Further contributions to some inequalities for normal distributions and their applications to simultaneous confidence bounds

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1. Introduction and summary. Let $\underset{\sim}{x} = (x_1,\dots,x_p)'$ be distributed as multivariate normal with zero means and covariance matrix $V(\underset{\sim}{x})$ and this will be denoted by $\underset{\sim}{x} \sim N(\underset{\sim}{0}, V(\underset{\sim}{x}))$. Dunn's conjucture [3], namely.

(1) $$P\left[|x_i| \le c_i,\ i=1,2,\dots,p\right] \ge \prod_{i=1}^{p} P\left[|x_i| \le c_i\right]$$

was established by Khatri [4], Sidak [6] and Scott [5] by using different approaches. Moreover, Khatri [4] conjuctured that

(2) $$P\left[|x_i| \ge c_i,\ i=1,2,\dots,p\right] \ge \prod_{i=1}^{p} P\left[|x_i| \ge c_i\right]$$

and the proof of (2) given by Scott [5] is incorrect. The purpose of this paper is to generalize (1) and (2) in the case of convex and symmetric regions about the origin. The generalized results are mentioned as under:

Let $\underset{\sim}{x}' = (\underset{\sim}{y}_1', \underset{\sim}{y}_2', \dots, \underset{\sim}{y}_q')$ where $y_i' = (x_{p_1+\dots+p_{i-1}+1}, \dots, x_{p_1+\dots+p_i})$, $i=1,2,\dots,q$ with $\sum_{i=1}^{q} p_i = p$. Moreover, let $\mathcal{D}_i(\underset{\sim}{y}_i)$ be convex and symmetric region in $\underset{\sim}{y}_i$ about the origin in p-dimensional space with $-\infty < x_j < \infty$, $j = 1,2,\dots,p_1+p_2+\dots+p_{i-1}, p_1+p_2+\dots+p_i+1,\dots,p$. Let $\overline{\mathcal{D}_i(y_i)}$ be the complementary region of $\mathcal{D}_i(y_i)$. Then, we have

(3) $$P\left(\bigcap_{i=1}^{q} \mathcal{D}_i(\underset{\sim}{y}_i)\right) \ge \prod_{i=1}^{q} P(\mathcal{D}_i(\underset{\sim}{y}_i))$$

and

(4) $$P\left(\bigcap_{i=1}^{q} \overline{\mathcal{D}_i(y_i)}\right) \ge \prod_{i=1}^{q} P(\overline{\mathcal{D}_i(y_i)}) .$$

Some applications of these results are given on simultaneous confidence intervals. All the results mentioned by Khatri [4] are now valid omitting the structure ℓ .

2. Inequalities for multivariate normal distributions.

The following lemmas will be used in establishing (3) and (4).

Lemma 1. Let $\underset{\sim}{x} \sim N(\underset{\sim}{0},V(\underset{\sim}{x}))$, $\underset{\sim}{x}^{(2)} = (x_2,x_3,\ldots,x_p)$ and let $\mathcal{D}_1(x_1)$ and $\mathcal{D}_2(\underset{\sim}{x}^{(2)})$ be two convex and symmetric regions in x_1 and $\underset{\sim}{x}^{(2)}$ respectively about the origin in p-dimensional space containing axes due to other variates. Then,

$$P(\mathcal{D}_1(x_1) \cap \mathcal{D}_2(\underset{\sim}{x}^{(2)})) \geq P(\mathcal{D}_1(x_1)\; P(\mathcal{D}_2(\underset{\sim}{x}^{(2)})) .$$

For proof, refer Khatri [4].

Lemma 2. Let $\underset{\sim}{x}$: p x 1 $\sim N(\underset{\sim}{0},V(\underset{\sim}{x}))$ and $\underset{\sim}{Z}$: p x 1 $\sim N(\underset{\sim}{0},V(\underset{\sim}{Z}))$. Then, if $V(\underset{\sim}{x}) - V(\underset{\sim}{Z})$ is positive semi-definite,

$$P(\mathcal{D}(\underset{\sim}{Z})) \geq P(\mathcal{D}(x))$$

where $\mathcal{D}(\underset{\sim}{w})$ is a convex and symmetric region in $\underset{\sim}{w}$ about the origin.

For proof, refer Anderson [1].

Theorem 1. Let $\underset{\sim}{x} \sim N(\underset{\sim}{0},V(\underset{\sim}{x}))$, $\underset{\sim}{x}' = (\underset{\sim}{y}_1',\ldots,\underset{\sim}{y}_q')$, $\underset{\sim}{y}_i' = (x_{p_1+\ldots+p_{i-1}},\ldots,x_{p_1+\ldots+p_i})$ and $\mathcal{D}_i(\underset{\sim}{y}_i)$ be convex and symmetric region in $\underset{\sim}{y}_i$ about the origin in p-dimensional space containing axes due to other variates, for $i=1,2,\ldots,q$. Then

$$P\left[\bigcap_{i=1}^{q} \mathcal{D}_i(\underset{\sim}{y}_i)\right] \geq P(\mathcal{D}_i(\underset{\sim}{y}_1))\, P(\bigcap_{i=2}^{q} \mathcal{D}_i(\underset{\sim}{y}_i)) \geq \prod_{i=1}^{q} P(\mathcal{D}_i(\underset{\sim}{y}_i)) .$$

Proof. When any $(q-1)$ values of p_i, $i=1,2,\ldots,q$ are at the most one, then theorem 1 is established by Khatri [4] or theorem 1 is equivalent to lemma 1. Here, we assume that $p_i > 1$, $i = 1,2,\ldots,q$. First of all, we shall consider the case when $V(\underset{\sim}{x})$ is positive definite. Without loss of generality, we can write $V(\underset{\sim}{x}) = \underset{\sim}{A}\underset{\sim}{A}'$ where $\underset{\sim}{A} = (\underset{\sim}{A}_{ii'})$ is nonsingular, $\underset{\sim}{A}_{ii'} = \underset{\sim}{0}$ for $i > i'$, $i,i' = 1,2,\ldots,q$. Let $\underset{\sim}{A}^{-1}\underset{\sim}{x} = \underset{\sim}{Z}$ and $\underset{\sim}{Z}' = (\underset{\sim}{w}_1',\ldots,\underset{\sim}{w}_q')$ with $\underset{\sim}{w}_i' = (Z_{p_1+\ldots+p_{i-1}+1},\ldots,Z_{p_1+\ldots+p_i})$. Then, it is easy to see that

$$\underset{\sim}{y}_i = \sum_{j=i}^{q} \underset{\sim}{A}_{ij}\,\underset{\sim}{w}_j \qquad \text{for} \qquad i = 1,2,\ldots,q$$

and $\underset{\sim}{Z} \sim N(\underset{\sim}{0},\underset{\sim}{I}_p)$ or $\underset{\sim}{w}_i \sim IN(\underset{\sim}{0},\underset{\sim}{I}_{p_i})$, $i = 1,2,\ldots,q$. It is easy to see that theorem 1 will be established for $V(\underset{\sim}{x})$ to be positive definite if we can establish

$$(5) \qquad P\left[\bigcap_{i=1}^{q} \mathcal{D}_i\left(\sum_{j=i}^{q} \underset{\sim}{A}_{ij}\,\underset{\sim}{w}_j\right) \,\middle|\, \underset{\sim}{Z} \in Q\right] \geq$$

$$P\left[\mathcal{D}_1(\underset{\sim}{y}_1)\right] P\left[\bigcap_{j=2}^{q} \mathcal{D}_i\left(\sum_{j=i}^{q} \underset{\sim}{A}_{ij}\,\underset{\sim}{w}_j\right) \,\middle|\, \underset{\sim}{Z} \in Q\right)$$

for every (p_1+1)-flat Q containing $(Z_1,Z_2,\ldots Z_{p_1})$-axes.

Let us take such a (p_1+1)-flat Q and let us suppose that this is determined by the set of linearly independent equations given by

$$\sum_{j=1}^{p-p_1} \ell_{kj} Z_{j+p_1} = 0 \quad \text{for} \quad k = 1,2,\ldots,p-p_1-1$$

where without loss of generality, take $\sum_{j=1}^{p-p_1} \ell_{kj}^2 = 1$ and $\sum_{j=1}^{p-p_1} \ell_{kj}\ell_{k'j} = 0$ for $k \neq k'$. Let $\underset{\sim}{L}_1 = (\ell_{kj}): (p-p_1-1) \times (p-p_1)$. Then, we can complete $\underset{\sim}{L}_1$ by a vector $\underset{\sim}{\ell}$ such that $L' = (\underset{\sim}{\ell}\ \underset{\sim}{L}_1')$ is an orthogonal matrix. Now use the transformation

$$L(\underset{\sim}{w}_2', \ldots, \underset{\sim}{w}_q')' = (v_1, v_2, \ldots, v_{p-p_1})' .$$

Then it is obvious that the (p_1+1)-flat Q will have the coordinate system given by $(\underset{\sim}{w}_1', v_1, v_i = 0 \text{ for } i = 2,\ldots,p-p_1)$ and $v_1 \sim N(0,1)$ and $\underset{\sim}{w}_1 \sim N(\underset{\sim}{0}, \underset{\sim}{I}_{p_1})$ and they are independently distributed. Hence, using this system of coordinates in the left side of (5), we get

$$(6) \quad I(Q) = P\left[\bigcap_{i=1}^{q} \mathscr{D}_i\left(\sum_{j=i}^{q} \underset{\sim}{A}_{ij}\, \underset{\sim}{w}_j\right) \Big| \underset{\sim}{Z} \epsilon Q\right] = P\left[\mathscr{D}_1(\underset{\sim}{A}_{11}\underset{\sim}{w}_1 + \underset{\sim}{\delta}_1 v_1) \cap \left(\bigcap_{j=2}^{q} \mathscr{D}_j(\underset{\sim}{\delta}_j v_1)\right)\right]$$

where if $\underset{\sim}{\ell}' = (\underset{\sim}{\ell}_2', \underset{\sim}{\ell}_3', \ldots, \underset{\sim}{\ell}_q')$ with $\underset{\sim}{\ell}_i'$: $p_i \times 1$, $\underset{\sim}{\delta}_i = \sum_{j=i}^{q} \underset{\sim}{A}_{ij}\, \underset{\sim}{\ell}_j$ for $i = 2,3,\ldots,q$ and $\underset{\sim}{\delta}_1 = \sum_{j=2}^{q} \underset{\sim}{A}_{1j}\, \underset{\sim}{\ell}_j$. Since $\underset{\sim}{\delta}_i$, $i = 2,\ldots,q$ are fixed vector and $\mathscr{D}_i(\underset{\sim}{\delta}_i v_1)$, $i=2,3,\ldots,q$ are convex and symmetric in $\underset{\sim}{\delta}_i v_1$ about the origin and hence $\mathscr{D}(v_1) = \bigcap_{i=2}^{q} \mathscr{D}_i'(v_1)$ is convex and symmetric in v_1 about the origin. Then, using this in (6) and then lemma 1, we get

$$(7) \qquad I(Q) \geq P\left[\mathscr{D}_1(\underset{\sim}{A}_{11}\underset{\sim}{w}_1 + \underset{\sim}{\delta}_1 v_1)\right] P(\mathscr{D}(v_1) = \bigcap_{i=2}^{q} \mathscr{D}_i(\underset{\sim}{\delta}_i v_1)) .$$

Note that

$$(8) \qquad P\left(\bigcap_{i=2}^{q} \mathscr{D}_i(\underset{\sim}{\delta}_i v_1)\right) = P\left(\bigcap_{i=2}^{q} \mathscr{D}_i\left(\sum_{j=i}^{q} \underset{\sim}{A}_{ij}\underset{\sim}{w}_j\right) \Big| \underset{\sim}{Z} \epsilon Q\right)$$

and by using lemma 2,

(9) $$P\left[\mathcal{S}_1(\underset{\sim}{A}_{11}\underset{\sim}{w}_1+\underset{\sim}{\delta}_1 v_1)\right] \geq P\left[\mathcal{S}_1(\underset{\sim}{y}_1)\right]$$

for $V(\underset{\sim}{y}_1) - V(\underset{\sim}{A}_{11}\underset{\sim}{w}_1+\underset{\sim}{\delta}_1 v_1) = \sum_{j=1}^{q} \underset{\sim}{A}_{1j}\underset{\sim}{A}'_{1j} - (\underset{\sim}{A}_{11}A'_{11} + \underset{\sim}{\delta}_1 \underset{\sim}{\delta}'_1) =$

$\sum_{j=2}^{q} \underset{\sim}{A}_{1j}(\underset{\sim}{I}_{p_j} - \underset{\sim}{\ell}_j\underset{\sim}{\ell}'_j)\underset{\sim}{A}'_{1j}$ is positive semi-definite.

Using (8) and (9) in (7), we get (5). Thus, theorem 1 is established when $V(\underset{\sim}{x})$ is nonsingular.

Let $V(\underset{\sim}{x})$ be positive semi-definite. Let $\underset{\sim}{u}: p \times 1 \sim N(\underset{\sim}{0}, h\,\underset{\sim}{I}_p)$ and let $\underset{\sim}{u}$ and $\underset{\sim}{x}$ be independently distributed. Then $\underset{\sim}{x} + \underset{\sim}{u} = \underset{\sim}{t} \sim N(\underset{\sim}{0}, h\underset{\sim}{I}_p + V(\underset{\sim}{x}))$ and $V(\underset{\sim}{x}) + h\,\underset{\sim}{I}_p$ is positive definite. Hence from the result for positive definite covariance matrix, we get

(10) $$P\left[\bigcap_{i=1}^{q} \mathcal{S}_i(\underset{\sim}{t}_i)\right] \geq \prod_{i=1}^{q} P\left[\mathcal{S}_i(\underset{\sim}{t}_i)\right] .$$

Taking limits as $h \to 0^+$, we get the result for the singular case, for

$$\lim_{h \to 0^+} P\left[\mathcal{S}(\underset{\sim}{t}_i)\right] = P\left[\mathcal{S}(\underset{\sim}{y}_i)\right]$$

if $\mathcal{S}(\underset{\sim}{y})$ is a convex and symmetric region in $\underset{\sim}{y}$ about the origin.

Theorem 2. Under the notations of theorem 1, we have

$$P\Big(\bigcap_{i=1}^{q} \overline{\mathcal{S}_i(\underset{\sim}{y}_i)}\Big) \geq P(\overline{\mathcal{S}_1(\underset{\sim}{y}_1)})\, P\Big(\bigcap_{i=2}^{q} \overline{\mathcal{S}_i(y_i)}\Big) \geq \prod_{i=1}^{q} P(\overline{\mathcal{S}_i(\underset{\sim}{y}_i)}),$$

where $\overline{\mathcal{S}(\underset{\sim}{w})}$ is the complement of $\mathcal{S}(\underset{\sim}{w})$.

Proof. Let us consider the case when $V(\underset{\sim}{x})$ is positive definite and we proceed in the same manner as in theorem 1 in considering

$$P\left[\mathcal{S}_1(\underset{\sim}{y}_1) \cap (\bigcup_{i=2}^{q} \mathcal{S}_i(\underset{\sim}{y}_i))\right] .$$

Using the same arguments as those in theorem 1, we get

$$(11) \qquad P\left[\mathcal{S}_1(\sum_{j=1}^{q} \underset{\sim}{A}_{1j} \underset{\sim}{w}_j) \cap (\bigcup_{i=2}^{q} \mathcal{S}_i(\sum_{j=i}^{q} \underset{\sim}{A}_{ij}\underset{\sim}{w}_j)) | \underset{\sim}{Z} \in Q\right]$$

$$= P\left[\mathcal{S}_1(\underset{\sim}{A}_{11}\underset{\sim}{w}_1+\underset{\sim}{\delta}_1 v_1) \cap (\bigcup_{i=2}^{q} \mathcal{S}_i' (v_1))\right] .$$

Now $\bigcup_{i=2}^{q} \mathcal{S}_i'(v_1) \Longleftrightarrow |v_1| \leq \alpha$ for some $\alpha \geq 0$. Hence, using this in (11) and using theorem 1, we get

$$(12) \qquad P\left[\mathcal{S}_1(\sum_{j=1}^{q} \underset{\sim}{A}_{1j}\underset{\sim}{w}_j) \cap (\bigcup_{i=2}^{q} \mathcal{S}_i(\sum_{j=i}^{q} \underset{\sim}{A}_{ij}\underset{\sim}{w}_j)) | \underset{\sim}{Z} \in Q\right]$$

$$\geq P\left[\mathcal{S}_1(\underset{\sim}{A}_{11}\underset{\sim}{w}_1+\underset{\sim}{\delta}_1 v_1)\right] P\left[|v_1| \leq \alpha\right]$$

and using $P\left[|v_1| \leq \alpha\right] = P\left[\bigcup_{i=2}^{q} \mathcal{S}_i'(v_1)\right] = P\left[\bigcup_{i=2}^{q} \mathcal{S}_i(\sum_{j=i}^{q} \underset{\sim}{A}_{ij}\underset{\sim}{w}_j) | \underset{\sim}{Z} \in Q\right]$ and $P\left[\mathcal{S}_1(\underset{\sim}{A}_{11}\underset{\sim}{w}_1+\underset{\sim}{\delta}_1 v_1)\right] \geq P(\mathcal{S}_1(\underset{\sim}{y}_1))$, we get

$$(13) \qquad P\left[\mathcal{S}_1(\sum_{j=1}^{q} \underset{\sim}{A}_{1j}\underset{\sim}{w}_j) \cap (\bigcup_{i=2}^{q} \mathcal{S}_i(\sum_{j=i}^{q} \underset{\sim}{A}_{1j}\underset{\sim}{w}_j)) | \underset{\sim}{Z} \in Q\right]$$

$$\geq P(\mathcal{S}_1(\underset{\sim}{y}_1)) \, P\left[\bigcup_{i=2}^{q} \mathcal{S}_i(\sum_{j=i}^{q} \underset{\sim}{A}_{1j}\underset{\sim}{w}_j) | \underset{\sim}{Z} \in Q\right] .$$

Then (13) gives us

$$(14) \qquad P\left[\mathcal{S}_1(\underset{\sim}{y}_1) \cap (\bigcup_{i=2}^{q} \mathcal{S}_i(\underset{\sim}{y}_i))\right] \geq P(\mathcal{S}_1(\underset{\sim}{y}_1)) \, P(\bigcup_{i=2}^{q} \mathcal{S}_i(\underset{\sim}{y}_i)) .$$

We note that if $\mathcal{R}_1$ and $\mathcal{R}_2$ be two regions, then

$$P(\mathcal{R}_1) = P(\mathcal{R}_1 \cap \mathcal{R}_2) + P(\mathcal{R}_1 \cap \bar{\mathcal{R}}_2) \quad .$$

Moreover, we have $\overline{\left\{\bigcup_{i=2}^{q} \mathcal{D}_i(\underset{\sim}{y}_i)\right\}} = \bigcap_{i=2}^{q} \overline{\mathcal{D}_i(y_i)}$. Using these in (14), we get

$$P\left[\mathcal{D}_i(\underset{\sim}{y}_1) \cap (\bigcap_{i=2}^{q} \overline{\mathcal{D}_i(y_i)}\right] \leq P(\mathcal{D}_1(\underset{\sim}{y}_1)\ P(\bigcap_{i=2}^{q} \overline{\mathcal{D}_i(\underset{\sim}{y}_i)}) \tag{15}$$

and this implies

$$P(\bigcap_{i=1}^{q} \overline{\mathcal{D}_i(\underset{\sim}{y}_i)}) \geq P(\overline{\mathcal{D}_1(y_1)})\ P(\bigcap_{i=2}^{q} \overline{\mathcal{D}_i(\underset{\sim}{y}_i)}) \ . \tag{16}$$

Thus, theorem 2 is proved when $V(\underset{\sim}{x})$ is positive definite. When $V(\underset{\sim}{x})$ is singular, we can argue in the same manner as in theorem 1. This completes the proof of theorem 2.

<u>Corollary 1</u>. Let $\underset{\sim}{x}_j \sim N(\underset{\sim}{0}, V(\underset{\sim}{x}_j))$, $j = 1,2,\ldots,n$ and let them be independent. Let $\mathcal{D}_i = \mathcal{D}_i(\underset{\sim}{y}_{i1}, \underset{\sim}{y}_{i2}, \ldots, \underset{\sim}{y}_{in})$ be convex and separately symmetric in $\underset{\sim}{y}_{i1}, \underset{\sim}{y}_{i2}, \ldots, \underset{\sim}{y}_{in}$ about the origin for $i = 1,2,\ldots,q$ and $\bar{\mathcal{D}}_i$ be the complement of $\mathcal{D}_i$. Then

$$P(\bigcap_{i=1}^{q} \mathcal{D}_i) \geq \prod_{i=1}^{q} P(\mathcal{D}_i) \quad \text{and} \quad P(\bigcap_{i=1}^{q} \bar{\mathcal{D}}_i) \geq \prod_{i=1}^{q} P(\bar{\mathcal{D}}_i) \ .$$

(For the definition of separately symmetric, see Khatri [4].)

<u>Proof</u>. We shall only indicate the proof for one case as under:

Let $\underset{\sim}{w}_{ij} \sim N(\underset{\sim}{0}, V(\underset{\sim}{y}_{ij}))$, $j = 1,2,\ldots,n$ and $i = 1,2,\ldots,q$ and let them be independent and independent of $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n$. By theorem 1, it is easy to see that

$$(17) \qquad P\left[\bigcap_{i=1}^{q} \mathcal{D}_i(\underset{\sim}{y}_{i1},\ldots,\underset{\sim}{y}_{in}) \mid \underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_{n-1}\right]$$

$$\geq P\left[\bigcap_{i=1}^{q} \mathcal{D}_i(\underset{\sim}{y}_{i1},\ldots,\underset{\sim}{y}_{in-1}) \mid \underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_{n-1}\right]$$

because $\mathcal{D}_i(\underset{\sim}{y}_{i1},\ldots,\underset{\sim}{y}_{in})$ is convex and symmetric in $\underset{\sim}{y}_{in}$ about the origin when $\underset{\sim}{y}_{i1},\ldots,\underset{\sim}{y}_{in-1}$ are fixed. From (17), we get

$$(18) \qquad P\left[\bigcap_{i=1}^{q} \mathcal{D}_i(\underset{\sim}{y}_{i1},\ldots,\underset{\sim}{y}_{in})\right] \geq P\left[\bigcap_{i=1}^{q} \mathcal{D}_i(\underset{\sim}{y}_{i1},\ldots,\underset{\sim}{y}_{in-1},\underset{\sim}{w}_{in})\right].$$

Proceeding in the same manner for $\underset{\sim}{x}_{n-1},\underset{\sim}{x}_{n-2},\ldots,\underset{\sim}{x}_1$, we get the final result as

$$(19) \qquad P\left[\bigcap_{i=1}^{q} \mathcal{D}_i\right] \geq P\left[\bigcap_{i=1}^{q} \mathcal{D}_i(\underset{\sim}{w}_{i1},\ldots,\underset{\sim}{w}_{in})\right] = \prod_{i=1}^{q} P\left[\mathcal{D}_i(\underset{\sim}{w}_{i1},\ldots,\underset{\sim}{w}_{in})\right]$$

$$= \prod_{i=1}^{q} P(\mathcal{D}_i)\ .$$

This proves the first part of corollary 1. The second part can be proved in the same manner.

Corollary 2. Let $\underset{\sim}{x}_j \sim N(\underset{\sim}{0},V(\underset{\sim}{x}_j))$, $j=1,2,\ldots,n$ and be independent. Let us suppose that $V(\underset{\sim}{x}_j) = (\underset{\sim}{A}_{ii',j})$, $\underset{\sim}{A}_{ii,j} = \sigma^2_{i,j}\, \underset{\sim}{I}_{\alpha_i}$ for $\alpha_i = 1,2,\ldots,r$

and $\displaystyle\sum_{\alpha=\alpha_1+\ldots+\alpha_{i-1}+1}^{\alpha_1+\ldots+\alpha_i} x^2_{\alpha,j}$ for $i = 1,2,\ldots,r$

$\underset{\sim}{y}'_{t,j} = (Z_{r_1+\ldots+t_{t-1}+1},\ldots,Z_{r_1+r_2+\ldots+r_t})$, $t = 1,2,\ldots,q$, $\displaystyle\sum_{t=1}^{q} r_t = r$ and $\mathcal{D}_t = \mathcal{D}_t(\underset{\sim}{y}_{t,1}, \underset{\sim}{y}_{t,2},\ldots, \underset{\sim}{y}_{t,n})$ about the origin for $t = 1,2,\ldots,q$. Then,

$$P\left(\bigcap_{t=1}^{q} \mathscr{D}_t\right) \geq \prod_{t=1}^{q} P(\mathscr{D}_t) \quad \text{and} \quad P\left(\bigcap_{t=1}^{q} \bar{\mathscr{D}}_t\right) \geq \prod_{t=1}^{q} P(\bar{\mathscr{D}}_t) .$$

This follows from corollary 1.

Note: In corollaries 1 and 2, if some observations are missing on $\underset{\sim}{y}_1$ or $\underset{\sim}{y}_2, \ldots$ or $\underset{\sim}{y}_q$, we have to omit these from the convex and symmetric regions $\mathscr{D}_i$, $i = 1,2,\ldots,q$. e.g. Suppose on $\underset{\sim}{y}_1$, the only observations available are $\underset{\sim}{y}_{1,j}$, $j = 1,2,\ldots,n_1$ $(n_1 < n)$. Then, $\mathscr{D}_1 = \mathscr{D}_1(\underset{\sim}{y}_{1,1},\ldots,\underset{\sim}{y}_{1,n_1})$ is convex and symmetric region in $\underset{\sim}{y}_{1,1},\ldots,\underset{\sim}{y}_{1,n_1}$ about the origin.

3. Direct applications.

(3.1) Confidence bounds for means.

Let us suppose that $\underset{\sim}{x}_j \sim N(\underset{\sim}{\xi}, V(\underset{\sim}{x}))$ for $j = 1,2,\ldots,n$ and let them be independent. Let us assume that $V(\underset{\sim}{x}) = (\underset{\sim}{A}_{ii'})$, $\underset{\sim}{A}_{ii} = \sigma_i^2 \underset{\sim}{I}_{\alpha_i}$, $i=1,2,\ldots,r$ and $\underset{\sim}{\xi}' = (\underset{\sim}{\xi}_1', \underset{\sim}{\xi}_2', \ldots, \underset{\sim}{\xi}_r')$, with $\underset{\sim}{\xi}_i : \alpha_i \times 1$.

Let $\underset{\sim}{x}_j' = (\underset{\sim}{y}_{1,j}', \ldots, \underset{\sim}{y}_{r,j}')$, $\underset{\sim}{y}_i : \alpha_i \times 1$, $\bar{\underset{\sim}{y}}_i = \sum_{j=1}^{n} \underset{\sim}{y}_{i,j}/n$ and

$\mathscr{S}_{ii} = \sum_{j=1}^{n} \underset{\sim}{y}_{i,j}' \underset{\sim}{y}_{i,j} - n\, \bar{\underset{\sim}{y}}_i' \bar{\underset{\sim}{y}}_i$. Then, by corollary 2, it is easy to see that

$$(20) \qquad P\left[(\bar{\underset{\sim}{y}}_i - \underset{\sim}{\xi}_i)'(\bar{\underset{\sim}{y}}_i - \underset{\sim}{\xi}_i) \leq c_i \mathscr{S}_{ii},\ i = 1,2,\ldots,r\right]$$

$$\geq \prod_{i=1}^{r} P\left[(\bar{\underset{\sim}{y}}_i - \underset{\sim}{\xi}_i)'(\bar{\underset{\sim}{y}}_i - \underset{\sim}{\xi}_i) \leq c_i \mathscr{S}_{ii}\right]$$

because $\bar{y}_i$ and $\mathscr{S}_{ii}$ are independently distributed and $\mathscr{S}_{ii} = \sum_{j=1}^{n-1} \underset{\sim}{Z}_{i,j}' \underset{\sim}{Z}_{i,j}$ with $(\bar{\underset{\sim}{y}}_1', \ldots, \bar{\underset{\sim}{y}}_r')' \sim N(\underset{\sim}{\xi}, V(\underset{\sim}{x})/n)$ and $\underset{\sim}{Z}_j = (\underset{\sim}{Z}_{1,j}', \ldots, \underset{\sim}{Z}_{r,j}')' \sim IN(\underset{\sim}{0}, V(\underset{\sim}{x}))$.

Now it is easy to see that $n(n-1)(\bar{y}_i-\xi_i)'(\bar{y}_i-\xi_i)/\mathscr{S}_{ii}$ is distributed as $F_{\alpha_i,(n-1)\alpha_i}$ with α_i and $(n-1)\alpha_i$ degrees of freedom for $i = 1,2,\ldots,r$. Hence, we can find $c_1,c_2,\ldots,c_r$ such that

(21) $$\prod_{i=1}^{r} P\left[(\bar{y}_i-\xi_i) \leq c_i\,\mathscr{S}_{ii}\right] = 1-\alpha .$$

One choice of choosing $c_1,c_2,\ldots,c_r$ is to take

$$P\left[(\bar{y}_i-\xi_i)'(\bar{y}_i-\xi_i) \leq c_i\,\mathscr{S}_{ii}\right] = (1-\alpha)^{1/r} .$$

Using (21) in (20), we can find simultaneous confidence bounds on ξ_i, $i = 1,2,\ldots,r$ with confidence greater than $(1-\alpha)$ as

(22) $$a_i'\bar{y}_i - \left\{c_i\mathscr{S}_{ii}(a_i'a_i)\right\}^{1/2} \leq a_i'\xi_i \leq a_i'\bar{y}_i + \left\{a_i'a_i)\right\}^{1/2}$$

for all $i = 1,2,\ldots,r$ and for all non-null vectors $a_i: \alpha_i \times 1$, $i = 1,2,\ldots,r$.

(3.2) <u>One sided confidence bounds on variances</u>.

Let us suppose that $x = (y_1',\ldots,y_q')' \sim N(0,V(x))$ and let us have n independent observations on x. Out of these n observations, it is found that n_i observations are missing on y_i, $i = 1,2,\ldots,q$. Let $V(x) = (A_{ii'})$, and S_i, the sample sum of squares matrix due to available observations on y_i, $i = 1,2,\ldots,q$.

If $\mathscr{D}_i = \mathscr{D}_i\left[ch_{max}(A_{ii}^{-1}S_i) \leq c_i\right]$, then $\mathscr{D}_i$ is section-wise convex and separately symmetric in available observations about the origin (see DasGupta, Mudholkar and Anderson [2]). Hence, by corollary 1, we get

(23) $$P\left[\bigcap_{i=1}^{q}\mathscr{D}_i\right] \geq \prod_{i=1}^{q} P(\mathscr{D}_i) \quad\text{and}\quad P\left[\bigcap_{i=1}^{q}\bar{\mathscr{D}}_i\right] \geq \prod_{i=1}^{q} P(\bar{\mathscr{D}}_i) .$$

In order to obtain the lower bounds on the parameters $\underset{\sim}{A}_{ii}$, $i = 1,2,\ldots,q$, we use the first part of (22). Let us choose $c_1, c_2, \ldots, c_q$ such that

$$\prod_{i=1}^{q} P\left[ch_{max} (\underset{\sim}{A}_{ii}^{-1} \underset{\sim}{S}_i) \leq c_i \right] = 1 - \alpha \ . \tag{24}$$

Using (24) in the first part of (23), we get simultaneous lower bounds on $\underset{\sim}{A}_{ii}$, $i = 1,2,\ldots,q$ with confidence greater than or equal to $(1 - \alpha)$ as

$$\underset{\sim}{a}_i' \underset{\sim}{A}_{ii} \underset{\sim}{a}_i \geq \underset{\sim}{a}_i' \underset{\sim}{S}_i \underset{\sim}{a}_i / c_i , \qquad i = 1,2,\ldots,q \tag{25}$$

for all non-null vectors $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_q$.

Similarly, by choosing c_i', $i = 1,2,\ldots,q$ from

$$\prod_{i=1}^{q} P\left[ch_{max} (A_{ii}^{-1} S_i) > c_i' \right] = 1 - \alpha , \tag{26}$$

we find the simultaneous upper bounds on $\underset{\sim}{A}_{ii}$, $i = 1,2,\ldots,q$ with confidence greater than or equal to $(1 - \alpha)$ as

$$\underset{\sim}{a}_i' \underset{\sim}{A}_{ii} \underset{\sim}{a}_i \leq c_i' (\underset{\sim}{a}_i' \underset{\sim}{S}_i \underset{\sim}{a}_i), \qquad i = 1,2,\ldots,q \tag{27}$$

for all non-null vectors $\underset{\sim}{a}_i$, $i = 1,2,\ldots,q$.

By combining (25) and (27), we get the simultaneous confidence bounds on $\underset{\sim}{A}_{ii}$, $i = 1,2,\ldots,q$ with confidence greater than or equal to $(1 - \alpha)^2$.

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