Non-central distributions of the smallest and second smallest roots of matrices in multivariate analysis*

by

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1. Introduction and summary: The non-central distributions of the largest roots of three matrices have been obtained by Pillai and Sugiyama [8] and those of the second largest roots by Al-Ani [1]. In this paper, the non-central distributions of the smallest and the second smallest root of a covariance matrix and those in the case of MANOVA, Canonical Correlation and test of equality of covariance matrices are considered. In the last section, the distributions of the sum of the two smallest and two largest roots of a covariance matrix and their ratio and sum are considered when $p = 4$, however, Pillai and Al-Ani [6] have obtained earlier the distribution of the sum of the two smallest roots for $p = 3, 4$ and 5.

2. The distribution of the smallest root of a covariance matrix. Let $\underset{\sim}{X}(p \times n)$ be a matrix variate with columns independently distributed as $N(\underset{\sim}{0},\underset{\sim}{\Sigma})$, then the distribution of the latent roots $0 < w_1 \leq \cdots \leq w_p < \infty$, of $\underset{\sim}{X}\,\underset{\sim}{X}'$ depends only upon the latent roots of $\underset{\sim}{\Sigma}$ and the density of $0 < g_1 \leq g_2 \leq \cdots \leq g_p < \infty$, where $g_i = w_i/2$, $i = 1,\ldots,p$, can be written in the following form [6]

$$(2.1) \quad k(p,n)|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}|\underset{\sim}{G}|^m e^{-\mathrm{tr}\,\underset{\sim}{G}} \prod_{i>j}(g_i - g_j) \; {}_0F_0((\underset{\sim}{I} - \underset{\sim}{\Sigma}^{-1}), \underset{\sim}{G}) ,$$

where $m = \frac{1}{2}(n-p-1)$, $\underset{\sim}{G} = \mathrm{diag}(g_1,\ldots,g_p)$ and $k(p,n) = \Pi^{\frac{1}{2}p^2}/\Gamma_p(\frac{1}{2}n)\,\Gamma_p(\frac{1}{2}p)$. Now transform $q_i = g_1/g_i$, $i=2,\ldots,p$, then the joint density of g_1 and $q_2,\ldots,q_p$ is given by:

$$(2.2)\quad k(p,n)\, g_1^{\frac{1}{2}np-1}\, e^{-g_1 \mathrm{tr}\underset{\sim}{Q}_1^{-1}}\, |\underset{\sim}{I}-\underset{\sim}{Q}|\, |\underset{\sim}{Q}|^{-m-p-1} \prod_{i>j}(q_j-q_i)\; {}_0F_0((\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1}),\; g_1\underset{\sim}{Q}_1^{-1}),$$

where $\underset{\sim}{Q}_1 = \mathrm{diag}(1,q_2,\ldots,q_p)$, $\underset{\sim}{Q} = \mathrm{diag}(q_2,\ldots,q_p)$. Now, by using the results of Constantine [2], namely, $C_\kappa(\underset{\sim}{L}^{-1}) = |\underset{\sim}{L}|^{-e}(C_\kappa(\underset{\sim}{I})/C_{\kappa^*}(\underset{\sim}{I}))C_{\kappa^*}(\underset{\sim}{L})$ where e_1 is any integer $\geq k_1$ and $\kappa^* = (e_1-k_1,\ldots,e_1-k_1)$ and $\kappa = (k_1,\ldots,k_p)$. Also expand $|\underset{\sim}{Q}_1|^{-m-p-1-e_1}$ as well as $C_\kappa(\underset{\sim}{I}-\underset{\sim}{Q}_1)$, then using the results of Khatri and Pillai [5] on the multiplication of two zonals polynomials, (2.2) can be written in the following form:

$$(2.3)\quad k(p,n)\, g_1^{\frac{1}{2}np-1}\, |\underset{\sim}{I}-\underset{\sim}{Q}| \prod_{i>j}(q_j-q_i) \sum_{k=0}^{\infty}\sum_{\kappa} \frac{C_\kappa(\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1})}{k!\, C_\kappa(\underset{\sim}{I})} \sum_{s=0}^{\infty}\sum_{\eta} \frac{(-1)^s}{s!}$$

$$g_1^{k+s} \sum_{\delta} g_{\eta,\kappa}^{\delta} \frac{C_\delta(\underset{\sim}{I})}{C_{\delta^*}(\underset{\sim}{I})} \sum_{t=0}^{\infty}\sum_{\tau} \frac{(m+p+e_1+1)_\tau}{t!} C_\tau(\underset{\sim}{I}) \sum_{d=0}^{t}\sum_{\nu} \frac{(-1)^d a_{\tau,\nu}}{C_\nu(\underset{\sim}{I})}$$

$$\sum_{\gamma} g_{\delta^*,\nu}^{\gamma}\, C_\gamma(\underset{\sim}{Q}_1),$$

where δ, γ are the partitions of $k+s$, and $d+pe-s-k$ and $\delta^* = (e-\delta_p,\ldots,e-\delta_1)$ where e_1 is any integer $\geq \delta_1$ and $\delta = (\delta_1,\ldots,\delta_p)$, the constants $g_{\eta,\kappa}^{\delta}$, $g_{\delta^*,\nu}^{\gamma}$ are defined in [5], and $a_{\tau,\nu}$ defined in [2]. Now integrate

(2.3) with respect to $1 > q_2 > q_3 > \dots > q_p$. The density function of g_1

$$(2.4)\quad \frac{\Gamma_p((p+1)/2)\, g_1^{\frac{1}{2}np-1}}{\Gamma_p(\frac{1}{2}n)} \sum_{k=o}^{\infty} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1})}{k!\, C_\kappa(\underset{\sim}{I})} \sum_{s=o}^{\infty} \sum_{\eta} \frac{(-1)^s g_1^{k+s}}{s!} \sum_{\delta}$$

$$g^{\delta}_{\tau,\kappa} \frac{C_\delta(\underset{\sim}{I})}{C_{\delta^*}(\underset{\sim}{I})} \sum_{t=o}^{\infty} \sum_{\tau} \frac{(m+p+e_1+1)_\tau}{t!} C_\tau(\underset{\sim}{I}) \sum_{d=o}^{t} \sum_{\nu} \frac{(-1)^d a_{\tau,\nu}}{C_\nu(\underset{\sim}{I})} \sum_{\gamma} g^{\gamma}_{\delta^*,\nu}$$

$$C_\gamma(\underset{\sim}{I})\ (p(p+1)/2 + d + pe_1 - s - k)(\Gamma_p((p+1)/2,\gamma)/\Gamma_p(p+1,\gamma)) \quad .$$

If $\underset{\sim}{\Sigma} = \underset{\sim}{I}$, in (2.1), then density function of g_1 can be written in the following form:

$$(2.5)\quad k_1(p,n)\, g_1^{\frac{1}{2}pn-1} e^{-g_1} \sum_{k=o}^{\infty} \sum_{\kappa} \frac{(-g_1)^k C_\kappa(\underset{\sim}{I})}{k!\, C_{\kappa^*}(\underset{\sim}{I})} \sum_{t=o}^{\infty} \sum_{\tau} \frac{(m+p+e_1+1)_\tau}{t!} C_\tau(\underset{\sim}{I})$$

$$\sum_{d=o}^{t} \sum_{\nu} \frac{(-1)^d a_{\tau,\nu}}{C_\nu(\underset{\sim}{I})} \sum_{\delta} g^{\delta}_{\nu,\kappa^*} C_\delta(\underset{\sim}{I})\ (\Gamma_{p-1}(p/2,\delta)/\Gamma_{p-1}(p+1,\delta)) \ ,$$

where $k_1(p,n) = (\Pi^{p-\frac{1}{2}} \Gamma_{p-1}(p/2+1)/\Gamma_p(n/2)\ \Gamma(p/2))$. The form (2.4) given above for the smallest root is simpler than the one given by Pillai and Chang [7].

3. <u>The distribution of the second smallest root</u>. Let $\underset{\sim}{\Sigma} = \underset{\sim}{I}$ in (2.1) and transform $q_i = g_2/g_i$, $i = 3,\dots,p$ and by the same method as in section (2), the joint density of g_1, g_2 can be written in the following form:

$$(3.1)\quad k_2(p,n)\, g_1^m\, g_2^{m(p-1)+\frac{1}{2}(p-2)(p+3)}\; e^{-(g_1+g_2)} (g_2-g_1) \sum_{k=o}^{\infty} \sum_{\kappa} \frac{(-g_2)^k\, C_\kappa(\underset{\sim}{I})}{k!\, C_{\kappa^*}(\underset{\sim}{I})}$$

$$\sum_{t=o}^{\infty} \sum_{\tau} \frac{(m+p+e+1)_\tau}{t!}\, C_\tau(\underset{\sim}{I}) \sum_{d=o}^{t} \sum_{\nu} \frac{(-1)^d\, a_{\tau,\nu}}{C_\nu(\underset{\sim}{I})} \sum_{\ell=o}^{p-2} \frac{C(\ell)\, g_1^\ell}{g_2^\ell} \sum_{\delta}$$

$$g^\delta_{(\kappa^*,\nu,1^\ell)}\, C_\delta(\underset{\sim}{I}_{p-2})\, (\Gamma_{p-2}((p-1)/2,\delta)/\Gamma_{p-2}(p,\delta))$$

where $C(\ell) = (-1)^\ell\, (2\ell)!/(\ell!)^2\, 2^\ell\, \chi_{[21^\ell]}(1)$, where $\chi_{[21^\ell]}(1)$ is the degree of of the representation $[21^\ell]$ of the symmetric group of 2ℓ symbols, and such that

$$\chi_{[\kappa]}(1) = k! \prod_{i<j}^{p} (k_i-k_j-i+j)/\prod_{i=1}^{p} (k_i+p-i)! \quad \text{and}$$

$$\kappa = (k_1 \geq k_2 \geq \ldots \geq k_p \geq 0) \quad \text{and} \quad k_2(p,n) = 2\Pi^{2p-3}\Gamma_{p-2}((p+1)/2)/\Gamma_p(n/2)$$

$$\Gamma_{p-2}((p-2)/2)\ .$$

Integrate (3.1) with respect to g_1, then the density function of g_2 is given by

$$(3.2)\quad k_2(p,n)\, g_2^{m(p-1)+\frac{1}{2}(p-2)(p+3)}\; e^{-g_2} \sum_{k=o}^{\infty} \sum_{\kappa} \frac{(-g_2)^k\, C_\kappa(\underset{\sim}{I})}{k!\, C_{\kappa^*}(\underset{\sim}{I})} \sum_{t=o}^{\infty} \sum_{\tau}$$

$$\frac{(m+p+e+1)_\tau}{t!}\, C_\tau(\underset{\sim}{I}) \sum_{d=o}^{t} \sum_{\nu} \frac{(-1)^d\, a_{\tau,\nu}}{C_\nu(\underset{\sim}{I})} \sum_{\ell=o}^{p-2} C(\ell) \sum_{\delta} g^\delta_{(\kappa^*,\nu,1^\ell)}\, C_\delta(\underset{\sim}{I}_{p-2})$$

$$(\Gamma_{p-2}((p-1)/2,\delta)/\Gamma_{p-2}(p,\delta))\, (g_2 I(o,g_2;m+\ell+1) - I(o,g_2;m+\ell+2))\ ,$$

where $I(a,b;c) = \int_a^b x^{c-1}\, e^{-x} dx$.

4. <u>Non-central distribution of the smallest root in MANOVA case</u>. Let $\underset{\sim}{X}$ be a $p \times n_1$ matrix variate $(p \leq n_1)$ and $\underset{\sim}{Y}$ a $p \times n_2$ matrix variate $(p \leq n_2)$ and the columns be all independently normally distributed with covariance matrix $\underset{\sim}{\Sigma}$, $E\underset{\sim}{X} = \underset{\sim}{M}$ and $E\underset{\sim}{Y} = \underset{\sim}{0}$. Then it is known that $\underset{\sim}{X}\underset{\sim}{X}' = \underset{\sim}{S}_1$ is non-central Wishart with n_1 degrees of freedom and $\underset{\sim}{Y}\underset{\sim}{Y}' = \underset{\sim}{S}_2$ is central Wishart with n_2 degrees of freedom and the covariance matrix $\underset{\sim}{\Sigma}$, respectively. Let $0 < \ell_1 < \ell_2 < \ldots < \ell_p < 1$ be the latent roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$, then the joint density of $\ell_1, \ldots, \ell_p$ is given by Constantine [3]

$$(4.1) \quad c(p,n_1,n_2) \exp(\mathrm{tr}\text{-}\underset{\sim}{\Omega}) \, |\underset{\sim}{L}|^m \, |\underset{\sim}{I}-\underset{\sim}{L}|^n \prod_{i>j} (\ell_i - \ell_j) \sum_{k=o}^{\infty} \sum_{\kappa} \frac{(\nu)_\kappa \, C_\kappa(\underset{\sim}{\Omega}) \, C_\kappa(\underset{\sim}{L})}{(n_1/2)_\kappa \, C_\kappa(\underset{\sim}{I}) k!} ,$$

where $\underset{\sim}{\Omega}$ is the non-centrality matrix, $\frac{1}{2} \underset{\sim}{M}' \underset{\sim}{\Sigma}^{-1} \underset{\sim}{M}$, and $\underset{\sim}{L} = \mathrm{diag}(\ell_1, \ldots, \ell_p)$ and $c(p,n_1,n_2) = \Pi^{p^2/2} \Gamma_p(\nu)/\Gamma_p(p/2)\Gamma_p(n_1/2)\Gamma_p(n_2/2)$, $m = \frac{1}{2}(n_1-p-1)$, $n = \frac{1}{2}(n_2-p-1)$ and $\nu = \frac{1}{2}(n_1+n_2)$. Now transform $\ell_i' = 1 - \ell_i$ and expand $C_\kappa(\underset{\sim}{I}-\underset{\sim}{L}')$. Then the joint density of $1 > \ell_1' > \ell_2' > \ldots > \ell_p' > 0$ can be written in the following form

$$(4.2) \quad c(p,n_1,n_2) \exp(\mathrm{tr}\text{-}\underset{\sim}{\Omega}) \, |\underset{\sim}{L}'|^n \, |\underset{\sim}{I}-\underset{\sim}{L}'|^m \prod_{i>j} (\ell_j' - \ell_i') \sum_{k=o}^{\infty} \sum_{\kappa} \frac{(\nu)_\kappa \, C_\kappa(\underset{\sim}{\Omega})}{(n_1/2)_\kappa \, k!}$$

$$\sum_{s=o}^{k} \sum_{\eta} (-1)^s \, a_{\kappa,\eta}(\underset{\sim}{L}')/C_\eta(\underset{\sim}{I}) \quad .$$

Now from the results of Pillai and Sugiyama [8], the density function of ℓ_1 can be written as

$$(4.3)\quad c_1(p,n_1,n_2)\exp(\mathrm{tr}\,\underset{\sim}{\Omega}) \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(\nu)_\kappa C_\kappa(\underset{\sim}{\Omega})}{(n_1/2)_\kappa\, k!} \sum_{s=0}^{k}\sum_{\tau} \frac{(-1)^s a_{\kappa,\eta}}{C_\eta(\underset{\sim}{I})}$$

$$\sum_{s=0}^{\infty} ((pn_2/2+s+t)/t!) \sum_{\sigma,\delta} g^{\delta}_{\eta,\sigma} \frac{((p+1-n_1)/2)_\sigma (n_2/2)}{((n_2+p+1)/2)_\delta} C_\delta(\underset{\sim}{I}_p)$$

$$(1-\ell_1)^{pn_2/2+s+t-1} \quad ,$$

where σ and δ are the partitions of t and $s+t$ respectively and $c_1(p,n_1,n_2) = \Gamma_p((p+1)/2)\,\Gamma_p(\nu)/\Gamma_p(n_1/2)\,\Gamma_p((n_2+p+1)/2)$. Also, from the results of Al-Ani [1], the density of ℓ_2 can be written in the following form

$$(4.4)\quad c_2(p,n_1,n_2)\exp(\mathrm{tr}\,\underset{\sim}{\Omega})(1-\ell_2)^{n(p-1)+(p-2)(p+1)/2} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(\nu)_\kappa C_\kappa(\underset{\sim}{\Omega})}{k!\,(n_1/2)_\kappa}$$

$$\sum_{s=0}^{\infty}\sum_{\eta} \frac{(-1)^s a_{\kappa,\eta}}{C_\eta(\underset{\sim}{I})} \sum_{t=0}^{\infty}\sum_{\sigma} \frac{(-m)_\sigma}{t!} \sum_{\ell=0}^{p-2} c(\ell)/\ell! \sum_{\tau,\nu} (1-\ell_2)^{t+\ell+s_2}$$

$$B((1-\ell_2),1;n+p+s_1-\ell;m+1) \sum_{\gamma} g^{\gamma}_{\ell\,(1,\sigma,\nu)} C_\gamma(\underset{\sim}{I}_{p-1})((n_2-1)(p-1)/2+t+\ell+s_2)$$

$$(\Gamma_{p-1}((n_2-1)/2,\gamma)/\Gamma_{p-1}(n_2+p-1)/2,\gamma))\ ,$$

where $B(a,b;c,d) = \int_a^b x^{c-1}(1-x)^{d-1}dx$;

$c_2(p,n_1,n_2) = \Pi^{p-1}\,\Gamma_p(\nu)\,\Gamma_{p-1}((p-1)/2)/\Gamma_p(n_1/2)\,\Gamma_p(n_2/2)$, and $a_{\tau,\nu}$ are constants defined in [2], τ and ν are the partitions of s_1 and s_2 such that $s_1+s_2 = s$ and γ is the partition of $\ell+s_2+t$.

5. <u>The distribution of the smallest root in the cononical correlation case</u>.

Let the columns of $\binom{\underset{\sim}{X}_1}{\underset{\sim}{X}_2}$ be n independent normal $(p+q)$-dimensional variates $(p \leq q)$ with zero means and covariance matrix

$$\underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}'_{12} & \underset{\sim}{\Sigma}_{22} \end{pmatrix}.$$

Let $\underset{\sim}{R} = \mathrm{diag}(r_1, r_2, \dots, r_p)$, where $r_1^2, \dots, r_p^2$ are the characteristics roots of the equation

$$|\underset{\sim}{X}_1\underset{\sim}{X}'_2(\underset{\sim}{X}_2\underset{\sim}{X}'_2)^{-1}\,\underset{\sim}{X}_2\,\underset{\sim}{X}'_1 - r^2\,\underset{\sim}{X}_1\,\underset{\sim}{X}'_1| = 0$$

and also $\underset{\sim}{P} = \mathrm{diag}(\rho_1, \rho_2, \dots, \rho_p)$ where $\rho_1^2, \dots, \rho_p^2$ are the characteristics roots of the equation

$$|\underset{\sim}{\Sigma}_{12}\,\underset{\sim}{\Sigma}_{22}^{-1}\,\underset{\sim}{\Sigma}'_{12} - \rho^2\,\underset{\sim}{\Sigma}_{11}| = 0 .$$

Then, the density function of $r_1^2, \dots, r_p^2$ is given by Constantine [3] in the following form

$$(5.1)\quad c\,(n,p,q)\,|\underset{\sim}{I}-\underset{\sim}{P}^2|^{n/2}\,|\underset{\sim}{R}^2|^{(q-p-1)/2}\,|\underset{\sim}{I}-\underset{\sim}{R}^2|^{(n-p-q-1)/2}\prod_{i>j}(r_i^2-r_j^2)$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(n/2)_\kappa(n/2)\;c_\kappa(\underset{\sim}{R}^2)\;c_\kappa(\underset{\sim}{P}^2)}{(q/2)_\kappa\;k!\;c_\kappa(\underset{\sim}{I}_p)}$$

where $$c(n,p,q)=\frac{\Gamma_p(n/2)\;\Pi^{p^2/2}}{\Gamma_p(q/2)\;\Gamma_p((n-q)/2)\;\Gamma_p(p/2)}\quad.$$

As before, transform $r_i^{2'} = 1 - r_i^2$, $i = 1,\ldots,P$, then the density function of r_1^2 can be written as

$$(5.2)\quad c_1(n,p,q)\,|\underset{\sim}{I}-\underset{\sim}{P}^2|^{n/2}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(n/2)_\kappa(n/2)_\kappa\;c_\kappa(\underset{\sim}{P}^2)}{(q/2)_\kappa\quad k!}\sum_{s=0}^{k}\sum_{\eta}\frac{(-1)^s\,a_{\kappa,\eta}}{C_\eta(\underset{\sim}{I})}$$

$$\sum_{t=0}^{\infty}((p(n-q)/2+s+t/t!)\sum_{\sigma,\delta}g^{\delta}_{\eta,\sigma}((p-q+1)/2)_\sigma\,\frac{((n-q)/2)_\delta}{((n-q+p+1)/2)_\delta}$$

$$C_\delta(\underset{\sim}{I}_p)\;(1-r_1^2)^{p(n-q)/2+s+t-1}\quad,$$

where $$c_1(n,p,q)=\frac{\Gamma_p(n/2)\;\Gamma_p(p+1)/2)}{\Gamma_p(q/2)\;\Gamma_p((n-q+p+1)/2)}\quad.$$

Also the density function of r_2^2 can be written in the following form

$$(5.3) \quad c_2(n,p,q)|\underset{\sim}{I}-\underset{\sim}{P}^2|^{n/2}(1-r_2^2)^{\{(n-q-p-1)(p-1)+(p-2)(p+1)\}/2} \sum_{k=0}^{\infty} \sum_{\kappa}$$

$$\frac{(n/2)_\kappa (n/2)_\kappa C_\kappa(\underset{\sim}{P}^2)}{(q/2)_\kappa \, k!} \sum_{s=0}^{k} \sum_{\eta} \frac{(-1)^s a_{\kappa,\eta}}{C_\eta(\underset{\sim}{I})} \sum_{t=0}^{\infty} \sum_{\sigma} \frac{((p-q+1)/2)_\sigma}{t!}$$

$$\sum_{\ell=0}^{p-2} C(\ell)/\ell! \sum_{\tau,\nu} a_{\tau,\nu} (1-r_2^2)^{t+\ell+s_2} B((1-r_2^2),1;(n-q+p-1)/2+k_1-\ell;(q-p+1)/2)$$

$$\sum_{\gamma} g_{\ell\,(1,\sigma,\nu)}^{\gamma} \quad C_\gamma(\underset{\sim}{I}_{p-1})((n-q-1)(p-1)/2+t+\ell+s_2) \; (\Gamma_{p-1}((n-q-1)/2,\gamma)/$$

$$\Gamma_{p-1}((n-q-1)/2,\gamma)) ,$$

where $c_2(n,p,q) = \Pi^{p-1} \, \Gamma_p(n/2) \, \Gamma_{p-1}((p-1)/2)/\Gamma_p(q/2) \, \Gamma_p((n-q)/2)$.

6. <u>Non-central distribution of the smallest root of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$</u> . Let $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ be independently distributed as Wishart $W(n_1,p,\underset{\sim}{\Sigma}_1)$ and $W(n_2,p,\underset{\sim}{\Sigma}_2)$, respectively. Let the characteristic roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ and $\underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_2^{-1}$ be denoted respectively by c_i and λ_i, $i = 1,\ldots,p$ and such that $0 < c_1 < c_2 < \ldots < c_p < \infty$ and $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_p < \infty$. Let $w_i = \delta' c_i/(1+\delta' c_i)$, $i = 1,\ldots,p$; $\delta' > 0$ and $\underset{\sim}{W} = \text{diag}(w_1,\ldots,w_p)$, and $\underset{\sim}{\Lambda} = \text{diag}(\lambda_1,\ldots,\lambda_p)$, then the distribution of $w_1,\ldots,w_p$ is given by Khatri [4] in the following form

$$(6.1)\quad c(p,m,n)|\delta'\Lambda|^{-\frac{1}{2}n_1}|W|^m|I-W|^n \prod_{i>j}(w_i-w_j)\sum_{k=o}^{\infty}\sum_{\kappa}\frac{(\nu)_\kappa\, c_\kappa(I-(\delta'\Lambda)^{-1}\, c_\kappa(W)}{k!\; c_\kappa(I)},$$

where m, n and ν as defined in section (3). Then, as before, the density function of w_1 can be written as

$$(6.2)\quad c_1(p,n_1,n_2)|\delta'\Lambda|^{-n_1/2}\sum_{k=o}^{\infty}\sum_{\kappa}\frac{(\nu)_\kappa\, c_\kappa(I-(\delta'\Lambda)^{-1})}{k!}\sum_{s=o}^{k}\sum_{\eta}\frac{(-1)^s a_{\kappa,\eta}}{C_\eta(I)}$$

$$\sum_{t=o}^{\infty}\sum_{\sigma,\delta}(pn_2/2+s+t)\; g^{\delta}_{\eta,\sigma}\frac{((p+1-n_1)/2)_\sigma (n_2/2)_\delta}{t!\;(n_2+p+1)_\delta}\quad c_\delta(I_p)(1-w_1)^{pn_2/2+s+t-1}.$$

Also, the density function of w_2 can be expressed in the form

$$(6.3)\quad c_2(p,n_1,n_2)|\delta'\Lambda|^{-n_1/2}(1-w_2)^{a}\sum_{k=o}^{\infty}\sum_{\kappa}\frac{(\nu)_\kappa\, c_\kappa(I-(\delta'\Lambda)^{-1})}{k!}\sum_{s=o}^{k}\sum_{\eta}$$

$$\frac{(-1)^s a_{\kappa,\eta}}{c_\eta(I)}\sum_{t=o}^{\infty}\sum_{\sigma}(-m)_\sigma/t!\sum_{\ell=o}^{p-2}c(\ell)/\ell!\sum_{\tau,\nu}a_{\tau,\nu}(1-w_2)^{t+\ell+s_2}$$

$$B(1-w_2),1;n+p+s_1-\ell;m+1)\quad\sum_{\gamma}g^{\gamma}_{\ell\,(1,\sigma,\nu)}\quad c_\gamma(I_{p-1})((n_2-1)(p-1)/2+t+\ell+s_2)$$

$$(\Gamma_{p-1}((n_2-1)/2,\gamma)/\Gamma_{p-1}((n_2+p-1)/2,\gamma)/\Gamma_{p-1}((n_2+p-1)/2,\gamma)\quad.$$

Where $a = n(p-1)+\frac{1}{2}(p-2)(p+1)$

7. The distribution of the sum of the two smallest (largest roots). Let $\Sigma=I$ in (2.1); p = 4 and transform $M_1 = g_1+g_2$, $M_2 = g_3+g_4$ and integrate g_1 and g_3 over the region $0 \le g_1 \le M_1/2$ and $M_1/2 \le g_3 \le M_2/2$ respectively, then the joint density of M_1 and M_2 can be written in the following form

$$(7.1)\quad k_1(4,n)\, e^{-M_1-M_2}\, M_1^{2m+1} \sum_{k=o}^{m} \binom{m}{k}(-2)^{-k} M_2^{m-k} \sum_{i=o}^{m} \binom{m}{i}(-2)^{i} \sum_{j=1}^{5} M_1^{j} M_2^{5-j}$$

$$(a_j M_2^{m+k+2} - b_j M_1^{m+k+2}) \quad ,$$

where $k_1(4,n) = k(4,n)/2^{2m+3}$ and

$a_1 = \{(m+1)^2 + 15(m+k+4)\}/8(m+i+1)_2(m+k+3)_4$,

$a_2 = -\ (m+k+6)/2(m+i+1)_2(m+k+2)_3$,

$a_3 = -\ (m+k+6)(m+i)/2(m+i+1)_2(m+k+2)_3 + [(m+i+2)(m+k+1)_2]^{-1} -$

$(3m+3i+13)(4(m+k+4)+(m+k+1)_2)/(m+i+3)_2(m+k+1)_2$,

$a_4 = -\ (m+i+6)/2(m+k+1)_2(m+i+2)_3$,

$a_5 = \{(4m+4i+25)(m+3+i)_2 - 8(m+i+1)(m+i+5)_2\}/(m+i+3)_4(m+k+1)_2$,

$b_1 = 0$,

$b_2 = \{(m+k+2)_3(m+i+1)^{(m+i+6)} - (m+i+3)_2(m+k+4)(m+k+1)/2(m+i+1)_4(m+k+1)_3\}$,

$b_3 = (m+k+6)/2(m+i+1)_2(m+k+3)_2 - (2m+2i+1)/(m+i+1)_2(m+k+2)$

$+ (m+i)/2(m+i+2)_2(m+k+1) + 3/(m+i+3)(m+k+2) + (m+k)/2(m+i+4)(m+k+1)_2$,

$b_4 = \sum_{i_1,i_2=1}^{6} c_{i_1,i_2}/(m+i+i_2)_1(m+k+i_2)$,

where $c_{1,1} = c_{1,2} = c_{2,1} = c_{3,5} = c_{4,5} = c_{5,i_2} = c_{6,i_2} = 0$, all $i_2 > 1$

$c_{13} = -3/2$, $c_{14} = -1/2$, $c_{15} = 5/8$, $c_{2,2} = 1$, $c_{2,3} = 3$, $c_{2,4} = 1/2$

$c_{2,5} = -5/8$, $c_{3,1} = 1/2$, $c_{3,2} = -3/2$, $c_{3,4} = -1/4$, $c_{4,1} = -1/2$

$c_{4,2} = 1/2$, $c_{4,3} = 3/4$, $c_{5,1} = 5/8$, $c_{6,1} = -1/8$

$b_5 = (3m+3k+20)/(m+i+1)_2(m+k+4)(m+k+6) + (m+i+2)/2\{(m+i+3)_2(m+k+2)\} +$

$(m+i+2)/2 \quad (m+i+3)_2(m+k+4) - (4m+4i+25)/8(m+i+5)_2(m+k+2)$,

where $(a)_b = a(a+1),\ldots,(a+b-1)$.

Integrate (7.1) with respect to M_1 then the density of M_2 can be written in the form

$$(7.2)\quad k_1(4,n)\, e^{-M_2} \sum_{k=o}^{\infty} \binom{m}{k}(-2)^{-k} M_2^{m-k} \sum_{i=o}^{m} \binom{m}{i}(-2)^{i} \sum_{j=1}^{5} M_2^{5-j}(a_j M_2^{m+k+2}$$

$$I(0,M_2;2m+2+j) - b_j I(0,M_2,3m+k+j+4)) \quad .$$

Also, integrate (7.1) with respect to M_2, then the density of M_1 is given by

$$(7.3)\quad k_1(4,n)\, e^{-M_1} M_1^{2m+1} \sum_{k=o}^{m} \binom{m}{k}(-2)^{-k} \sum_{i=o}^{m} \binom{m}{i}(-2)^{-i} \sum_{j=1}^{5} M_1^{j}(a_j I(M_1,\infty;$$

$$2m-j+8) - b_j M_1^{m+k+2}\, I(M_1,\infty;m+6-j-k) \quad .$$

Now, let $T = M_1 + M_2$ in (7.1), and integrate M_1, then the density function of T reduces

$$(7.4)\quad \frac{1}{\Gamma(4m+10)} T^{4m+9} \quad e^{-T} \quad .$$

Further, transform $R_1 = M_1/M_2$ in (7.1) and integrate M_2, then the density function of R_1 can be written in the form

$$k_2(4,n)(1+R_1)^{-(4m+10)} \quad R_1^{2m+1} \sum_{k=o}^{m} \binom{m}{k}(-2)^{-k} \sum_{i=o}^{m} \binom{m}{i}(-2)^{-i} \sum_{j=1}^{5}$$

$$R_1^{j}(a_j - b_j R_1^{m+k+2}) \quad ,$$

where $k_2(4,n) = \Gamma(4m+10)\, k_1(4,n)$.

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