On the distributions of Hotelling's T_0^2 for three latent roots and the smallest root of a covariance matrix*

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1. Introduction and Summary. In this paper, first, the distribution of $U^{(s)}$, the sum of the s non-null characteristic roots of a matrix (which is a constant times Hotelling's T_o^2) is derived for $s = 3$, starting with the joint density of the s roots given by Roy [10] (see Section 2). The C.D.F. of $U^{(3)}$ thus obtained is used to compute upper 5 per cent points for selected values of two sample parameters which show that the approximate percentage points given by Pillai [8] are generally accurate to the three decimals provided. The distribution of the sum of the three smallest roots of a sample covariance matrix is obtained next for $p = 4$, where p is the number of variables, taking the population covariance matrix $\underset{\sim}{\Sigma} = \underset{\sim}{I}$. Further, the distribution of the smallest characteristic root of a sample covariance matrix is derived for an arbitrary $\underset{\sim}{\Sigma}$. For tests based on the sum of the i smallest of p roots and the smallest root alone of a covariance matrix, reference may be made to [1], [9], [10].

2. Exact distribution of $U^{(3)}$. The distribution of non-null characteristic roots of a matrix derived from sample observations taken from multivariate normal populations, given by Roy [10], is of the form

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$$(2.1)\quad f(\lambda_1,\lambda_2,\dots,\lambda_s;m,n) = C(s,m,n)\prod_{i=1}^{s}\left\{\frac{\lambda_i^m}{(1+\lambda_i)^{m+n+s+1}}\right\}\prod_{i<j}(\lambda_i-\lambda_j)$$

$$0<\lambda_1\le\lambda_2\le\cdots\le\lambda_s<\infty$$

where $C(s,m,n) = \pi^{\frac{1}{2}s}\prod_{i=1}^{s}\left\{\Gamma[\frac{1}{2}(2m+2n+s+i+2)]\Gamma[\frac{1}{2}(2m+i+1)]\Gamma[\frac{1}{2}(2n+i+1)]\Gamma(\frac{1}{2}i)\right\}$

and m and n are defined differently for various situations described by Pillai [7] and [8]. In this section, we will obtain the density of $U^{(3)} = \lambda_1 + \lambda_2 + \lambda_3$ with $s = 3$. First put $s = 3$ in (2.1) and let $\ell_i = \lambda_i/\lambda_3$, $i = 1,2$, then we have

$$(2.2)\quad C(3,m,n)\,\lambda_3^{3m+5}\,(\ell_2-\ell_1)\prod_{i=1}^{2}\ell_i^m(1-\ell_i)/\left\{(1+\lambda_3)^{3(m+n+4)}(1-d)^{m+n+4}\right\}$$

$$0<\lambda_3<\infty,\qquad 0<\ell_1\le\ell_2<1$$

where $d = (\frac{\lambda_3}{1+\lambda_3})(2-\ell_1-\ell_2) - (\frac{\lambda_3}{1+\lambda_3})^2(1-\ell_1)(1-\ell_2)$. It can be shown that $0<d<1$ and we expand (2.2) in the following series form:

$$(2.3)\quad C(3,m,n)\,\frac{\lambda_3^{3m+5}\,(\ell_2-\ell_1)\prod_{i=1}^{2}\ell_i^m(1-\ell_i)}{(1+\lambda_3)^{3(m+n+4)}}\sum_{k=0}^{\infty}(m+n+4)_k\,\frac{d^k}{k!}$$

where $(a)_k = a(a+1)\dots(a+k-1)$ and $(a)_0 = 1$. Now transform $M = \ell_1 + \ell_2$ and $G = \ell_1\ell_2$, then the joint density of M, G and λ_3 is given by

$$(2.4)\quad C(3,m,n)\,\frac{\lambda_3^{3m+5}}{(1+\lambda_3)^{3(m+n+4)}}\sum_{k=0}^{\infty}(m+n+4)_k\sum_{j=0}^{k}\frac{(-1)^{k-j}}{(k-j)!\,j!}(2-M)^j\left(\frac{\lambda_3}{1+\lambda_3}\right)^{2k-j}$$

$$(1-M+G)^{k-j+1}\,G^M\,.$$

(2.4) is true only if both m and n are non-negative integers. We may integrate by parts term by term with respect to G from 0 to $M^2/4$ for $0 < M \le 1$ and from $M-1$ to $M^2/4$ for $1 < M \le 2$. Further, transform $U^{(3)} = \lambda_3(M+1)$ and integrate with respect to λ_3 from $U^{(3)}/2$ to $U^{(3)}$ for $0 < M \le 1$ and from $U^{(3)}/3$ to $U^{(3)}/2$ for $1 < M < 2$, we have finally the density of $U^{(3)}$

$$(2.5)\quad C(3,m,n)m! \sum_{k=0}^{\infty} (m+n+4)_k \sum_{j=0}^{k} \frac{(-1)^{k-j}}{(k-j)!\, j!} \Big\{ \sum_{v=0}^{m} \sum_{p=0}^{2m-2v} \sum_{q=0}^{2k-j+2v+4} \eta_1(m,k,j,v,p,q)$$

$$u^{b-7}\, B\Big(\frac{u}{3+u}, \frac{u}{1+u}; 3m+a, 3n+b\Big) + \sum_{s=0}^{j} \sum_{t=0}^{k-j+m+2} \eta_2(m,k,j,s,t)$$

$$u^{(m+d-7)}\, B\Big(\frac{u}{2+u}, \frac{u}{1+u}; 2m+c, m+3n+d\Big)\Big\}$$

where

$$\eta_1(m,k,j,v,p,q) = \binom{2m-2v}{p}\binom{2k-j+2v+4}{q} \frac{(-1)^{k+v-p-q}\, 3^q}{(k-j+2)_{v+1}\, 4^{k-j+2+m}(m-v)!} ,$$

$$\eta_2(m,k,j,s,t) = \binom{j}{s}\binom{k-j+m+2}{t} \frac{3^s \cdot 2^t\, (-1)^{1-j-s-t}}{(k-j+2)_{m+1}} ,$$

$a = 1-p-2v+q,$ $\qquad$ $b = 2k-j+p-q+2v+11,$

$c = k-j+t+s+3,$ $\qquad$ $d = k-t-s+9$,

and $\quad B(x_1,x_2;\, p,q) = \int_{x_1}^{x_2} y^{p-1}(1-y)^{q-1}dy, \qquad 0 \le x_1 \le x_2 \le 1$. Although (2.5) is expressed in a series form, it converges for all values of $0 < u < \infty$. Further, the C.D.F. of $U^{(3)}$ obtained from (2.5) is of the form

$$(2.6)\quad P\{U^{(3)} \le x\} = C(3,m,n)m! \sum_{k=o}^{\infty} (m+n+k)_k \sum_{j=o}^{k} \frac{(-1)^{k-j}}{(k-j)!\ j!} \left\{ \sum_{v=o}^{m} \sum_{p=o}^{2m-2v} \sum_{q=o}^{2k-j+2v+4} \right.$$

$$\frac{\eta_1(m,k,j,v,p,q)}{b-6} \left[x^{b-6} B\left(\frac{x}{3+x}, \frac{x}{1+x}; 3m+a, 3n+b\right) + 3^{b-6} B\left(0, \frac{x}{3+x};\right.\right.$$

$$\left.\left. 3m+2k-j+6, 3n+6\right) - B\left(0, \frac{x}{1+x}; 3m+2k-j+6, 3n+6\right) \right] + \sum_{s=o}^{j} \sum_{t=o}^{k-j+2+m}$$

$$\frac{\eta_2(m,k,j,s,t)}{m+d-6} \left[x^{m+d-6} B\left(\frac{x}{2+x}, \frac{x}{1+x}; 2m+c, m+3n+d\right) + 2^{m+d-6} B\left(0, \frac{x}{2+x};\right.\right.$$

$$\left.\left. 3m+2k-j+6, 3n+6\right) - B\left(0, \frac{x}{1+x}; 3m+2k-j+6, 3n+6\right) \right], \qquad 0 < x < \infty .$$

The C.D.F. of $U^{(3)}$ in (2.6) has been used to compute upper 5 percent points for selected values of n and $m = o$ and 1. These values are given along with the approximate values obtained from the Pearson type approximation (Pillai [8]) for comparison.

Table 1

Exact and approximate upper 5 percent points of $U^{(3)}$ for $m = o$ and 1 and selected values of n.

n	m = o		m = 1	
	Exact	Approximate	Exact	Approximate
15	0.747	.747	1.03	1.02
20	0.547	.546		
25	0.437	.437		
30	0.362	0.362	0.500	0.499

The table shows that the approximate values (Pillai [8]) are generally accurate to the three decimals provided. The exact values from (2.6) were computed on CDC 6500 and terms of the series up to $k = 25$ were generally used.

3. <u>The distribution of the sum of the three smallest roots of a covariance matrix when $p = 4$, $\underset{\sim}{\Sigma}=\underset{\sim}{I}$.</u> We may start with the following density which will be discussed in detail in the next section.

$$(3.1)\quad K_1(p,n) \prod_{i=1}^{p} (g_i^m e^{-g_i}) \prod_{i>j} (g_i - g_j) \qquad 0 < g_1 \le g_2 \le \cdots \le g_p < \infty ,$$

where $K_1(p,n) = \Pi^{\frac{1}{2}p^2}/\Gamma_p(\frac{1}{2}n)\,\Gamma_p(\frac{1}{2}p)$.

First put $p = 4$ in (3.1) and integrate with respect to g_4. Next transform $M_1 = \ell_1' + \ell_2'$, $G_1 = \ell_1'\ell_2'$ where $\ell_i' = g_i/g_3$, $i = 1,2$ and integrate with respect to G_1. Then the joint density of M_1 and g_3 is of the form:

$$f(M_1,g_3) = f_1(M_1,g_3) + f_2(M_1,g_3)$$

where

$$(3.2)\quad f_1(M_1 g_3) = K_2(4,n)\, e^{-g_3(2+M_1)} M_1^{2m+2} \sum_{r=0}^{m+2} (r+1) g_3^{4m+7-r} \Big\{(a-bM_1)\left[\left(1-\frac{M_1}{2}\right)^2 - M_1^2/4(m+2)\right] + dcM_1^2\left[\left(1-\frac{M_1}{2}\right)^2 - \frac{M_1^2}{4(m+3)}\right]\Big\} ,$$

$0 < g_3 < \infty$, $0 < M_1 \le 1$,

and where

$a = (m+2)!/(m+2-r)!$, $b = (m+1)!/(m+1-r)!$, $c = m!/(m-r)!$, $d = (m+1)/4(m+2)$,

$K_2(4,n) = K_1(4,n)/\left[(m+1)\cdot 2^{2m+2}\right]$.

$$(3.3)\quad f_2(M_1,g_3) = K_1(4,n)\, e^{-g_3(2+M_1)} \sum_{r=o}^{m+2} (r+1)\, g_3^{4m+7-r} \frac{1}{(m+1)} \Big\{ \left(\frac{M_1}{2}\right)^{2m+2}$$

$$\left[a-bM_1+C\left(\frac{M_1}{2}\right)^2\right] \frac{(2-M_1)^2}{4} - \frac{C\left(\frac{M_1}{2}\right)^{2m+4}(2-M_1)^2}{4(m+2)} - \left(\frac{M_1}{2}\right)^{2m+4}$$

$$\frac{\left[a-bM_1+C\left(\frac{M_1}{2}\right)^2\right]}{(m+2)} + \frac{(M_1-1)^{m+2}\left[a-bM_1+C(M_1-1)\right]}{(m+2)} + \frac{2C\left(\frac{M_1}{2}\right)^{2m+6}}{(m+2)(m+3)}$$

$$- \frac{2C(M_1-1)^{m+3}}{(m+2)(m+3)} \Big\}, \qquad 0 < g_3 < \infty,\quad 1 < M_1 < 2 \;.$$

Now we make the following transformation $T = g_3(M_1+1)$ in (3.2) and (3.3) and integrate with respect to g_3 from $\frac{1}{2}T$ to T and $\frac{1}{3}T$ to $\frac{1}{2}T$ respectively. Finally the density of T is given by

$$(3.4)\quad K_2(4,n)\, e^{-T} \sum_{r=o}^{m+2} (r+1) \Big\{ \sum_{j=o}^{4} \sum_{i=o}^{2m+2} \binom{2m+2}{i} (-1)^i \Big[\frac{C_j}{4} \cdot I\left(\frac{T}{2},T;4m+6-r-j-i\right) +$$

$$K_j \cdot I\left(\frac{T}{3},\frac{T}{2};4m+6-r-j-i\right)\Big] T^{j+i} + \sum_{\ell=o}^{m+2} \binom{m+2}{\ell} (-2)^{m+2-\ell}\, T^{\ell} \Big[K_5 \cdot I\left(\frac{T}{3},\frac{T}{2};\right.$$

$$\left. 4m+6-r-\ell\right) + K_6 \cdot T \cdot I\left(\frac{T}{3},\frac{T}{2};\ 4m+5-r-\ell\right)\Big]\Big\},\ 0 < T < \infty ,$$

where

$$I(x_1,x_2;n) = \int_{x_1}^{x_2} e^{-y}\, y^n dy, \qquad 0 \le x_1 \le x_2 < \infty ,$$

and constant coefficients are:

$$C_o = (9-\frac{1}{m+2})(a+b) + dc(9-\frac{1}{m+3}), \qquad C_1 = -3(2a+5b+8dc) + \frac{2a+3b}{m+2} + \frac{4dc}{m+3} ,$$

$$C_2 = (a+7b+22dc) - (\frac{a+3b}{m+2} + \frac{6dc}{m+3}), \quad C_3 = \frac{b}{m+2} + \frac{4dc}{m+3} -(b+8dc), \quad C_4 = dc(1-\frac{1}{m+3}) ,$$

$$K_o = 9(a+b+\frac{C}{4}) - \frac{9C}{16(m+2)} - \frac{(a+b+\frac{C}{4})}{4(m+2)} + \frac{C}{8(m+2)(m+3)} ,$$

$$K_1 = -[6a+15b+6C] + \frac{3C}{2(m+2)} + \frac{2a+3b+C}{4(m+2)} - \frac{C}{2(m+2)(m+3)} ,$$

$$K_2 = (a+7b+\frac{11}{2}C) - \frac{11C}{8(m+2)} - \frac{a+3b+\frac{3}{2}C}{4(m+2)} + \frac{3C}{4(m+2)(m+3)} ,$$

$$K_3 = -(b+2C) + \frac{C}{2(m+2)} + \frac{b+C}{4(m+2)} - \frac{C}{2(m+2)(m+3)} ,$$

$$K_4 = \frac{C}{4} - \frac{C}{8(m+2)} + \frac{C}{8(m+2)(m+3)} ,$$

$$K_5 = \frac{[2^{2m+2}(a+b-2C)(m+3) + 2^{2m+4}C]}{(m+2)(m+3)} , \quad \text{and}$$

$$K_6 = \frac{[2^{2m+2}(c-b)(m+3) - 2^{2m+3}C]}{(m+2)(m+3)} .$$

4. <u>The distribution of the smallest characteristic root of the sample covariance matrix</u>. Let $\underset{\sim}{X}(p \times n)$ be a matrix variate with columns independently distributed as $N(\underset{\sim}{0},\underset{\sim}{\Sigma})$, then the distribution of the characteristic roots, $0 < w_1 \leq w_2 \leq \cdots \leq w_p < \infty$ of $\underset{\sim}{X}\,\underset{\sim}{X}'$ depends only upon the characteristic roots of $\underset{\sim}{\Sigma}$ and can be given in the form (James [4])

$$(4.1) \qquad K(p,n)|\underset{\sim}{\Sigma}|^{-\frac{n}{2}}|\underset{\sim}{W}|^m \prod_{i>j}(w_i-w_j)\int_{o(p)}\exp(-\tfrac{1}{2}\mathrm{tr}\,\underset{\sim}{\Sigma}^{-1}\underset{\sim}{H}\,\underset{\sim}{W}\,\underset{\sim}{H}')d(\underset{\sim}{H})$$

$$0 < w_1 \leq w_2 \leq \cdots \leq w_p < \infty ,$$

where the integral is taken over the orthogonal group of $(p \times p)$ orthogonal matrices $\underset{\sim}{H}$; $m = \frac{1}{2}(n-p-1)$ and $K(p,n) = \Pi^{\frac{1}{2}p^2}/2^{\frac{1}{2}pn}\, \Gamma_p(\frac{1}{2}n)\, \Gamma_p(\frac{1}{2}p)$ and $\underset{\sim}{W} = \mathrm{diag}(w_p, \ldots, w_1)$. (4.1) can also be written in the form (James [4])

$$(4.2)\quad K(p,n)|\underset{\sim}{\Sigma}|^{-\frac{n}{2}}|\underset{\sim}{W}|^m \left\{\exp(-\tfrac{1}{2}\mathrm{tr}\,\underset{\sim}{W})\right\} \prod_{i>j}(w_i-w_j)\; {}_0F_0(\tfrac{1}{2}(\underset{\sim}{I}_p - \underset{\sim}{\Sigma}^{-1}), \underset{\sim}{W}),$$

$$0 < w_1 \le \cdots \le w_p < \infty,$$

where ${}_pF_q(a_1,\ldots,a_p;\, b_1,\ldots,b_q;\, \underset{\sim}{\mathscr{S}},\, \underset{\sim}{T}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa,\ldots,(a_p)_\kappa}{(b_1)_\kappa,\ldots,(b_q)_\kappa} \frac{C_\kappa(\underset{\sim}{\mathscr{S}})C_\kappa(\underset{\sim}{T})}{C_\kappa(\underset{\sim}{I}_p)k!}$

and $a_1,\ldots,a_p;\, b_1,\ldots,b_q$ are real or complex constants and the multivariate coefficient $(a)_\kappa$ is given by $(a)_\kappa = \prod_{i=1}^{p}(a-\frac{1}{2}(i-1))k_i$ and where partition κ of k is such that $\kappa = (k_1,\ldots,k_p)$, $k_1 \ge k_2 \ge \cdots \ge k_p \ge 0$, $k_1+\cdots+k_p = k$ and the zonal polynomials, $C_\kappa(\underset{\sim}{\mathscr{S}})$, are expressible in terms of elementary symmetric functions of the characteristic roots of $\underset{\sim}{\mathscr{S}}$, (James [5]). If we let $\underset{\sim}{\Sigma} = \underset{\sim}{I}_p$ in (4.2) and transform $g_i = \frac{1}{2}w_i$, $i = 1,\ldots,p$, we obtain the joint density of $g_1,\ldots,g_p$ in the form [see (3.1)]

$$(4.3)\quad K_1(p,n) \prod_{i=1}^{p}(g_i^m e^{-g_i}) \prod_{i>j}(g_i-g_j) \qquad 0 < g_1 \le g_2 \le \cdots \le g_p < \infty .$$

Expanding (4.2) as a power series, we have

$$(4.4)\quad K(p,n)|\underset{\sim}{\Sigma}|^{-\frac{n}{2}}|\underset{\sim}{W}|^m \exp(-\tfrac{1}{2}\mathrm{tr}\,\underset{\sim}{W}) \prod_{i>j}(w_i-w_j) \sum_{k=0}^{\infty}\sum_{\kappa} \frac{C_\kappa(\frac{1}{2}(\underset{\sim}{I}_p-\underset{\sim}{\Sigma}^{-1}))C_\kappa(\underset{\sim}{W})}{C_\kappa(\underset{\sim}{I}_p)\,k!} .$$

Let $u_i' = \frac{w_i}{w_p}$, $i = 1,2,\ldots,p-1$, and make use of the known equality (Khatri and Pillai [6]), $C_\kappa({}^1\underset{\sim}{U}') = \sum_{n=0}^{k}\sum_{\eta} b_{\kappa,\eta}\, C_\eta(\underset{\sim}{U}')$ where $\underset{\sim}{U}' = \mathrm{diag}(u_{p-1}', \ldots, u_1')$

and $b_{\kappa,\eta}$'s are constants depending on κ and η, then we have

$$(4.5)\quad K(p,n)|\Sigma|^{-\frac{n}{2}}|U'|^{m}\exp(-\tfrac{w_p}{2})\sum_{s=0}^{\infty}\sum_{\mathscr{S}}\frac{(-\frac{1}{2})^{s}\,C_{\mathscr{S}}(U')}{s!}\,w_p^{pm+s+\frac{1}{2}(p+2)(p-1)}$$

$$|I_{p-1}-U'|\prod_{i>j}(u_i'-u_j')\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\frac{1}{2}(I_p-\Sigma^{-1}))w_p^{k}\sum_{n=0}^{k}\sum_{\eta} b_{\kappa,\eta}\,C_\eta(U')}{C_\kappa(I_p)\,k!}$$

$$0<w_p<\infty,\quad 1>u'_{p-1}\geq u'_{p-2}\geq\cdots\geq u'_1>0\,,$$

we need only to consider

$$(4.6)\quad |U'|^{m}\,C_{\mathscr{S}}(U')\,|I_{p-1}-U'|\prod_{i>j}(u_i'-u_j')\,C_\eta(U')\,.$$

Now apply the result (Khatri and Pillai [6], Hayakawa [3]), $C_{\mathscr{S}}(U')\,C_\eta(U')=\sum_\theta d^{\theta}_{\mathscr{S}\eta}\,C_\theta(U')$ where the summation is over all partition θ of q satisfying $n+s=q$ and $d^{\theta}_{\mathscr{S}\eta}$ are constant depending on $\theta,\mathscr{S}$ and η. In (4.5) transform $u_i=1-u_i'$, $i=1,2,\ldots,p-1$, i.e., $U=I-U'$ where $U=\mathrm{diag}(u_1,\ldots,u_{p-1})$, then (4.6) becomes

$$(4.7)\quad |I_{p-1}-U|^{m}|U|\prod_{i>j}(u_i-u_j)\sum_\theta d^{\theta}_{\mathscr{S}\eta}\,C_\theta(I_{p-1}-U)\qquad 1>u_1\geq u_2\geq\cdots\geq u_{p-1}>0$$

Applying Constantine's result [2], $C_\theta(I_{p-1}-U)=C_\theta(I_{p-1})\sum_{z=0}^{q}\sum_{v}(-1)^{z}\dfrac{A_{\theta,v}\,C_v(U)}{C_v(I_{p-1})}$ and making use of the following equality (Khatri and Pillai [6]), $|I_{p-1}-U|^{m}C_v(U)=\sum_{t=0}^{\infty}\sum_{\sigma\delta}(-m)_\sigma\dfrac{g^{\delta}_{v,\sigma}\,C_\delta(U)}{t!}$ where $A_{\theta,v}$'s are constants depending on θ and v and $g^{\delta}_{v,\sigma}$ is the coefficient of $C_\delta(U)$,

$$(4.10)\quad K(p,n)|\underset{\sim}{\Sigma}|^{-\frac{n}{2}} \exp(-\frac{w_p}{2}) \sum_{s=o}^{\infty} \sum_{\mathscr{s}} \frac{(-\frac{1}{2})^s}{s!} \sum_{k=o}^{\infty} \sum_{\kappa}$$

$$\frac{C_\kappa(\frac{1}{2}(\underset{\sim}{I}_p-\underset{\sim}{\Sigma}^{-1}))w_p^{pm+\frac{1}{2}(p+2)(p-1)+s+k}}{C_\kappa(\underset{\sim}{I}_p)\ k!} \sum_{n=o}^{k} \sum_{\eta} b_{\kappa,\eta} \sum_{\theta} d^{\theta}_{\mathscr{s}\eta}\, C_\theta(\underset{\sim}{I}_{p-1})$$

$$\sum_{z=o}^{q} \sum_{v} \frac{(-1)^z A_{\theta,v}}{C_v(\underset{\sim}{I}_{p-1})} \sum_{t=o}^{\infty} \sum_{\sigma\delta}(-m)_\sigma \frac{g^{\delta}_{v,\sigma}}{t!} F(p,\delta)(1-\frac{w_1}{w_p})^{\frac{1}{2}p(p+1)+h+2} ,$$

$$\infty > w_p > w_1 > 0 .$$

Note that the series in t is actually only a finite summation and (4.10) converges for all values of $\infty > w_p > w_1 > 0$. So if we integrate (4.10) with respect w_p, we have the density of the smallest characteristic root

$$(4.11)\quad K(p,n)|\underset{\sim}{\Sigma}|^{-\frac{n}{2}} \sum_{s=o}^{\infty} \sum_{\mathscr{s}} \frac{(-\frac{1}{2})^s}{s!} \sum_{k=o}^{\infty} \sum_{\kappa} \frac{C_\kappa(\frac{1}{2}(\underset{\sim}{I}_p-\underset{\sim}{\Sigma}^{-1}))}{C_\kappa(\underset{\sim}{I}_p)\ k!} \sum_{n=o}^{k} \sum_{\eta}$$

$$b_{\kappa,\eta} \sum_{\theta} d^{\theta}_{\mathscr{s}\eta}\, C_\theta(\underset{\sim}{I}_{p-1}) \sum_{z=o}^{q} \sum_{v}(-1)^z \frac{A_{\theta,v}}{C_v(\underset{\sim}{I}_{p-1})} \sum_{t=o}^{\infty} \sum_{\sigma\delta} (-m)_\sigma$$

$$\frac{g^{\delta}_{v,\sigma}}{t!} F(p,\delta) \sum_{\ell=o}^{\frac{p(p+1)}{2}+h+2} \binom{\frac{1}{2}p(p+1)+h+2}{\ell} (-w_1)^\ell\, 2^{pm+\frac{1}{2}(p+2)(p-1)+s+k-\ell+1}$$

$$I(\frac{w_1}{2},\infty;pm+\tfrac{1}{2}(p+2)(p-1)+s+k-\ell),\quad 0 < w_1 < \infty .$$

Also note that if we expand (4.1) as a power series and proceed as above, we will obtain the joint density of the largest and smallest characteristic roots.

$$(4.12)\quad K(p,n)|\underset{\sim}{\Sigma}|^{-\frac{n}{2}} \sum_{k=o}^{\infty} \sum_{\kappa} \frac{C_\kappa(-\frac{1}{2}\underset{\sim}{\Sigma}^{-1})}{k!\, C_\kappa(\underset{\sim}{I}_p)} \; w_p^{pm+\frac{1}{2}(p+2)(p-1)+k} \sum_{n=o}^{k} \sum_{\eta}$$

$$b_{\kappa,\eta}\, C_\eta(\underset{\sim}{I}_{p-1}) \sum_{z=o}^{n} \sum_{v} \frac{(-1)^z}{C_v(\underset{\sim}{I}_{p-1})} A_{\eta,v} \sum_{t=o}^{\infty} \sum_{\sigma\delta} (-m)_\sigma \frac{g^{\delta}_{v,\sigma}}{t!}$$

$$F(p,\delta)(1-w_1/w_p)^{\frac{1}{2}p(p+1)+h+2} , \quad \infty > w_p > w_1 > 0 ,$$

where $A_{\eta,v}$ is defined similar to $A_{\theta,v}$. We may also set $u_1=1-w_1/w_p$ in (4.10), then u_1 and w_p are independently distributed. If we integrate with respect to w_p, then the density of u_1, i.e. the ratio of the smallest root to the largest root is given by

$$(4.13)\quad K(p,n)|\underset{\sim}{\Sigma}|^{-\frac{n}{2}} \sum_{s=o}^{\infty} \sum_{\mathscr{S}} \frac{(-\frac{1}{2})^s}{s!} \sum_{k=o}^{\infty} \sum_{\kappa} \frac{C_\kappa(\frac{1}{2}(\underset{\sim}{I}_p-\underset{\sim}{\Sigma}^{-1}))}{C_\kappa(\underset{\sim}{I}_p)\, k!} \sum_{n=o}^{k} \sum_{\eta}$$

$$b_{\kappa,\eta} \sum_{\theta} d^{\theta}_{\mathscr{S}\eta}\, C_\theta(\underset{\sim}{I}_{p-1}) \sum_{z=o}^{q} \sum_{v} \frac{(-1)^z \cdot A_{\theta,v}}{C_v(\underset{\sim}{I}_{p-1})} \sum_{t=o}^{\infty} \sum_{\sigma\delta} (-m)_\sigma$$

$$\frac{g^{\delta}_{v,\sigma}}{t!} \quad F_1(p,s,k;\delta) \quad u_1^{\frac{1}{2}p(p+1)+h+2} , \quad 0 < u_1 < 1$$

where $F_1(p,s,k;\delta) = 2^{\frac{1}{2}(p+2)(p-1)+pm+s+k+1}\, \Gamma(\frac{1}{2}(p+2)(p-1)+pm+s+k+1) \cdot F(p,\delta)$.

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