Monotonicity of the Variance Under Truncation

and Variations of Jensen's Inequality*

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Let X be a random variable and $Y = XI_{[X < b]}$ for $-\infty < b < \infty$. Then it is easy to construct examples for which $\sigma^2 X < \sigma^2 Y$. However, if $Y = \max(a, \min(X,b))$ for $-\infty \leq a, b \leq \infty$, then we always have $\sigma^2 X \geq \sigma^2 Y$. The motivation for this note was proving a conditional version of this result which was used in [2]; see Theorem 2 and Corollary 4. We will prove the above facts and some of their generalizations which provide intermediate terms in Jensen's inequality. The results and the methods of proof given below are actually special cases or slight modifications of more general inequalities involving duals of cones of generalized convex functions. See for example Karlin and Novikoff [3], Karlin and Studden [4], Ziegler [5], Barlow, Marshall and Proschan [1] and references therein. The methods used below are quite elementary and produce the desired results without recourse to dual cones.

Theorem 1. Let h and g be two Baire functions on the real line and X a random variable such that (a) g is nondecreasing, (b) $Eg(x)$ and $Eh(X)$ exist and $Eg(X) = Eh(X)$, (c) the function $h-g$ has one sign change from negative to positive, i.e., there exists t_o such that $(h(t)-g(t))(t-t_o) \geq 0$ for every t. If $E\varphi(h(X))$ and $E\Phi(g(X))$ exist, then

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$$(1) \qquad E\varphi(h(X)) \geq E\varphi(g(X))$$

for all continuous, convex φ on $(-\infty,\infty)$.

Proof. By the convexity of φ,

$$(2) \qquad \varphi(h(t))-\varphi(g(t)) \geq \varphi'(g(t))(h(t)-g(t)),$$

where φ' denotes the right (or left) derivative of φ. If $t > t_o$, then $h(t)-g(t) \geq 0$ and $\varphi'(g(t)) \geq \varphi'(g(t_o))$, so that

$$(3) \qquad \varphi(h(t))-\varphi(g(t)) \geq \varphi'(g(t_o))(h(t)-g(t)).$$

The above equation also holds for $t < t_o$, since $h(t)-g(t) \leq 0$ and $\varphi'(g(t)) \leq \varphi'(g(t_o))$. Therefore (3) holds for all t. (1) follows immediately from (3).

Corollary 1. Suppose that f is nondecreasing, φ is continuous, convex and for a random variable X, EX and $Ef(X)$ exist, and $t-f(t)+Ef(X)-EX$ has one sign change from negative to positive. Then

$$(4) \qquad E\varphi(X-EX) \geq E\,\varphi(f(X)-Ef(X)),$$

$$(5) \qquad E\varphi(X) \geq E\varphi(f(X)-Ef(X)+EX) \geq \varphi(EX),$$

provided that all the expectations involved in (4) and (5) exist.

All of the inequalities in (4) and (5) follow from (1) by taking the obvious choices of g and h in Theorem 1.

The two extremes $E\varphi(X)$ and $\varphi(EX)$ in (5) produce Jensen's inequality and (5) provides intermediate terms between these two extremes. Note that $\varphi(EX)$ is attained for $f(t) \equiv$ constant and $E\varphi(X)$ results if $f(t) \equiv t +$ constant. Useful particular cases of (4) and (5) are provided by:

Corollary 2. Equations (4) and (5) hold for

$$f(t) = f_{a,b}(t) = \begin{cases} b, & t \geq b \\ t, & a < t < b, \\ a, & t \leq a, \end{cases} \tag{6}$$

where $-\infty \leq a \leq b \leq \infty$, or

$$f(t) = f_c(t) = ct, \quad 0 \leq c \leq 1. \tag{7}$$

Moreover, $E\varphi(f_{a,b}(X)-Ef_{a,b}(X)+EX)$ is nonincreasing in a, nondecreasing in b, and continuous in both variables; while $E\varphi(f_c(X)-Ef_c(X)+EX)$ is nondecreasing and continuous in c.

Corollary 3. If $E|X|^\alpha < \infty$ for some $\alpha \geq 1$, then

$$E|X-EX|^\alpha \geq E|f_{a,b}(X)-Ef_{a,b}(X)|^\alpha, \tag{8}$$

where $f_{a,b}$ is given by (6).

As mentioned earlier the motivation for this note was in proving a conditional version of (8). This is the content of the following theorem. Various other extensions can be established by the same methods.

Theorem 2. Let $\mathcal{G}$ be a σ-field of measurable sets and a,b be $\mathcal{G}$-measurable random variables. Define $X_{a,b} = \max(a,\min(X,b))$. If $E|X| < \infty$ and $E\varphi(X)$ exists for a continuous, convex function φ on $(-\infty,\infty)$, then

$$(9) \qquad E(\varphi(X)|\mathcal{G}) \geq E(\varphi(X_{a,b}-E(X_{a,b}|\mathcal{G})+E(X|\mathcal{G}))|\mathcal{G}),$$

$$(10) \qquad E(\varphi(X-E(X|\mathcal{G}))|\mathcal{G}) \geq E(\varphi(X_{a,b}-E(X_{a,b}|\mathcal{G}))|\mathcal{G}).$$

Proof. By (3),

$$\varphi(X-E(X|\mathcal{G}))-\varphi(X_{a,b}-E(X_{a,b}|\mathcal{G}))$$
$$\geq \varphi'(Y)(X-E(X|\mathcal{G})-X_{a,b}+E(X_{a,b}|\mathcal{G})),$$

where

$$Y = \begin{cases} b, & \text{if } E(X|\mathcal{G}) \leq E(X_{a,b}|\mathcal{G}), \\ a, & \text{if } E(X|\mathcal{G}) > E(X_{a,b}|\mathcal{G}). \end{cases}$$

Put $A = [|Y| < M]$ for $0 < M < \infty$. Then $A \in \mathcal{G}$ and on A, (10) holds; letting $M \to \infty$, we have (10). The proof of (9) is similar.

Corollary 4. Under the assumptions of Theorem 2,

$$E((X-E(X|\mathcal{G}))^2|\mathcal{G}) \geq E((X_{a,b}-E(X_{a,b}|\mathcal{G}))^2|\mathcal{G}).$$

References

[1] Barlow, R.E., A.W. Marshall and F. Proschan (1967). Some inequalities for starshaped and convex functions, Boeing Scientific Research Laboratories. Document D1-82-0643.

[2] Chow, Y.S. (1966). On the moments of some one-sided stopping rules. Ann. Math. Statist. 37, 382-387.

[3] Karlin, S. and A. Novikoff (1963). Generalized convex inequalities. Pacific J. Math. 13, 1251-1279.

[4] Karlin, S. and W.J. Studden (1966). Tchebycheff Systems: with Applications in Analysis and Statistics. Interscience, New York.

[5] Ziegler, Zvi (1966). Generalized convexity cones. Pacific J. Math. 16, 561-580.