S)}{k!}$$

$a_1 \ldots a_p$, $b_1, \ldots, b_q$ are real or complex constants and the multivariate hypergeometric coefficient $(a)_\kappa$ is given by

$$(a)_\kappa = \prod_{i=1}^{m} (a - \tfrac{1}{2}(i-1))_{k_i}$$

Definition 2.

$$(1.4.2.3) \quad {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; S, T) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa(S) C_\kappa(T)}{C_\kappa(I_m) k!}$$

Relationship between hypergeometric functions, zonal polynomials and Laguerre polynomials has been discussed in later sections.

1.4.3 Zonal Polynomials

The definition of zonal polynomials requires a few concepts from group representation theory. Let V_k be the vector space of homogeneous polynomials $\varphi(S)$ of degree k in the $n = \frac{1}{2} m(m+1)$ different elements of the $m \times m$ symmetric matrix S. The dimension N of V_k is the number $N = (n-k-1)! / (n-1)! \, k!$ of monomials

$$\prod_{i \leq j}^{m} S_{ij}^{k_{ij}} , \quad \text{of degree} \quad \sum_{i \leq j}^{m} k_{ij} = k .$$

Corresponding to any congruence transformation

$$(1.4.3.1) \qquad S \to L S L'$$

by a non-singular $m \times m$ matrix L, we can define a linear transformation of the space V_k of polynomials $\varphi(S)$, namely

$$(1.4.3.2) \qquad \varphi \to L\varphi : (L\varphi)(S) = \varphi(L^{-1} S L^{-1'})$$

A subspace $V' \subset V_k$ is called invariant if $LV' \subset V'$ for all non-singular matrices L. V' is called an irreducible invariant subspace if it has no proper invariant subspace. Thrall (1942), Theorem 3, p. 378 proved that V_k decomposes into a direct sum of irreducible invariant subspaces V_κ corresponding to each partition κ of k into not more than m parts

$$(1.4.3.3) \qquad V_k = \bigoplus_{\kappa} V_\kappa .$$

The polynomial $(\mathrm{tr}\ S)^k \in V_k$ then has a unique decomposition

$$(1.4.3.4) \qquad (\mathrm{tr}\ S)^k = \sum_{\kappa} C_\kappa(S)$$

into polynomials, $C_\kappa(S) \in V_\kappa$, belonging to the respective invariant subspaces.

The zonal polynomial $C_\kappa(S)$ is defined as the component of $(\mathrm{tr}\ S)^k$ in the subspace V_κ. It is a symmetric homogeneous polynomial of degree k in the latent roots of S.

Equation (1.4.3.4) holds for all m, and the zonal polynomials look the same for all m, but if the partition κ has more than m parts, the corresponding zonal polynomial $C_\kappa(S)$ will be identically zero.

Zonal polynomials, denoted by $Z_\kappa(S)$ because they are given a different normalizing constant, are listed up to $k = 6$ in James (1964). Zonal polynomials for $k = 7$ to 11 (unpublished) were communicated to us by James. General methods of calculating them have been described by James (1964). The $C_\kappa(S)$ and $Z_\kappa(S)$ are related as

$$(1.4.3.5) \quad C_\kappa(S) = [\, \chi_{[2\kappa]}(1)\, 2^k\, k! \,/\, (2k)! \,]\, Z_\kappa(S)$$

where $\chi_{[2\kappa]}(1)$ is the dimension of the representation $[2\kappa]$ of the symmetric group on $2k$ symbols. It is found by substituting $(2\kappa) = (2k_1, \ldots 2k_p)$ for $\kappa = (k_1, \ldots k_p)$ in the well known formula (Weyl (1946), p. 213. Theorem 7.7.B) that

$$(1.4.3.6)\quad \chi_{[\kappa]}(1) = k!\prod_{i<j}^{p}(k_i - k_j - i + j)/\prod_{i=1}^{p}(k_i+p-i)!$$

From equation (38) of Constantine (1963), namely

$$(1.4.3.7)\qquad Z_\kappa(I_m) = 2^k(\tfrac{1}{2}m)_\kappa$$

we have the value of the zonal polynomial at the unit matrix;

$$(1.4.3.8)\quad C_\kappa(I_m) = 2^{2k}\, k!\, (\tfrac{1}{2}m)_\kappa \prod_{i<j}^{p}(2k_i - 2k_j - i+j)/\prod_{i=1}^{p}(2k_i + p-i)\,!$$

Note that if $m = 1$, equation (1.4.3.4) which defines the zonal polynomials becomes $x^k = C_{(k)}(x)$. Thus zonal polynomials of a matrix variable are analogous to powers of a single variable.

1.4.4 Generalized Laguerre Polynomials

Let S be a complex symmetric matrix. The generalized Laguerre polynomials are polynomials in the elements of S and are extensions of the classical Laguerre polynomials, to which they reduce when $m = 1$. Many of the results for the classical polynomials generalize to the case of matrix variables. The reader is referred to Chapter 10 of "Higher Transcendental Functions" by Erdelyi et al. for the case $m = 1$, especially Section 12.

For each Romogeneous, symmetric polynomial $\sigma(R)$ in the $m \times m$ matrix R, Herz (1955) defines the function $L_\sigma^\gamma(S)$ by

$$(1.4.4.1)\quad e^{-\mathrm{tr}\ S}\, L_\sigma^\gamma(S) = \int_{R>0} e^{-\mathrm{tr}\ R}\, |R|^\gamma\, \sigma(R)\, A_\gamma(RS)\, dR\ ,$$

where $\gamma > -1$ and the generalized "Bessel" function $A_\gamma(R)$ has the expansion (Constantine 1963)

$$(1.4.4.2)\quad A_\gamma\ (R) = [1/\Gamma_m(\gamma+\tfrac{1}{2}(m+1))] \sum_{k=0}^{\infty} [C_\kappa(-R)/(\gamma+\tfrac{1}{2}(m+1))_\kappa\ k!]\ ,$$

where

$$(1.4.4.3)\qquad \Gamma_m(a,\kappa) = \Pi^{\frac{1}{4}m(m-1)}\ \Pi_{i=1}^{m}\ \Gamma(a+k_i - \tfrac{1}{2}\ (i-1))$$

and

$$(1.4.4.4)\qquad \Gamma_m\ (a) = \Gamma_m\ (a,0)$$

He showed that $L_\sigma^\gamma\ (S)$ is a polynomial of the same degree as σ .

Constantine (1966) took the zonal polynomials as a basis for symmetric functions, and defined

$$(1.4.4.5)\quad e^{-\mathrm{tr}\ S}\, L_\kappa^\gamma\ (S) = \int_{R>0} e^{-\mathrm{tr}\ R}\, |R|^\gamma\, C_\kappa(R)\, A_\gamma(RS)\, dR\ .$$

Now the Bessel function has the integral definition

$$(1.4.4.6)\quad A_\gamma(R) = [2^{\frac{1}{2}m(m-1)}/(2\pi i)^{\frac{1}{2}mp}] \int_{R(Z)>0} e^{\mathrm{tr}\ Z}\, e^{-\mathrm{tr}\ R\ Z^{-1}}\, |Z|^{-\gamma-p}\, d\ Z\ ,$$

where

$$p = \tfrac{1}{2}\,(m+1)\ .$$

Substituting (1.4.4.6) in (1.4.4.5) and reversing the order of integration,

$$(1.4.4.7)\quad L_{\kappa}^{\gamma}\,(S) = \Gamma_m(\gamma+p,\kappa)[2^{\frac{1}{2}m(m-1)}/(2\pi i)^{mp}]$$

$$\cdot \int_{R(Z)>o} e^{\mathrm{tr}\ Z}\ |Z|^{-\gamma-p}\ C_{\kappa}(I-SZ^{-1})\ dZ\quad .$$

Equation (1.4.4.7) allows the calculation of the Laguerre polynomials. The inverse Laplace transform of the zonal polynomial is (Constantine 1963)

$$(1.4.4.8)\quad [2^{\frac{1}{2}m(m-1)}/(2\pi i)^{\frac{1}{2}mp}] \int_{R(Z)>0} e^{\mathrm{tr}\ R\ Z}\ |Z|^{-\gamma}\ C_{\kappa}(Z^{-1})\ dZ$$

$$= [1/\Gamma_m(\gamma,\kappa)]\ |R|^{\gamma-p}\ C_{\kappa}\ (R)\quad .$$

Expanding $C_{\kappa}\ (I-S\ Z^{-1})$,

$$(1.4.4.9)\quad C_{\kappa}\ (I-S\ Z^{-1})/C_{\kappa}(I) = \sum_{n=o}^{k} \sum_{\nu} (-1)^n\ a_{\kappa,\nu}\ C_{\nu}(S\ Z^{-1})/C_{\nu}(I)$$

and performing integration in (1.4.4.7) using (1.4.4.8), we obtain

$$(1.4.4.10)\ L_{\kappa}^{\nu}\ (S) = (\gamma+p)_{\kappa}\ C_{\kappa}(I) \sum_{n=o}^{k} \sum_{\nu} (-1)^n\ [a_{\kappa,\nu}/(\gamma+p)_{\nu}][C_{\nu}(S)/C_{\nu}(I)]$$

An explicit formula for $a_{\kappa,\nu}$ is not known, but they may be readily calculated from (1.4.4.9). They are tabulated up to order $k = 4$ in Constantine (1966). (1.4.4.10) shows that, in general $L_\kappa^\nu(S)$ is a polynomial of degree k in S, unless S is singular when the degree may be less than k.

CHAPTER II

DISTRIBUTION OF $W_2^{(s)}$

2.1. Introduction

Distribution problems in multivariate anaylsis are often related to the joint distribution of the characteristic roots of a matrix derived from sample observations taken from multivariate normal populations. This joint distribution (under certain null hypotheses) of s non-null characteristic roots given by Fisher (1939), Girshick (1939), Hsu (1939), and Roy (1939) can be expressed in the form

$$(2.1.1) \quad f(\theta_1,\ldots,\theta_s)=C(s,m,n) \prod_{i=1}^{s} \theta_i^m(1-\theta_i)^n \prod_{i>j}(\theta_i-\theta_j)$$

$$0 < \theta_1 \leq \ldots \leq \theta_s < 1$$

where

$$(2.1.2) \quad C(s,m,n) = \pi^{s/2} \prod_{i=1}^{s} \Gamma(\frac{2m+2n+s+i+2}{2})/[\Gamma(\frac{2m+i+1}{2})\ \Gamma(\frac{2n+i+1}{2})\Gamma(i/2)]\ .$$

and m and n are defined differently for various situations described by Pillai (1955, 1960). Nanda (1948a) has shown that if $\xi_i=n\theta_i(i=1,\ldots,s)$, then the limiting distribution of ξ_i's as n tends to infinity is given by

$$(2.1.3) \qquad f_1(\xi_1,\xi_2,\ldots,\xi_s) = K(s,m) \prod_{i=1}^{s} \xi_i^m e^{-\xi_i} \prod_{i>j} (\xi_i-\xi_j)$$

$$0 < \xi_1 \leq \cdots \leq \xi_s < \infty$$

where

$$(2.1.4) \qquad K(s,m) = \Pi^{s/2}/[\prod_{i=1}^{s} \Gamma(\tfrac{2m+i+1}{2})\, \Gamma(i/2)].$$

The distribution (2.1.3) can also be arrived at as that of $\xi_i = \frac{1}{2}\gamma_i$ $(i=1,2,\ldots,s)$ where γ_i's are the roots of the equation $|S- \gamma\, \Sigma| = 0$ where S is the variance-covariance matrix computed from a sample taken from an s-variate normal population with dispersion matrix Σ. In this chapter the first four moments of $W_2^{(s)}$, the second elementary symmetric function (esf) in s ξ's, have been obtained and approximations to its distribution suggested. In addition, the variances of the third and fourth esf's are also obtained. An example is given to illustrate the use of $W_2^{(s)}$ as a test criterion.

2.2. Formulae for the First Four Moments of $W_2^{(s)}$

The joint distribution (2.1.3) can be thrown into a determinantal form of the Vandermonde type and integrated over the range R, $0 < \xi_1 \leq \cdots \leq \xi_s < \infty$, giving

$$(2.2.1)\quad \int_R f_1(\xi_1,\xi_2,\ldots,\xi_s)\prod_{i=1}^{s} d\xi_i = K(s,m)\begin{vmatrix} \int_0^\infty \xi_s^{m+s-1} e^{-\xi_s} d\xi_s \cdots \int_0^\infty \xi_s^{m} e^{-\xi_s} d\xi_s \\ \\ \int_0^{\xi_2} \xi_1^{m+s-1} e^{-\xi_1} d\xi_1 \cdots \int_0^{\xi_2} \xi_1^{m} e^{-\xi_1} d\xi_1 \end{vmatrix} .$$

Now denote by $W(s-1,s-2,\ldots,1,0)$ the determinant on the right side of (2.2.1). Using Lemma 1 in (Pillai, 1964), the first four moments of $W_2^{(s)}$ can be obtained as follows: (denoting $E(W_2^{(s)})^r$ by μ_r') .

$$(2.2.2)\qquad \mu_1' = K(s,m)\, W(s,s-1,s-3,\ldots,1,0)$$

$$(2.2.3)\qquad \mu_2' = K(s,m)[W(s+1,s,s-3,\ldots,1,0) + W(s+1,s-1,s-2,s-4,\ldots,1,0) + W(s,s-1,s-2,s-3,s-5,\ldots,1,0)]$$

$$(2.2.4)\qquad \mu_3' = K(s,m)[W(s+2,s+1,s-3,\ldots,1,0) + 2W(s+2,s,s-2,s-4,\ldots,1,0) + 3W(s+1,s,s-2,s-3,s-5,\ldots,1,0) + W(s+2,s-1,s-2,s-3,s-5,\ldots,1,0) + W(s+1,s,s-1,s-4,\ldots,1,0) + 2W(s+1,s-1,s-2,s-3,s-4,s-6,\ldots,1,0) + W(s,s-1,s-2,s-3,s-4,s-5,s-7,\ldots,1,0)]$$

and

$$(2.2.5)\quad \mu_4' = K(s,m)[W(s+3,s+2,s-3,\ldots,1,0) + 3W(s+3,s+1,s-2,s-4,\ldots,1,0)$$
$$+ 6W(s+2,s+1,s-2,s-3,s-5,\ldots,1,0) + 2W(s+3,s,s-1,s-4,\ldots,1,0)$$
$$+ 3W(s+3,s,s-2,s-3,s-5,\ldots,1,0) + 3W(s+2,s+1,s-1,s-4,\ldots,1,0)$$
$$+ 7W(s+2,s,s-1,s-3,s-5,\ldots,1,0)$$
$$+ 8W(s+2,s,s-2,s-3,s-4,s-6,\ldots,1,0)$$
$$+ 3W(s+1,s,s-1,s-2,s-5,\ldots,1,0)$$
$$+ 6W(s+1,s,s-1,s-3,s-4,s-6,\ldots,1,0)$$
$$+ 6W(s+1,s,s-2,s-3,s-4,s-5,s-7,\ldots,1,0)$$
$$+ W(s+3,s-1,s-2,s-3,s-4,s-6,\ldots,1,0)$$
$$+ 3W(s+2,s-1,s-2,s-3,s-4,s-5,s-7,\ldots,1,0)$$
$$+ 3W(s+1,s-1,s-2,s-3,s-4,s-5,s-6,s-8,\ldots,1,0)$$
$$+ W(s,s-1,s-2,s-3,s-4,s-5,s-6,s-7,s-9,\ldots,1,0)] .$$

2.3. A Method of Evaluation of the W-Determinants

Let us denote by $V(q_s,q_{s-1},\ldots,q_1)$ the determinant which could be obtained from $W(q_s,q_{s-1},\ldots,q_1)$ by replacing ξ_i by θ_i in (2.1.1) $e^{-\xi_i}$ by $(1-\theta_i)^n$ and the range of integration by that in (2.1.1). Pillai (1954,1956) has given a method of reducing the sth order determinant $V(q_s,q_{s-1},\ldots,q_1)$ in terms of (s-2)th order determinants and an sth order determinant with q_s changed to q_s-1, the last one being zero if $q_s-1 = q_{s-1}$. The method of reduction for $W(q_s,\ldots,q_1)$ can be deduced from that for $V(q_s,\ldots,q_1)$ in (Pillai, 1956) and we obtain the following:

$$(2.3.1)\quad W(q_s,q_{s-1},\ldots,q_1) = 2\sum_{j=s-1}^{1}(-1)^{s-j-1}I(q_s+q_j;2)W(q_{s-1},\ldots q_{j+1}, q_{j-1},\ldots,q_1) + (m+q_s)\,W(q_s-1,q_{s-1},\ldots,q_1)$$

where

(2.3.2) $$I(p;2) = \int_0^\infty x^p e^{-2x} dx = \Gamma(p+1)/2^{p+1} .$$

The values of the W-determinants involved in (2.2.2) - (2.2.5) are obtained using (2.3.1) and presented in the following section.

2.4. Values of the W-Determinants

Let us set

(2.4.1) $$(2m+a)(2m+b) \ldots = M(a,b,\ldots) .$$

Then for the first moment, we get

(2.4.2) $$K(s,m)\, W(s,s-1,s-3,\ldots,1,0) = s(s-1)\, M(s,s+1)/2^3 .$$

In fact, in general

(2.4.3) $$K(s,m)W(s,s-1,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0) = \binom{s}{i}M(s-i+2,\ldots,s+1)/2^i .$$

For the second raw moment, we get

(2.4.4) $$K(s,m)W(s+1,s,s-3,\ldots,1,0)$$
$$= \left[\binom{s}{2}M(s,s+1)/2^4 3!\right]\left[4s(s+1)m^2 + 2s(2s^2+5s+9)m+s^4 + 4s^3 + 11s^2 + 8s+12\right] .$$

(2.4.5) $$K(s,m)\, W(s+1,s-1,s-2,s-4,\ldots,1,0)$$
$$= \left[\binom{s}{3}\, M(s-1,s,s+1)/2^6\right]\left[2(3s-1)m+3s^2+s+10\right]$$
$$+ (m+s+1)\, K(s,m)\, W(s,s-1,s-2,s-4,\ldots,1,0) .$$

The last determinant on the right side of (2.4.5) is evaluated by putting $i = 3$ in (2.4.3). In general

(2.4.6) $$K(s,m)\, W(s+1,s-1,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0)$$
$$= \left[i\binom{s}{i}M(s-i+2,\ldots,s+1)/2^{i+1}(i+1)\right]\left[2(s+1)m+(s+1)(s+2)+i+1\right] .$$

$K(s,m)W(s,s-1,s-2,s-3,s-5,\ldots,1,0)$ is obtained from (2.4.3) by putting $i = 4$.

For the <u>third raw moment</u>, we get

(2.4.7) $$K(s,m)\, W(s+2,s+1,s-3,\ldots,1,0)$$
$$= [\binom{s+2}{4}M(s,s+1,s+2,s+3)/2^6\, 3!][4s(s+1)m^2$$
$$+ 2s(2s^2+5s+21)m+s^4+4s^3+23s^2+20s+72] .$$

In fact, in general

(2.4.8) $$K(s,m)\, W(s+2,s+1,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0)$$
$$= [i(i-1)\binom{s+2}{i+2}M(s-i+2,\ldots,s+3)/2^{i+5}3!][4s(s+1)m^2$$
$$+2s(2s^2+5s+4i+13)m + s(s+1)(s^2+3s+4i+12)+6(i+1)(i+2)] .$$

(2.4.9) $$K(s,m)W(s+2,s,s-2,s-4,\ldots,1,0)$$
$$= [\binom{s+1}{4}M(s-1,s,s+1,s+2)/2^5.15][2s(8s+1)m^2+s(16s^2+19s+109)m$$
$$+ 4s^4+9s^3+59s^2+54s+180]$$
$$+(m+s+2)K(s,m)\, W(s+1,s,s-2,s-4,\ldots,1,0) .$$

The value of the determinant in the last term on the right side of (2.4.9) is obtained by putting $i = 3$ in the following general result:

$$(2.4.10)\quad K(s,m)\, W(s+1,s,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0)$$
$$= [(i-1)\binom{s}{i}M(s-i+2,\ldots,s+1)/2^{i+3}(i+1)][4s(s+1)m^2$$
$$+2s(2s^2+5s+2i+5)m+s^4+4s^3+(2i+7)s^2+(2i+4)s+2i(i+1)] .$$

Now $K(s,m)\, W(s+1,s,s-2,s-3,s-5,\ldots,1,0)$ is obtained from (2.4.10) by putting $i = 4$.

$$(2.4.11)\quad K(s,m)\, W(s+2,s-1,s-2,s-3,s-5,\ldots,1,0)$$
$$= [\binom{s+1}{5}M(s-2,\ldots,s+2)/2^5 3!][2(5s-2)m+5s^2+s+42]$$
$$+ (m+s+2)\, K(s,m)\, W(s+1,s-1,s-2,s-3,s-5,\ldots,1,0) .$$

In fact, in general

$$(2.4.12)\quad K(s,m)\, W(s+j,s-1,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0)$$
$$= [\binom{i+j-2}{j-1}\binom{s+j-1}{i+j-1}\, M(s-i+2,\ldots,s+j)/2^{i+j}j(i+j)]$$
$$\times [2\{(i+j-1)s-j\}\, m+(i+j-1)s^2+(i-j-1)s+j(i-1)(i+j+1)]$$
$$+ (m+s+j)\, K(s,m)\, W(s+j-1,s-1,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0).$$

The value of the last determinant on the right side of (2.4.11) is obtained easily from (2.4.6) by putting $i = 4$.

$$(2.4.13)\quad K(s,m)\, W(s+1,s,s-1,s-4,\ldots,1,0)$$
$$= [\binom{s}{3}M(s-1,s,s+1)/2^6 4!][8(s-1)s(s+1)m^3+12(s-1)s(s^2+2s+5)m^2$$
$$+ 2(s-1)(3s^4+9s^3+32s^2+14s+72)m$$
$$+ s(s-1)(s^4+4s^3+17s^2+14s+72)+144] .$$

Now, $K(s,m)\,W(s+1,s-1,s-2,s-3,s-4,s-6,\ldots,1,0)$ is obtained from (2.4.6) by putting $i = 5$ and $K(s,m)W(s,s-1,s-2,s-3,s-4,s-5,s-7,\ldots,1,0)$ from (2.4.3) with $i = 6$.

For the <u>fourth raw moment</u> , we get

$$(2.4.14)\quad K(s,m)\,W(s+3,s+2,s-3,\ldots,1,0) = [\tbinom{s+2}{4}M(s,s+1,s+2,s+3)/2^8 5!]$$
$$\times [16s(s+1)(s+2)(s+3)m^4+8s(s+2)(s+3)(4s^2+14s+46)m^3$$
$$+ 4(s+1)(s+2)(6s^4+48s^3+233s^2+609s+720)m^2$$
$$+ 2(s+2)(4s^6+46s^5+310s^4+1320s^3+3542s^2+5802s+6480)m$$
$$+ (s+2)(s+3)(s^6+11s^5+81s^4+373s^3+1118s^2+2256s+4320)+2880] .$$

$$(2.4.15)\quad K(s,m)W(s+3,s+1,s-2,s-4,\ldots,1,0)$$
$$= [\tbinom{s+2}{5}M(s-1,\ldots,s+3)/2^8 4!][8s(s+1)(5s+3)m^3$$
$$+ 4s(15s^3+47s^2+189s+213)m^2+2(15s^5+70s^4+403s^3+966s^2+1842s+1440)m$$
$$+ 5s^6+31s^5+217s^4+769s^3+2210s^2+4512s+5760]$$
$$+ (m+s+3)K(s,m)W(s+2,s+1,s-2,s-4,\ldots,1,0) .$$

The value of the determinant in the last term of the right side of (2.4.15) is obtained from (2.4.8) by putting $i = 3$. $K(s,m)W(s+2,s+1,s-2,s-3,s-5,\ldots,1,0)$ is deduced from (2.4.8) with $i = 4$.

$$(2.4.16)\quad K(s,m)W(s+3,s,s-1,s-4,\ldots,1,0) = [\tbinom{s+2}{5}\, M(s-1,\ldots,s+3)/2^8 3!]$$
$$\times [8s^2(s-1)m^3+4s(s-1)(3s^2+2s+24)m^2+2(s-1)(3s^4+4s^3+49s^2+24s+180)m$$
$$+ s(s-1)(s^4+2s^3+25s^2+24s+180)+360]$$
$$+ (m+s+3)K(s,m)W(s+2,s,s-1,s-4,\ldots,1,0) .$$

The value of the determinant in the last term on the right side of (2.4.16) is obtained from the following result by putting $i = 3$.

$$(2.4.17) \quad K(s,m)W(s+2,s,s-1,s-3,\ldots,s-i+1,s-i-1,\ldots,1,0)$$
$$= [(i-2)\binom{s+1}{i+1} M(s-i+2,\ldots,s+2)/2^{i+4}(i+2)4!]$$
$$\times [8s(s-1)\{3(i+1)s+2(i-1)\}m^3+4s(s-1)\{9(i+1)s^2+2(7i-1)s + 2(6i^2+20i+4)\}m^2$$
$$+ 2(s-1)\{9(i+1)s^4+2(11i+1)s^3+(24i^2+85i+17)s^2+2(6i^2+19i+5)s + 24i(i+1)(i+2)\}m$$
$$+ s(s-1)\{3(i+1)s^4+2(5i+1)s^3+3(4i^2+15i+3)s^2+2(6i^2+19i+5)s + 24i(i+1)(i+2)\} + 24(i-1)i(i+1)(i+2)]$$
$$+ (m+s+2)K(s,m)W(s+1,s,s-1,s-3,\ldots,s-i+1,s-i-1,\ldots,1,0) ,$$

where the values of the last determinant on the right side of (2.4.17) is obtained from the following:

$$(2.4.18) \quad K(s,m)W(s+1,s,s-1,s-3,\ldots,s-i+1,s-i-1,\ldots,1,0)$$
$$= [(i-2)\binom{s}{i} M(s-i+2,\ldots,s+1)/2^{i+1}(i+1)4!]$$
$$\times [8(s+1)s(s-1)m^3+12s(s-1)(s^2+2s+i+2)m^2$$
$$+ 2(s-1)\{3s^4+9s^3+(6i+14)s^2+(3i+5)s+6i(i+1)\}m$$
$$+ s(s-1)\{s^4+4s^3+(3i+8)s^2+(3i+5)s+6i(i+1)\}+6(i-1)i(i+1)] .$$

$K(s,m)W(s+3,s,s-2,s-3,s-5,\ldots,1,0)$ is deduced from the following result by putting $i = 4$.

(2.4.19) $$K(s,m)W(s+3,s,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0)$$
$$= [(i^2-1)\binom{s+2}{i+2}M(s-i+2,\ldots,s+3)/2^{i+4}(i+3)4!]$$
$$\times [4s\{3(i+2)s+(i-6)\}m^2+2s\{6(i+2)s^2+3(3i-2)s+12i^2+49i-6\}m$$
$$+ s(s+1)\{3(i+2)s^2+(5i-6)s+12i(i+4)\}+12i(i+2)(i+3)]$$
$$+ (m+s+3)K(s,m)W(s+2,s,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0)$$

where the value of the determinant in the last term on the right side of (2.4.19) is given by

(2.4.20) $$K(s,m)W(s+2,s,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0)$$
$$= [(i-1)\binom{s+1}{i+1} M(s-i+2,\ldots,s+2)/2^{i+3}(i+2)3!]$$
$$\times [4s\{2(i+1)s+i-2\}m^2+2s\{4(i+1)s^2+(7i-2)s+(6i^2+19i-2)\}m$$
$$+ s(s+1)\{2(i+1)s^2+2(2i-1)s+6i(i+3)+6i(i+1)(i+2)]$$
$$+ (m+s+2)K(s,m)W(s+1,s,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0) ,$$

where the value of the determinant in the last term on the right side of (2.4.20) is obtained from (2.4.10).

(2.4.21) $$K(s,m)W(s+2,s+1,s-1,s-4,\ldots,1,0) = [\binom{s+2}{5}M(s-1,\ldots,s+3)/2^{11}]$$
$$\times [8(s+1)s(s-1)m^3+4s(s-1)(3s^2+6s+31)m^2$$
$$+2(s-1)(3s^4+9s^3+64s^2+30s+240)m+s(s-1)(s^4+4s^3+33s^2+30s+240)+480].$$

$K(s,m)W(s+2,s,s-1,s-3,s-5,\ldots,1,0)$ is obtained from (2.4.17) by putting $i = 4$ and $K(s,m)W(s+2,s,s-2,s-3,s-4,s-6,\ldots,1,0)$ from (2.4.20) with $i = 5$.

(2.4.22)
$$K(s,m)W(s+1,s,s-1,s-2,s-5,\ldots,1,0)$$
$$= [\tbinom{s}{4}M(s-2,\ldots,s+1)/2^8 5!]$$
$$\times [16(s+1)(s)(s-1)(s-2)m^4+16s(s-1)(s-2)(2s^2+3s+11)m^3$$
$$+ 4(s-1)(s-2)(6s^4+12s^3+65s^2-s+240)m^2$$
$$+ 2(s-2)(4s^6+6s^5+54s^4-68s^3+462s^2-698s+1680)m$$
$$+ s^8+14s^6-60s^5+269s^4-900s^3+2596s^2-4800s+5760] \ .$$

(2.4.23)
$$K(s,m)W(s+3,s,s-3,\ldots,1,0) = [\tbinom{s+2}{4}M(s,\ldots,s+3)/5.2^7]$$
$$\times [4s(3s-1)m^2+2s(6s^2+3s+35)m + s(s+1)(2s^2+39s-1)+120]$$
$$+ (m+s+3)K(s,m)W(s+2,s,s-3,\ldots,1,0) \ .$$

The value of the determinant in the last term on the right side of (2.4.23) is presented in the Appendix A.

$K(s,m)W(s+1,s,s-1,s-3,s-4,s-6,\ldots,1,0)$ is obtained from (2.4.18) by putting $i = 5$; $K(s,m)W(s+1,s,s-2,s-3,s-4,s-5,s-7,\ldots,1,0)$ from (2.4.10) with $i = 6$; $K(s,m)W(s+3,s-1,s-2,s-3,s-4,s-6,\ldots,1,0)$ from (2.4.12) by putting $j = 3$ and $i = 5$; $K(s,m)W(s+2,s-1,s-2,s-3,s-4,s-5,s-7,\ldots,1,0)$ from (2.4.12) by putting $j = 2$ and $i = 6$;

$K(s,m)W(s+1,s-1,s-2,s-3,s-4,s-5,s-6,s-8,\ldots,1,0)$ from (2.4.6) with $i = 7$; and $K(s,m)W(s,s-1,s-2,s-3,s-4,s-5,s-6,s-7,s-9,\ldots,1,0)$ from (2.4.3) by putting $i = 8$.

2.5. Moments of the Second, Third and Fourth ESF's

Using the values of the determinants evaluated in the preceding section, the raw moments of the second esf, $W_2^{(s)}$, are obtained as follows:

$$\mu_1'\{W_2^{(s)}\} = \binom{s}{2} M(s,s+1)/2^2 , \tag{2.5.1}$$

$$\mu_2'\{W_2^{(s)}\} = [\binom{s}{2}M(s,s+1)/2^5]s(s-1)(2m)^2+(2s^3-s^2+7s-8)(2m) + s^4+7s^2-8s+12] , \tag{2.5.2}$$

$$\begin{aligned}\mu_3'\{W_2^{(s)}\} = [\binom{s}{2}M(s,s+1)/2^8][&s^2(s-1)^2(2m)^4+2s(s-1)^2(2s^2+s+12)\\ &(2m)^3 + (s-1)(6s^5+67s^3-49s^2+172s-160)(2m)^2\\ &+ 2(2s^7-s^6+33s^5-47s^4+185s^3-314s^2+462s-368)(2m)\\ &+ s^8+22s^6-24s^5+173s^4-296s^3+764s^2-832s+672] ,\end{aligned} \tag{2.5.3}$$

and

$$(2.5.4)\qquad \mu_4'\{W_2^{(s)}\} = [\binom{s}{2}M(s,s+1)/2^{11}][s^3(s-1)^3(2m)^6$$

$$+ s^2(s-1)^2(6s^3-3s^2+45s-48)(2m)$$

$$+ s(s-1)^2(15s^5+228s^3-147s^2+808s-832)(2m)^4$$

$$+ (s-1)(20s^8-10s^7+452s^6-569s^5+3386s^4-5679s^3+10080s^2$$

$$-13440s+5376)(2m)^3$$

$$+ (15s^{10}-15s^9+453s^8-840s^7+5295s^6-12105s^5+32517s^4$$

$$-62960s^3+92536s^2-104048s+50304)(2m)^2$$

$$+ (6s^{11}-3s^{10}+225s^9-330s^8+3340s^7-7155s^6+27645s^5$$

$$-58688s^4+128240s^3-198448s^2+213456s-132480)(2m)$$

$$+ s^{12}+45s^{10}-48s^9+811s^8-1568s^7+8415s^6-18416s^5+54520s^4$$

$$-99776s^3+164304s^2-161280s+93312].$$

Using the raw moments given above, the first four central moments of $W_2^{(s)}$ are obtained in the following simple forms:

$$(2.5.5)\qquad \mu_2\{W_2^{(s)}\} = [\binom{s}{2}M(s,s+1)/2^3][4(s-1)m+2s^2-2s+3]\quad ,$$

$$(2.5.6)\qquad \mu_3\{W_2^{(s)}\} = [\binom{s}{2}M(s,s+1)/2^3][5(s-1)^2(2m)^2$$

$$+ (10s^3-20s^2+30s-23)(2m)+5s^4-10s^3+25s^2-26s+21]\quad ,$$

and

$$(2.5.7) \qquad \mu_4\{W_2^{(s)}\} = [3\tbinom{s}{2}M(s,s+1)/2^7][4s(s-1)^3(2m)^4$$
$$+ 4(s-1)^2(4s^3-3s^2+30s-28)(2m)^3$$
$$+ (24s^6-60s^5+408s^4-1056s^3+1833s^2-2173s+1048)(2m)^2$$
$$+ (16s^7-36s^6+384s^5-1036s^4+2634s^3-4209s^2+4503s-2760)(2m)$$
$$+ 4s^8-8s^3+124s^2-340s^5+1145s^4-2148s^3+3479s^2-3360s+1944].$$

Further, the results of the preceding section can also be used to obtain the first two raw moments (and hence the central moments) of $W_3^{(s)}$ and $W_4^{(s)}$, the third and fourth esf's respectively in the s, ξ's. It may be observed in general that

$$(2.5.8) \qquad \mu_1'\{W_i^{(s)}\} = K(s,m)W(s,s-1,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0)\ ,$$
$$i = 1,\ldots,s$$

and the value of the right side of (2.5.8) is given in (2.4.3). Now using the methods in section 2.3, we get

$$(2.5.9) \qquad \mu_2'\{W_3^{(s)}\} = K(s,m)[W(s+1,s,s-1,s-4,\ldots,1,0)$$
$$+ W(s+1,s,s-2,s-3,s-5,\ldots,1,0)$$
$$+ W(s+1,s-1,s-2,s-3,s-4,s-6,\ldots,1,0)$$
$$+ W(s,s-1,s-2,s-3,s-4,s-5,s-7,\ldots,1,0)]$$

and

(2.5.10) $$\mu_2'\{W_4^{(s)}\} = K(s,m)[W(s+1,s,s-1,s-2,s-5,\ldots,1,0) + W(s+1,s,s-1,s-3,s-4,s-6,\ldots,1,0) + W(s+1,s,s-2,s-3,s-4,s-5,s-7,\ldots,1,0) + W(s+1,s-1,s-2,s-3,s-4,s-5,s-6,s-8,\ldots,1,0) + W(s,s-1,s-2,s-3,s-4,s-5,s-6,s-7,s-9,\ldots,1,0)] .$$

It may be pointed out that the values of the determinants on the right side of (2.5.9) and (2.5.10) are available in the preceding section and using these values and (2.5.8), the variances of $W_3^{(s)}$ and $W_4^{(s)}$ were obtained and are given below.

(2.5.11) $$\mu_2\{W_3^{(s)}\} = [3\tbinom{s}{3}M(s-1,s,s+1)/2^6][(s-1)(s-2)(2m)^2 + (s-2)(2s^2-3s+7)2m+s^4-4s^3+11s^2-20s+20] ,$$

and

(2.5.12) $$\mu_2\{W_4^{(s)}\} = [\tbinom{s}{4}M(s-2,\ldots,s+1)/4!2^4][2(s-1)(s-2)(s-3)(2m)^3 + 3(s-2)(s-3)(2s^2-4s+11)(2m)^2 + (s-3)(6s^4-30s^3+106s^2-225s+314)2m + 2s^6-18s^5+89s^4-318s^3+845s^2-1500s+1368] .$$

2.6. Approximations to the Distribution of $W_2^{(s)}$

Using the results on moments of $W_2^{(s)}$ given in (2.5.1), (2.5.5), (2.5.6) and (2.5.7) the following approximation to the distribution of

$W_2^{(s)}$ is suggested:

$$(2.6.1) \qquad f(W_2^{(s)}) = \frac{\alpha^\nu}{2\Gamma(\nu)} e^{-\alpha(W_2^{(s)})^{\frac{1}{2}}} (W_2^{(s)})^{\frac{1}{2}\nu-1}, 0 < W_2^{(s)} < \infty \quad ,$$

where

$$(2.6.2) \qquad \nu = s(2m+s+1)/2$$

and

$$(2.6.3) \qquad \alpha^2 = 2[s(2m+s+1)+2]/(s-1)(2m+s) \quad .$$

It may be pointed out that the first moment is the same for the exact and approximate distributions. For further comparison, numerical values of the first four moments from the exact and approximate distributions and the ratios of the respective approximate and exact moments and the moment quotients are presented in Tables 1 to 2 for values of $s = 3,4,5,7$ and 10 and selected values of m. The tables show that the ratio of the respective approximate to the exact moments tends to unity as m increases or s increases or both. On the basis of these ratios the approximate distribution might be recommended for $m = 5$ and above when $s = 3$, $m = 3$ and above for $s = 4$, $m = 2$ and above for $s = 5$ and $m = 0$ and above for $s = 7$ and all values of m and all values of s beyond 7. The values of the approximate and exact standard deviations, β_1's and β_2's practically agree in the first two places at the smallest values of m recommended for each value of s and this in turn almost guarantees sufficient accuracy for upper or

Table 1

Ratios of moments (central) of $W_2^{(s)}$ from the exact and approximate distributions for $s = 3$ and different values of m

Moments	m = 2 Exact	m = 2 Approximate	Ratio (A/E)	m = 5 Exact	m = 5 Approximate	Ratio (A/E)
μ_1'	$.42000000 \times 10^{2}$	$.42000000 \times 10^{2}$	1.0000	$.13650000 \times 10^{3}$	$.13650000 \times 10^{3}$	1.0000
μ_2	$.65100000 \times 10^{3}$	$.61061538 \times 10^{3}$	.9380	$.37537500 \times 10^{4}$	$.36296590 \times 10^{4}$	.9669
μ_3	$.26271000 \times 10^{5}$	$.22869372 \times 10^{5}$	.8705	$.26433224 \times 10^{6}$	$.24565092 \times 10^{6}$	.9293
μ_4	$.31091445 \times 10^{7}$	$.25952045 \times 10^{7}$	.8347	$.74113715 \times 10^{8}$	$.67887768 \times 10^{8}$	.9160
$\sqrt{\mu_2}$	$.25514701 \times 10^{2}$	$.24710632 \times 10^{2}$	.9685	$.61267854 \times 10^{2}$	$.60246652 \times 10^{2}$	.9833
β_1	$.25015561 \times 10$	$.22972342 \times 10$	.9183	$.13210043 \times 10$	$.12619417 \times 10$	.9553
β_2	$.73363312 \times 10$	$.69604304 \times 10$	.9488	$.52597837 \times 10$	$.51529966 \times 10$	.9797

Moments	m = 10 Exact	m = 10 Approximate	Ratio (A/E)	m = 20 Exact	m = 20 Approximate	Ratio (A/E)
μ_1'	$.41400000 \times 10^{3}$	$.414000000 \times 10^{3}$	1.0000	$.14190000 \times 10^{4}$	$.14190000 \times 10^{4}$	1.0000
μ_2	$.19665000 \times 10^{5}$	$.19301351 \times 10^{5}$	.9815	$.12416250 \times 10^{6}$	$.12294470 \times 10^{6}$	.9902
μ_3	$.23687010 \times 10^{7}$	$.22738869 \times 10^{7}$	.9600	$.27374639 \times 10^{8}$	$.26789059 \times 10^{8}$	.9786
μ_4	$.16445728 \times 10^{10}$	$.15718388 \times 10^{10}$	.9558	$.564473347 \times 10^{11}$	$.55202852 \times 10^{11}$	.9780
$\sqrt{\mu_2}$	$.14023194 \times 10^{3}$	$.13892930 \times 10^{3}$	.9907	$.35236699 \times 10^{3}$	$.35063471 \times 10^{3}$	.9951
β_1	.73779996	.71907568	.9746	.39149430	.38617617	.9864
β_2	$.42527045 \times 10$	$.42192239 \times 10$	.9921	$.36615303 \times 10$	$.36520935 \times 10$	.9974

Moments	m = 40 Exact	m = 40 Approximate	Ratio (A/E)	m = 100 Exact	m = 100 Approximate	Ratio (A/E)
μ_1'	$.52290000 \times 10^{4}$	$.52290000 \times 10^{4}$	1.0000	$.31059000 \times 10^{5}$	$.31059000 \times 10^{5}$	1.0000
μ_2	$.87585749 \times 10^{6}$	$.87143137 \times 10^{6}$	.9949	$.12656542 \times 10^{8}$	$.12630491 \times 10^{8}$	.9979
μ_3	$.36828631 \times 10^{9}$	$.36421020 \times 10^{9}$	.9889	$.12915931 \times 10^{11}$	$.12857552 \times 10^{11}$	.9955
μ_4	$.25623342 \times 10^{13}$	$.25345972 \times 10^{13}$	.9892	$.502735540 \times 10^{15}$	$.50060149 \times 10^{15}$	.9958
$\sqrt{\mu_2}$	$.93587258 \times 10^{3}$	$.93350488 \times 10^{3}$	.9975	$.35576034 \times 10^{4}$	$.35539402 \times 10^{4}$	.9990
β_1	.20186954	.20044934	.9930	$.82282265 \times 10^{-1}$	$.82045702 \times 10^{-1}$	.9971
β_2	$.33401722 \times 10$	$.33376636 \times 10$	.9992	$.31384072 \times 10$	$.31379906 \times 10$	.9999

Table 2

Ratios of moments (central) of $W_2^{(s)}$ from the exact and approximate distributions for $s = 4, 5, 7$ and 10 and different values of m .

Moments	$s = 4 \quad m = 0$			$s = 4 \quad m = 3$		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	.30000000X10^{2}	.30000000X10^{2}	1.0000	.16500000X10^{3}	.16500000X10^{3}	1.0000
μ_2	.40500000X10^{3}	.37636364X10^{3}	.9293	.51975000X10^{4}	.50576086X10^{4}	.9731
μ_3	.14354999X10^{5}	.12228099X10^{5}	.8518	.41901749X10^{6}	.39426890X10^{6}	.9409
μ_4	.13799024X10^{7}	.11130166X10^{7}	.8066	.13880369X10^{9}	.12914122X10^{9}	.9304
$\sqrt{\mu_2}$	.20124611X10^{2}	.19400093X10^{2}	.9640	.72093689X10^{2}	.71116866X10^{2}	.9865
β_1	.31019966X10	.28047550X10	.9042	.12504917X10	.12015708X10	.9609
β_2	.84127571X10	.78575356X10	.9340	.51382120X10	.50486403X10	.9826

Moments	$s = 4 \quad m = 20$			$s = 4 \quad m = 100$		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	.29700000X10^{4}	.29700000X10^{4}	1.0000	.62730000X10^{5}	.62730000X10^{5}	1.0000
μ_2	.39649499X10^{6}	.39419406X10^{6}	.9942	.38484855X10^{8}	.38437464X10^{8}	.9988
μ_3	.13311094X10^{9}	.13137293X10^{9}	.9869	.59102856X10^{11}	.58938151X10^{11}	.9972
μ_4	.54702853X10^{12}	.53999866X10^{12}	.9871	.45958868X10^{16}	.45842690X10^{16}	.9975
$\sqrt{\mu_2}$	.62967848X10^{3}	.62784875X10^{3}	.9971	.62036163X10^{4}	.61997955X10^{4}	.9994
β_1	.28425910	.28176142	.9912	.61284040X10^{-1}	.61168642X10^{-1}	.9981
β_2	.34796418X10	.34751417X10	.9987	.31030560X10	.31028490X10	.9999

Moments	$s = 5 \quad m = 0$			$s = 5 \quad m = 2$		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	.75000000X10^{2}	.75000000X10^{2}	1.0000	.22500000X10^{3}	.22500000X10^{3}	1.0000
μ_2	.16125000X10^{4}	.15468749X10^{4}	.9593	.84375000X10^{4}	.82557692X10^{4}	.9785
μ_3	.89662500X10^{5}	.81738281X10^{5}	.9116	.80763750X10^{6}	.76890088X10^{6}	.9520
μ_4	.16395075X10^{8}	.14589294X10^{8}	.8899	.34549841X10^{9}	.32636309X10^{9}	.9446
$\sqrt{\mu_2}$	.40155946X10^{2}	.39330331X10^{2}	.9794	.91855864X10^{2}	.90861263X10^{2}	.9892
β_1	.19174432X10	.18050338X10	.9414	.10859043X10	.10506742X10	.9676
β_2	.63054191X10	.60971074X10	.9670	.48530915X10	.47883505X10	.9867

Table 2 (Cont'd.)

Moments	s = 5 m = 20 Exact	Approximate	Ratio (A/E)	s = 7 m = 0		Ratio (A/E)
μ_1'	$.51750000X10^{4}$	$.51750000X10^{4}$	1.0000	$.29400000X10^{3}$	$.29400000X10^{3}$	1.0000
μ_2	$.93926249X10^{6}$	$.93551508X10^{6}$	.9960	$.12789000X10^{5}$	$.12560896X10^{5}$	.9822
μ_3	$.42815620X10^{9}$	$.42425269X10^{9}$	.9909	$.14169329X10^{7}$	$.13600565X10^{7}$	.9599
μ_4	$.29756611X10^{13}$	$.29497474X10^{13}$	.9913	$.75814050X10^{9}$	$.72364864X10^{9}$	.9545
$\sqrt{\mu_2}$	$.96915556X10^{3}$	$.96722028X10^{3}$	.9980	$.11308846X10^{3}$	$.11207540X10^{3}$	.9910
β_1	.22122975	.21983500	.9937	.95981795	.93336613	.9724
β_2	.33729468X10	.33704139X10	.9992	.46352859X10	.45865535X10	.9895

Moments	s = 7 m = 5 Exact	Approximate	Ratio (A/E)	s = 7 m = 20 Exact	Approximate	Ratio (A/E)
μ_1'	$.16065000X10^{4}$	$.16065000X10^{4}$	1.0000	$.11844000X10^{5}$	$.11844000X10^{5}$	1.0000
μ_2	$.16627274X10^{6}$	$.16514318X10^{6}$	.9932	$.33577739X10^{7}$	$.33498896X10^{7}$	.9977
μ_3	$.43382729X10^{8}$	$.42705234X10^{8}$	.9844	$.23872706X10^{10}$	$.23742615X10^{10}$	.9946
μ_4	$.10207592X10^{12}$	$.10046898X10^{12}$	.9843	$.36682109X10^{14}$	$.36498082X10^{14}$	.9950
$\sqrt{\mu_2}$	$.40776555X10^{3}$	$.40637812X10^{3}$	.9966	$.18324229X10^{4}$	$.18302704X10^{4}$	.9988
β_1	.40942136	.40493031	.9890	.15053897	.14995660	.9961
β_2	.36921655X10	.36839245X10	.9978	.32535042X10	.32524380X10	.9997

Moments	s = 10 m = 0 Exact	Approximate	Ratio (A/E)	s = 10 m = 500 Exact	Approximate	Ratio (A/E)
μ_1'	$.12375000X10^{4}$	$.12375000X10^{4}$	1.0000	$.11487487X10^{8}$	$.11487487X10^{8}$	1.0000
μ_2	$.11323125X10^{6}$	$.11236942X10^{6}$	.9924	$.10443849X10^{12}$	$.10443158X10^{12}$	.9999
μ_3	$.26148994X10^{8}$	$.25690653X10^{8}$	.9825	$.23740000X10^{16}$	$.23736303X10^{16}$	.9998
μ_4	$.48682272X10^{11}$	$.47808616X10^{11}$	.9821	$.32812861X10^{23}$	$.32808512X10^{23}$	.9999
$\sqrt{\mu_2}$	$.33649851X10^{3}$	$.33521548X10^{3}$	.9962	$.32316944X10^{6}$	$.32315876X10^{6}$	1.0000
β_1	.47098932	.46516379	.9876	$.49474294X10^{-2}$	$.49468701X10^{-2}$	.9999
β_2	.37969793X10	.37862553X10	.9972	.30083123X10	.30083112X10	1.0000

lower percentage points from the approximate distribution. It may further be observed that an interesting feature of the distribution of $W_2^{(s)}$ is that it is asymptotically normal for large values of m or s.

An alternate approximation (which is exact for $s = 2$) is obtained by replacing the value of ν in (2.6.2) by $s(2m+s)/2$ and α^2 in (2.6.3) by $2[s(2m+s)+2]/(s-1)(2m+s+1)$. But this second approximation is not as good as the one suggested in (2.6.1) even for $s = 3$.

2.7 Some Remarks

It may be pointed that $2 \sum_{i=1}^{s} \xi_i$ is distributed (Pillai, 1954) as a chi-square with $s(2m+s+1)$ degrees of freedom and hence the distribution problem in this case is very simple. The results of this paper show that we can also have a simple approximation to the distribution of the second esf in the s ξ's. While the former chi-square distribution can be interpreted as the limiting distribution of Pillai's $V^{(s)}$ criterion (Pillai, 1954, 1960; Pillai and Mijares, 1959), the same is also true in the present case that the distribution of $W_2^{(s)}$ can also be considered as the limiting distribution of the second esf in the s θ's following the joint density (2.1.1). It might also be pointed out that the distribution problem studied in this paper has great use since it has been shown that several tests based on the esf's of the characteristic roots have been observed to have monotonicity of power and other optimum properties (Anderson and DasGupta, 1964a, 1964b; DasGupta, Anderson and Mudholkar, 1964; Kiefer and Schwartz, 1965).

2.8. An Example

The criterion $W_2^{(s)}$ may now be used to test the equality of p-dimensional vectors of ℓ p-variate normal populations having a common covariance matrix. The values of m and n in (2.1.1) appropriate for this test are given by

$$m = \tfrac{1}{2}|\ell-p-1|-\tfrac{1}{2}\ , \qquad n = \tfrac{1}{2}(N-\ell-p-1)\ , \tag{2.8.1}$$

where N is the total of ℓ sample sizes. The data studied by Rao (1952, p. 263) may be used for the test, which consist of measurements on (1) head length (2) height and (3) weight of 140 school boys of almost the same age belonging to six different schools in an Indian city. The problem is to test the equality of the three mean characters from the six different schools. Let S^* and S be the sum of product matrices 'between' and 'within' schools for the three characters. These are available in Pillai and Samson (1959). Now

$$(S + S^*)^{-1} = \begin{pmatrix} .0^{4}78984 & -.0^{4}61351 & .0^{5}11246 \\ & .001388366 & -.0^{3}269036 \\ & & .0^{4}97857 \end{pmatrix} \tag{2.8.2}$$

and

$$S^*(S + S^*)^{-1} = \begin{pmatrix} .04684095 & .11100391 & -.00576916 \\ .00808723 & .08898122 & -.00120418 \\ .01837418 & .09115586 & .05046243 \end{pmatrix} . \tag{2.8.3}$$

Now, from (2.8.3), $V_2^{(3)} = \theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3 = 0.010340$. Further from (2.8.1), $m = 0.5$ and $n = 65$. For, this value of m and $s = 3$, the first four moments, β_1, and β_2 of $W_2^{(3)}$ were computed using the results of sections 4 and 5 and their values are as follows:

$$\mu_1' = 15, \qquad \mu_2 = 142.5 \qquad \mu_3 = 3600, \qquad \mu_4 = 221400$$
$$\sqrt{\mu_2} = 11.9373 \qquad \beta_1 = 4.4788, \qquad \sqrt{\beta_1} = 2.1163, \qquad \beta_2 = 10.9030 \quad .$$

Now using the above values of $\sqrt{\beta_1}$ and β_2 and extrapolating from "Tables of percentage points of Pearson curves, for given $\sqrt{\beta_1}$ and β_2, expressed in standard measure" (Johnson et al, 1963), the upper 5 per cent point of $W_2^{(3)}$ was determined as 38. Further, taking $\xi_i = n\theta_i (i = 1,\ldots,s)$,

$$W_2^{(3)} = n^2 V_2^{(3)} = 51 \quad ,$$

which shows that the test rejects the null hypothesis of equality of the mean characters of boys from six different schools. However the test does not reject the null hypothesis at the upper 1% level. This agrees with (a) the findings of Rao (1952) who examined the data using the Λ criterion of Pearson and Wilks which is the product, $\prod_{i=1}^{s} (1-\theta_i)$; (b) the findings of Pillai and Samson (1959) who tested the same hypothesis based on the criterion $U^{(s)} = \sum_{i=1}^{s} [\theta_i/(1-\theta_i)] = \sum_{i=1}^{s} \lambda_i$, and (c) the findings of Pillai (1965) who further considered the test of this hypothesis using the criterion $U_{s-1}^{(s)}$, the (s-1)th elementary symmetric function in the s λ's. Foster (1957), however, finds that the largest root is significant only at the upper 15% level.

CHAPTER III

NON-CENTRAL DISTRIBUTION OF $W_2^{(s)}$

3.1. Introduction

Let X be a $p \times f$ matrix variate $(p \leq f)$ whose columns are independently normally distributed with $E(X) = M$ and covariance matrix Σ. Let $w_1, \ldots, w_p$ be the characteristic roots of $|XX' - w\,\Sigma| = 0$, then the distribution of $W = \text{diag}\,(w_i)$ is given by (James, 1961, 1964)

$$(3.1.1) \quad e^{-\frac{1}{2}\text{tr}\Omega}\ {}_0F_1(\tfrac{1}{2}f;\ \tfrac{1}{4}\Omega,\ W)\ \kappa(p,f)\ e^{-\frac{1}{2}\text{tr}W}|W|^{\frac{1}{2}(f-p-1)} \prod_{i>j} (w_i - w_j)\ ,$$

$$0 < w_1 \leq \cdots \leq w_p < \infty$$

where

$$(3.1.2) \qquad \kappa(p,f) = \pi^{\frac{1}{2}p^2} / \{2^{\frac{1}{2}pf}\Gamma_p(\tfrac{1}{2}f)\ \Gamma_p(\tfrac{1}{2}p)\}\ ,$$

$\Omega = \text{diag}(\omega_i)$ where ω_i, $i = 1, \ldots, p$, are the characteristic roots of $|M\,M' - \omega\Sigma| = 0$ and ${}_0F_1$ is the hypergeometric function of matrix argument (see Section 2) defined in James (1964). The above distribution of non-central means with known covariance matrix was obtained by James (1961). But (3.1.1) has also been shown, (James, 1964), to be the limiting distribution as $n \to \infty$ of $n\,R^2 = W$ such that $0 < nP^2 = \Omega < \infty$, where $R^2 = \text{diag}\,(r_i^2)$ and $P^2 = \text{diag}\,(\rho_i^2)$ and where the canonical correlation coefficients $r_1^2, \ldots, r_p^2$ between a p-set and a q-set of variates $(p \leq q)$ following a $(p + q)$ variate normal distribution, are calculated from a sample of $n + 1$ observations and $\rho_1^2, \ldots, \rho_p^2$ are population canonical correlation coefficients. Further $q = f$.

In this chapter, the first two non-central moments of $W_2^{(p)}$, the second elementary symmetric function (esf) in $\frac{1}{2}w_1, \frac{1}{2}w_2, \ldots, \frac{1}{2}w_p$ (note in this chapter $s = p$) have been obtained first by evaluating certain integrals involving zonal polynomials and then by using generalized Laguerre polynomials. These moments were used to suggest an approximation to the non-central distribution of $W_2^{(p)}$. The approximation is observed to be good even for small values of f.

3.2 The Moments of $W_2^{(p)}$

First let us recall a lemma due to Constantine (1963) which will be used later in this section.

Lemma 1. Let $Z: m \times m$ be a complex symmetric matrix whose real part $R(Z)$ is p.d. and let $T: m \times m$ be an arbitrary complex symmetric matrix. Then

$$(3.2.1) \quad \int_{S>0} \exp(-\mathrm{tr}\, Z\, S)|S|^{t-\frac{1}{2}(m+1)} C_\kappa(T\, S)dS = \Gamma_m(t,\kappa)|Z|^{-t} C_\kappa(T\, Z^{-1}) ,$$

where $R(t) > \frac{1}{2}(m-1)$ and $\Gamma_m(t,\kappa) = \pi^{\frac{1}{4}m(m-1)} \prod_{j=1}^{m} \Gamma(t + k_j - \frac{1}{2}(j-1))$ where $\kappa = (k_1, \ldots, k_s), k_1 \geq k_2 \geq \cdots \geq k_s > 0, k_1 + \ldots + k_s = k$. (See Khatri (1966)). Now let us note that

$$(3.2.2) \qquad C_\kappa(S) = [\chi[2\kappa](1)\ 2^k k!/(2k)!] Z_\kappa(S)$$

where $\chi[2\kappa](1)$ is defined in James (1960). Hence one can either work with the zonal polynomials $C_\kappa(S)$ or $Z_\kappa(S)$ which differ only in their normalizing constants. Now since $Z_{(1^2)} = 2\, a_2$, where a_2 is the second esf in the roots of S, $W_2^{(p)}$ can be expressed in terms of the zonal polynomials $C_{(1^2)}(W)$ or $Z_{(1^2)}(W)$. Further let us note that (James, 1964)

$$(3.2.3) \quad {}_0F_1(\tfrac{1}{2}f;\ \tfrac{1}{4}\Omega, W) = \sum_{k=0}^{\infty} \sum_{\kappa} C_\kappa(\tfrac{1}{4}\Omega)\ C_\kappa(W)/\{(\tfrac{1}{2}f)_\kappa\ C_\kappa(I_p)\ k!\}\ .$$

Now since

$$(3.2.4) \qquad C_\kappa(W)\ C_\eta(W) = \sum_{\delta} g^{\delta}_{\kappa,\eta}\ C_\delta(W)\ ,$$

where δ is a partition of $k + n = d$ and g's are constants, it is easy to see that using (3.2.3) and (3.2.4) in the product of (3.1.1) by $(3/4)^2 C_{(1^2)}(W)$, we can obtain $E(W_2^{(p)})$ by using lemma 1. Similarly the higher order moments can be obtained successively. Thus the first moment of $W_2^{(p)}$ is given by

$$(3.2.5)\ E(W^{(p)}) = (3/4^2)e^{-\frac{1}{2}\mathrm{tr}\Omega} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{\delta} 2^{\delta}\ \frac{\Gamma_p(\frac{1}{2}f,\delta)}{\Gamma_p(\frac{1}{2}f,\kappa)}\ \frac{C_\delta(I_p)}{C_\kappa(I_p)}\ \frac{g^{\delta}_{\kappa,(1^2)}}{k!}\ C_\kappa(\tfrac{1}{4}\Omega)\ ,$$

where $k + 2 = d$ such that $n = 2$. Similarly the rth moment of $W_2^{(p)}$ is given by

$$(3.2.6)\ E(W_2^{(p)})^r = (3/4^2)^r e^{-\frac{1}{2}\mathrm{tr}\Omega} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{\delta} 2^{\delta}\ \frac{\Gamma_p(\frac{1}{2}f,\delta)}{\Gamma_p(\frac{1}{2}f,\kappa)}\ \frac{C_\delta(I_p)}{C_\kappa(I_p)}\ \frac{g^{\delta}_{\kappa,\eta}}{k\ !}\ C_\kappa(\tfrac{1}{4}\Omega)\ ,$$

where now $k + 2r = d$ such that $n = 2r$. The g-coefficients in (3.2.5) and (3.2.6) may be computed using (3.2.4).

The first two moments of $W_2^{(p)}$ obtained from (3.2.5) and (3.2.6) are given at the end of this section. Following are some intermediate results on the expected values of certain expressions in the central case i.e. when $\Omega = 0$ which have been used to obtain $E(W_2^{(p)})$ and $E(W_2^{(p)})^2$ in the non-central case. These results were obtained with the help of lemma 1. Noting that

(3.2.7) $$a_2 = \tfrac{1}{2} Z_{(1^2)}$$

(3.2.8) $$E(a_2 Z_{(k)}) = B_\kappa (p,f)(p-1)(f-1)(pf+4k), k = 1,2,\ldots .$$

(3.2.9) $$E(a_2 Z_{(k-1,1)}) = B_\kappa (p,f) [(p-1)(f-1)(pf+4k) + 4(2k-1)]$$

$$k = 2,3,\ldots .$$

(3.2.10) $$E(a_2 Z_{(k-2,1^2)}) = B_\kappa (p,f)[(p-1)(f-1)(pf+4k) + 4(4k-3)]$$

$$k = 3,4,\ldots .$$

(3.2.11) $$E(a_2 Z_{(k-3,1^3)}) = B_\kappa (p,f)[(p-1)(f-1)(pf+4k) + 4(6k-6)]$$

$$k = 4,5,\ldots .$$

(3.2.12) $$E(a_2 Z_{(2^2)}) = B_\kappa(p,f)[p(p-1)f^2 - (p-1)(p-16)f - 8(2p-7)] .$$

where $B_\kappa(p,f) = 2^{2k-1}(\frac{1}{2}p)_\kappa\ (\frac{1}{2}f)_\kappa$, κ denoting the specific partition of k given on the left side of each equation involving $B_\kappa(p,f)$. Further noting that

$$a_2^2 = \frac{1}{24} Z_{(2^2)} + \frac{2}{15} Z_{(2,1^2)} + \frac{3}{40} Z_{(1^4)} \tag{3.2.13}$$

$$\begin{aligned} E(a_2^2\ Z_{(k)}) = 2^{2k-2}(\tfrac{p}{2})_\kappa\ (\tfrac{f}{2})_\kappa\ (p-1)(f-1)[&p^2(p-1)f^3 - p(p-1)\{p-8(k+1)\}f^2 \\ &-4\ \{2(k+1)p^2 - (4k^2 + 14k + 5)p + 4k(k+3)\}f \\ &-16k\{(k+3)\ p-(3k+7)\}], k = 1,2,\ldots \quad . \end{aligned} \tag{3.2.14}$$

$$\begin{aligned} E(a_2^2\ Z_{(k-1,1)}) = 2^{2k-2}(\tfrac{p}{2})_\kappa(\tfrac{f}{2})_\kappa[&p^2(p-1)^2f^4 - 2p(p-1)^2\{p-4(k+1)\}f^3 \\ &+(p-1)\{p^3-(16k+17)p^2 + 4(4k^2 + 20k+5)p - 16k(k+3)\}f^2 \\ &+4\{2(k+1)p^3-(8k^2+32k+5)p^2+(40k^2+94k-13)p \\ &\qquad - (32k^2+80k-24)\}f \\ &+16\{k(k+3)p^2 - 2(4k^2+10k-3)p + (15k^2+25k-12)\}] \\ &\qquad k = 2,3,\ldots \end{aligned} \tag{3.2.15}$$

where

$$(a)_\kappa = \prod_{i=1}^{s} (a - \tfrac{1}{2}(i-1))_{k_i} \tag{3.2.16}$$

$$(a)_k = a(a+1)\ \ldots\ (a+k-1) \quad . \tag{3.2.17}$$

In addition, expressions for $E(a_2^2 Z_{(1^3)})$, $E(a_2^2 Z_{(3,2)})$, $E(a_2^2 Z_{(3,1^2)})$, $E(a_2^2 Z_{(2^2,1)})$, $E(a_2^2 Z_{(2,1^3)})$, and $E(a_2^2 Z_{(1^5)})$ were also obtained (which are presented in Appendix C) and all these were used to compute the following two moments of $W_2^{(p)}$.

$$(3.2.18) \qquad E(W_2^{(p)}) = [1/2^3][(p-1)(f-1)(pf+4b_1) + 8b_2]$$

$$\begin{aligned}(3.2.19)\ E(W_2^{(p)})^2 = {} & [(p-1)(f-1)/2^6][p^2(p-1)f^3 - p(p-1)\{p-8(b_1+1)\}f^2 \\ & -4\{2(b_1+1)p^2-(4b_1^2+18b_1+5)p+4b_1(b_1+4)\}f \\ & - 16b_1\{(b_1+4)p - (3b_1 + 10)\}] \\ & + \frac{b_2}{4}[p(p-1)f^2 - \{p^2 - (4b_1+17)p + 4(b_1+5)\}f \\ & - 4\{(b_1+5)p - 3(b_1+4)\}] + b_2^2 + 6b_3 \quad .\end{aligned}$$

where b_i is the ith esf in $\frac{1}{2}\omega_1, \ldots, \frac{1}{2}\omega_p$.

It may be pointed out that (3.2.18) and (3.2.19) were obtained after summation of infinite series arising from the use of (3.2.5) or (3.2.6). For example, (3.2.18) was obtained from the following expression:

$$(3.2.20)\quad E(W_2^{(p)}) = e^{-\frac{1}{2}\mathrm{tr}\Omega}\left\{ \sum_{i=o}^{\infty} b_1^i\,[(p-1)(f-1)(pf+4i)\,/\,2^3 i!] + b_2\left[\sum_{i=o}^{\infty}(b_1^i\,/\,i!)\right]\right\} \quad .$$

The coefficients of b_1^i and $b_2 b_1^i$ in (3.2.20) were obtained by the use of (3.2.8) to (3.2.12) since computing the coefficients for a few small values of i easily yielded the generalization. It was further observed that the coefficients of terms $b_1^{\nu_1} b_2^{\nu_2} b_3^{\nu_3} \cdots$ other than given in (3.2.20) reduced to zero. The method used for obtaining (3.2.19) was similar.

3.3. Alternate Method

Alternately the moments of $W_2^{(p)}$ can be obtained in terms of the generalized Laguerre polynomials in the sense of Constantine (1966). In chapter I we noted two properties of these polynomials, denoted by $L_\kappa^\nu(S)$.

$$(3.3.1)\quad L_\kappa^\nu(S) = e^{\mathrm{tr}\, S} \int_{R>0} e^{-\mathrm{tr}\, R} |R|^\gamma C_\kappa(R) A_\gamma(RS)\, dS$$

and

$$(3.3.2)\quad L_\kappa^\nu(S) = (\gamma + \tfrac{1}{2}(m+1))_\kappa C_\kappa(I) \sum_{n=0}^{k} \sum_{\nu} (-1)^n \Big[a_{\kappa\nu}/(\gamma + \tfrac{1}{2}(m+1))_\nu\Big] \cdot \Big[C_\nu(S)/C_\nu(I)\Big]$$

where $a_{\kappa\nu}$ are tabulated only up to order $k = 4$ in Constantine (1966).

Now a_2^i can be expressed as a linear compound of the zonal polynomials as follows

$$(3.3.3)\quad a_2 = \frac{3}{4} C_{(1^2)}$$

$$(3.3.4)\quad a_2^2 = \frac{1}{16} [5\, C_{(2^2)} + 4\, C_{(21^2)} + 9\, C_{(1^4)}]$$

$$(3.3.5)\quad a_2^3 = \frac{7}{64} C_{(3^2)} + \frac{1}{12} C_{(321)} + \frac{57}{320} C_{(2^2 1^2)} + \frac{3}{40} C_{(31^3)} + \frac{1}{8} C_{(2^3)} + \frac{9}{40} C_{(21^4)} + \frac{27}{64} C_{(1^6)} \quad .$$

$$(3.3.6)\quad a_2^4 = \frac{9}{256} C_{(4^2)} + \frac{5}{192} C_{(431)} + \frac{1}{42} C_{(42^2)} + \frac{11}{480} C_{(421^2)} + \frac{3}{140} C_{(41^4)} + \frac{1}{24} C_{(3^2 2)} + \frac{293}{5376} C_{(3^2 1^2)} + \frac{11}{192} C_{(32^2 1)} + \frac{153}{2240} C_{(321^3)} + \frac{9}{112} C_{(31^5)} + \frac{171}{1792} C_{(2^4)} + \frac{231}{2240} C_{(2^3 1^2)} + \frac{1179}{8960} C_{(2^2 1^4)} + \frac{81}{448} C_{(21^6)} + \frac{81}{256} C_{(1^8)} \quad .$$

Now the moments of $W_2^{(p)}$ can be obtained from (3.3.1) by substituting

$$(3.3.7)\qquad {}_0F_1\left(\frac{f}{2}, \frac{1}{2}\,\Omega W\right) = \Gamma_p\left(\frac{f}{2}\right) A_\gamma\left(-\tfrac{1}{2}\Omega W\right)$$

where

$$(3.3.8)\qquad \gamma = \tfrac{1}{2}\,(f-p-1)\ .$$

We get

$$(3.3.9)\quad E(W_2^{(p)}) = \frac{3}{4} \cdot \frac{e^{-\mathrm{tr}\,\Omega}}{\Gamma_p\left(\frac{f}{2}\right)} \int_{W>0} e^{-\mathrm{tr}\,W}\, |W|^\gamma\, C_{(1^2)}(W)\, {}_0F_1\left(\frac{f}{2}, \frac{1}{2}\,\Omega W\right) dW = \frac{3}{4} L^\gamma_{(1^2)}\left(-\tfrac{1}{2}\Omega\right)\ .$$

Similarly, writing L_κ for $L_\kappa^\nu(-\frac{1}{2}\Omega)$ we get

$$(3.3.10)\quad E(W_2^{(p)})^2 = \frac{1}{16}[5\, L_{(2^2)} + 4\, L_{(21^2)} + 9\, L_{(1^4)}]$$

$$(3.3.11)\quad E(W_2^{(p)})^3 = \frac{7}{64} L_{(3^2)} + \frac{1}{12} L_{(321)} + \frac{57}{320} L_{(2^2 1^2)} + \frac{3}{40} L_{(31^3)} + \frac{1}{8} L_{(2^3)} + \frac{9}{40} L_{(21^4)} + \frac{27}{64} L_{(1^6)}$$

and

$$(3.3.12)\quad E(W_2^{(p)})^4 = \frac{9}{256} L_{(4^2)} + \frac{5}{192} L_{(431)} + \frac{1}{42} L_{(42^2)} + \frac{11}{480} L_{(421^2)} + \frac{3}{40} L_{(41^4)} + \frac{1}{24} L_{(3^2 2)} + \frac{293}{5376} L_{(3^2 1^2)} + \frac{11}{192} L_{(32^2 1)} + \frac{153}{2240} L_{(321^3)} + \frac{9}{112} L_{(31^5)} + \frac{171}{1792} L_{(2^4)} + \frac{231}{2240} L_{(2^3 1^2)} + \frac{1179}{8960} L_{(2^2 1^4)} + \frac{81}{448} L_{(21^6)} + \frac{81}{256} L_{(1^8)}$$

But, since $a_{\kappa\nu}$ in (3.3.2) have been tabulated only up to order $k = 4$, only the first two moments could be evaluated explicitly, which correspond to the expressions (3.2.18) and (3.2.19) respectively. However the third and the fourth moments are now available in terms of the generalized Laguerre polynomials as in (3.3.11) and (3.3.12). For explicit evaluation of third and fourth moment we need coefficients $a_{\kappa\nu}$ for $k = 6$ and $k = 8$ respectively.

3.4. An Approximation to the Non-Central Distribution of $W_2^{(p)}$

In the previous chapter we have suggested an approximation to the central distribution of $W_2^{(p)}$ in the following form after obtaining the first four moments:

$$(3.4.1) \quad f(W_2^{(p)}) = [\alpha^{\nu}/2\Gamma(\nu)]e^{-\alpha(W_2^{(p)})^{\frac{1}{2}}}(W_2^{(p)})^{\frac{1}{2}\nu-1} \quad 0 < W_2^{(p)} < \infty$$

where

$$(3.4.2) \qquad \nu = \tfrac{1}{2}\,pf$$

and

$$(3.4.3) \qquad \alpha^2 = 2(pf+2) \;/\; [(p-1)(f-1)] \quad .$$

From a comparison of the exact and approximate moments and moment quotients the approximation (3.4.1) was recommended for $f = 14$ and above when $p = 3$, $f = 11$ and above when $p = 4$, $f = 10$ and above when $p = 5$, $f = 8$ and above for $p = 7$ and all values of f and p beyond 7. However, since the lowest value f can take is p and small values of f are quite important from a practical point of view, the approximation to the non-central distribution of $W_2^{(p)}$ given below is believed to serve that purpose. The new approximation satisfies (3.4.1) with

(3.4.4) $$\nu = 2[2(\mu_1')^3 / \mu_2((\mu_2')^{\frac{1}{2}} - \mu_1')]^{\frac{1}{2}}$$

and

(3.4.5) $$\alpha^2 = \nu(\nu+1) / \mu_1' \quad ,$$

where μ_1' and μ_2' are the first two (non-central)moments given in (3.2.18) and (3.2.19) respectively and μ_2 is the variance of $W_2^{(p)}$.

3.5. Accuracy Comparisons

The approximation to the non-central distribution of $W_2^{(p)}$ has the first moment the same as that of the exact. An idea of the closeness of the approximate to the exact second moment can be had from Table 3.

Table 3

Values of exact and approximate variances

$p = 3 \quad f = 3$					
$\frac{1}{2}\omega_1$	$\frac{1}{2}\omega_2$	$\frac{1}{2}\omega_3$	μ_2 Exact	μ_2 Approx.	Ratio (A/E)
0	0	0	24.75	24.56	.9924
1	2	0	156.75	155.89	.9945
25	0	0	1824.75	1820.46	.9976
5	5	5	5154.75	5148.41	.9988
5	5	25	31194.75	31180.31	.9995
15	15	15	98214.75	98194.92	.9998

Table 3 (cont'd.)

					p = 5 f = 5		
$\frac{1}{2}\omega_1$	$\frac{1}{2}\omega_2$	$\frac{1}{2}\omega_3$	$\frac{1}{2}\omega_4$	$\frac{1}{2}\omega_5$		μ_2	
					Exact	Approx.	Ratio (A/E)
0	0	0	0	0	875.00	874.27	.9992
1	2	1	2	1	4821.00	4817.81	.9993
25	0	2	2	10	85963.00	85948.64	.9998
10	10	10	10	10	256875.00	256846.38	.9999

The values of the exact and approximate variances tend to be closer for larger values for a given p and hence the tabulation has been confined to the smallest value of f in each case. It may further be noted from Table 3 that ratios of the (approximate to exact) variances are closer to unity for larger values of p, for example, in the null case.

Further, the approximation to the non-central distribution is better even in the null case than that given in Chapter II just for the null case which is the same as in (3.4.1) - (3.4.3). The moments of $W_i^{(p)}$ when $\Omega = 0$ which were evaluated in Chapter II are presented in the notation of this chapter in a much simpler form below.

(3.5.1) $$\mu_2^{(0)}\{W_2^{(p)}\} = [(\tfrac{p}{2})(\tfrac{f}{2})|2^2][2(p-1)f-2p+5] ,$$

(3.5.2) $$\mu_3^{(0)}\{W_2^{(p)}\} = [(\tfrac{p}{2})(\tfrac{f}{2})|2^2][5(p-1)^2f^2-(10p^2-40p+33)f+5p^2-33p+49] ,$$

(3.5.3) $$\mu_4^{(0)}\{W_2^{(p)}\} = [3(\tfrac{p}{2})(\tfrac{f}{2})|2^6]$$

$$\cdot[4p(p-1)^3f^4-4(p-1)^2(3p^2-34p+28)f^3+(12p^4-396p^3+1917p^2-2893p+1384)f^2$$

$$-(4p^4-360p^3+2893p^2-7129p+5192)f-(112p^3-1384p^2+5192p-5864)] ,$$

$$(3.5.4)\quad \mu_2^{(0)}\{W_3^{(p)}\} = [9\tbinom{p}{3}\tbinom{f}{3}|2^5][(p-1)(p-2)f^2-3(p-2)(p-3)f+2(p^2-9p+18)]$$

$$(3.5.5)\quad \mu_2^{(0)}\{W_4^{(p)}\} = [\tbinom{p}{4}\tbinom{f}{4}|2^4][2(p-1)(p-2)(p-3)f^3-3(p-2)(p-3)(4p-13)f^2$$
$$+(p-3)(22p^2-201p+458)f-6(2p^3-39p^2+229p-420)]$$

where $\mu_i^{(0)}$ denotes the ith moment in the central case. The moments (3.5.1) - (3.5.5) were obtained by evaluating linear compounds of certain determinants (see Pillai 1964). That the approximation suggested for the non-central case (see eqns. (3.4.1), (3.4.4) and (3.4.5)) works very well for the null case for all values of p and f, can be inferred from Table 4.

Table 4

Ratios of moments of $W_2^{(p)}$ ($\Omega = 0$) from the exact and approximate distributions for $p = 3$ and $f = 3$ and 10

Moments	f = 3			f = 10		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	$.45000000\text{X}10$	$.45000000\text{X}10$	1.0000	$.67500000\text{X}10^2$	$.67500000\text{X}10^2$	1.0000
μ_2	$.24750000\text{X}10^2$	$.24561937\text{X}10^2$	.9924	$.13162500\text{X}10^4$	$.13154799\text{X}10^4$	.9994
μ_3	$.37350000\text{X}10^3$	$.36307500\text{X}10^3$	.9721	$.66318750\text{X}10^5$	$.65752496\text{X}10^5$	.9915
μ_4	$.12067312\text{X}10^5$	$.11468028\text{X}10^5$	.9503	$.10954364\text{X}10^8$	$.10835868\text{X}10^8$	.9892
$\sqrt{\mu_2}$	$.49749371\text{X}10$	$.49560001\text{X}10$	.9962	$.36280159\text{X}10^2$	$.36269546\text{X}10^2$	.9997
β_1	$.92014357\text{X}10$	$.88962059\text{X}10$	.9668	$.19286681\text{X}10$	$.18992046\text{X}10$	.9847
β_2	$.19699724\text{X}10^2$	$.19009186\text{X}10^2$	.9649	$.63228139\text{X}10$	$.62617429\text{X}10$	.9903

Table 5 provides some comparison of the closeness of the approximate to the exact moments when $\Omega = 0$ for a) the earlier approximation for the null case (Eqns. (3.4.1) - (3.4.3))and b) the new approximation for the non-null case (Eqns. (3.4.1), (3.4.4) and (3.4.5)).

Table 5

Ratios of moments of $W_2^{(p)}$ $(\Omega = 0)$ from the exact and approximate distributions using a) earlier approximation b) new approximation for $p = 4$ and $f = 5$

Moments		a)earlier approximation		b)new approximation	
	Exact	Approximate	Ratio(A/E)	Approximation	Ratio(A/E)
μ_1'	$.30000000\text{X}10^2$	$.30000000\text{X}10^2$	1.0000	$.30000000\text{X}10^2$	1.0000
μ_2	$.40500000\text{X}10^3$	$.37636364\text{X}10^3$	.9293	$.40445819\text{X}10^3$	.9987
μ_3	$.14354999\text{X}10^3$	$.12228099\text{X}10^5$	.8518	$.14153057\text{X}10^5$	.9859
μ_4	$.13799024\text{X}10^7$	$.11130166\text{X}10^7$	.8066	$.13501215\text{X}10^7$	.9784
$\sqrt{\mu_2}$	$.20124611\text{X}10^2$	$.19400093\text{X}10^2$	.9640	$.20111145\text{X}10^2$	.9993
β_1	.31019966X10	.28047550X10	.9042	.30274682X10	.9760
β_2	.84127571X10	.78575356X10	.9340	.82532614X10	.9810

Thus it may be seen that the new approximation can be used in the null case even for the very small values of f for which the earlier approximation was not recommended.

CHAPTER IV

EXACT DISTRIBUTION OF WILKS' Λ

4.1. Introduction

Wilks (1932), following the likelihood ratio method (Neyman and Pearson, 1928, 1931; Pearson and Neyman, 1930), obtained suitable generalizations in the analysis of variance applicable to several variables. The statistic Λ proposed by him has been found useful in a variety of problems. Bartlett (1934) applied it for testing the significance of treatments with respect to two variables in a varietal trial and indicated its general use in multivariate tests of significance. Wilks (1935) and Hotelling (1936) found it useful in testing the independence of several groups of variates. Wilks' statistic supplied some of the basic tests in multivariate analysis, but the problem of tabulation has not been tackled except in some limited cases. The problem of finding the percentage points of this statistic becomes difficult because its exact distribution is either unknown or is too involved.

The problem of multivariate analysis of variance can be reduced to the following form (e.g. see S.N. Roy, 1957): the joint probability density function (p.d.f.) of the elements of the random matrices $X(p \times f_1)(f_1 \geq p)$ and $Y(p \times f_2)$ is given by

$$(4.1.1) \qquad (2\Pi)^{-\frac{1}{2}p(f_1+f_2)} |\Sigma|^{-\frac{1}{2}(f_1+f_2)} \exp[-\tfrac{1}{2}\mathrm{tr}\Sigma^{-1}\{X X'+(Y-\mu)(Y-\mu)'\}] ,$$

where the matrix $\Sigma(p \times p)$ is positive definite, symmetrical and unknown, and the problem is to test the null hypothesis H_0 that the matrix $\mu(p \times f_2)$ is a null-matrix, i.e.

$$H_0 : \mu = 0 \; (p \times f_2) \qquad \text{against} \qquad H_1 : \mu \neq 0$$

Let the non-zero roots of the determinantal equation

$$(4.1.2) \qquad |A_2 - \lambda A_1| = 0$$

be denoted by $\lambda_1 \leq \dots \leq \lambda_s$, where $s = \min(p, f_2)$ and

$$A_1(p \times p) = X X'$$

$$A_2(p \times p) = Y Y' \quad .$$

The likelihood ratio statistic for testing H_0, given by Wilks (1932) is

$$(4.1.3) \qquad \Lambda = \frac{|A_1|}{|A_1 + A_2|} = \prod_{i=1}^{s} \left(\frac{1}{1+\lambda_i}\right) .$$

In this model of multivariate analysis of variance, p is the number of variates, A_1 and A_2 are the sums of squares and products matrices for error and hypothesis respectively, and f_1 and f_2 are the corresponding degrees of freedom.

Wilks (1935) has obtained the exact null hypothesis distribution of Λ in the form of a $(p-1)$ fold multiple integral, which he was able to evaluate for $p = 1,2$; $p = 3$ with $f_2 = 3,4$ and for $p = 4$ with $f_2 = 4$ only. A number of asymptotic approximations have been given for general p and f_2. Bartlett (1938), observing the asymptotic behaviour of likelihood ratio statistics, obtained a chi-square approximation to $-f_2 \log \Lambda$, for testing independence of several groups of variates as an infinite series of chi-square distributions. Wilks' Λ criterion is a special case of the statistics considered by Wald and Brookner (1941), when the number of groups is equal to two. Rao (1948), using $-\{f_1 - \frac{1}{2}(p-f_2+1)\} \log \Lambda$ obtained the first three terms of a more rapidly convergent series. Finally, Rao's approximation was shown to be a special case of a more general result of Box (1949), who gave asymptotic approximations to functions of general likelihood ratio statistics.

Schatzoff (1966) has given a method for obtaining the exact distribution of Λ but has not given explicitly the distribution function. We are giving the same here in explicit form which he was not able to give. He has been able to tabulate the correction factors for converting chi-square percentiles to exact percentiles of $-\{f_1 - \frac{1}{2}(p-f_2+1)\} \log \Lambda$ for $p = 3(1)\,8$ and values of f_2 such that $p\,f_2 \leq 70$, using certain recurrence relations on IBM 7094. We are able to give the exact expression for the distribution function overcoming the restriction $p\,f_2 \leq 70$ because our method is by far simpler than his. Unlike Schatzoff who could not suggest a suitable method even for handling the distribution problem for odd values of f_2 (unless by some indirect methods in some special cases) we are giving here the distribution also for that case. We give here exact distributions of Λ for $p = 1(1)6$ and extend Schatzoff's tables to $f_2 = 12(2)22$ for $p = 3$, to $f_2 = 11(1)13(2)23$ for $p = 4$ and to $f_2 = 11(1)13$ for $p = 6$. Additional 17 tables can be obtained by interchanging p and f_2.

4.2. Distributional Properties of Λ

For purposes of notational ease, the symbol Λ will be replaced by U. Let us denote by $B[a,b;X]$ the density function

$$(4.2.1) \qquad [1/B(a,b)]X^{a-1}(1-X)^{b-1} \qquad 0 \le X \le 1$$

of a Beta variable X. The following theorems, which we state without proof, appear in Anderson (1958, Chapter 8) and have been used in the next section.

Theorem 4.2.2. The distribution of U_{p,f_2,f_1} is the same as that of U_{f_2,p,f_1+f_2-p}.

This implies that without loss of generality we need consider only values $f_2 \ge p$.

Theorem 4.2.3. U_{p,f_2,f_1} is distributed like $X_1 \ldots X_p$ where X_i are independently distributed as $B[\frac{1}{2}(f_1-i+1), \frac{1}{2}f_2; X_i]$.

Theorem 4.2.4. U_{2r,f_2,f_1} is distributed like $Y_1^2 \ldots Y_r^2$, where Y_i are independently distributed as $B[f_1+1-2i, f_2; Y_i]$; U_{2s+1,f_2,f_1} is distributed as $Z_1^2 \ldots Z_s^2 \cdot Z_{s+1}$, where $Z_i (i = 1,\ldots,s)$ are independently distributed as $B[f_1+1-2i, f_2; Z_i]$ and Z_{s+1} is independently distributed as $B[\frac{1}{2}(f_1+1-p); \frac{f_2}{2}; Z]$.

Theorems 4.2.3 and 4.2.4 directly provide us with the two special cases.

Case 1: $p = 1$. U_{1,f_2,f_1} has the density

$$(4.2.5) \qquad [1/B(\tfrac{f_1}{2}, \tfrac{f_2}{2})]\, U^{\frac{f_1}{2}-1} (1-U)^{\frac{f_2}{2}-1}$$

and hence

$$(4.2.6) \qquad \frac{1-U_{1,f_2,f_1}}{U_{1,f_2,f_1}} \cdot \frac{f_1}{f_2} = F_{f_2,f_1} \quad .$$

Case 2. $p = 2$. $X = \sqrt{U_{2,f_2,f_1}}$ has the density

$$(4.2.7) \qquad [1/B(f_1-1,f_2)]\, X^{f_1-2}\, (1-X)^{f_2-1} \quad ,$$

and thus the density of U_{2,f_2,f_1} is

$$(4.2.8) \qquad [1/2B(f_1-1,f_2)]\, U^{\frac{1}{2}(f_1-3)}\, (1-\sqrt{U})^{f_2-1} \quad ,$$

and hence

$$(4.2.9) \qquad \frac{1-\sqrt{U_{2,f_2,f_1}}}{\sqrt{U_{2,f_2,f_1}}} \cdot \frac{f_1-1}{f_2} = F_{2f_2,2(f_1-1)} \quad ,$$

where $F_{m,n}$ is Snedecor's F with m and n degrees of freedom. Thus the percentage points in both these cases can be obtained from the tables of F-Distribution (e.g. Biometrika Tables for Statisticians) with respective degrees of freedom.

By interchanging p and f_2, we will get the corresponding results for $f_2 = 1$ and $f_2 = 2$.

In order to derive the exact distribution for cases of higher dimensionality we use the method described in section 4.3.

4.3. Method of Derivation

An immediate consequence of Theorem 4.2.3 is that the distribution of $\operatorname{Log} U_{p,f_2,f_1}$ is the distribution of $\sum_{i=1}^{p} \log X_i$, so that the problem is transformed to one of finding the distribution of a sum of independently distributed random variables. Such a problem can be handled by taking successive convolutions provided that the process yields expressions which can be easily integrated at each stage. Schatzoff (1966) has proved that this is in fact the case. But whereas he invokes Theorem 4.2.3 we make use of Theorem 4.2.4. And by doing so we can get exact distribution for $p = 3,4,5,6$, in much simpler form than otherwise is possible. For example, for $p = 4$, we convolute just once as against three times, as Schatzoff (1966) has done. This results, for general f_2, in a simple expression involving just one double but finite series. Similarly for $p = 5$ and 6 we convolute only two times, as against 4 and 5 times. This saves lot of work, which is not easy by any means, and for the first time it makes possible to write down explicitly the distribution of Wilks' statistic in a closed form for $p = 4,6$ for general f_2 and for $p = 3,5$ when f_2 is even. The case when p and f_2 both are odd involves infinite series, however the same formulae, derived for f_2 even, work.

Consider the beta random variable of the Theorem 4.2.3. The density of X_i is given by

$$(4.3.1) \qquad B[\tfrac{1}{2}(f_1-i+1), \tfrac{f_2}{2} ; X_i] = K_i \; X_i^{\frac{f_1-i-1}{2}} (1-X_i)^{\frac{f_2-2}{2}}$$

$$0 < X_i < 1, \quad f_1 \geq i$$

where

$$(4.3.2) \qquad K_i = [1/B(\frac{f_1-i+1}{2}, \frac{f_2}{2})] = \Gamma(\frac{f_1-i+1+f_2}{2}) / \Gamma(\frac{f_1-i+1}{2}) \Gamma(\frac{f_2}{2})$$

When f_2 is even, $\frac{f_2-2}{2}$ is an integer, the expression (4.3.1) can be expanded using binomial theorem, resulting in

$$(4.3.3) \qquad B[\frac{f_1-i+1}{2}, \frac{f_2}{2}; X_i] = K_i \sum_{\ell=0}^{b} (-1)^{\ell} \binom{b}{\ell} X_i^{\frac{1}{2}(f_1-i+1+2\ell)}$$

where

$$(4.3.4) \qquad b = \frac{f_2-2}{2} \quad .$$

Let us now make the transformation

$$(4.3.5) \qquad Y_i = -\log_e X_i \quad , \quad dY_i = -\frac{dX_i}{X_i} \quad .$$

Then the density of Y_i is given by

$$(4.3.6) \qquad Y_i \sim K_i \sum_{\ell=0}^{b} (-1)^{\ell} \binom{b}{\ell} e^{-\frac{1}{2}Y_i(f_1-i+1+2\ell)} , Y_i > 0, i=1,\ldots,p \quad .$$

Similarly in the light of the Theorem 4.2.4 we consider the random variable defined by

$$(4.3.7) \qquad Z_i = X_{2i-1} \cdot X_{2i}$$

then the density of Z_i is given by

(4.3.8) $$Z_i \sim C_i \cdot Z_i^{\frac{1}{2}(f_1-2i-1)} (1-\sqrt{Z_i})^{f_2-1}$$

where

(4.3.9) $$C_i = [1/2B(f_1-2i+1,f_2)] \quad .$$

Note that in this case to apply binomial theorem to get a finite series, unlike the previous case, f_2 does not have to be even. This is important for our method.

Making the transformation (Schatzoff, 1966)

(4.3.10) $$Y_i' = -\log Z_i$$

and expanding by binomial theorem, we get the density of Y_i' as

(4.3.11) $$Y_i' \sim C_i \sum_{\ell=0}^{f_2-1} (-1)^{\ell} \binom{f_2-1}{\ell} e^{-\frac{Y_i'}{2}(f_1+\ell-2i+1)}, \quad Y_i' > 0 \quad .$$

Finally, we consider the density and distributions of random variables like

(4.3.12) $$V = V_1 + V_2$$

where apart from normalizing constants

(4.3.13) $$V_1 \sim v_1^k e^{av_1} \quad , \quad v_1 > 0 \ , \quad k = \text{non-negative integer}$$

and

$$(4.3.13) \qquad V_2 \sim e^{bv_2} \quad , \quad v_2 > 0 \quad .$$

The density function of V is easily found by forming the convolution integral

$$(4.3.14) \qquad V_1^k e^{aV_1} * e^{bV_2} = \int_o^v v_1^k \; e^{av_1} \; e^{b(v-v_1)} \; dv_1$$

$$= e^{bv} \int_o^v v_1^k \; e^{(a-b)v_1} \; dv_1$$

where the asterisk denotes the convolution operator. We have two cases.

Case I. $a = b$, then (4.3.14) is simply

$$(4.3.15) \qquad e^{bv} \int_o^v v_1^k \, dv_1 = e^{bv} \frac{v^{k+1}}{k+1} \quad .$$

Case II. $a \neq b$, then we have

$$(4.3.16) \qquad e^{bv} \int_o^v v_1^k \, e^{(a-b)v_1} \, dv_1$$

$$= e^{av} \left[\sum_{r=1}^{k+1} (-1)^{r+1} \frac{k!}{(k-r+1)!} \frac{v^{k-r+1}}{(a-b)^r} \right] + e^{bv} \left(\frac{-1}{a-b}\right)^{k+1} k!$$

which can be easily varified by performing k successive integrations by parts.

The distribution functions are readily obtainable by direct integration of the density functions. It should be noted that whether we are integrating the right hand side of (4.3.15) or (4.3.16), we have only to evaluate integrals of the form given by the left hand side of (4.3.16).

4.4. Exact Probability and Distribution Function of U_{p,f_2,f_1}

4.4.1. $p = 3$

We write the Wilks' statistic as a product of independent variables

$$U_{3,f_2,f_1} = X_1 \cdot X_2 \cdot X_3 = Z_1 \cdot X_3 \tag{4.4.1.1}$$

So that, in the notation of section 4.3, we have

$$-\log U_{3,f_2,f_1} = -\log Z_1 - \log X_3 = Y_1' + Y_3 = W_3 \quad \text{(say)} \tag{4.4.1.2}$$

where

$$Y_1' \sim [1|2\, B(f_1-1, f_2)] \sum_{\ell=0}^{f_2-1} (-1)^{\ell} \binom{f_2-1}{\ell} e^{-\frac{Y_1}{2}(f_1+\ell-1)} \tag{4.4.1.3}$$

and

$$Y_3 \sim [1|B(\tfrac{f_1-2}{2}, \tfrac{f_2}{2})] \sum_{m=0}^{\frac{f_2}{2}-1} (-1)^m \binom{\frac{f_2}{2}-1}{m} e^{-\frac{Y_3}{2}(f_1+2m-2)} \tag{4.4.1.4}$$

Then

$$(4.4.1.5)\quad W_3 = -\log U_{3,f_2,f_1} \sim \left\{[1/2\ B(f_1-1,f_2)]\sum_{\ell=0}^{f_2-1} (-1)^{\ell}\binom{f_2-1}{\ell} e^{-\frac{Y_1'}{2}(f_1+\ell-1)}\right\} *$$

$$\left\{[1/B(\tfrac{f_1-2}{2}, \tfrac{f_2}{2})] \sum_{m=0}^{\frac{f_2}{2}-1} (-1)^m \binom{\frac{f_2}{2}-1}{m} e^{-\frac{Y_3}{2}(f_1+2m-2)}\right\} .$$

Thus we get the probability density function of $-\log U_{3,f_2,f_1}$

$$(4.4.1.6)\quad W_3 \sim [1/2\ B(f_1-1,f_2)B(\tfrac{f_1-2}{2}, \tfrac{f_2}{2})]\Big[\sum_{m=1}^{\frac{f_2}{2}-1} (-1)^{m-1}\binom{f_2-1}{2m-1}\binom{\frac{f_2}{2}-1}{m} W_3\, e^{-\frac{W_3}{2}(f_1+2m-2)}$$

$$+ 2\sum_{\substack{\ell=0 \\ \ell\neq 2m-1}}^{f_2-1} \sum_{m=0}^{\frac{f_2}{2}-1} \frac{(-1)^{\ell+m}}{(2m-\ell-1)}\binom{f_2-1}{\ell}\binom{\frac{f_2}{2}-1}{m}\left(e^{-\frac{W_3}{2}(f_1+\ell-1)} - e^{-\frac{W_3}{2}(f_1+2m-2)}\right)\Big]$$

$$W_3 > 0 ,$$

which may be easily verified by reference to (4.3.14) - (4.3.16).

If now we make the substitution

$$(4.4.1.7)\qquad W = -\log U , \qquad dW = -\frac{dU}{U}$$

we obtain

$$(4.4.1.8)\quad U_{3,f_2,f_1} \sim [1/2B(f_1-1,f_2)B(\tfrac{f_1-2}{2}, \tfrac{f_2}{2})]\Big[\sum_{m=1}^{\frac{f_2}{2}-1}(-1)^{m-1}\binom{f_2-1}{2m-1}\binom{\frac{f_2}{2}-1}{m}(-\log U)\, U^{\frac{f_1+2m-4}{2}}$$

$$+\, 2\sum_{\substack{\ell=0\\ \ell\neq 2m-1}}^{f_2-1}\ \sum_{m=0}^{\frac{f_2}{2}-1}\frac{(-1)^{\ell+m}}{(2m-\ell-1)}\binom{f_2-1}{\ell}\binom{\frac{f_2}{2}-1}{m}\Big(U^{\frac{f_1+\ell-3}{2}} - U^{\frac{f_1+2m-4}{2}}\Big)\Big]$$

$$0 \le U \le 1$$

which is a closed form for f_2 even and an infinite series for f_2 odd.

To find the distribution function of U_{3,f_2,f_1} ; we integrate (4.4.1.8) between the limits (o,u) ; $o \le u \le 1$, obtaining

$$(4.4.1.9)\quad P[U_{3,f_2,f_1} \le u] = [1|B(f_1-1,f_2)B(\tfrac{f_1-2}{2}, \tfrac{f_2}{2})]\Big[\sum_{m=0}^{\frac{f_2}{2}-1}\frac{(-1)^{m-1}\, u^{\frac{f_1+2m-2}{2}}}{(f_1+2m-2)^2}\binom{\frac{f_2}{2}-1}{m}\binom{f_2-1}{2m-1}\cdot(2-(f_1+2m-2)\log u)$$

$$+\, 2\sum_{\substack{\ell=0\\ \ell\neq 2m-1}}^{f_2-1}\ \sum_{m=0}^{\frac{f_2}{2}-1}\frac{(-1)^{\ell+m}}{(2m-\ell-1)}\binom{f_2-1}{\ell}\binom{\frac{f_2}{2}-1}{m}\Big(\frac{u^{\frac{f_1+\ell-1}{2}}}{f_1+\ell-1} - \frac{u^{\frac{f_1+2m-2}{2}}}{f_1+2m-2}\Big)\Big] .$$

For comparison's sake, we list below the exact probability and distribution functions of U_{3,f_2,f_1} for the particular cases $f_2=3$ and 4.

Special Cases.

Case I. For $f_2 = 3$, the exact probability density of $U_{3,3,f_1}$ is given by

$$(4.4.1.10)\quad U_{3,3,f_1} \sim \frac{U^{\frac{f_1-4}{2}}}{6B(f_1-1,3)B(\frac{f_1-2}{2},\frac{3}{2})}\Big[2-9\,U^{\frac{1}{2}}+3(2-\log U)U+U^{3/2}$$

$$+\,6\sum_{m=2}^{\frac{1}{2}}(-1)^m\binom{\frac{1}{2}}{m}\left\{\frac{U^{3/2}}{2m-3}-\frac{U}{m-1}+\frac{U^{\frac{1}{2}}}{2m-1}-\frac{U^m}{(2m-1)(m-1)(2m-3)}\right\}\Big]$$

which is an infinite series. Integrating (2.4.1.10) between the limits (o,u), we get the distribution function of $U_{3,3,f_1}$

$$(4.4.1.11)\quad P[U_{3,3,f_1}\le u] = \frac{u^{\frac{f_1-2}{2}}}{B(f_1-1,3)B(\frac{f_1-2}{2},\frac{3}{2})}\Big[\frac{u^{3/2}}{3(f_1+1)}+\frac{u}{f_1^2}\{2+f_1(2-\log u)\}$$

$$-\frac{3u^{\frac{1}{2}}}{(f_1-1)}+\frac{2}{3(f_1-2)}+2\sum_{m=2}^{\frac{1}{2}}(-1)^m\binom{\frac{1}{2}}{m}\left\{\frac{u^{3/2}}{(2m-3)(f_1+1)}\right.$$

$$\left.-\frac{u}{(m-1)f_1}+\frac{u^{\frac{1}{2}}}{(2m-1)(f_1-1)}-\frac{u^m}{(2m-1)(2m-3)(m-1)(f_1+2m-2)}\right\}\Big]$$

Case II. For $f_2 = 4$, the exact probability density function of $U_{3,4,f_1}$ becomes

$$(4.4.1.12)\quad U_{3,4,f_1}\sim\Big[U^{\frac{f_1-4}{2}}/4B(f_1-1,4)B(\tfrac{f_1-2}{2},2)\Big]\Big[1+8U^{\frac{1}{2}}-8U^{3/2}-U^2-6U\log U\Big]$$

$$0\le U\le 1$$

and the distribution function of $U_{3,4,f_1}$ becomes

$$(4.4.1.13)\quad P[U_{3,4,f_1} \leq u] = \frac{u^{\frac{f_1-2}{2}}}{2B(f_1-1,4)B(\frac{f_1-2}{2},2)}\left[\frac{1}{f_1-2} - \frac{8u^{\frac{1}{2}}}{f_1-1} + \frac{6u}{f_1^2}(2-f_1\log u) + \frac{8u^{3/2}}{f_1+1} - \frac{u^2}{f_1+2}\right].$$

Wilks (1935) and Anderson (1958) obtained the distribution functions of $U_{3,3,f_1}$ and $U_{3,4,f_1}$ in the form of algebraic expressions. Our result agrees with that of Anderson (1958) and Consul (1966) who used different methods. Thus the expression obtained by Wilks (1935) was definitely incorrect.

4.4.2. $p = 4$

We write Wilks' statistic as

$$(4.4.2.1)\quad U_{4,f_2,f_1} = X_1 \; X_2 \; X_3 \; X_4 = Z_1 \; Z_2$$

So that, in the notation of section 4.3, we have

$$(4.4.2.2)\quad -\log U_{4,f_2,f_1} = -\log Z_1 - \log Z_2 = Y_1' + Y_2' = W_4 \quad \text{(say)}$$

where

$$(4.4.2.3)\quad Y_1' \sim [1|2B(f_1-1,f_2)] \sum_{\ell=0}^{f_2-1} (-1)^{\ell} \binom{\frac{f_2-1}{2}}{\ell} e^{-\frac{Y_1'}{2}(f_1+\ell-1)}$$

and

$$(4.4.2.4)\quad Y_2' \sim [1|2B(f_1-3,f_2)] \sum_{m=0}^{f_2-1} (-1)^{m} \binom{\frac{f_2-1}{2}}{m} e^{-\frac{Y_2'}{2}(f_1+m-3)}$$

Then by way of convolution, we have

$$(4.4.2.5)\quad W_4 = -\log U_{4,f_2,f_1} \sim \left[\frac{1}{2B(f_1-1,f_2} \sum_{\ell=0}^{f_2-1} (-1)^{\ell} \binom{\frac{f_2-1}{2}}{\ell} e^{-\frac{Y_1'}{2}(f_1+\ell-1)} \right] *$$

$$\left[\frac{1}{2B(f_1-3,f_2)} \sum_{m=0}^{f_2-1} (-1)^{m} \binom{\frac{f_2-1}{2}}{m} e^{-\frac{Y_2'}{2}(f_1+m-3)}\right]$$

which gives us the probability density of $-\log U_{4,f_2,f_1}$ as

$$(4.4.2.6)\quad W_4 \sim \prod_{i=1}^{2} \frac{1}{2B(f_1-2i+1,f_2)} \left[\sum_{\ell=0}^{f_2-3} \binom{\frac{f_2-1}{2}}{\ell} \binom{\frac{f_2-1}{2}}{\ell+2} W_4\, e^{-\frac{W_4}{2}(f_1+\ell-1)} \right.$$

$$\left. + 2 \sum_{\substack{\ell=0 \\ \ell\neq m-2}}^{f_2-1} \sum_{m=0}^{f_2-1} \frac{(-1)^{\ell+m}}{(m-\ell-2)} \binom{\frac{f_2-1}{2}}{\ell} \binom{\frac{f_2-1}{2}}{m} \left(e^{-\frac{W_4}{2}(f_1+\ell-1)} - e^{-\frac{W_4}{2}(f_1+m-3)} \right) \right].$$

Now substitute

$$(4.4.2.7)\qquad W = -\log U$$

to get probability density function of U_{4,f_2,f_1} as

$$(4.4.2.8)\quad U_{4,f_2,f_1} \sim \prod_{i=1}^{2} \frac{1}{2B(f_1-2i+1,f_2)}\Big[\sum_{\ell=0}^{f_2-3} \binom{f_2-1}{\ell}\binom{f_2-1}{\ell+2}(-\log U)\, U^{\frac{f_1+\ell-3}{2}}$$

$$+ 2\sum_{\ell=0}^{f_2-1}\sum_{\substack{m=0\\ \ell\neq m-2}}^{f_2-1} \frac{(-1)^{\ell+m}}{(m-\ell-2)}\binom{f_2-1}{\ell}\binom{f_2-1}{m}\Big(U^{\frac{f_1+\ell-3}{2}} - U^{\frac{f_1+m-5}{2}}\Big)\Big]$$

which is a closed form for f_2 even or odd. Integrating (4.4.2.8) between limits (o,u) we get the distribution function of U_{4,f_2,f_1}.

$$(4.4.2.9)\quad P[U_{4,f_2,f_1} \leq u] = \frac{1}{2}\prod_{i=1}^{2} \frac{1}{B(f_1-2i+1,f_2)}\Big[\sum_{\ell=0}^{f_2-3} \frac{u^{\frac{f_1+\ell-1}{2}}}{(f_1+\ell-1)^2}\binom{f_2-1}{\ell}\binom{f_2-1}{\ell+2}\cdot(2-(f_1+\ell-1)\log u)$$

$$+ 2\sum_{\ell=0}^{f_2-1}\sum_{\substack{m=0\\ \ell\neq m-2}}^{f_2-1} \frac{(-1)^{\ell+m}}{(m-\ell-2)}\binom{f_2-1}{\ell}\binom{f_2-1}{m}\Big\{\frac{u^{\frac{f_1+\ell-1}{2}}}{f_1+\ell-1} - \frac{u^{\frac{f_1+m-3}{2}}}{f_1+m-3}\Big\}\Big]$$

A special case, which has been obtained by Wilks (1935) and Anderson (1958) is $f_2 = 4$. The algebraic forms of the exact probability and distribution functions for this case are respectively,

$$(4.4.2.10)\quad U_{4,4,f_1} \sim \frac{2\, U^{\frac{f_1-5}{2}}}{5B(f_1-1,4)B(f_1-3,4)} [1-15U^{\frac{1}{2}}-10U(8+3\log U)+10U^{3/2}$$

$$(8-3\log U) + 15U^2-U^{5/2}] \quad , 0 \leq U \leq 1$$

and

$$(4.4.2.11)\quad P[U_{4,4,f_1} \leq u] = \frac{4\, u^{\frac{f_1-3}{2}}}{5B(f_1-1,4)B(f_1-3,4)} \left[\frac{1}{f_1-3} - \frac{15\, u^{\frac{1}{2}}}{f_1-2} + \frac{10\, u}{(f_1-1)^2}\right.$$

$$\left.(6-(f_1-1)(3\log u+8)) + \frac{10u^{3/2}}{f_1^2}(6+f_1(8-3\log u)) + \frac{15u^2}{f_1+1} - \frac{u^{5/2}}{f_1+2}\right]$$

The expression (4.4.2.11) is much simpler than the result obtained by Wilks (1935) and Anderson (1958) and agrees with the result of Consul (1966). Further, Consul (1966) has shown that Anderson's result can be simplified to (4.4.2.11).

4.4.3. $p = 5$

Writing the Wilks' statistic as

$$(4.4.3.1)\qquad U_{5,f_2,f_1} = X_1\, X_2\, X_3\, X_4\, X_5$$

$$= Z_1\, Z_2\, X_5$$

and then taking the logarithm of the inverse, we get

$$(4.4.3.2)\quad -\log U_{5,f_2,f_1} = -\log Z_1 - \log Z_1 - \log X_5$$

$$= Y_1' + Y_2' + Y_5$$

$$= W_4 + Y_5 = W_5 \quad \text{(say)}$$

where W_4 is distributed as in (4.4.2.6) and

$$(4.4.3.3)\quad Y_5 \sim \left[1/B\left(\frac{f_1-4}{2}, \frac{f_2}{2}\right)\right] \sum_{n=0}^{\frac{f_2}{2}-1} (-1)^n \binom{\frac{f_2}{2}-1}{n} e^{-\frac{Y_5}{2}(f_1+2n-4)} .$$

Then the convolution technique yields the algebraic forms of the exact probability and distribution functions for U_{5,f_2,f_1} respectively as follows.

$$(4.4.3.4)\quad U_{5,f_2,f_1} \sim K\, U^{\frac{f_1-6}{2}} \Big[\sum_{n=2} (-1)^n f(2n-3,2n-1,n)\, U^n (\log U)^2$$

$$+ 4 \sum_{\substack{\ell \\ \ell \neq 2n-3}}^{f_2-3} \sum_{n} \frac{(-1)^n}{(2n-\ell-3)} f(\ell,\ell+2,n) \left\{ \frac{2}{(2n-\ell-3)} (U^n - U^{\ell+3}) - U^{\frac{\ell+3}{2}} \log U \right\}$$

$$- 4 \sum_{\substack{m \\ m \neq 2n-1}} \sum_{n=2} \frac{(-1)^{m+n-1}}{(m-2n+1)} f(2n-3,m,n)\, U^n \log U$$

$$- 8 \sum_{\substack{\ell \\ \ell \neq m-2,2n-3}} \sum_{m} \sum_{n} \frac{(-1)^{\ell+m+n}}{(m-\ell-2)(2n-\ell-3)} f(\ell,m,n) \left(U^n - U^{\frac{\ell+3}{2}}\right)$$

$$+ 4 \sum_{\substack{\ell \\ \ell \neq 2n-3}} \sum_{n=1} \frac{(-1)^{\ell+n+1}}{(2n-\ell-3)} f(\ell,2n-1,n)\, U^n \log U$$

$$+ 8 \sum_{\substack{\ell \\ \ell \neq m-2,\, m \neq 2n-1}} \sum_{m} \sum_{n} \frac{(-1)^{\ell+m+n}}{(m-\ell-2)(2n-m-1)} f(\ell,m,n) \left(U^n - U^{\frac{m+1}{2}}\right) \Big]$$

and

(4.4.3.5) $P[U_{5,f_2,f_1} \le u]$

$$= 2\,K\,u^{\frac{f_1-4}{2}} \Big[\sum_{n=2} \frac{(-1)^n\, u^n}{(f_1+2n-4)^3}\, f(2n-3,2n-1,n)\{(f_1+2n-4)^2(\log u)^2 - 4(f_1+2n-4)\log u + 8\}$$

$$+ 4 \sum_{\substack{\ell \\ \ell \neq 2n-3}}^{f_2-3} \sum_{n} \frac{(-1)^n \quad u^{\frac{\ell+3}{2}}}{(2n-\ell-3)(f_1+\ell-1)^2}\, f(\ell,\ell+2,n)\Big\{2-(f_1+\ell-1)\log u + \frac{2(f_1+\ell-1)}{(2n-\ell-3)}\Big(\frac{(f_1+\ell-1)u^{\frac{2n-\ell-3}{2}}}{(f_1+2n-4)} - 1\Big)\Big\}$$

$$+ 4 \sum_{\substack{m \\ m \neq 2n-1}} \sum_{n=2} \frac{(-1)^{m+n-1} \quad u^n}{(m-2n+1)(f_1+2n-4)^2}\, f(2n-3,m,n)(2-(f_1+2n-4)\log u)$$

$$+ 8 \sum_{\substack{\ell \\ \ell \neq m-2,\,2n-3}} \sum_{m} \sum_{n} \frac{(-1)^{\ell+m+n}}{(m-\ell-2)(2n-\ell-3)}\, f(\ell,m,n)\Big(\frac{u^n}{(f_1+2n-4)} - \frac{u^{\frac{\ell+3}{2}}}{(f_1+\ell-1)}\Big)$$

$$- 4 \sum_{\substack{\ell \\ \ell \neq 2n-3}} \sum_{n=1} \frac{(-1)^{\ell+n-1} \quad u^n}{(2n-\ell-3)(f_1+2n-4)^2}\, f(\ell,2n-1,n)(2-(f_1+2n-4)\log u)$$

$$+ 8 \sum_{\substack{\ell \\ \ell \neq m-2,\, m \neq 2n-1}} \sum_{m} \sum_{n} \frac{(-1)^{\ell+m+n}}{(m-\ell-2)(2n-m-1)}\, f(\ell,m,n)\Big(\frac{u^n}{(f_1+2n-4)} - \frac{u^{\frac{m+1}{2}}}{(f_1+m-3)}\Big)\Big]$$

where

$$(4.4.3.6) \qquad f(\ell,m,n) = \binom{f_2-1}{\ell}\binom{f_2-1}{m}\binom{\frac{f_2}{2}-1}{n}$$

and

$$(4.4.3.7) \qquad K = \left[1/2B\left(\frac{f_1-4}{2}, \frac{f_2}{2}\right)\right] \prod_{i=1}^{2} [1/2B(f_1-2i+1, f_2)]$$

and the indices ℓ,m and n run from 0 to f_2-1, f_2-1 and $\frac{f_2}{2}$ - 1 respectively unless specified otherwise.

4.4.4 p = 6

As in (4.4.3) we have

$$(4.4.4.1) \qquad U_{6,f_2,f_1} = X_1 X_2 X_3 X_4 X_5 X_6 = Z_1 Z_2 Z_3$$

and

$$(4.4.4.2) \qquad -\log U_{6,f_2,f_1} = -\log Z_1 - \log Z_2 - \log Z_3 = Y_1' + Y_2' + Y_3' = W_4 + Y_3' = W_6 \quad \text{(say)}$$

where W_4 is distributed as in (4.4.2.6) and

$$(4.4.4.3)\qquad Y_3' \sim [1/2B(f_1-5,f_2)]\sum_{n=0}^{f_2-1}(-1)^n\binom{f_2-1}{n}e^{-\frac{Y_3'}{2}(f_1+n-5)}\quad .$$

By convoluting we get the algebraic forms of the probability and distribution functions for U_{6,f_2,f_1} respectively as follows,

$$(4.4.4.4)\quad U_{6,f_2,f_1} \sim K\,U^{\frac{f_1-3}{2}}\Big[\sum_{\ell}^{f_2-5}(-1)^{\ell} f(\ell,\ell+2,\ell+4)\,U^{\frac{\ell}{2}}(\log U)^2$$

$$+ 4\sum_{\substack{\ell\\ \ell\neq n-4}}^{f_2-3}\sum_{n}\frac{(-1)^n}{(n-\ell-4)^2}f(\ell,\ell+2,n)\Big(2U^{\frac{n-4}{2}} - U^{\frac{\ell}{2}}(2+(n-\ell-4)\log U)\Big)$$

$$+ 8\sum_{\substack{\ell\\ \ell\neq m-2,\,n-4}}\sum_{m}\sum_{n}\frac{(-1)^{\ell+m+n}}{(m-\ell-2)(n-\ell-4)}f(\ell,m,n)\Big(U^{\ell/2} - U^{\frac{n-4}{2}}\Big)$$

$$- 4\sum_{\substack{\ell\\ \ell\neq m-2}}^{f_2-5}\sum_{m}\frac{(-1)^m}{(m-\ell-2)}f(\ell,m,\ell+4)\,U^{\ell/2}\log U$$

$$- 8\sum_{\substack{\ell\\ \ell\neq m-2,\,m\neq n-2}}\sum_{m}\sum_{n}\frac{(-1)^{\ell+m+n}}{(m-\ell-2)(n-m-2)}f(\ell,m,n)\Big(U^{\frac{m-2}{2}} - U^{\frac{n-4}{2}}\Big)$$

$$+ 4\sum_{\substack{\ell\\ \ell\neq m-2}}^{f_2-3}\sum_{m}\frac{(-1)^{\ell}}{(m-\ell-2)}f(\ell,m,m+2)\,U^{\frac{m-2}{2}}\log U\Big]$$

and

(4.4.4.5) $P\,[U_{6,f_2,f_2} \le u]$

$$= 2\,K\,u^{\frac{f_1-1}{2}} \Big[\sum_{\ell}^{\frac{f_2-5}{2}} \frac{(-1)^{\ell}\, u^{\frac{\ell}{2}}}{(f_1+\ell-1)^3}\, f(\ell,\ell+2,\ell+4)\{(f_1+\ell-1)^2(\log u)^2 - 4(f_1+\ell-1)\log u + 8\}$$

$$+ 4 \sum_{\substack{\ell\; n \\ \ell \neq n-4}}^{f_2-3} \frac{(-1)^n\, u^{\frac{\ell}{2}}}{(n-\ell-4)(f_1+\ell-1)^2}\, f(\ell,\ell+2,n)\Big\{2-(f_1+\ell-1)\log u + \frac{2(f_1+\ell-1)}{(n-\ell-4)} \Big(\frac{(f_1+\ell-1)u^{\frac{n-\ell-4}{2}}}{(f_1+n-5)} - 1\Big)\Big\}$$

$$+ 8 \sum_{\substack{\ell\; m\; n \\ \ell \neq m-2,\, n-4}} \frac{(-1)^{\ell+m+n}}{(m-\ell-2)(n-\ell-4)}\, f(\ell,m,n) \Big(\frac{u^{\ell/2}}{f_1+\ell-1} - \frac{u^{\frac{n-4}{2}}}{f_1+n-5}\Big)$$

$$+ 4 \sum_{\substack{\ell\; m \\ \ell \neq m-2}}^{f_2-5} \frac{(-1)^m\, u^{\ell/2}}{(m-\ell-2)(f_1+\ell-1)^2}\, f(\ell,m,\ell+4)(2-(f_1+\ell-1)\log u)$$

$$- 8 \sum_{\substack{\ell\; m\; n \\ \ell \neq m-2,\, m \neq n-2}} \frac{(-1)^{\ell+m+n}}{(m-\ell-2)(n-m-2)}\, f(\ell,m,n) \Big(\frac{u^{\frac{m-2}{2}}}{f_1+m-3} - \frac{u^{\frac{n-4}{2}}}{f_1+n-5}\Big)$$

$$- 4 \sum_{\substack{\ell\; m \\ \ell \neq m-2}}^{f_2-3} \frac{(-1)^{\ell}\, u^{\frac{m-2}{2}}}{(m-\ell-2)(f_1+m-3)^2}\, f(\ell,m,m+2)(2-(f_1+m-3)\log u)\Big]$$

where

(4.4.4.5) $$f(\ell,m,n) = \binom{f_2-1}{\ell}\binom{f_2-1}{m}\binom{f_2-1}{n}$$.

and

$$(4.4.4.6) \qquad K = \tfrac{1}{2} \prod_{i=1}^{3} [1/2B(f_1-2i+1,f_2)]$$

and the indices ℓ,m and n run from 0 to f_2-1 unless specified otherwise.

4.4.5. $p > 6$

Since by theorem 4.2.2 the distirbution of U_{p,f_2,f_1} is the same as of U_{f_2,p,f_1+f_2-p}, the distribution of U_{p,f_2,f_1} for $p > 6$ and $f_2 \le 6$ can be obtained from the previous results. For example the probability density function of U_{7,f_2,f_1} and U_{9,f_2,f_1} for $f_2 = 4,6$ can be obtained from (4.4.2.8) and (4.4.4.4) by substituting $f_2 = 7$ and 8 respectively. And probability density function of U_{8,f_2,f_1} and U_{10,f_2,f_1} for $f_2 = 3,4,5,6$ can be obtained from (4.4.1.8), (4.4.2.8), (4.4.3.4) and (4.4.4.4) by substituting $f_2 = 8$ and 10 respectively. The distribution functions are obtained similarly. However the exact distribution functions for $p > 6$ and $f_2 > 6$ become too involved for presentation as well as for computational purposes.

4.5 Computation of Percentage Points

The number of terms in the expressions for the probability and distribution functions increases very rapidly as p or f_2 increases. For the pair $(p,f_2) = (4,4)$ the number of terms are 8 and 10 respectively. Since Schatzoff's (1966) derivation proceeded in stages,

and

$$(4.4.4.6) \qquad K = \tfrac{1}{2} \prod_{i=1}^{3} [1/2B(f_1-2i+1,f_2)]$$

and the indices ℓ,m and n run from 0 to f_2-1 unless specified otherwise.

4.4.5, p > 6

Since by theorem 4.2.2 the distirbution of U_{p,f_2,f_1} is the same as of U_{f_2,p,f_1+f_2-p}, the distribution of U_{p,f_2,f_1} for $p > 6$ and $f_2 \leq 6$ can be obtained from the previous results. For example the probability density function of U_{7,f_2,f_1} and U_{9,f_2,f_1} for $f_2 = 4,6$ can be obtained from (4.4.2.8) and (4.4.4.4) by substituting $f_2 = 7$ and 8 respectively. And probability density function of U_{8,f_2,f_1} and U_{10,f_2,f_1} for $f_2 = 3,4,5,6$ can be obtained from (4.4.1.8), (4.4.2.8), (4.4.3.4) and (4.4.4.4) by substituting $f_2 = 8$ and 10 respectively. The distribution functions are obtained similarly. However the exact distribution functions for $p > 6$ and $f_2 > 6$ become too involved for presentation as well as for computational purposes.

4.5 Computation of Percentage Points

The number of terms in the expressions for the probability and distribution functions increases very rapidly as p or f_2 increases. For the pair $(p,f_2) = (4,4)$ the number of terms are 8 and 10 respectively. Since Schatzoff's (1966) derivation proceeded in stages, an error at any stage would promulgate errors at all succeeding stages. We have no such problem, since we have derived explicitly the distribution at each stage separately.

The method developed in the preceding section was used in the preparation of the tables of percentage points of Λ. Tabulation of exact percentiles of Λ requires solving for the roots of $F(u) = \alpha$, where $F(u)$ is the distribution function.

Each table represents a particular pair (p,f_2), with arguments $M = f_1-p+1$ and significance level α. By defining M(Schatzoff,1966) in this manner, each Table starts with argument unity. Further, since by theorem 4.2.2 the distribution of U_{p,f_2,f_1} is the same as that of U_{f_2,p,f_1+f_2-p}, we see that interchanging the roles of p and f_2 does not affect the computation of M. For example, the distribution of $U_{4,7,12}$ and $U_{7,4,15}$ are identical, and in either case, $M = 9$. Thus, to use the tables when f_2 is odd and p is even, enter the table with arguments f_2, p and $M = f_1-p+1$. Table 6 gives the percentage points for $p = 3$, $f_2 = 12(2)22$ and $\alpha = .10, .05, .025, .010, .005$. Table 7 gives the percentage points for $p = 4$, $f_2 = 11(1)13(2)23$ and $\alpha = .10, .05, .025, .010, .005$. Table 8 gives the percentage points for $p = 6$, $f_2 = 11$ and $\alpha = .05, .01$. Table 8 also gives the percentage points for $p = 6$, $f_2 = 12, 13$ and $\alpha = .05$ only. Because of the inherent decimal accuracy problem, all calculations were carried out on IBM 7094, using double precision arithmetic.

Table 6

Percentage Points of Wilks' Criterion U

for $p = 3$ and $f_2 = 12(2)22$

			$f_2 = 12$		
M \ α	.100	.050	.025	.010	.005
1	.92705(-5)	.21622(-5)	.52231(-6)	.81868(-7)	.20327(-7)
2	.28648(-3)	.12464(-3)	.56787(-4)	.20971(-4)	.10078(-4)
3	.13777(-2)	.74172(-3)	.41669(-3)	.20280(-3)	.12023(-3)
4	.36674(-2)	.22136(-2)	.13880(-2)	.77837(-3)	.51298(-3)
5	.73198(-2)	.47566(-2)	.31986(-2)	.19606(-2)	.13820(-2)
6	.12342(-1)	.84534(-2)	.59725(-2)	.38959(-2)	.28695(-2)
7	.18647(-1)	.13291(-1)	.97466(-2)	.66607(-2)	.50747(-2)
8	.26099(-1)	.19198(-1)	.14496(-1)	.10274(-1)	.80369(-2)
9	.34542(-1)	.26069(-1)	.20157(-1)	.14714(-1)	.11757(-1)
10	.43818(-1)	.33785(-1)	.26644(-1)	.19930(-1)	.16207(-1)
12	.64294(-1)	.51298(-1)	.41737(-1)	.32444(-1)	.27125(-1)
14	.86493(-1)	.70784(-1)	.58958(-1)	.47171(-1)	.40258(-1)
16	.10964	.91534(-1)	.77661(-1)	.63558(-1)	.55127(-1)
18	.13312	.11297	.97292(-1)	.81094(-1)	.71261(-1)
20	.15655	.13466	.11742	.99360(-1)	.88259(-1)
24	.20218	.17764	.15791	.13683	.12360
30	.26561	.23866	.21653	.19235	.17686
40	.35589	.32748	.30364	.27696	.25948
60	.48791	.46057	.43707	.41009	.39199
120	.68727	.66690	.64892	.62770	.61309

p = number of variates; f_2 = hypothesis d.f.; f_1 = error d.f.;

$M = f_1 - p + 1$; the numbers in parentheses indicate the power of 10 by which tabulated values are to be multiplied.

Table 6 (Cont'd.)

		$f_2 = 14$			
M \ α	.100	.050	.025	.010	.005
1	.59470(-5)	.13855(-5)	.33448(-6)	.52408(-7)	.13011(-7)
2	.18808(-3)	.81585(-4)	.37096(-4)	.13675(-4)	.65660(-5)
3	.92497(-3)	.49596(-3)	.27779(-3)	.13481(-3)	.79789(-4)
4	.25136(-2)	.15100(-2)	.94341(-3)	.52713(-3)	.34664(-3)
5	.51128(-2)	.33052(-2)	.22137(-2)	.13512(-2)	.94849(-3)
6	.87710(-2)	.59747(-2)	.42030(-2)	.27292(-2)	.20044(-2)
7	.13462(-1)	.95416(-2)	.69654(-2)	.47372(-2)	.35983(-2)
8	.19116(-1)	.13981(-1)	.10508(-1)	.74103(-2)	.57785(-2)
9	.25637(-1)	.19237(-1)	.14805(-1)	.10751(-1)	.85632(-2)
10	.32957(-1)	.25268(-1)	.19838(-1)	.14768(-1)	.11973(-1)
12	.49366(-1)	.39166(-1)	.31716(-1)	.24527(-1)	.20436(-1)
14	.67683(-1)	.55066(-1)	.45652(-1)	.36336(-1)	.30907(-1)
16	.87181(-1)	.72382(-1)	.61131(-1)	.49774(-1)	.43029(-1)
18	.10737	.90628(-1)	.77701(-1)	.64441(-1)	.56443(-1)
20	.12786	.10941	.94987(-1)	.79989(-1)	.70824(-1)
24	.16868	.14748	.13057	.11261	.10141
30	.22713	.20318	.18365	.16245	.14895
40	.31327	.28717	.26540	.24119	.22542
60	.44455	.41841	.39606	.37054	.35351
120	.65286	.63241	.61444	.59334	.57887

Table 6 (Cont'd)

$f_2 = 16$					
M \ α	.100	.050	.025	.010	.005
1	.40404(-5)	.94046(-6)	.22695(-6)	.35550(-7)	.88247(-8)
2	.13006(-3)	.56292(-4)	.25556(-4)	.94083(-5)	.45141(-5)
3	.65086(-3)	.34790(-3)	.19441(-3)	.94136(-4)	.55643(-4)
4	.17977(-2)	.10760(-2)	.67034(-3)	.37348(-3)	.24518(-3)
5	.37121(-2)	.23900(-2)	.15956(-2)	.97074(-3)	.68004(-3)
6	.64571(-2)	.43795(-2)	.30703(-2)	.19865(-2)	.14557(-2)
7	.10039(-1)	.70830(-2)	.51520(-2)	.34904(-2)	.26449(-2)
8	.14425(-1)	.10501(-1)	.78628(-2)	.55228(-2)	.42957(-2)
9	.19558(-1)	.14607(-1)	.11198(-1)	.80989(-2)	.64336(-2)
10	.25405(-1)	.93867(-1)	.15162(-1)	.11239(-1)	.90886(-2)
12	.38771(-1)	.30597(-1)	.24682(-1)	.19006(-1)	.15793(-1)
14	.53979(-1)	.43711(-1)	.36097(-1)	.28608(-1)	.24266(-1)
16	.70506(-1)	.58268(-1)	.49021(-1)	.39745(-1)	.34263(-1)
18	.87914(-1)	.73870(-1)	.63094(-1)	.52108(-1)	.45515(-1)
20	.10585	.90179(-1)	.78000(-1)	.65413(-1)	.57762(-1)
24	.14229	.12388	.10929	.93888(-1)	.84334(-1)
30	.19587	.17454	.15726	.13860	.12677
40	.27738	.25342	.23354	.21155	.19730
60	.40641	.38151	.36032	.33625	.32024
120	.62093	.60054	.58270	.56181	.54753

Table 6 (Cont'd.)

	f_2 = 18				
M \ α	.100	.050	.025	.010	.005
1	.28692(-5)	.66739(-6)	.16100(-6)	.25214(-7)	.62584(-8)
2	.93663(-4)	.40465(-4)	.18348(-4)	.67476(-5)	.32357(-5)
3	.47525(-3)	.25340(-3)	.14134(-3)	.68318(-4)	.40342(-4)
4	.13300(-2)	.79368(-3)	.49336(-3)	.27424(-3)	.17979(-3)
5	.27802(-2)	.17841(-2)	.11881(-2)	.72087(-3)	.50418(-3)
6	.48917(-2)	.33061(-2)	.23113(-2)	.14898(-2)	.10906(-2)
7	.76863(-2)	.54031(-2)	.39185(-2)	.26464(-2)	.20013(-2)
8	.11154(-1)	.80890(-2)	.60383(-2)	.42272(-2)	.32811(-2)
9	.15263(-1)	.11355(-1)	.86780(-2)	.62547(-2)	.49578(-2)
10	.19966(-1)	.15176(-1)	.11828(-1)	.87342(-2)	.70446(-2)
12	.31003(-1)	.24371(-1)	.19594(-1)	.15034(-1)	.12464(-1)
14	.43758(-1)	.35293(-1)	.29050(-1)	.22940(-1)	.19413(-1)
16	.57853(-1)	.47623(-1)	.39935(-1)	.32261(-1)	.27746(-1)
18	.72922(-1)	.61036(-1)	.51965(-1)	.42762(-1)	.37264(-1)
20	.88659(-1)	.75246(-1)	.64877(-1)	.54215(-1)	.47763(-1)
24	.12118	.10513	.92464(-1)	.79161(-1)	.70945(-1)
30	.17018	.15114	.13579	.11929	.10888
40	.24689	.22489	.20673	.18673	.17381
60	.37267	.34901	.32897	.30628	.29124
120	.59123	.57099	.55334	.53275	.51871

Table 6 (Cont'd.)

			$f_2 = 20$		
M \ α	.100	.050	.025	.010	.005
1	.21103(-5)	.49061(-6)	.11832(-6)	.18526(-7)	.45983(-8)
2	.69675(-4)	.30058(-4)	.13616(-4)	.50029(-5)	.23980(-5)
3	.35758(-3)	.19027(-3)	.10597(-3)	.51145(-4)	.30176(-4)
4	.10116(-2)	.60219(-3)	.37362(-3)	.20729(-3)	.13575(-3)
5	.21362(-2)	.13671(-2)	.908845(-3)	.55000(-3)	.38415(-3)
6	.37947(-2)	.25571(-2)	.17836(-2)	.11477(-2)	.83827(-3)
7	.60161(-2)	.42161(-2)	.30501(-2)	.20545(-2)	.15512(-2)
8	.88038(-2)	.63643(-2)	.47386(-2)	.33081(-2)	.25633(-2)
9	.12142(-1)	.90035(-2)	.68627(-2)	.49321(-2)	.39023(-2)
10	.16003(-1)	.12122(-1)	.94221(-2)	.69370(-2)	.55845(-2)
12	.25186(-1)	.19731(-1)	.15820(-1)	.12101(-1)	.10013(-1)
14	.35974(-1)	.28916(-1)	.23734(-1)	.18684(-1)	.15780(-1)
16	.48071(-1)	.39437(-1)	.32978(-1)	.26558(-1)	.22794(-1)
18	.61175(-1)	.51032(-1)	.43327(-1)	.35543(-1)	.30910(-1)
20	.75023(-1)	.63463(-1)	.54566(-1)	.45459(-1)	.39968(-1)
24	.10410	.90017(-1)	.78961(-1)	.67399(-1)	.60285(-1)
30	.14885	.13180	.11812	.10347	.94265(-1)
40	.22078	.20058	.18397	.16575	.15402
60	.34268	.32024	.30129	.27992	.26581
120	.56353	.54351	.52611	.50586	.49210

Table 6 (Cont'd.)

			$f_2 = 22$		
M \ α	.100	.050	.025	.010	.005
1	.15972(-5)	.37114(-6)	.89487(-7)	.14010(-7)	.34771(-8)
2	.53229(-4)	.22936(-4)	.10381(-4)	.38116(-5)	.18263(-5)
3	.27576(-3)	.14649(-3)	.81484(-4)	.39280(-4)	.23160(-4)
4	.78725(-3)	.46769(-3)	.28972(-3)	.16049(-3)	.10500(-3)
5	.16768(-2)	.10707(-2)	.71020(-3)	.42918(-3)	.29943(-3)
6	.30030(-2)	.20186(-2)	.14052(-2)	.90240(-3)	.65824(-3)
7	.47974(-2)	.33533(-2)	.24208(-2)	.16270(-2)	.12267(-2)
8	.70709(-2)	.50979(-2)	.37874(-2)	.26378(-2)	.20409(-2)
9	.97599(-2)	.72389(-2)	.55214(-2)	.39586(-2)	.31272(-2)
10	.13018(-1)	.98360(-2)	.76282(-2)	.56022(-2)	.45368(-2)
12	.20742(-1)	.16202(-1)	.12960(-1)	.98874(-2)	.81672(-2)
14	.29938(-1)	.23994(-1)	.19646(-1)	.15425(-1)	.13004(-1)
16	.40385(-1)	.33034(-1)	.27556(-1)	.22131(-1)	.18962(-1)
18	.51835(-1)	.43114(-1)	.36514(-1)	.29873(-1)	.25933(-1)
20	.64063(-1)	.54034(-1)	.46347(-1)	.38507(-1)	.33795(-1)
24	.90099(-1)	.77695(-1)	.67991(-1)	.57881(-1)	.51681(-1)
30	.13098	.11567	.10343	.90375(-1)	.82195(-1)
40	.19829	.17972	.16450	.14786	.13719
60	.31591	.29464	.27674	.25663	.24338
120	.53763	.51789	.50078	.48092	.46745

Table 7

Percentage Points of Wilks' Criterion U

for p = 4 and f_2 = 11(1)13(2)23

$f_2 = 11$

M \ α	.100	.050	.025	.010	.005
1	.13773(-7)	.34196(-8)	.85196(-9)	.13603(-9)	.33984(-10)
2	.85307(-4)	.36251(-4)	.16238(-4)	.59027(-5)	.28129(-5)
3	.47887(-3)	.24866(-3)	.13684(-3)	.65205(-4)	.38171(-4)
4	.14153(-2)	.83203(-3)	.51073(-3)	.27993(-3)	.18190(-3)
5	.30973(-2)	.19622(-2)	.12922(-2)	.77412(-3)	.53690(-3)
6	.56328(-2)	.37660(-2)	.26078(-2)	.16636(-2)	.12077(-2)
7	.90658(-2)	.63160(-2)	.45437(-2)	.30393(-2)	.22833(-2)
8	.13388(-1)	.96379(-2)	.71462(-2)	.49622(-2)	.38297(-2)
9	.18556(-1)	.13721(-1)	.10428(-1)	.74647(-2)	.58882(-2)
10	.24502(-1)	.18530(-1)	.14376(-1)	.10555(-1)	.84781(-2)
12	.38413(-1)	.30106(-1)	.24131(-1)	.18441(-1)	.15242(-1)
14	.54468(-1)	.43865(-1)	.36044(-1)	.28393(-1)	.23981(-1)
16	.72053(-1)	.59289(-1)	.49683(-1)	.40085(-1)	.34437(-1)
18	.90666(-1)	.75925(-1)	.64647(-1)	.53184(-1)	.46326(-1)
20	.10986	.93340(-1)	.80530(-1)	.67324(-1)	.59315(-1)
24	.14880	.12934	.11393	.97669(-1)	.87598(-1)
30	.20580	.18326	.16500	.14527	.13277
40	.29154	.26639	.24550	.22236	.20734
60	.42465	.39889	.37693	.35192	.33527
120	.63909	.61854	.60052	.57939	.56491

p = number of variates; f_2 = hypothesis d.f.; f_1 = error d.f.;

$M = f_1 - p + 1$; the numbers in parentheses indicate the power of 10 by which tabulated values are to be multiplied.

Table 7 (Cont'd.)

	f_2 = 12				
M \ α	.100	.050	.025	.010	.005
1	.84954(-8)	.21102(-8)	.52584(-9)	.83970(-10)	.20979(-10)
2	.63094(-4)	.26748(-4)	.11961(-4)	.43414(-5)	.20672(-5)
3	.35698(-3)	.18660(-3)	.10244(-3)	.48701(-4)	.28470(-4)
4	.10826(-2)	.63421(-3)	.38821(-3)	.21217(-3)	.13761(-3)
5	.24039(-2)	.15172(-2)	.99610(-3)	.67899(-3)	.41170(-3)
6	.44295(-2)	.29498(-2)	.20362(-2)	.12945(-2)	.93772(-3)
7	.72144(-2)	.50059(-2)	.35894(-2)	.23925(-2)	.17934(-2)
8	.10770(-1)	.77214(-2)	.57062(-2)	.39479(-2)	.30398(-2)
9	.15074(-1)	.11102(-1)	.84089(-2)	.59973(-2)	.47195(-2)
10	.20085(-1)	.15128(-1)	.11698(-1)	.85568(-2)	.68569(-2)
12	.31993(-1)	.24975(-1)	.19953(-1)	.15192(-1)	.12527(-1)
14	.45983(-1)	.36890(-1)	.30215(-1)	.23717(-1)	.19984(-1)
16	.61567(-1)	.50477(-1)	.42170(-1)	.33909(-1)	.29067(-1)
18	.78242(-1)	.65286(-1)	.55421(-1)	.45440(-1)	.39493(-1)
20	.95654(-1)	.80992(-1)	.69674(-1)	.58057(-1)	.51040(-1)
24	.13152	.11396	.10010	.85555(-1)	.76577(-1)
30	.18508	.16434	.14759	.12958	.11821
40	.26751	.24383	.22424	.20262	.18863
60	.39897	.37406	.35290	.32889	.31294
120	.61760	.59710	.57917	.55819	.54385

Table 7 (Cont'd.)

	f_2 = 13				
M \ α	.100	.050	.025	.010	.005
1	.54329(-8)	.13500(-8)	.33647(-9)	.53737(-10)	.13426(-10)
2	.47670(-4)	.20168(-4)	.98728(-5)	.32643(-5)	.15533(-5)
3	.27388(-3)	.14277(-3)	.78222(-4)	.37111(-4)	.21669(-4)
4	.84238(-3)	.49193(-3)	.30038(-3)	.19782(-3)	.10609(-3)
5	.18946(-2)	.11917(-2)	.78034(-3)	.46469(-3)	.32109(-3)
6	.35318(-2)	.23438(-2)	.16133(-2)	.10226(-2)	.73937(-3)
7	.58137(-2)	.40197(-2)	.28739(-2)	.19096(-2)	.14285(-2)
8	.87633(-2)	.62603(-2)	.46128(-2)	.31812(-2)	.24444(-2)
9	.12375(-1)	.90810(-2)	.68580(-2)	.48752(-2)	.38285(-2)
10	.16624(-1)	.12476(-1)	.96184(-2)	.70128(-2)	.56077(-2)
12	.26867(-1)	.20899(-1)	.16647(-1)	.12634(-1)	.10395(-1)
14	.39100(-1)	.31259(-1)	.25529(-1)	.19974(-1)	.16794(-1)
16	.52923(-1)	.43244(-1)	.36026(-1)	.28877(-1)	.24702(-1)
18	.67895(-1)	.56468(-1)	.47805(-1)	.39075(-1)	.33892(-1)
20	.83700(-1)	.70648(-1)	.60614(-1)	.50357(-1)	.44182(-1)
24	.11672	.10083	.88353(-1)	.75300(-1)	.67272(-1)
30	.16695	.14784	.13248	.11602	.10566
40	.24597	.22368	.20531	.18511	.17207
60	.37528	.35124	.33087	.30783	.29256
120	.59710	.57669	.55888	.53809	.52391

Table 7 (Cont'd.)

	$f_2 = 15$				
M \ α	.100	.050	.025	.010	.005
1	.24292(-8)	.60402(-9)	.15059(-9)	.24056(-10)	.60104(-11)
2	.28697(-4)	.12100(-4)	.53905(-5)	.19498(-5)	.92671(-6)
3	.16910(-3)	.87758(-4)	.47920(-4)	.22659(-4)	.13202(-4)
4	.53248(-3)	.30936(-3)	.18815(-3)	.10215(-3)	.65990(-4)
5	.12237(-2)	.76546(-3)	.49900(-3)	.29578(-3)	.20378(-3)
6	.23268(-2)	.15352(-2)	.10517(-2)	.66327(-3)	.47804(-3)
7	.39004(-2)	.26806(-2)	.19072(-2)	.12605(-2)	.93980(-3)
8	.59784(-2)	.42447(-2)	.31120(-2)	.21960(-2)	.16230(-2)
9	.85737(-2)	.62526(-2)	.46981(-2)	.33213(-2)	.25988(-2)
10	.11683(-1)	.87137(-2)	.66834(-2)	.48455(-2)	.38605(-2)
12	.19368(-1)	.14973(-1)	.11866(-1)	.89545(-2)	.73407(-2)
14	.28842(-1)	.22923(-1)	.18631(-1)	.14499(-1)	.12149(-1)
16	.39770(-1)	.32306(-1)	.26781(-1)	.21349(-1)	.18197(-1)
18	.51895(-1)	.42915(-1)	.36156(-1)	.29395(-1)	.25405(-1)
20	.64950(-1)	.54517(-1)	.46556(-1)	.38474(-1)	.33639(-1)
24	.92927(-1)	.79860(-1)	.69666(-1)	.59078(-1)	.52605(-1)
30	.13698	.12072	.10774	.93919(-1)	.85270(-1)
40	.20914	.18941	.17325	.15557	.14423
60	.33313	.31079	.29196	.27078	.25681
120	.55878	.53866	.52117	.50084	.48702

Table 7 (Cont'd.)

$f_2 = 17$					
M \ α	.100	.050	.025	.010	.005
1	.11950(-8)	.29730(-9)	.74142(-10)	.11846(-10)	.29598(-11)
2	.18297(-4)	.76948(-5)	.34215(-5)	.12356(-5)	.58662(-6)
3	.11000(-3)	.56889(-4)	.30980(-4)	.14610(-4)	.85000(-5)
4	.35298(-3)	.20424(-3)	.12382(-3)	.67000(-4)	.43199(-4)
5	.82552(-3)	.51407(-3)	.33394(-3)	.19718(-3)	.13552(-3)
6	.15953(-2)	.10476(-2)	.71495(-3)	.44905(-3)	.32281(-3)
7	.27146(-2)	.18565(-2)	.13156(-2)	.86579(-3)	.64375(-3)
8	.42191(-2)	.29806(-2)	.21763(-2)	.14861(-2)	.11346(-2)
9	.61293(-2)	.44473(-2)	.33277(-2)	.23417(-2)	.18270(-2)
10	.84529(-2)	.62724(-2)	.47907(-2)	.34572(-2)	.27462(-2)
12	.14321(-1)	.11015(-1)	.86920(-2)	.65284(-2)	.53351(-2)
14	.21739(-1)	.17191(-1)	.13913(-1)	.10777(-1)	.90026(-2)
16	.30481(-1)	.24637(-1)	.20340(-1)	.16139(-1)	.13714(-1)
18	.40370(-1)	.33222(-1)	.27876(-1)	.22559(-1)	.19438(-1)
20	.51203(-1)	.42774(-1)	.36382(-1)	.29931(-1)	.26092(-1)
24	.74951(-1)	.64121(-1)	.55724(-1)	.47051(-1)	.41776(-1)
30	.11351	.99630(-1)	.88602(-1)	.76927(-1)	.69657(-1)
40	.17908	.16159	.14734	.13184	.12194
60	.29690	.27619	.25883	.23937	.22659
120	.52369	.50395	.48686	.46707	.45365

Table 7 (Cont'd.)

	$f_2 = 19$				
M \ α	.100	.050	.025	.010	.005
1	.63378(-9)	.15774(-9)	.39347(-10)	.62872(-11)	.15711(-11)
2	.12212(-4)	.51246(-5)	.22753(-5)	.82048(-6)	.38935(-6)
3	.74628(-4)	.38483(-4)	.20912(-4)	.98399(-5)	.57181(-5)
4	.24320(-3)	.14024(-3)	.84799(-4)	.45762(-4)	.29452(-4)
5	.57704(-3)	.35802(-3)	.23186(-3)	.13651(-3)	.93640(-4)
6	.11303(-2)	.73931(-3)	.50299(-3)	.31486(-3)	.22587(-3)
7	.19477(-2)	.13266(-2)	.93700(-3)	.61444(-3)	.45582(-3)
8	.30629(-2)	.21548(-2)	.15680(-2)	.10667(-2)	.81251(-3)
9	.44987(-2)	.32503(-2)	.24236(-2)	.16990(-2)	.13223(-2)
10	.62683(-2)	.46314(-2)	.35248(-2)	.25337(-2)	.20076(-2)
12	.10820(-1)	.82857(-2)	.65152(-2)	.48738(-2)	.39728(-2)
14	.16702(-1)	.13151(-1)	.10606(-1)	.81821(-2)	.68172(-2)
16	.23764(-1)	.19127(-1)	.15734(-1)	.12434(-1)	.10539(-1)
18	.31890(-1)	.26133(-1)	.21851(-1)	.17613(-1)	.15137(-1)
20	.40928(-1)	.34050(-1)	.28862(-1)	.23652(-1)	.20565(-1)
24	.61150(-1)	.52108(-1)	.45133(-1)	.37966(-1)	.33625(-1)
30	.94917(-1)	.83001(-1)	.73585(-1)	.63663(-1)	.57512(-1)
40	.15430	.13878	.12620	.11256	.10389
60	.26557	.24641	.23041	.21254	.20085
120	.49145	.47218	.45554	.43632	.42334

Table 7 (Cont'd.)

			$f_2 = 21$		
M \ α	.100	.050	.025	.010	.005
1	.35718(-9)	.88930(-10)	.22187(-10)	.35455(-11)	.88600(-12)
2	.84599(-5)	.35439(-5)	.15715(-5)	.56607(-6)	.26846(-6)
3	.52404(-4)	.26959(-4)	.14623(-4)	.68686(-5)	.39874(-5)
4	.17299(-3)	.99481(-4)	.60021(-4)	.32320(-4)	.20773(-4)
5	.41548(-3)	.25699(-3)	.16605(-3)	.97500(-4)	.66776(-4)
6	.82318(-3)	.53667(-3)	.36416(-3)	.22731(-3)	.16277(-3)
7	.14337(-2)	.97321(-3)	.68548(-3)	.44814(-3)	.33181(-3)
8	.22775(-2)	.15966(-2)	.11590(-2)	.78560(-3)	.59720(-3)
9	.33769(-2)	.24310(-2)	.18073(-2)	.12628(-2)	.98069(-3)
10	.47472(-2)	.34947(-2)	.26516(-2)	.18997(-2)	.15020(-2)
12	.83280(-2)	.63543(-2)	.49818(-2)	.37133(-2)	.30204(-2)
14	.13047(-1)	.10235(-1)	.82287(-2)	.63265(-2)	.52594(-2)
16	.18806(-1)	.15080(-1)	.12367(-1)	.97398(-2)	.82360(-2)
18	.25533(-1)	.20847(-1)	.17378(-1)	.13959(-1)	.11970(-1)
20	.33119(-1)	.27454(-1)	.23200(-1)	.18947(-1)	.16437(-1)
24	.50403(-1)	.42801(-1)	.36963(-1)	.30990(-1)	.27387(-1)
30	.80007(-1)	.69735(-1)	.61653(-1)	.53173(-1)	.47930(-1)
40	.13371	.11992	.10877	.96746(-1)	.89123(-1)
60	.23836	.22064	.20589	.18948	.17878
120	.46177	.44300	.42685	.40825	.39571

Table 7 (Cont'd.)

f_2 = 23					
M \ α	.100	.050	.025	.010	.005
1	.21164(-9)	.52711(-10)	.13153(-10)	.21020(-11)	.52530(-12)
2	.60449(-5)	.25286(-5)	.11201(-5)	.40309(-6)	.19108(-6)
3	.37873(-4)	.19445(-4)	.10531(-4)	.49393(-5)	.28649(-5)
4	.12640(-3)	.72517(-4)	.43672(-4)	.23471(-4)	.15069(-4)
5	.30675(-3)	.18924(-3)	.12202(-3)	.71488(-4)	.48897(-4)
6	.61373(-3)	.39900(-3)	.27014(-3)	.16822(-3)	.12027(-3)
7	.10789(-2)	.73017(-3)	.51308(-3)	.33457(-3)	.24732(-3)
8	.17288(-2)	.12082(-2)	.87450(-3)	.66789(-3)	.44882(-3)
9	.25845(-2)	.18547(-2)	.13753(-2)	.95831(-3)	.74290(-3)
10	.36614(-2)	.26869(-2)	.20334(-2)	.14527(-2)	.11464(-2)
12	.65354(-2)	.49724(-2)	.38892(-2)	.28927(-2)	.23494(-2)
14	.10343(-1)	.80872(-2)	.64842(-2)	.49704(-2)	.41239(-2)
16	.15080(-1)	.12053(-1)	.98576(-2)	.77402(-2)	.65322(-2)
18	.20690(-1)	.16838(-1)	.13998	.11210(-1)	.95937(-2)
20	.27094(-1)	.22388(-1)	.18868(-1)	.15363(-1)	.13302(-1)
24	.41930(-1)	.35495(-1)	.30574(-1)	.25558	.22542
30	.67934	.59038	.52066	.44779	.40292
40	.11648	.10419	.94290(-1)	.83652(-1)	.76929(-1)
60	.21460	.19822	.18462	.16955	.15974
120	.43439	.41616	.40051	.38253	.37043

Table 7a

Percentage Points of Wilks' Criterion U

for p = 5 and f_2 = 12(2)16

	f_2= 12		f_2= 14		f_2= 16	
M \ α	.05	.01	.05	.01	.05	.01
1	.88625(-7)	.32921(-8)	.44258(-7)	.16412(-8)	.24071(-7)	.89138(-9)
2	.70478(-5)	.11143(-5)	.36386(-5)	.57161(-6)	.20307(-5)	.31744(-6)
3	.55426(-4)	.13976(-4)	.29579(-4)	.73895(-5)	.16943(-4)	.42022(-5)
4	.20850(-3)	.67186(-4)	.11477(-3)	.36578(-4)	.67377(-4)	.21289(-4)
5	.54353(-3)	.20511(-3)	.30788(-3)	.11477(-3)	.18491(-3)	.68284(-4)
6	.11374(-2)	.48053(-3)	.66121(-3)	.27583(-3)	.40570(-3)	.16752(-3)
7	.20567(-2)	.94709(-3)	.12249(-2)	.55661(-3)	.76633(-3)	.34462(-3)
8	.33523(-2)	.16536(-2)	.20420(-2)	.99323(-3)	.13008(-2)	.62611(-3)
9	.50608(-2)	.26404(-2)	.31530(-2)	.16184(-2)	.20379(-2)	.10374(-2)
10	.72010(-2)	.39384(-2)	.45587(-2)	.24596(-2)	.30015(-2)	.16015(-2)
12	.12792(-1)	.75419(-2)	.83743(-2)	.48709(-2)	.56724(-2)	.32626(-2)
14	.20090(-1)	.12552(-1)	.13544(-1)	.83515(-2)	.94090(-2)	.57385(-2)
16	.28867(-1)	.18873(-1)	.19958(-1)	.12881(-1)	.14169(-1)	.90467(-2)
18	.38943(-1)	.26423(-1)	.27527(-1)	.18445(-1)	.19924(-1)	.13210(-1)
20	.50090(-1)	.35054(-1)	.36112(-1)	.24966(-1)	.26594(-1)	.18196(-1)
24	.74754(-1)	.54931(-1)	.55726(-1)	.40482(-1)	.42273(-1)	.30409(-1)
30	.11508	.89082(-1)	.89207(-1)	.68338(-1)	.70090(-1)	.53209(-1)
40	.18367	.15036	.14912	.12098	.12222	.98374(-1)
60	.30592	.26607	.26257	.22678	.22659	.19449
120	.53618	.49827	.49255	.45584	.45340	.41803

p = number of variates; f_2 = hypothesis d.f.; f_1 = error d.f.; $M = f_1 - p + 1$; the numbers in parentheses indicate the power of 10 by which tabulated values are to be multiplied.

Table 8

Percentage Points of Wilks' Criterion U

for p = 6 and f_2 = 11(1)13

	f_2= 11		f_2= 12		f_2= 13	
M \ α	.05	.01	.05	.01	.05	.01
1	.38408(-7)	.14198(-8)	.24498(-7)	.90441(-9)	.16131(-7)	.59584(-9)
2	.33078(-5)	.51481(-6)	.21601(-5)	.33466(-6)	.14517(-5)	.22401(-6)
3	.27938(-4)	.69021(-5)	.18672(-4)	.45832(-5)	.12806(-4)	.31255(-5)
4	.11166(-3)	.35196(-4)	.76238(-4)	.23854(-4)	.53285(-4)	.16560(-4)
5	.30639(-3)	.11307(-3)	.21335(-3)	.78091(-4)	.15172(-3)	.55132(-4)
6	.66973(-3)	.27681(-3)	.47472(-3)	.19456(-3)	.34265(-3)	.13951(-3)
7	.12577(-2)	.56693(-3)	.90501(-3)	.40486(-3)	.66217(-3)	.29450(-3)
8	.21190(-2)	.10235(-2)	.15474(-2)	.74176(-3)	.11465(-2)	.54676(-3)
9	.32926(-2)	.16826(-2)	.24388(-2)	.12365(-2)	.18343(-2)	.92266(-3)
10	.48110(-2)	.25779(-2)	.36037(-2)	.19169(-2)	.27413(-2)	.14467(-2)
12	.89024(-2)	.51578(-2)	.68301(-2)	.39229(-2)	.53017(-2)	.30213(-2)
14	.14479(-1)	.89032(-2)	.11325(-1)	.69057(-2)	.89485(-2)	.54155(-2)
16	.21403(-1)	.13788(-1)	.17016(-1)	.10873(-1)	.13650(-1)	.86579(-2)
18	.29571(-1)	.19790(-1)	.23849(-1)	.15835(-1)	.19388(-1)	.12781(-1)
20	.38818(-1)	.26819(-1)	.31706(-1)	.21739(-1)	.26082(-1)	.17758(-1)
24	.59867(-1)	.43496(-1)	.49954(-1)	.36037(-1)	.41924(-1)	.30047(-1)
30	.95541(-1)	.73251(-1)	.81720(-1)	.62254(-1)	.70207(-1)	.53168(-1)
40	.15865	.12889	.13970	.11287	.12338	.99183(-1)
60	.27605	.23880	.25160	.21674	.22971	.19711
120	.50789	.47064	.48256	.44607	.45880	.42313

p = number of variates; f_2 = hypothesis d.f.; f_1 = error d.f.; $M = f_1 - p + 1$; the numbers in parentheses indicate the power of 10 by which tabulated values are to be multiplied.

CHAPTER V

DISTRIBUTION OF WILKS' Λ, IN THE NONCENTRAL LINEAR CASE

5.1. Introduction

In this chapter we continue our investigation into the problem of the distribution of Wilks' Λ. The work already done in Chapter IV concerns the distribution of Λ when the hypothesis to be tested is true (null case). Here we investigate the non-null distribution of Λ in the linear case i.e. when the alternative hypothesis is of unit rank. Thus using the notation and terminology of Chapter IV, it is our specific purpose to derive exact non-null probability and distribution functions of Λ for $p = 2(1)5$ and general f_1 and f_2.

A first step towards the derivation of the non-null distribution, was taken when Anderson and Girshick (1944) attacked the problem of finding the distribution of the Wishart matrix in the noncentral case. The result let Anderson (1946) to the derivation of the moments of the criterion for testing the hypothesis of the general multivariate regression problem (or rather the equivalent Wilks-Lawley hypothesis). Anderson's results were valid for a certain class of alternatives, which he called linear and planar (corresponding to one or two noncentrality parameters). The moments for the linear alternative involve an infinite sum of expressions containing gamma functions and for the planar alternative, involve a triple infinite sum of the same type of expressions.

J. Roy (1960) obtained gamma-series expansion for the power function of Wilks test, which is convenient to use when the error d.f. is large and the noncentrality parameter is small. However, more recently Posten and Bargman (1964) obtained an approximation to the power of the likelihood-ratio test of the multivariate general linear hypothesis by obtaining the characteristic function of a test statistic and expanding this function in a series , the terms of which were of a form which could be easily inverted.

The non-null distribution of Λ when $f_2 = 1$ was derived by Bose and Roy (1938) and Hsu (1938) and the probability density function in this case can be written as:

(5.1.1) $$\sum_{j=o}^{\infty} p_j(\tfrac{1}{2}\lambda^2)\, B\left[\frac{f_1-p+1}{2}, \frac{p}{2}+j ; L\right]$$

where $p_j(\theta)$ is the Poisson probability function

(5.1.2) $$p_j(\theta) = e^{-\theta}\, \theta^j \,|j\,!$$

and $B[r,s;L]$ is the Beta density function

(5.1.3) $$B[r,s;L] = [1/B(r,s)]\, |L|^{r-1}\, |I-L|^{s-1}$$

and

(5.1.4) $$\lambda^2 = \mu'\, \Sigma^{-1}\, \mu \quad .$$

Anderson (1946) has shown that in general, when the rank of the matrix μ is $s, s = \min(p, f_2)$, the distribution of Λ can involve at most s parameters $\lambda_1, \lambda_2, \cdots \lambda_s$ defined as the positive square-roots of the non-zero roots of the determinantal equation

(5.1.5)
$$|\mu \mu' - \lambda \Sigma| = 0$$

5.2. Distributional Properties of Λ

In the notation of Chapter IV A_1 has the $W(A_1/\Sigma/f_1)$ distribution, where $W(S/\Sigma/f)$ stands for the density function of a pxp Wishart matrix S , based on f d.f. and population covariance matrix Σ. But A_2 has $W(A_2/\Sigma/f_2)$ distribution if and only if H_0 is true, otherwise A_2 has noncentral Wishart distribution. Note that under H_0, the statistic Λ is distributed independently of the elements of $(A_1 + A_2)$. The following result is known.

Let A_1 and A_2 be two positive definite symmetric matrices of order p, A_1 having Wishart distribution with f_1 d.f. and A_2 having an independent noncentral Wishart distribution with f_2 d.f., corresponding to the linear case (Anderson 1946; Anderson and Girshick 1944). If we write

(5.2.1)
$$A_1 = C\, L\, C' ,$$

where C is a lower triangular matrix such that

(5.2.2) $$A_1 + A_2 = C\,C'$$

it has been shown (Kshirsagar 1961) that the density of L is given by

(5.2.3) $$f(L) = k \exp(-\tfrac{\lambda^2}{2})\ {}_1F_1\{\tfrac{1}{2}(f_1+f_2), \tfrac{1}{2}f_2, \tfrac{1}{2}\lambda^2(1-\ell_{11})\}\,|L|^{\frac{1}{2}(f_1-p-1)} \cdot |I-L|^{\frac{1}{2}(f_2-p-1)} ,$$

where

(5.2.4) $$k = \pi^{-p(p-1)/4} \prod_{i=1}^{p} \Gamma\,[\tfrac{1}{2}(f_1+f_2+1-i)]/\{\Gamma[\tfrac{1}{2}(f_1+1-i)]\,\Gamma\,[\tfrac{1}{2}(f_1+1-i)]\}$$

λ^2 is the single noncentrality parameter in the linear case, ℓ_{11} is the element in the top left corner of the matrix L, and ${}_1F_1$ denotes the confluent hypergeometric function, defined below.

(5.2.5) $${}_1F_1(a,b,Z) = \sum_{0}^{\infty} \frac{(a)n}{(b)n} \cdot \frac{Z^n}{n!}$$

where

(5.2.6) $$(a)_n = a(a+1) \cdots (a+n-1)$$

Note that $|L| = \Lambda$.

Further, we can write

(5.2.7) $$L = T\,T'$$

where T is a lower triangular matrix $[t_{ij}]$ of order p. Then it has been shown by Kshirsagar (1961) that the diagonal elements t_{ii} are independently distributed and that t_{ii}^2 $(i = 2,\ldots p)$ follows the distribution

$$(5.2.8)\quad f(t_{ii}^2) = [1/B(\frac{f_1-i+1}{2}, \frac{f_2}{2})](t_{ii}^2)^{\frac{1}{2}(f_1+1-i)-1}(1-t_{ii}^2)^{\frac{f_2}{2}-1},$$

$$0 \le t_{ii}^2 \le 1 \quad ,$$

while t_{11}^2 is distributed as

$$(5.2.9)\ f(t_{11}^2) = \frac{\exp(-\lambda^2/2)}{B(\frac{f_1}{2}, \frac{f_2}{2})}\ (t_{11}^2)^{\frac{f_1}{2}-1}(1-t_{11}^2)^{\frac{f_2}{2}-1}$$

$${}_1F_1(\tfrac{1}{2}(f_1+f_2), \tfrac{1}{2}f_2, \tfrac{1}{2}\lambda^2(1-t_{11}^2)),\quad 0 \le t_{11}^2 \le 1 .$$

Observe that

$$(5.2.10)\qquad \Lambda = |L| = \prod_{i=1}^{p} t_{ii}^2 = \prod_{i=1}^{p} X_i \qquad \text{(say)}$$

where $X_i \sim B[\frac{1}{2}(f_1+1-i), \frac{1}{2} f_2; X_i]$.

Now we state a theorem which is a direct consequence of above result and theorem 4.2.4. We shall use this result quite often. However a proof is not given, since its so obvious (See Anderson 1958).

<u>Theorem 5.2.11</u>. In the noncentral linear case, U_{2r,f_2,f_1} is distributed like $X_1\ Y_1^2 \ldots Y_{r-1}^2\ X_{2r}$ where X_1 is independently distributed

as in (5.2.9), $Y_i(i=1,\ldots,r-1)$ are independently distributed as $B[f_1-2i,\ f_2;\ Y]$ and X_{2r} is independently distributed as $B[\frac{f_1+1-p}{2},\ \frac{1}{2}f_2;\ X]$. And U_{2s+1,f_2,f_1} is distributed like $X_1\ Y_1^2 \ldots Y_s^2$ where X_1 is independently distributed as in (5.2.9) and $Y_i(i=1,\ldots,s)$ are independently distributed as $B[f_1-2i,f_2;\ Y]$.

5.3. Method of Derivation

The method used in deriving the probability and distribution functions of U_{p,f_2,f_1} in the noncentral linear case is the same as described in 4.3. We use the convolution technique to derive the algebraic forms of the exact probability and distribution functions.

Consider the distribution of X_1

$$(5.3.1)\quad f(X_1) = \frac{e^{-\lambda^2/2}}{B(\frac{1}{2}f_1,\frac{1}{2}f_2)} X_1^{\frac{1}{2}f_1-1}(1-X_1)^{\frac{1}{2}f_2-1}\,{}_1F_1(\tfrac{1}{2}(f_1+f_2),\tfrac{1}{2}f_2;\tfrac{1}{2}\lambda^2(1-X_1))$$

$$= \frac{e^{-\lambda^2/2}}{B(\frac{1}{2}f_1,\frac{1}{2}f_2)} X_1^{\frac{1}{2}f_1-1} \sum_{j=0}^{\infty} \frac{(\frac{f_1+f_2}{2})_j}{(\frac{f_2}{2})_j} \frac{(\frac{\lambda^2}{2})^j}{j!} (1-X_1)^{\frac{f_2}{2}-1+j}$$

by substituting for ${}_1F_1$ from (5.2.5).

Now make the transformation

$$(5.3.2)\qquad Y_1 = -\log X_1\ ,\quad dY_1 = -\frac{dX_1}{X_1}$$

We get

$$(5.3.3)\ f(Y_1) = \frac{e^{-\lambda^2/2}}{B(\frac{1}{2}f_1,\frac{1}{2}f_2)} \sum_{j=0}^{\infty} \frac{(\frac{f_1+f_2}{2})}{(\frac{f_2}{2})_j} \frac{(\lambda^2/2)^j}{j!} \sum_{k=0}^{b+j} (-1)^k \binom{b+j}{k} e^{-(\frac{f_1+2k}{2})Y_1}$$

$$Y_1 \geq 0$$

where $b = \frac{f_2}{2} - 1$.

Also we have the probability density function of X_i , $(i = 2,\ldots p)$

$$(5.3.4) \qquad f(X_i) = k_i X_i^{\frac{1}{2}(f_1-1-i)} (1-X_i)^{\frac{f_2}{2}-1}$$

$$0 \leq X_i \leq 1$$

where

$$(5.3.5) \qquad k_i = [1|B(\frac{f_1-i+1}{2}, \frac{f_2}{2})]$$

Transform

$$(5.3.6) \qquad Y_i = -\log X_i \ , \quad dY_i = -\frac{dX_i}{X_i} \ .$$

We get, after applying binomial theorem, the density of Y_i as

$$(5.3.7)\ f(Y_i) = k_i \sum_{\ell=0}^{b} (-1)^{\ell} \binom{b}{\ell} e^{-\frac{Y_i}{2}(f_1+1+2\ell-i)} \ , \quad Y_i \geq 0 \ .$$

Similarly in the light of the theorem 5.2.11 let us consider the random variables defined by

$$(5.3.8) \qquad Z_i = X_{2i} \quad X_{2i+1}$$

then the density function of Z_i is given by

$$(5.3.9) \qquad f(Z_i) = C_i \quad Z_i^{\frac{1}{2}(f_1-2i-2)} (1 - \sqrt{Z_i})^{f_2-1}$$

where

$$(5.3.10) \qquad C_i = [\ 1 \mid 2B(f_1-2i, f_2)] \quad .$$

Making the transformation

$$(5.3.11) \qquad Y_i' = - \log Z_i$$

and expanding by binomial theorem, we get the density of Y_i' as

$$(5.3.12)\ f(Y_i') = C_i \sum_{\ell=0}^{f_2-1} (-1)^{\ell} \binom{f_2-1}{\ell} e^{-\frac{Y_i}{2}(f_1+\ell-2i)}, \qquad Y_i' \geq 0 \quad .$$

Finally, since the distribution of $\text{Log}\ U_{p,f_2,f_1}$ is the distribution of $\sum_{i=1}^{b} \log X_i$, we evaluate the convolution integrals of the type (4.3.14) discussed in section 4.3. The convolution is performed at successive stages to obtain the probability density functions. The distribution function is obtained by integrating the probability function thus derived.

5.4 Exact Probability and Distribution Functions of U_{p,f_2,f_1}

5.4.1. p = 2

We write the Wilks' statistic as a product of independent variables

$$U_{2,f_2,f_1} = X_1 X_2$$

or taking logarithm of the inverse we get

$$(5.4.1.1) \quad -\log U_{2,f_2,f_1} = -\log X_1 - \log X_2$$

$$= Y_1 + Y_2 = W_2 \quad \text{(say)}$$

where Y_1 is distributed as in (5.3.3) and Y_2 is distributed as

$$(5.4.1.2) \quad Y_2 \sim [1|B(\frac{f_1-1}{2}, \frac{f_2}{2})] \sum_{\ell=0}^{b} (-1)^{\ell} \binom{b}{\ell} e^{-\frac{Y_2}{2}(f_1+2\ell-1)} .$$

Then

$$(5.4.1.3) \quad W_2 \sim f(Y_1) * f(Y_2)$$

$$\sim k\, e^{-\lambda^2/2} \sum_{j=0}^{\infty} \frac{(\frac{\nu}{2})_j}{(\frac{f_2}{2})_j} \frac{(\frac{\lambda^2}{2})}{j!} \left\{ \sum_{k=0}^{b+j} \sum_{\ell=0}^{b} \frac{(-1)^{\ell+k}}{(2\ell-2k-1)} \binom{b+j}{k} \binom{b}{\ell} \right.$$

$$\left. \cdot \left(e^{-\frac{f_1+2k}{2} W_2} - e^{-\frac{f_1+2\ell-1}{2} W_2}\right) \right\} ,$$

where

(5.4.1.4) $$\nu = f_1 + f_2$$

and

(5,4.1.5) $$k = \left[2 | B(\frac{f_1}{2}, \frac{f_2}{2}) \; B(\frac{f_1-1}{2}, \frac{f_2}{2}) \right],$$

which may be easily varified by reference to (4.3.14) - (4.3.16) .

If now we make the transformation $W = - \log U$ we obtain

(5.4.1.6) $$U_{2,f_2,f_1} \sim k \, e^{-\lambda^2/2} \sum_{j=0}^{\infty} \frac{(\frac{\nu}{2})_j}{(\frac{f_2}{2})_j} \frac{(\lambda^2/2)^j}{j!} \left\{ \sum_{k=0}^{b+j} \sum_{\ell=0}^{b} \frac{(-1)^{\ell+k}}{(2\ell-2k-1)} \binom{b+j}{k}\binom{b}{\ell} \cdot \left(U^{\frac{f_1+2k-2}{2}} - U^{\frac{f_1+2\ell-3}{2}} \right) \right\}$$

To find the distribution function of U_{2,f_2,f_1}, integrate (5.4.1.6) between the limits (o,u), $o \leq u \leq 1$, we get

(5.4.1,7) $$P[U_{2,f_2,f_1} \leq u] = 2 \, k \, e^{-\lambda^2/2} \sum_{j=0}^{\infty} \frac{(\frac{\nu}{2})_j}{(\frac{f_2}{2})_j} \frac{(\frac{\lambda^2}{2})^j}{j!} \left\{ \sum_{k=0}^{b+j} \sum_{\ell=0}^{b} \frac{(-1)^{\ell+k}}{(2\ell-2k-1)} \binom{b+j}{k}\binom{b}{k} \cdot \left(\frac{u^{\frac{f_1+2k}{2}}}{f_1+2k} - \frac{u^{\frac{f_1+2\ell-1}{2}}}{f_1+2\ell-1} \right) \right\}$$

Let us consider a special case when $f_2 = 4$. In this case we get the probability and distribution functions of $U_{2,4,f_1}$ respectively;

$$(5.4.1.8)\quad U_{2,4,f_1} \sim k\, e^{-\frac{\lambda^2}{2}}\, U^{\frac{f_1-3}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{f_1+4}{2}\right)_j}{(2)_j} \frac{\left(\frac{\lambda^2}{2}\right)^j}{j!} \sum_{k=0}^{j+1} \frac{(-1)^k}{4k^2-1} \binom{j+1}{k} \left(2U^{\frac{2k+1}{2}} - (2k+1)\,U + (2k-1)\right)$$

and

$$(5.4.1.9)\quad P[U_{2,4,f_1} \le u] = 2\,k\, e^{-\lambda^2/2}\, u^{\frac{f_1-1}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{f_1+4}{2}\right)_j}{(2)_j} \frac{(\lambda^2/2)^j}{j!} \sum_{k=0}^{j+1} \frac{(-1)^k}{(4k^2-1)} \binom{j+1}{k} \left(\frac{2k-1}{f_1-1} - \frac{(2k+1)u}{f_1+1} + \frac{2u^{\frac{2k+1}{2}}}{f_1+2k} \right)$$

where value of k becomes

$$k = [2|B(\frac{f_1}{2}, 2)\, B(\frac{f_1-1}{2}, 2)]$$

5.4.2 $p = 3$

We write Wilks' statistic as

$$(5.4.2.1)\qquad U_{3,f_2,f_1} = X_1 \cdot X_2 \cdot X_3$$

$$= X_1 \cdot Z_1$$

Taking the logarithm of the inverse

$$(5.4.2.2)\qquad -\log U_{3,f_2,f_1} = -\log X_1 - \log Z_1$$

$$= Y_1 + Y_1' = W_3 \quad \text{(say)},$$

where Y_1 is distributed as in (5.3.3) and Y_1' is distributed as

$$(5.4.2.3)\quad Y_1' \sim [1/2B(f_1-2,f_2)] \sum_{\ell=0}^{f_2-1} (-1)^{\ell} \binom{f_2-1}{\ell} e^{-\frac{f_1+\ell-2}{2} Y'} .$$

Then the convolution technique gives the exact probability density function of W_3 as follows

$$(5.4.2.4)\ W_3 \sim k\, e^{-\lambda^2/2} \sum_{j=0}^{\infty} \frac{(\frac{\nu}{2})_j}{(\frac{f_2}{2})_j} \frac{(\lambda^2/2)^j}{j!} \Big\{ \sum_{k=0}^{b+j} (-1)^k \binom{b+j}{k} \binom{f_2-1}{2k+2}$$

$$W_3\, e^{-\frac{f_1+2k}{2} W_3} + 2 \sum_{k=0}^{b+j} \sum_{\substack{\ell=0 \\ \ell \neq 2k+2}}^{f_2-1} \frac{(-1)^{\ell+k}}{(\ell-2k-2)} \binom{f_2-1}{\ell} \binom{b+j}{k}$$

$$\Big(e^{-\frac{f_1+2k}{2} W_3} - e^{-\frac{f_1+\ell-2}{2} W_3} \Big) \Big\} ,$$

where

$$(5.4.2.5)\qquad K = [1 \,/\, 2B\,(\tfrac{f_1}{2}, \tfrac{f_2}{2})\, B(f_1-2,f_2)] \quad .$$

Now transform

$$W = -\log U$$

to get the probability density function of U_{3,f_2,f_1} as

(5.4.2.6) $$U_{3,f_2,f_1} \sim K\, e^{-\lambda^2/2}\, U^{\frac{f_1-2}{2}} \sum_{j=0}^{\infty} \frac{(\frac{\nu}{2})_j}{(\frac{f_2}{2})_j} \frac{(\lambda^2/2)^j}{j!} \Big\{ \sum_{k=0}^{b+j} (-1)^k \binom{f_2-1}{2k+2} \binom{b+j}{k} (-\log U) U^k + 2 \sum_{k=0}^{b+j} \sum_{\substack{\ell=0 \\ \ell \neq 2k+2}}^{f_2-1} \frac{(-1)^{\ell+k}}{(\ell-2k-2)} \binom{f_2-1}{\ell} \binom{b+j}{k} \Big(U^k - U^{\frac{\ell-2}{2}} \Big) \Big\} .$$

Integrating (5.4.2.6) between limits (o,u), $o \le u \le 1$, we get the distribution function of U_{3,f_2,f_1} as follows.

(5.4.2.7) $$P[U_{3,f_2,f_1} \le u] = 2k\, e^{-\lambda^2/2}\, u^{\frac{f_1}{2}} \sum_{j=0}^{\infty} \frac{(\nu/2)_j}{(\frac{f_2}{2})_j} \frac{(\lambda^2/2)^j}{j!} \Big\{ \sum_{k=o}^{b+j} \frac{(-1)^k}{(f_1+2k)^2} \binom{f_2-1}{2k+2} \binom{b+j}{k} u^k (2-(f_1+2k)\log u) + 2 \sum_{k=0}^{b+j} \sum_{\substack{\ell=0 \\ \ell \neq 2k+2}}^{f_2-1} \frac{(-1)^{\ell+k}}{(\ell-2k-2)} \binom{f_2-1}{\ell} \binom{b+j}{k} \Big(\frac{u^k}{f_1+2k} - \frac{u^{\frac{\ell-2}{2}}}{f_1+\ell-2} \Big) \Big\} .$$

We consider a special case when $f_2 = 4$. In this case we get the probability and distribution functions of $U_{3,4,f}$ respectively,

$$(5.4.2.8)\quad U_{3,4,f_1} \sim K\, e^{-\lambda^2/2}\, U^{\frac{f_1-4}{2}} \sum_{j=0}^{\infty} \frac{(\frac{f_1+4}{2})_j}{(2)_j} \frac{(\lambda^2/2)^j}{j!} \Big[1-6U^{1/2}+2U^{3/2}+3U$$

$$(1-\log U) + \sum_{k=1}^{j+1} (-1)^k \binom{j+1}{k} \Big\{\frac{1}{k+1} - \frac{6U^{1/2}}{2k+1} - \frac{2U^{3/2}}{2k-1} + \frac{3U}{K}$$

$$\Big(1 + \frac{U^k}{(k+1)(2k-1)(2k+1)}\Big)\Big\}\Big]$$

and

$$(5.4.2.9)\quad P[U_{3,4,f_1} \le u] = 2K\, e^{-\lambda^2/2}\, u^{\frac{f_1-2}{2}} \sum_{j=0}^{\infty} \frac{(\frac{f_1+4}{2})_j}{(2)_j} \frac{(\lambda^2/2)^j}{j!} \Big[\frac{1}{f_1-2} - \frac{6u^{1/2}}{f_1-1}$$

$$+ \frac{2u^{3/2}}{f_1+1} + \frac{3u^{\frac{f_1}{2}}}{f_1^2}(2+f_1(1-\log u)) + \sum_{k=1}^{j+1} (-1)^k \binom{j+1}{k}$$

$$\Big\{\frac{1}{(f_1-2)(k+1)} - \frac{6u^{1/2}}{(2k+1)(f_1-1)} - \frac{2u^{3/2}}{(2k-1)(f_1+1)} + \frac{3u}{k}$$

$$\Big(\frac{1}{f_1} + \frac{u^k}{(k+1)(2k-1)(2k+1)(f_1+2k)}\Big)\Big\}\Big],$$

where k now becomes,

$$k = [1/\ 2B(\tfrac{f_1}{2}, 2)\ B(f_1-2, 4)]\ .$$

5.4.3 $p = 4$

Writing Wilks' statistic as

$$(5.4.3.1) \qquad U_{4,f_2,f_1} = X_1 \cdot X_2 \cdot X_3 \cdot X_4$$

$$= X_1 \cdot Z_1 \cdot X_4$$

and taking the logarithm of the inverse, we get

$$(5.4.3.2) \quad -\log U_{4,f_2,f_1} = -\log X_1 - \log Z_1 - \log X_4$$

$$= Y_1 + Y_1' + Y_4$$

$$= W_3 + Y_4 = W_4 \quad \text{(say)} \quad ,$$

where W_3 is distributed as in (5.4.2.4) and

$$(5.4.3.3) \quad Y_4 \sim [1 \mid B(\frac{f_1-3}{2}, \frac{f_2}{2})] \sum_{m=0}^{b} (-1)^m \binom{b}{m} e^{-\frac{Y_4}{2}(f_1+2m-3)} \quad .$$

Then by evaluating the convolution integral we obtain the exact probability and distribution functions for U_{4,f_2,f_1} respectively as follows:

$$(5.4.3.4) \quad U_{4,f_2,f_1} \sim k\, e^{-\lambda^2/2}\, U^{\frac{f_1-2}{2}} \sum_{j=0}^{\infty} \frac{(\nu/2)_j}{(\frac{f_2}{2})_j} \frac{(\lambda^2/2)^j}{j!}$$

$$\cdot \Big[\sum_{k} \sum_{m} \frac{(-1)^{k+m}}{(2m-2k-3)} f(2k+2,m,k) \{ \frac{2}{(2m-2k-3)} (U^{m-3/2} - u^k)$$

$$- U^k \log U \} + 2 \sum_{k} \sum_{\substack{\ell \\ \ell \neq 2k+2}} \sum_{m} \frac{(-1)^{\ell+k+m}}{(\ell-2k-2)(2m-2k-3)} f(\ell,m,k) (U^k - U^{m-3/2})$$

$$+ \sum_{k} \sum_{m=1} \frac{(-1)^{k+m-1}}{(2m-2k-3)} f(2m-1,m,k)\, U^{m-3/2} \log U$$

$$- 2 \sum_{k} \sum_{\substack{\ell \\ \ell \neq 2k+2,2m-1}} \sum_{m} \frac{(-1)^{\ell+k+m}}{(\ell-2k-2)(2m-\ell-1)} f(\ell,m,k) (U^{\frac{\ell-2}{2}} - U^{m-3/2}) \Big]$$

and

$$(5.4.3.5)\quad P[U_{4,f_2,f_1} \le u] = 2\,k\,e^{-\lambda^2/2}\,u^{\frac{f_1}{2}} \sum_{j=0}^{\infty} \frac{(\nu/2)_j}{(\frac{f_2}{2})_j}\,\frac{(\lambda^2/2)^j}{j!}$$

$$\cdot \Big[\sum_k \sum_m \frac{(-1)^{k+m} u^k}{(2m-2k-3)(f_1+2k)^2}\; f(2k+2,m,k) \Big\{ 2-(f_1+2k)\log u + \frac{2(f_1+2k)}{(2m-2k-3)} \Big(\frac{(f_1+2k)u^{\frac{2m-2k-3}{2}}}{f_1+2m-3} - 1 \Big) \Big\}$$

$$+ 2 \sum_{\substack{k\\ \ell \ne 2k+2}} \sum_{\ell} \sum_m \frac{(-1)^{\ell+k+m}}{(\ell-2k-2)(2m-2k-3)}\, f(\ell,m,k) \Big(\frac{u^k}{f_1+2k} - \frac{u^{m-3/2}}{f_1+2m-3} \Big)$$

$$- \sum_k \sum_{m=1} \frac{(-1)^{k+m-1}\, u^{m-3/2}}{(2m-2k-3)(f_1+2m-3)^2}\, f(2m-1,m,k)(2-(f_1+2m-3)\log u)$$

$$- \sum_{\substack{k\\ \ell \ne 2k+2\\ \ell \ne 2m-1}} \sum_{\ell} \sum_m \frac{(-1)^{\ell+k+m}}{(\ell-2k-2)(2m-\ell-1)}\, f(\ell-m,k) \Big(\frac{u^{\frac{\ell-2}{2}}}{f_1+\ell-2} - \frac{u^{m-3/2}}{f_1+2m-3} \Big) \Big]$$

where

$$(5.4.3.6)\quad k = \Big[1 \big| B\,(\tfrac{f_1}{2}, \tfrac{f_2}{2})\; B\,(f_1-2, f_2)\; B\,(\tfrac{f_1-3}{2}, \tfrac{f_2}{2}) \Big],$$

$$(5.4.3.7) \qquad f(\ell,m,k) = \binom{f_2-1}{\ell}\binom{b}{m}\binom{b+j}{k}$$

and the indices ℓ, m and k run from o to f_2-1, b and b+j respectively unless specified otherwise.

A special case when $f_2 = 4$ is now given. The probability and distribution functions of $U_{4,4,f_1}$ respectively, obtained as before, are as follows:

$$(5.4.3.8)\; U_{4,4,f_1} \sim \frac{1}{6}\, k\, e^{-\lambda^2/2}\, U^{\frac{f_1-5}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{f_1+4}{2}\right)_j}{(f_2/2)_j} \frac{(\lambda^2/2)^j}{j!} \Big[1-12U^{\frac{1}{2}}-18U(2+\log U)$$

$$+ 4U^{3/2}(11-3\log U) + 3U^2 + 3\sum_{k=1}^{j+1} (-1)^k \binom{j+1}{k} \Big\{ \frac{1}{2k+3} - \frac{4U^{\frac{1}{2}}}{k+1}$$

$$- \frac{6U}{(2k+1)^2}(2+(2k+1)\log U) - \frac{1}{2k-1}\; U^2 + \frac{4}{k}\, U^{3/2}$$

$$\left(1 + \frac{3\, U^k}{(2k+1)^2(2k+3)(2k-1)(k+1)}\right)\Big\}\Big]$$

and

$$(5.4.3.9)\quad P[U_{4,4,f_1} \le u] = \frac{1}{3}\, k\, e^{-\lambda^2/2}\, u^{\frac{f_1-3}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{f_1+f_2}{2}\right)_j}{(f_2/2)_j} \frac{(\lambda^2/2)^j}{j!}$$

$$\left[\frac{1}{f_1-3} - \frac{12u^{\frac{1}{2}}}{f_1-2} + \frac{6u}{(f_1-1)^2}\,(6-(f_1-1)(2+3\log u))\right.$$

$$+ \frac{4}{f_1^2}\, u^{3/2}(6+f_1\,(11 - 3\log u)) + \frac{3u^2}{f_1+1}$$

$$+ 3\sum_{k=1}^{j+1} (-1)^k \binom{j+1}{k} \left\{ \frac{1}{(2k+3)(f_1-3)} - \frac{4\,u^{\frac{1}{2}}}{(k+1)(f_1-2)} - \frac{6\,u}{(2k+1)(f_1-1)}\right.$$

$$\left(\log u + \frac{2}{2k+1} - \frac{2}{f_1-1}\right)$$

$$\left.\left. - \frac{u^2}{(2k-1)(f_1+1)} + \frac{4}{k}\, u^{3/2}\left(\frac{1}{f_1} + \frac{3u^k}{(k+1)(2k-1)(2k+1)^2(2k+3)(f_1+2k)}\right)\right\}\right]$$

where now k becomes

$$k = [\,1 \mid B\left(\frac{f_1}{2}, 2\right) B\,(f_1-2, 4)\, B\left(\frac{f_1-3}{2}, 2\right)] .$$

5.4.4 $p = 5$

Writing Wilks' statistic as

$$(5.4.4.1) \quad U_{5,f_2,f_1} = X_1 \; X_2 \; X_3 \; X_4 \; X_5$$

$$= X_1 \; Z_1 \; Z_2$$

and taking the logarithm of the inverse, we get

$$(5.4.4.2) \quad -\log U_{5,f_2,f_1} = -\log X_1 - \log Z_1 - \log Z_2$$

$$= Y_1 + Y_1' + Y_2'$$

$$= W_3 + Y_2' = W_5 \quad \text{(say)}$$

where W_3 is distributed as in (5.4.2.4) and

$$(5.4.4.3) \quad Y_2' \sim [1 \mid 2B(f_1-4,f_2)] \sum_{n=0}^{f_2-1} (-1)^n \binom{f_2-1}{n} e^{-\frac{Y_2'}{2}(f_1+n-4)} .$$

Then by evaluating the convolution integral we obtain the exact probability and distribution functions for U_{5,f_2,f_1} respectively as follows:

$$(5.4.4.4) \quad U_{5,f_2,f_1} \sim k\, e^{-\lambda^2/2}\, U^{\frac{f_1-2}{2}} \sum_{j=0}^{\infty} \frac{(\nu/2)_j}{(f_2/2)_j} \frac{(\lambda^2/2)^j}{j!}$$

$$\Big[\sum_k (-1)^k f(2k+2,2k+4,k)(\log U)^2 U^k$$

$$+ 4 \sum_k \sum_{\substack{n \\ n \neq 2k+4}} \frac{(-1)^{n+k}}{(n-2k-4)} f(2k+2,n,k) \Big(\frac{2}{(n-2k-4)} \big(U^{\frac{n-4}{2}} - U^k\big) - U^k \log U\Big)$$

$$- 4 \sum_k \sum_{\substack{\ell \\ \ell \neq 2k+2}} \frac{(-1)^{\ell+k}}{(\ell-2k-2)} f(\ell,2k+4,k)\, U^k \log U$$

$$+ 8 \sum_k \sum_\ell \sum_{\substack{n \\ \ell \neq 2k+2 \\ n \neq 2k+4}} \frac{(-1)^{\ell+k+n}}{(\ell-2k-2)(n-2k-4)} f(\ell,n,k)\big(U^k - U^{\frac{n-4}{2}}\big)$$

$$+ 4 \sum_k \sum_{\substack{\ell \\ \ell \neq 2k+2}}^{f_2-3} \frac{(-1)^k}{(\ell-2k-2)} f(\ell,\ell+2,k)\, U^{\frac{\ell-2}{2}} \log U$$

$$- 8 \sum_k \sum_\ell \sum_{\substack{n \\ \ell \neq 2k+2 \\ \ell \neq n-2}} \frac{(-1)^{\ell+k+n}}{(\ell-2k-2)(n-\ell-2)} f(\ell,n,k)\big(U^{\frac{\ell-2}{2}} - U^{\frac{n-4}{2}}\big) \Big]$$

and

(5.4.4.5) $$P[U_{5,f_2,f_1} \le u] = 2\ k\ e^{-\lambda^2/2}\ u^{f/2} \sum_{j=0}^{\infty} \frac{(\nu/2)_j}{j!} \frac{(\lambda^2/2)^j}{j!}$$

$$\Big[\sum_k \frac{(-1)^k u^k}{(f_1+2k)^3} f(2k+2,2k+4,k)\{(f_1+2k)^2(\log u)^2-4(f_1+2k)\log u+8\}$$

$$+ 4 \sum_k \sum_{\substack{n \\ n\neq 2k+4}} \frac{(-1)^{n+k}\ u^k}{(n-2k-4)(f_1+2k)^2} f(\ell,n,k)\Big\{2-(f_1+2k)\log u+ \frac{2(f_1+2k)}{(n-2k-4)} \Big(\frac{(f_1+2k)}{(f_1+n-4)} u^{\frac{n-2k-4}{2}} -1\Big)\Big\}$$

$$+ 4 \sum_k \sum_{\substack{\ell \\ \ell\neq 2k+2}} \frac{(-1)^{\ell+k}\ u^k}{(\ell-2k-2)(f_1+2k)^2} f(\ell,2k+4,k)(2-(f_1+2k)\log u)$$

$$+ 8 \sum_k \sum_{\ell} \sum_{\substack{n \\ \ell\neq 2k+2 \\ n\neq 2k+4}} \frac{(-1)^{\ell+k+n}}{(\ell-2k-2)(n-2k-4)} f(\ell,n,k) \Big(\frac{u^k}{f_1+2k} - \frac{u^{\frac{n-4}{2}}}{f_1+n-4}\Big)$$

$$- 4 \sum_k \sum_{\substack{\ell \\ \ell\neq 2k+2}} \frac{(-1)^k}{(\ell-2k-2)(f_1+\ell-2)^2} f(\ell,\ell+2,k)(2-(f_1+\ell-2)\log u)$$

$$- 8 \sum_k \sum_{\ell} \sum_{\substack{n \\ \ell\neq 2k+2 \\ \ell\neq n-2}} \frac{(-1)^{\ell+k+n}}{(\ell-2k-2)(n-\ell-2)} f(\ell,n,k) \Big(\frac{u^{\frac{\ell-2}{2}}}{f_1+\ell-2} - \frac{u^{\frac{n-4}{2}}}{f_1+n-4}\Big)\Big]$$

where

$$(5.4.4.6)\quad k = [1|8\, B(\tfrac{f_1}{2}, \tfrac{f_2}{2})\, B\,(f_1-2, f_2) B(f_1-4, f_2)]\quad ,$$

$$(5.4.4.7)\qquad f(\ell, n, k) = \binom{f_2-1}{\ell}\binom{f_2-1}{n}\binom{b+j}{k}\quad ,$$

and the indices ℓ, n, and k run from 0 to f_2-1, f_2-1 and $b+j$ respectively unless specified otherwise.

A special case when $f_2 = 4$ is now given. The probability and distribution functions of $U_{5,4,f_1}$ respectively, obtained as before, are as follows:

$$(5.4.4.8)\quad U_{5,4,f_1} \sim k\, e^{-\lambda^2/2}\, U^{\frac{f_1-6}{2}} \sum_{j=0}^{\infty} \frac{(\frac{f_1+4}{2})_j}{(\frac{f_2}{2})_j} \frac{(\lambda^2/2)^j}{j\,!}$$

$$\cdot \Big[\sum_{k=0}^{j+1} (-1)^k \binom{j+1}{k} \Big\{ \frac{2}{k+2} - \frac{60}{2k+3} U^{\frac{1}{2}} - \frac{20(8k+1)}{(k+1)^2} U + \frac{80(8k+1)}{(2k+1)^2} U^{3/2}$$

$$- \frac{4}{(2k-1)} U^{5/2} - 60\, U \log U \Big(\frac{2U^{\frac{1}{2}}}{2k+1} - \frac{1}{k+1} \Big) \Big\}$$

$$+ 5\, U^2 (31 - 6 \log U)$$

$$+30\, U^2 \sum_{k=1}^{j+1} \frac{(-1)^k}{k} \binom{j+1}{k} \Big(1 + \frac{6\, U^k}{(k+1)^2(k+2)(2k-1)(2k+1)^2(2k+3)}\Big) \Big]$$

and

$$(5.4.4.9)\quad P[U_{5,4,f_1} \leq u] = 2\,k\,e^{-\lambda^2/2}\,u^{\frac{f_1-4}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{f_1+4}{2}\right)_j}{(f_2/2)_j} \frac{(\lambda^2/2)^j}{j!}$$

$$\cdot \Bigg[\sum_{k=0}^{j+1} (-1)^k \binom{j+1}{k} \Bigg\{ \frac{2}{(k+2)(f_1-4)} - \frac{60\,u^{\frac{1}{2}}}{(2k+3)(f_1-3)} + \frac{20u}{(k+1)(f_1-2)}$$

$$\left(\frac{6}{f_1-2} - \frac{8k+11}{k+1} - 3\log u\right) + \frac{40\,u^{3/2}}{(2k+1)(f_1-1)}$$

$$\left(\frac{6}{f_1-1} + \frac{6k+2}{2k+1} - 3\log u\right) - \frac{4\,u^{5/2}}{(2k-1)(f_1+1)} \Bigg\}$$

$$+ \frac{5\,u^2}{f_1}\left(\frac{12}{f_1} + 31 - 6\log u\right)$$

$$+ 30\,u^2 \sum_{k=1}^{j+1} \frac{(-1)^k}{k} \binom{j+1}{k} \left(\frac{1}{f_1} + \frac{6\,u^k}{(k+1)^2(k+2)(2k-1)(2k+1)^2(2k+3)(f_1+2k)}\right)\Bigg]$$

where now k becomes

$$k = [1|30\,B\left(\frac{f_1}{2}, 2\right) B\,(f_1-2, 4)\, B\left(\frac{f_1-3}{2}, 2\right) B\left(\frac{f_1-4}{2}, 2\right)]$$

5.4.5 General p

The techniques of the preceding sections and that of Chapter IV lead us to give the form of the noncentral distribution of U_{p,f_2,f_1} in the linear case for general p and general f_2.

The probability density function of $Y = -\log U_{p,f_2,f_1}$ in the linear case is of the form.

$$(5.4.4.1)\quad f(Y) = \left(\prod_{i=1}^{p} k_i\right) e^{-\lambda^2/2} \sum_{j=0}^{\infty} \frac{(\nu/2)_j}{(f_2/2)_j} \frac{(\lambda^2/2)^j}{j!} \sum_{k=0}^{m_j} C_{jk}\, e^{-\frac{Y}{2}(f_1-b_k)}\, Y^{d_k},\ Y>0$$

where

$$(5.4.5.2)\qquad k_i = [1|B\,(\tfrac{1}{2}(f_1+1-i),\tfrac{1}{2} f_2)] \quad .$$

For example when $p = 1$

$$m_j = \frac{f_2}{2} - 1+j\ ,\ C_{jk} = (-1)^k \binom{\frac{f_2}{2}-1+j}{k}\ ,\ b_k = -2k,\ d_k = 0\ .$$

By substituting

$$Y = -\log U$$

we get the form of the probability density function of U_{p,f_2,f_1} in the linear case as follows

$$(5.4.5.3)\quad U_{p,f_2,f_1} \sim \left(\prod_{i=1}^{p} k_i\right) e^{-\lambda^2/2} \sum_{j=0}^{\infty} \frac{(\nu/2)_j}{(\frac{f_2}{2})_j} \frac{(\lambda^2/2)^j}{j\,!} \sum_{k=0}^{m_j} C_{jk} U^{\frac{1}{2}(f_1-bk)} (-\log U)^{d_k},$$

$$0 \le U \le 1\ .$$

The corresponding distribution function, by straight forward integration, is obtained as

$$(5.4.5.4)\quad P[U_{p,f_2,f_1} \le u] = (\prod_{i=1}^{b} k_i)\, e^{-\lambda^2/2} \sum_{j=0}^{\infty} \frac{(\nu/2)_j}{(f_2/2)_j} \frac{(\lambda^2/2)^j}{j!} \cdot \sum_{k=0}^{m_j} c_{jk} u^{\frac{1}{2}(f_1-b_k)} \cdot \sum_{r_k=1}^{d_k+1} \left(\frac{2}{f_1-b_k}\right)^{r_k} \frac{(-\log u)^{d_k-r_{k+1}}(d_k)!}{(d_k-r_k+1)!}$$

$$= (\prod_{i=1}^{p} k_i)\, e^{-\lambda^2/2} \sum_{j=0}^{\infty} \frac{(\nu/2)_j}{(\frac{f_2}{2})_j} \frac{(\lambda^2/2)^j}{j!} \sum_{h=1}^{M_j} a_{hj}\, u^{\frac{1}{2}(f_1-b_h)} \left(\frac{2}{f_1-b_h}\right)^{r_k} (-\log u)^{s_h} ,$$

where c_{jk}, b_k, d_k, r_k, a_{hj} and s_h are constants and can be determined from p, f_2 and f_1. However it is not known explicitly how to find the values of these constants, a task which is by no means easy for large values of p or f_2.

5.5 Power Function of Wilks' Test

Formulae derived in section 5.4 can be used to evaluate the power function of the analysis of dispersion test when the alternative hypothesis is of rank one. However, the calculations using above formulae are limited to $p = 2(1)5$ and for all f_2. J. Roy (1966) has obtained exact expression for $p = 2$ and has suggested two approximations for $p > 2$. We shall leave this topic for future work.

CHAPTER VI

SUMMARY AND CONCLUSION

6.1 Summary

The study of some central and non-central distribution problems in multivariate analysis has been carried out in this dissertation. The primary objective has been to investigate the distributions of $W_2^{(s)}$, the second e s f in the s non-zero roots of a determinantal equation, and that of Wilks' Λ . The two test statistics are related in the sense of being symmetric functions of the characteristic roots of a determinantal equation.

The approach used for the study of $W_2^{(s)}$ is to derive its moments and with the help of moments suggest an approximation to its distribution, and further carry out accuracy comparisions by comparing moments of the exact and approximate distributions. In the central case the first four moments were evaluated using certain recurrence relations on Vandermonde determinants but in the non-central case only the first two moments. The non-central moments, however, were obtained by alternate methods, first by evaluating certain integrals involving zonal polynomials and then by using generalized Laguerre polynomials. The second method is simpler since for these cases, certain $a_{\kappa,\nu}$ coefficients involved in the generalized Laguerre polynomials are available in Constantine (1966). But his tabulations are limited up to order $k = 4$.

Consequently, the third and fourth moments could not be evaluated explicitly, instead they are presented as linear functions of certain generalized Laguerre polynomials. During the course of this study certain determinants were evaluated, which are presented in Chapter II and some of them in Appendix A in a consolidated form. These results helped to evaluate the variances of $W_3^{(s)}$ and $W_4^{(s)}$, the third and fourth e s f's respectively. Certain coefficients $g^{\delta}_{\kappa,\eta,i}$ obtained in the expansion of $a_2^i Z_\kappa$ and certain integrals involving zonal polynomials, are presented in appendices B and C respectively.

For the study of Wilks' Λ, convolution techniques have been used to derive the exact probability density function in the central as well as the non-central linear case. In chapter IV the exact probability density functions of U_{p,f_2,f_1} have been derived for $p = 3(1)6$ and for all f_2. These results are generally in finite series form whenever p or f_2 is even. However, for larger values of p the formulae become too involved for presentation as well as for programming purposes. Some special cases when $f_2 = 3,4$ have been worked out. Respective distribution functions are obtained by direct integration of the density functions. The results confirm that for $p = 3$ with $f_2 = 4$ Anderson's result is correct and Wilks' result is incorrect. The formulae derived in this chapter helped to calculate exact percentage points of Λ for $p = 3,4,6$ and extensive tabulation for various values of f_2 is achieved.

The non-central probability density function of Λ, necessary for the study of the power function, has been derived in Chapter V for $p = 2(1)5$. Distribution functions are obtained by direct integration of the probability density functions. Special cases when $f_2 = 4$ are also discussed.

This is well known (Gihash 1964, Kiefer & Schwartz 1965) that Wilk's Λ is unbiased, consistent and admissible. The power function of Wilks' Λ is a monotonic - increasing function of each of the noncentrality parameters (DasGupta, Anderson and Mudholkar 1964). Very little, however, is known about the actual magnitude of the power, and further studies have to be done. A thorough investigation has to be carried out to look for these properties for $W_2^{(s)}$. Also power computations of $W_2^{(s)}$ have to be done in order to see how it stands as compared to Wilks' Λ or other existing tests of multivariate hypotheses, which have been shown to have some desirable properties.

6.2 Suggestions for Further Research

Some of the problems, whose solutions promise new insight into the realm of the multivariate distribution and testing theory, are listed below.

(i) It is seen that there are many reasonable criteria to be used. One has been suggested in Chapter II. Unfortunately, there is no theory to tell what is a good class of tests nor how to choose among the proposed tests when we want power against certain kind of alternatives.

(ii) The exact distribution of $W_2^{(s)}$ in the null as well as non-null case has to be investigated.

(iii) The properties of $W_2^{(s)}$ namely, unbiasedness, consistency, odmissibility, some good lower bound to the power and possibility of getting suitable confidence intervals; have to be explored. Also the computation of power of $W_2^{(s)}$ and its comparison with the existing tests has to be carried out.

(iv) We have evaluated first two moments of $W_3^{(s)}$ and $W_4^{(s)}$. Their third and fourth moments have to be evaluated and approximations to their distribution have to be suggested. It will be nice to explore their exact distribution and for that matter exact distribution of $W_i^{(s)}$.

(v) The non-null distribution of $W_i^{(s)}$, $i = 2,3,..s$ is unexplored. And even for $i = 2$, we could evaluate only the first two moments. Evaluation of higher moments will require evaluation of $a_{\kappa,\nu}$ in the expansion of zonal polynomials beyond $k = 4$.

(vi) Exact power computations of Wilks' Λ can be carried out using the results of Chapter V for $p = 2(1)5$ for general f_2 and f_1. This will also provide a check of the existing approximations for the same.

APPENDIX A

CONSOLIDATED VALUES OF DETERMINANTS WITH $q_s > q_{s-1}+1$

It has been shown in Chapter II that the moments of $W_i^{(s)}(i=1,\ldots s)$, the ith esf in the s ξ's, can be obtained as linear compounds of determinants of the type $W(q_s,\ldots,q_1)$, $q_i \geq 0$. Further we have evaluated in that chapter, the values of each determinant involved in the first four moments of $W_2^{(s)}$. However, the evaluation of the determinants was done in successive stages using a reduction formula (Pillai,1965) which reduced the original determinant into two parts, the first part consisting of a linear compound of lower order determinants and the second a determinant of the same order with q_s changed to q_s-1. The second part vanishes if $q_s = q_{s-1}+1$, but otherwise successive reductions should be carried on the second part as for the original determinant. In that chapter the values of the determinants were presented giving the results for each stage separately but these are now consolidated for determinants with $q_s > q_{s-1}+1$ and presented below.

(A1) $K(s,m)W(s+2,s,s-3,\ldots,1,0)$

$$= \binom{s}{2}\frac{M(s,s+1)}{2^6 \cdot 3!}[8s(s+1)(s+2)m^3+12s(s+2)(s^2+3s+6)m^2$$

$$+2(s+2)(3s^4+15s^3+47s^2+59s+24)m+(s^6+9s^5+43s^4+123s^3+196s^2+204s+144)],$$

(A2) $K(s,m)W(s+2,s+1,s,s-4,\ldots,1,0)$

$$= \binom{s+2}{5} \frac{M(s-1,\ldots,s+3)}{2^{11} \cdot 18} [16(s-1)s^2(s+1)m^4$$

$$+ 16s^2(s-1)(s^2+2s+10)m^3$$

$$+4s(s-1)(6s^4+18s^3+125s^2+77s+432)m^2$$

$$+2(s-1)(4s^6+17s^5+107s^4+353s^3+169s^2+1278s+2520)m$$

$$+s(s+1)\{s(s-1)(s^4+4s^3+41s^2+38s+432)+1440\}] \quad ,$$

(A3) $K(s,m)W(s+2,s+1,s,s-3,s-5,\ldots 1,0)$

$$= \binom{s+2}{6} \frac{M(s-2,\ldots,s+3)}{2^{11} \cdot 12} [16(s-1)s^2(s+1)m^4$$

$$+8(s-1)s^2(2s^2+4s+23)m^3$$

$$+4s(s-1)(6s^4+18s^3+143s^2+89s+624)m^2$$

$$+2(s-1)(4s^6+17s^5+125s^4+373s^3+643s^2+838s+5880)m$$

$$+s(s+1)\{s(s-1)(s^4+4s^3+47s^2+44s+624)+2880\}],$$

(A4) $K(s,m)W(s+2,s,s-2,s-4,\ldots 1,0) = \left[\binom{s}{3} \frac{(s+2)M(s,s+1)}{2^2 \cdot 5!}\right] [16s(s+1)m^4$

$$+8s(4s^2+9s+20)m^3+4(6s^4+21s^3+65s^2+50s+45)m^2$$

$$+2(4s^5+19s^4+70s^3+95s^2+70s+90)m$$

$$+(s^6+6s^5+25s^4+45s^3+19s^2+39s-135)] \quad ,$$

(A5) $K(s,m)W(s+2,s-1,s-2,s-3,s-5,\ldots 1,0)$

$$= \left[\binom{s}{4}\frac{(s+2)M(s-2,s-1)}{2^5\cdot 3!}\right]\left[4(s+1)m^2+2(2s^2+7s+17)m\right.$$

$$\left.+(s^3+6s^2+23s+42)\right] ,$$

(A6) $K(s,m)W(s+3,s+1,s-2,s-4,\ldots 1,0)$

$$= 5\binom{s+2}{5}\frac{M(s-1,\ldots,s+3)}{2^8\cdot 4!}\left[8s(s+1)(s+3)m^3\right.$$

$$+4s(3s^3+19s^2+69s+117)m^2$$

$$+2(3s^5+26s^4+143s^3+450s^2+786s+576)m$$

$$\left.+(s^6+11s^5+77s^4+341s^3+946s^2+1824s+2880)\right] ,$$

(A7) $K(s,m)W(s+3,s,s-1,s-4,\ldots,1,0)$

$$= \binom{s}{3}\frac{M(s-1,\ldots,s+1)}{2^8\cdot 5!}\left[32(s-1)s(s+1)(s+2)(s+3)m^5\right.$$

$$+16s(5s^5+35s^4+105s^3+145s^2-50s-240)m^4$$

$$+8(s-1)(10s^6+100s^5+505s^4+1550s^3+2785s^2+2670s+1080)m^3$$

$$+4(10s^8+110s^7+635s^6+2265s^5+4765s^4+5125s^3+1550s^2-4020s-6120)m^2$$

$$+2(5s^9+65s^8+450s^7+2000s^6+5679s^5+10115s^4+11770s^3+7860s^2$$

$$+1656s+1440)m$$

$$+(s^{10}+15s^9+120s^8+630s^7+2193s^6+5115s^5+8630s^4+10800s^3$$

$$\left.+12096s^2+25200s+30240)\right] ,$$

(A8) $K(s,m)W(s+3,s,s-2,s-3,s-5,\ldots 1,0)$

$$= 3\binom{s}{4}\frac{M(s-2,\ldots,s+1)}{2^{11}\cdot 7}\Big[16s(s+1)(s+2)(s+3)m^4$$

$$+8s(4s^4+34s^3+132s^2+274s+228)m^3$$

$$+4(6s^6+66s^5+365s^4+1220s^3+2357s^2+2226s+672)m^2$$

$$+2(4s^7+54s^6+378s^5+1692s^4+4862s^3+8670s^2+9428s+5152)m$$

$$+s^8+16s^7+134s^6+740s^5+2757s^4+6924s^3+12212s^2+15040s+9408\Big],$$

(A9) $K(s,m)W(s+2,s,s-1,s-3,s-5,\ldots,1,0)$

$$=\binom{s}{4}\frac{M(s-2,\ldots,s+1)}{2^{8}\cdot 4!}\Big[16(s-1)s(s+1)(s+2)m^4$$

$$+8s(4s^4+14s^3+32s^2+10s-60)m^3$$

$$+4(6s^6+30s^5+113s^4+160s^3+97s^2-22s-384)m^2$$

$$+2s(4s^6+26s^5+126s^4+288s^3+458s^2+598s+228)m$$

$$+(s+2)(s^7+6s^6+34s^5+72s^4+157s^3+306s^2+1728)\Big],$$

(A10) $K(s,m)W(s+2,s,s-2,s-3,s-4,s-6,\ldots,1,0)$

$$=\binom{s}{5}\frac{M(s-3,\ldots,s+1)}{2^{6}\cdot 7\cdot 3}\Big[8s(s+1)(s+2)m^3$$

$$+4s(3s^3+15s^2+45s+54)m^2$$

$$+2(3s^5+21s^4+95s^3+225s^2+295s+210)m$$

$$+s^6+9s^5+52s^4+177s^3+376s^2+663s+630\Big],$$

(A11) $K(s,m)W(s+3,s-1,s-2,s-3,s-4,s-6,\ldots 1,0)$

$$=5\binom{s}{5}\frac{M(s-3,\ldots,s+1)}{2^{12}\cdot 3}\Big[8(s+1)(s+2)(s+3)m^3$$

$$+4(3s^4+27s^3+111s^2+237s+198)m^2$$

$$+2(3s^5+36s^4+215s^3+780s^2+1570s+1260)m$$

$$+(s^6+15s^5+115s^4+573s^3+1804s^2+3108s+2160)\Big] ,$$

(A12) $K(s,m)W(s+2,s-1,s-2,s-3,s-4,s-5,s-7,\ldots 1,0)$

$$= 3\cdot\binom{s}{6}\frac{M(s-4,\ldots,s+1)}{2^{11}}\Big[4(s+1)(s+2)m^2$$

$$+2(2s^3+11s^2+35s+42)m$$

$$+s^4+8s^3+39s^2+108s+108\Big]$$

and

(A13) $K(s,m)W(s+1,s,s-1,s-2,s-5,\ldots,1,0)$

$$=\Big[\binom{s}{4}M(s-2,\ldots,s+1)/2^8.5!\Big]\Big[16(s+1)s(s-1)(s-2)m^4$$

$$+16s(s-1)(s-2)(2s^2+3s+11)m^3$$

$$+4(s-1)(s-2)(6s^4+12s^3+65s^2-s+240)m^2$$

$$+2(s-2)(4s^6+6s^5+54s^4-68s^3+462s^2-698s+1680)m$$

$$+s^8+14s^6-60s^5+269s^4-900s^3-2596s^2-4800s+5760\Big] .$$

APPENDIX B

EXPANSION IN TERMS OF ZONAL POLYNOMIALS

Note that

$$\text{(B1)} \qquad a_2^i \, Z_\kappa(W) = \sum_\delta g^\delta_{\kappa,\eta,i} \, Z_\delta(W) \quad , \; i = 1,2$$

where W is the symmetric matrix, and $Z_\kappa(W)$ is a zonal polynomial which corresponds to a partition κ of k and the summation is over all partitions δ of $2 + k = d$ and g's are constants. Explicit formula for $g^\delta_{\kappa,\eta,i}$ are not known, but we give tables of $g^\delta_{\kappa,\eta,i}$ in lower orders which we came across while evaluating the first two non-central moments of $W_2^{(p)}$. We must note here that

$$\text{(B2)} \qquad C_\kappa(W) = \frac{c(\kappa)}{1.3\ldots(2k-1)} \; Z_\kappa(W)$$

where $c(\kappa)$ is the degree of the representation $[2\kappa]$ of the symmetric group of $2k$ symbols. We read the tables as follows

$$\text{(B3)} \qquad a_2 \, Z_{(2)}(W) = \frac{1}{6} \, Z_{(31)}(W) + \frac{1}{3} \, Z_{(21^2)}(W) \, .$$

These tables are calculated from the tables of zonal polynomials of degree one to six in James (1964) and of degree seven to eleven which he and A. Purkhurst sent to Prof. K. C. S. Pillai in a private communication.

Table 6

$g^{\delta}_{\kappa,\eta,1}$ Coefficients

a_2	$Z_{(21)}$	$Z_{(1^3)}$
$Z_{(1)}$	1/4	1/4

a_2	$Z_{(31)}$	$Z_{(2^2)}$	$Z_{(21^2)}$	$Z_{(1^4)}$
$Z_{(2)}$	1/6		1/3	
$Z_{(1^2)}$		1/12	4/15	3/20

a_2	$Z_{(41)}$	$Z_{(32)}$	$Z_{(31^2)}$	$Z_{(2^21)}$	$Z_{(21^3)}$	$Z_{(1^5)}$
$Z_{(3)}$	1/8		3/8			
$Z_{(21)}$		1/18	8/63	5/36	5/28	
$Z_{(1^3)}$				3/20	1/4	1/10

Table 7

$g^{\delta}_{\kappa,\eta,2}$ Coefficients

a_2^2	$Z_{(52)}$	$Z_{(51^2)}$	$Z_{(43)}$	$Z_{(421)}$	$Z_{(41^3)}$	$Z_{(3^21)}$	$Z_{(32^2)}$	$Z_{(321^2)}$	$Z_{(31^4)}$	$Z_{(2^31)}$	$Z_{(2^21^3)}$	$Z_{(21^5)}$	$Z_{(1^7)}$
$Z_{(3)}$	1/240	2/231		61/1260	57/1100		1/48	16/225	9/200				
$Z_{(21)}$			1/720	2/135	2/225	11/1008	1/63	893/10800	7/225	13/504	5/112	1/72	
$Z_{(1^3)}$						1/168	1/105	13/240	1/45	9/280	9/112	29/720	3/560

Table 7 (cont'd.)

		$Z_{(62)}$	$Z_{(61^2)}$	$Z_{(53)}$	$Z_{(521)}$	$Z_{(51^3)}$	$Z_{(4^2)}$	$Z_{(431)}$	$Z_{(42^2)}$	$Z_{(421^2)}$	$Z_{(41^4)}$
a_2^2	$Z_{(4)}$	$\frac{1}{360}$	$\frac{2}{351}$		$\frac{97}{2376}$	$\frac{31}{728}$			$\frac{1}{40}$	$\frac{32}{385}$	$\frac{1}{20}$
	$Z_{(31)}$			$\frac{1}{1200}$	$\frac{9}{1100}$	$\frac{1}{210}$		$\frac{61}{6075}$	$\frac{11}{972}$	$\frac{18871}{311850}$	$\frac{37}{1650}$
	$Z_{(22)}$						$\frac{1}{1680}$	$\frac{22}{1701}$	$\frac{64}{8505}$	$\frac{8}{405}$	
	$Z_{(21^2)}$							$\frac{5}{1944}$	$\frac{5}{1701}$	$\frac{13}{810}$	$\frac{1}{165}$
	$Z_{(1^4)}$										

		$Z_{(3^22)}$	$Z_{(3^21^2)}$	$Z_{(32^21)}$	$Z_{(321^3)}$	$Z_{(31^5)}$	$Z_{(2^4)}$	$Z_{(2^31^2)}$	$Z_{(2^21^4)}$	$Z_{(21^6)}$	$Z_{(1^8)}$
a_2^2	$Z_{(4)}$										
	$Z_{(31)}$	$\frac{7}{1296}$	$\frac{16}{1215}$	$\frac{91}{2430}$	$\frac{3}{50}$	$\frac{7}{440}$					
	$Z_{(22)}$	$\frac{4}{567}$	$\frac{73}{1944}$	$\frac{23}{486}$	$\frac{1}{15}$		$\frac{1}{168}$	$\frac{1}{42}$	$\frac{1}{48}$		
	$Z_{(21^2)}$	$\frac{13}{2268}$	$\frac{23}{1701}$	$\frac{4399}{97200}$	$\frac{19}{280}$	$\frac{19}{1100}$	$\frac{19}{560}$	$\frac{2}{525}$	$\frac{1}{35}$	$\frac{9}{1400}$	
	$Z_{(1^4)}$		$\frac{1}{112}$	$\frac{1}{50}$	$\frac{11}{189}$	$\frac{4}{225}$	$\frac{1}{200}$	$\frac{3}{70}$	$\frac{73}{1080}$	$\frac{2}{75}$	$\frac{1}{336}$

Table 7 (cont'd.)

		$Z_{(3^2 1^3)}$	$Z_{(32^2 1^2)}$	$Z_{(321^4)}$	$Z_{(31^6)}$	$Z_{(2^4 1)}$	$Z_{(2^3 1^3)}$	$Z_{(2^2 1^5)}$	$Z_{(21^7)}$	$Z_{(1^9)}$
a_2^2	$Z_{(1^5)}$	$\frac{5}{432}$	$\frac{4}{135}$	$\frac{33}{560}$	$\frac{10}{693}$	$\frac{3}{280}$	$\frac{145}{3024}$	$\frac{61}{1080}$	$\frac{57}{3080}$	$\frac{1}{560}$

APPENDIX C

EXPECTED VALUES OF $a_2^2 Z_\kappa$

The expressions for $E(a_2^2 Z_{(1^3)})$, $E(a_2^2 Z_{(32)})$, $E(a_2^2 Z_{(31^2)})$, $E(a_2^2 Z_{(2^2 1)})$, $E(a_2^2 Z_{(21^3)})$ and $E(a_2^2 Z_{(1^5)})$ which were used in the evaluation of $E(W_2^{(p)})^2$ are given below:

(C1) $$E(a_2^2 Z_{(1^3)}) = 2^9 D_\kappa(p,f)[p^2(p-1)^2 f^4 - 2p(p-1^2(p-16) f^3$$
$$+ (p^4-66p^3+501p^2-724p+288)f^2$$
$$+ 4(8p^3-181p^2+797p-696)f$$
$$+ 96(3p^2-29p+65)]$$

(C2) $$E(a_2^2 Z_{(32)}) = D_\kappa(p,f)[p^2(p-1)^2 f^4 - 2p(p-1)^2(p-24)f^3$$
$$+ (p^4-98p^3+957p^2-1500p+640)f^2$$
$$+ 4(12p^3-375p^2+1747p-1496)f$$
$$+ 32(20p^2-187p+384)]$$

$$(C3)\quad E(a_2^2 Z_{(31^2)}) = D_K(p,f)[p^2(p-1)^2f^4-2p(p-1)^2(p-24)f^3$$
$$+ (p^4-98p^3+981p^2-1524p+640)f^2$$
$$+ 4(12p^3-381p^2+1921p-1688)f$$
$$+ 32(20p^2-211p+516)]$$

$$(C4)\quad E(a_2^2 Z_{(2^21)}) = D_K(p,f)[p^2(p-1)^2f^4-2p(p-1)^2(p-24)f^3$$
$$+ (p^4-98p^3+1005p^2-1548p+640)f^2$$
$$+ 4(12p^3-387p^2+2095p-1880)f$$
$$+ 32(20p^2-235p+630)]$$

$$(C5)\quad E(a_2^2 Z_{(21^3)}) = D_K(p,f)[p^2(p-1)^2f^4-2p(p-1)^2(p-24)f^3$$
$$+ (p^4-98p^3+1037p^2-1580p+640)f^2$$
$$+ 4(12p^3-395p^2+2327p-2136)f$$
$$+32(20p^2-267p+838)]$$

$$(C6)\quad E(a_2^2 Z_{(1^5)}) = D_K(p,f)[p^2(p-1)^2f^4-2p(p-1)^2(p-24)f^3$$
$$+ (p^4-98p^3+1085p^2-1628p+640)f^2$$
$$+ 4(12p^3-407p^2+2675p-2520)f$$
$$+ 32(20p^2 - 315p +1180)]$$

BIBLIOGRAPHY

Anderson, T.W. (1946). The noncentral Wishart distribution and certain problems of multivariate statistics. Ann. Math. Statist., 17, 409-431.

——————— (1958). An Introduction to Multivariate Statistical Analysis. Wiley, New York.

Anderson, T.W. and DasGupta, S. (1964a). Monotonicity of the power functions of some tests of independence between two sets of variates. Ann. Math. Statist. 35, 206-208.

——————————————— (1964b). A monotonicity property of the power functions of some tests of the equality of two covariance matrices. Ann. Math. Statist. 35, 1059-1063.

Anderson, T.W. and Girshick, M.A. (1944). Some extensions of the Wishart distribution. Ann. Math. Statist. 15, 345-357.

Bartlett, M.S. (1934). The vector representation of a sample. Proc. Camb. Phil. Soc. 30, 327-340.

——————— (1938). Further aspects of the theory of multiple regression. Proc. Camb. Phil. Soc. 34, 33-40.

——————— (1947a). The general canonical correlation distribution. Ann. Math. Statist. 18, 1-17.

——————— (1947b). Multivariate Analysis. J. Roy. Statist. Soc. Supp. 9, 176-190.

Bose, R.C. and Roy, S.N. (1938). The exact distribution of the studentised D^2-statistics. Sankhyā 4, 19-38.

Box, G.E.P. (1949). A general distribution theory for a class of likelihood criteria. Biometrika. 36, 317-346.

Constantine, A.G. (1963). Some noncentral distribution problems in multivariate analysis. Ann. Math. Statist. 34, 1270-1285.

———————— (1966). The distribution of Hotelling's generalized T_0^2. Ann. Math. Statist. 37, 215-225.

Consul, P.C. (1966). Exact distributions of the likelihood ratio. Ann. Math. Statist. 37, 1319-1330.

DasGupta, S. , Anderson, T.W. and Mudholkar, G.S. (1964). Monotonocity of power functions of some tests of the multivariate linear hypothesis. Ann. Math. Statist. 35, 200-205.

Erdelyi, A. ET AL. (1953). Higher Transcendental Function 2, Chapter 10. McGraw Hill, New York.

Fisher, R.A. (1939). The sampling distribution of some statistics obtained from non-linear equations. Ann. Eugenics. 9, 238-249.

Foster, F. G. (1957). Upper percentage points of the generalized beta distribution. Biometrika. 44, 441-453.

Ghosh, M.N. (1964). On the admissibility of some tests of MANOVA. Ann. Math. Statist. 35, 789-794.

Girshick, M.A. (1939). On the sampling theory of roots of determinantal equation. Ann. Math. Statist. 10, 203-224.

Herz, C.S. (1955). Bessel functions of matrix argument. Ann. of Math. 61, 474-523.

Hotelling, H. (1936). Relation between two sets of variates. Biometrika. 28, 321-377.

——————— (1951). A generalized T test and measure of multivariate dispersion. Proc. Second Berkeley Symp. Math. Statist. Prob. University of California Press. 23-41.

Hsu, P.L. (1938). Notes on Hotelling's generalized T. Ann. Math. Statist. 9, 231-243.

————— (1939). On the distribution of the roots of certain determinantal equations. Ann. Eugenics 9, 250-258.

Ito, K. (1956). Asymptotic formula for the distribution of Hotelling's generalized T-statistic. Ann. Math. Statist. 27, 1091-1105.

James, A.T. (1954). Normal multivariate analysis and the orthogonal group. Ann. Math. Statist. 25, 40-75.

—————— (1955). The noncentral Wishart distribution. Proc. Roy. Soc. Ser. A. 229, 364-366.

—————— (1960). The distribution of the latent roots of the covariance matrix. Ann. Math. Statist. 31, 151-158.

——————— (1961). The distribution of noncentral means with known covariances. Ann. Math. Statist. 32, 874-882.

——————— (1964). Distributions of matrix variates and latent roots derived from normal samples. Ann. Math. Statist. 35, 475-501.

Johnson, N.L., Nixon, E., Amos, D.E. and Pearson, E.S. (1963). Table of percentage points of pearson curves, for given $\sqrt{\beta_1}$ and β_2 expressed in standard measure. Biometrika 50, 459-498.

Khatri, C.G. (1966). On certain distribution problems based on positive definite quadratic functions in normal vectors. Ann. Math. Statist. 37, 468-479.

Kiefer, J. and Schwartz, R. (1965). Admissible Bayes character of T^2-, R^2- and other fully invariant tests for classical multivariate normal problems. Ann. Math. Statist. 36, 747-770.

Kshirsagar, A.M. (1961). Noncentral multivariate Beta distribution. Ann. Math. Statist. 32, 104-110.

Nanda, D.N. (1948a). Distribution of a root of a determinantal equation. Ann. Math. Statist. 19, 47-57.

——————— (1948b). Limiting distribution of a root of a determinantal equation. Ann. Math. Statist. 19, 340-350.

——————— (1950). Probability distribution tables of the largest root of a determinantal equation with two roots. J. Indian Soc. Agric. Statist. 3, 175-177.

Neyman, J. and Pearson, E.S. (1928). On the use and interpretation of certain test criteria for purpose of statistical inference. Biometrika. 20A, 175-240.

——————————————— (1931). On the problem of k samples. Bull. Int. Acad. Cracovie. A. p. 460.

Pearson, E.S. and Hartley, H.O. (1956). Biometrika tables for statisticians Vol. 1. Cambridge Univ. Press., London.

——————— and Neyman, J. (1930). On the problem of two samples. Bull. Int. Acad. Cracovie, A, p. 73.

Pillai, K.C.S. (1954). On some distribution problems in multivariate analysis, Mimeo. Series, No. 88. Institute of Statistics, University of North Carolina.

——— (1955). Some new test criteria in multivariate analysis. Ann. Math. Statist. 26, 117-121.

——— (1956). Some results useful in multivariate analysis. Ann. Math. Statist. 27, 1106-1114.

——— (1960). Statistical tables for tests of multivariate hypotheses. The statistical center, University of the Phillippines, Manila.

——— (1964). On the moments of elementary symmetric functions of the roots of two matrices. Ann. Math. Statist. 35, 1704-1712.

——— (1965). On the distribution of the largest characteristic root of a matrix in multivariate analysis. Biometrika 52, 405-411.

——— (1967). Upper percentage points of the largest root of a matrix in multivariate analysis. Biometrika 54, 189-194.

——— and Mijares, T.A. (1959). On the moments of the trace of a matrix and approximations to its distribution. Ann. Math. Statist. 30, 1135-1140.

——— and Samson, Jr., P. (1959). On Hotelling's generalization of T^2. Biometrika. 46, 160-165.

Posten, H.O. and Bargmann, R.E. (1964). Power of the likelihood-ratio test of the general linear hypothesis in multivariate analysis. Biometrika. 51, 467-476.

Rao, C.R. (1948). Tests of significance in multivariate analysis. Biometrika 35, 58-79.

——— (1951). An asymptotic expansion of the distribution of Wilks' criterion. Bull. Inst. Internat. Statist. 33(2). 177-180.

——— (1952). Advanced statistical methods in Biometric Research. John Wiley, New York.

Roy, J. (1951). The distribution of a certain class of likelihood-ratio criteria useful in multivariate analysis. Bull. Inst. Internat. Statist. 33(2), 219-230.

——— (1960). Nonnull distribution of the likelihood-ratio in analysis of dispersion. Sankhyā 22, 289-294.

——— (1966). Power of the likelihood-ratio test used in analysis of dispersion. Multivariate Analysis. Academic Press, New York. 105-127.

Roy, S. N. (1939). p-statistics or some generalizations in analysis of variance appropriate to multivariate problems. Sankhyā, 4, 381-396.

——— (1942). Analysis of variance for multivariate normal population: The sampling distribution of the requisite p-statistic on the null and nonnull hypothesis. Sankhyā 6, 35-50.

——— (1942). The sampling distribution of p-statistic and certain allied statistics on the nonnull hypothesis. Sankhyā 6, 15-34.

——— (1945). The individual sampling distribution of the max., min., and any intermediate of the p-statistics on the null hypothesis. Sankhyā 7, 133-138.

——— (1953). On a heuristic method of test construction and its use in multivariate analysis. Ann. Math. Statist. 24, 220-238.

——— (1957). Some Aspects of Multivariate Analysis. Wiley, New York.

Schatzoff, M. (1966). Exact distribution of Wilks' likelihood-ratio criterion. Biometrika, 53, 347-358.

Thrall, R.M. (1942). On symmetrized Kronecker powers and the structure of the free Lie ring. Amer. J. of Math. 64, 371-388.

Wald, A. and Brookner, R.J. (1941). On the distribution of Wilks' statistic for testing independence of several groups of variates. Ann. Math. Statist. 12, 137-152.

Wilks, S.S. (1932). Certain generalizations in the analysis of variance. Biometrika, 24, 471-494.

——— (1935). On the independence of k sets of normally distributed variables. Econometrica 3, 309-326.

Weyl, H. (1946). The Classical Groups. 2nd. Ed. Princeton.