On the moments of elementary symmetric functions of the roots of two matrices

by

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1. Introduction and Summary. Let $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ be two symmetric matrices of order p, $\underset{\sim}{A}_1$, positive definite and having a Wishart distribution [2], [19], with f_1 degrees of freedom and $\underset{\sim}{A}_2$, at least positive semi-definite and having a non-central (linear) Wishart distribution [1], [3], [8], [19], [20] with f_2 degrees of freedom. Now let

$$\underset{\sim}{A}_2 = \underset{\sim}{C}\,\underset{\sim}{Y}\,\underset{\sim}{Y}'\,\underset{\sim}{C}'$$

where $\underset{\sim}{Y}$ is $p \times f_2$ and $\underset{\sim}{C}$ is a lower triangular matrix such that

$$\underset{\sim}{A}_1 + \underset{\sim}{A}_2 = \underset{\sim}{C}\,\underset{\sim}{C}' \quad .$$

Now consider the $s(= \text{minimum } (f_2,p))$ non-zero characteristic roots of the matrix $\underset{\sim}{Y}\,\underset{\sim}{Y}'$. It can be shown that the density function of the characteristic roots of $\underset{\sim}{Y}\,\underset{\sim}{Y}'$ for $f_2 \leq p$ can be obtained from that of the characteristic roots of $\underset{\sim}{Y}\,\underset{\sim}{Y}'$ for $f_2 \geq p$ if in the latter case the following changes are made [9], [19] .

$$(1.1) \qquad (f_1, f_2, p) \rightarrow (f_1 + f_2 - p, p, f_2) \quad .$$

Now define $U_i^{(s)} = tr_i(\underset{\sim}{I}_p - \underset{\sim}{Y}\,\underset{\sim}{Y}')^{-1} - p = tr_i(\underset{\sim}{I}_{f_2} - \underset{\sim}{Y}'\,\underset{\sim}{Y})^{-1} - f_2$, where $tr_i\underset{\sim}{A}$ denotes the ith elementary symmetric function (esf) of the characteristic roots

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of $\underset{\sim}{A}$. In view of (1.1) we only consider $U_i^{(s)}$ when $s = p$ i.e. $U_i^{(p)}$ based on the density function [14] of $\underset{\sim}{L} = \underset{\sim}{Y}\,\underset{\sim}{Y}'$ for $f_2 \geq p$. Now define $V_i^{(p)} = \mathrm{tr}_i \underset{\sim}{L}$ and further $\underset{\sim}{U} = (\underset{\sim}{I}_p - \underset{\sim}{Y}\,\underset{\sim}{Y}')^{-1} - p$. Khatri and Pillai [13] have obtained some results towards finding the moments of $U_i^{(p)}$ and $V_i^{(p)}$ and in this paper an attempt is made to give general expressions of the first three moments of $U_i^{(p)}$ and the first two moments of $V_i^{(p)}$. Further, the moments of the second esf of a matrix in the non-central means case (James [6]) have been considered and tabulation of certain constants made which arose in this context.

2. Results on the ith esf of the roots of a matrix. The lemma below is proved by Khatri and Pillai [13] and is used to obtain the results of section 3.

Lemma: Let $\underset{\sim}{L} = \begin{array}{cc} \begin{pmatrix} \ell_{11} & \underset{\sim}{\ell}' \\ \underset{\sim}{\ell} & \underset{\sim}{L}_{11} \end{pmatrix} & \begin{array}{c} 1 \\ p-1 \end{array} \\ \begin{array}{cc} 1 & p-1 \end{array} & \end{array}$ be a symmetric matrix of order p,

$\underset{\sim}{L}_{22} = \underset{\sim}{L}_{11} - \underset{\sim}{\ell}\,\underset{\sim}{\ell}' / \ell_{11}$, $\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22}$ be positive definite and

$\underset{\sim}{u} = (\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22})^{-\frac{1}{2}} \underset{\sim}{\ell} / \{\ell_{11}(1-\ell_{11})\}^{\frac{1}{2}}$. Further let $\underset{\sim}{U} = (\underset{\sim}{I}_p - \underset{\sim}{L})^{-1} - \underset{\sim}{I}_p$ and

$\underset{\sim}{M} = (\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22})^{-1} - \underset{\sim}{I}_{p-1}$. Then

$$\mathrm{tr}_i \underset{\sim}{U} = \ell_{11}\{(1-\ell_{11})(1-\underset{\sim}{u}'\underset{\sim}{u})\}^{-1}\, \mathrm{tr}_{i-1}\underset{\sim}{M} + \mathrm{tr}_i\, \underset{\sim}{M}$$

$$+ (1-\underset{\sim}{u}'\,\underset{\sim}{u})^{-1} \sum_{j=0}^{i-1} (-1)^j\, \underset{\sim}{u}'(\underset{\sim}{M}^j + \underset{\sim}{M}^{j+1})\, \underset{\sim}{u}\, (\mathrm{tr}_{i-1-j}\underset{\sim}{M}) \quad \text{for} \quad i < p$$

$$= \ell_{11}\{(1-\ell_{11})(1-\underset{\sim}{u}'\underset{\sim}{u})\}^{-1}\, |\underset{\sim}{M}| \quad \text{for} \quad i = p \,.$$

Notice that the distributions of ℓ_{11}, $\underset{\sim}{u}$ and $\underset{\sim}{L}_{22}$ are available in [11, 12] except that the non-centrality parameter, which is involved in the density of ℓ_{11} above, will be denoted here by λ in place of $2\lambda^2$ given there.

3. Moments of $\mathrm{tr}_i\underset{\sim}{U}$. Let $\underset{\sim}{U}_o$ be a $\underset{\sim}{U}$ matrix when $\lambda = 0$, let $\ell_{11,0}$ be the top left corner element of $\underset{\sim}{L}_o$. ($\underset{\sim}{L}$ matrix where $\lambda = 0$) and let

(3.1) $$y_1 = E(1-\underset{\sim}{u}'\underset{\sim}{u})^{-1}[E\{\ell_{11}/(1-\ell_{11})\} - E\{\ell_{11,0}/(1-\ell_{11,0})\}] = \lambda/(a-1),$$

(3.2) $$y_2 = E(1-\underset{\sim}{u}'\underset{\sim}{u})^{-2}[E\{\ell_{11}/(1-\ell_{11})\}^2 - E\{\ell_{11,0}/(1-\ell_{11,0})\}^2]$$
$$= \{2(f_2+2)\lambda + \lambda^2\}/\{(a-1)(a-3)\},$$

(3.3) $$y_3 = E(1-\underset{\sim}{u}'\underset{\sim}{u})^{-3}[E\{\ell_{11}/(1-\ell_{11})\}^3 - E\{\ell_{11,0}/(1-\ell_{11,0})\}^3]$$
$$= \{3(f_2+2)(f_2+4)\lambda + 3(f_2+4)\lambda^2 + \lambda^3\}/\{(a-1)(a-3)(a-5)\},$$

and

(3.4) $$y_4 = E(1-\underset{\sim}{u}'\underset{\sim}{u})^{-4}[E\{\ell_{11}/(1-\ell_{11})\}^4 - E\{\ell_{11,0}/(1-\ell_{11,0})\}^4]$$
$$= \{4(f_2+2)(f_2+4)(f_2+6)\lambda + 6(f_2+4)(f_2+6)\lambda^2 + 4(f_2+6)\lambda^3 + \lambda^4\}$$
$$/\{(a-1)(a-3)(a-5)(a-7)\}$$

where $a = f_1 - p$.

Now let

$$\beta_i = \mathrm{tr}_{i-1}\underset{\sim}{M} \quad \text{and} \quad \alpha_i = \mathrm{tr}_i\underset{\sim}{M} + \sum_{j=0}^{i-1} (-1)^j(1-\underset{\sim}{u}'\underset{\sim}{u})^{-1}\underset{\sim}{u}'(\underset{\sim}{M}^j + \underset{\sim}{M}^{j+1})\underset{\sim}{u}\ \mathrm{tr}_{i-1-j}\underset{\sim}{M} \ .$$

Then

$$\text{(3.5)} \qquad E[\mathrm{tr}_i\underset{\sim}{U}] = E[\mathrm{tr}_i\underset{\sim}{U}_o] + y_1 E\ \beta_i ,$$

$$\text{(3.6)} \qquad E[\mathrm{tr}_i\underset{\sim}{U}]^2 = E[\mathrm{tr}_i\underset{\sim}{U}_o]^2 + y_2 E\ \beta_i^2 + 2y_1 E\ \beta_i\ \alpha_i ,$$

$$\text{(3.7)} \qquad E[\mathrm{tr}_i\underset{\sim}{U}]^3 = E[\mathrm{tr}_i\underset{\sim}{U}_o]^3 + y_3 E\ \beta_i^3 + 3y_2 E\ \beta_i^2\ \alpha_i + 3y_1 E\ \beta_i\ \alpha_i^2 ,$$

and

$$\text{(3.8)} \qquad E[\mathrm{tr}_i\underset{\sim}{U}]^4 = E[\mathrm{tr}_i\underset{\sim}{U}_o]^4 + y_4 E\ \beta_i^4 + 4y_3 E\ \beta_i^3\ \alpha_i + 6y_2 E\ \beta_i^2\ \alpha_i^2 + 4y_1 E\ \beta_i\ \alpha_i^3 \ .$$

In order to evaluate the right sides of (3.5) - (3.8), it appears that general results are obtainable in terms of functions of esf's of latent roots of $\underset{\sim}{M}$. Hence we suggest the following general form for $E\ \beta_i\ \alpha_i$

$$\text{(3.9)} \qquad E\ \beta_i\ \alpha_i = \frac{1}{a-1} E[\mathrm{tr}_{i-1}\underset{\sim}{M}\{(p-1)\mathrm{tr}_{i-1}\underset{\sim}{M} + (a+i-1)\mathrm{tr}_i\underset{\sim}{M}\}] \ .$$

The above result as well as others in this section and the next have been suggested by computing special cases for $i = 1,2,3,4$ and further checking the result for $i = 5$.

Similarly

$$\text{(3.10)} \qquad E\ \beta_i^2\ \alpha_i = \frac{1}{a-1} E[(\mathrm{tr}_{i-1}\underset{\sim}{M})^2\{(p-i)\mathrm{tr}_{i-1}\underset{\sim}{M} + (a+i-1)\mathrm{tr}_i\underset{\sim}{M}\}] \ ,$$

and

$$(3.11)\qquad E\,\beta_i\,\alpha_i^2 = \frac{1}{(a-1)(a-3)}\,E[\mathrm{tr}_{i-1}\underset{\sim}{M}\{(p-i)(p-i+2)(\mathrm{tr}_{i-1}\underset{\sim}{M})^2$$

$$+\,2[(a+i-3)(p-i)+2]\,\mathrm{tr}_{i-1}\underset{\sim}{M}\,\mathrm{tr}_i\underset{\sim}{M} + (a+i-3)(a+i-1)(\mathrm{tr}_i\underset{\sim}{M})^2$$

$$+\sum_{k=0}^{i-1}\ \sum_{j=2i-2-k}^{2i-k} a_{kj}\,\mathrm{tr}_k\underset{\sim}{M}\,\mathrm{tr}_j\underset{\sim}{M}\}]\quad ,$$

where

$$a_{kj} = \begin{cases} 0 & \text{if } j-k \le 1 \\ -2(j-k) & \text{if } j-k > 1 \text{ and } j-k \text{ is even} \\ 4(j-k) & \text{if } j-k > 1 \text{ and } j-k \text{ is odd} \end{cases} .$$

Now noting that $E[\mathrm{tr}_i\underset{\sim}{U}_o]$, $E[\mathrm{tr}_i\underset{\sim}{U}_o]^2$ and $E[\mathrm{tr}_i\underset{\sim}{U}_o]^3$ are available in Pillai [15, 16], using (3.9) - (3.11) in (3.5) - (3.7) and the fact that $E\,\beta_i^j = E(\mathrm{tr}_{i-1}\underset{\sim}{M})^j$, we can obtain the first three moments of $U_i^{(p)} = \mathrm{tr}_i\{(\underset{\sim}{I}-\underset{\sim}{L})^{-1}-\underset{\sim}{I}\}$ (which are suggested based on computations for $i = 1,2,3,4,5$). Expected values of functions of $\mathrm{tr}_i\underset{\sim}{M}$ can be obtained in individual cases by use of zonal polynomials [7] or by Pillai's lemma on multiplication of a basic Vandermonde determinant by monomials of esf's [15].

4. <u>Moments of $\mathrm{tr}_i\underset{\sim}{L}$.</u> Khatri and Pillai have shown [13] that

$$(4.1)\qquad E[\mathrm{tr}_i\underset{\sim}{L}] = E[\mathrm{tr}_i\underset{\sim}{L}_o] + x_1 E\,\beta_{1(i)}\,,$$

$$(4.2) \qquad E[\mathrm{tr}_i \underset{\sim}{L}]^2 = E[\mathrm{tr}_i \underset{\sim}{L}_o]^2 - x_2 E\, \beta^2_{1(i)} + 2x_1 E\, \alpha_{1(i)}\, \beta_{1(i)},$$

$$(4.3) \qquad E[\mathrm{tr}_i \underset{\sim}{L}]^3 = E[\mathrm{tr}_i \underset{\sim}{L}_o]^3 + x_3 \beta^3_{1(i)} - 3x_2 E\, \beta_{1(i)}\alpha_{1(i)} + 3x_1 E\, \beta_{1(i)}\alpha^2_{1(i)},$$

and

$$(4.4) \qquad E[\mathrm{tr}_i \underset{\sim}{L}]^4 = E[\mathrm{tr}_i \underset{\sim}{L}_o]^4 - x_4 \beta^4_{1(i)} + 4x_3 \beta_{1(i)}\alpha_{1(i)} - 6x_2 \beta^2_{1(i)}\alpha^2_{1(i)}$$
$$+ 4x_1\, \beta_{1(i)}\, \alpha^3_{1(i)},$$

where x_1, x_2, x_3, x_4, $\alpha_{1(i)}$, $\beta_{1(i)}$ and $\underset{\sim}{L}_o$ are defined in [13] and are functions similar to y_i's, α_i's and β_i's in the preceding section.

Using the results of section 2 of [12] gives

$$(4.5) \qquad E[\beta_{1(i)}] = \frac{1}{f_1} E[(a+i)\, \mathrm{tr}_{i-1} \underset{\sim}{L}_{22} + i\, \mathrm{tr}_i \underset{\sim}{L}_{22}] \quad ,$$

where $a = f_1 - p$ and $\mathrm{tr}_o \underset{\sim}{L}_{22} = 1$.

Similarly

$$(4.6) \qquad E[\alpha_{1(i)}\beta_{1(i)}] = \frac{1}{f_1} E[(a+2i)\mathrm{tr}_{i-1}\underset{\sim}{L}_{22}\mathrm{tr}_i\underset{\sim}{L}_{22} + (a+i)(\mathrm{tr}_{i-1}\underset{\sim}{L}_{22})^2 + i(\mathrm{tr}_i\underset{\sim}{L}_{22})^2]$$

and

$$(4.7) \qquad E[\beta^2_{1(i)}] = \frac{1}{f_1(f_1+2)} E[(a+i)(a+i+2)(tr_{i-1}\underset{\sim}{L}_{22})^2$$

$$+ [2i(a+i+1) + 2(i-2)]tr_{i-1}\underset{\sim}{L}_{22}tr_i\underset{\sim}{L}_{22}$$

$$+ i(i+2)(tr_i\underset{\sim}{L}_{22})^2 - \sum_{k=0}^{i-1} \sum_{j=2i-2-k}^{2i-k} a_{kj}tr_k\underset{\sim}{L}_{22}tr_j\underset{\sim}{L}_{22}] \quad ,$$

where

$$a_{kj} = \begin{cases} 0 & \text{if } j-k \leq 1 \\ 2(j-k) & \text{if } j-k > 1 \text{ and even} \\ 4(j-k) & \text{if } j-k > 1 \text{ and odd} \end{cases} \quad .$$

Now noting that $E[tr_i\underset{\sim}{L}_o]$ and $E[tr_i\underset{\sim}{L}_o]^2$ are available in Pillai [15,16] and using the above results, we can obtain the first two moments of $V_i^{(p)} = tr_i\underset{\sim}{L}$ ((4.5) - (4.7) being suggested based on computations for $i = 1,2,3,4$ and 5). Further expected values of functions of $tr_i\underset{\sim}{L}_{22}$ can be obtained by methods suggested at the end of the preceeding section.

5. Moments of the second esf of a matrix. Let $\underset{\sim}{X}$ be a $p \times f$ matrix variate $(p \leq f)$ whose columns are independently normally distributed with $E(\underset{\sim}{X}) = \underset{\sim}{M}$ and covariance matrix $\underset{\sim}{\Sigma}$. Let $w_1,\ldots, w_p$ be the characteristic roots of $|\underset{\sim}{X}\,\underset{\sim}{X}' - w\,\underset{\sim}{\Sigma}| = 0$, then the distribution of $\underset{\sim}{W} = \text{diag } (w_i)$ is given by James [6], [7]

$$(5.1) \qquad e^{-\frac{1}{2}tr\underset{\sim}{\Omega}} K(p,f) \; {}_0F_1(\tfrac{1}{2}f;\tfrac{1}{4}\underset{\sim}{\Omega},\underset{\sim}{W}) \; e^{-\frac{1}{2}tr\underset{\sim}{W}}|\underset{\sim}{W}|^{\frac{1}{2}(f-p-1)} \prod_{i>j} (w_i - w_j)$$

$$0 < w_1 \leq \ldots \leq w_p < \infty$$

where

$$\kappa(p,f) = \Pi^{\frac{1}{2}p^2} / \{2^{\frac{1}{2}pf}\Gamma_p(\tfrac{1}{2}f)\ \Gamma_p(\tfrac{1}{2}p)\} \quad , \tag{5.2}$$

$\underset{\sim}{\Omega} = \text{diag}\ (\omega_i)$ where $\omega_i, i = 1,\ldots, p$ are the characteristic roots of $|\underset{\sim}{M}\,\underset{\sim}{M}' - \omega\,\underset{\sim}{\Sigma}| = \underset{\sim}{0}$, ${}_0F_1$ is the hypergeometric function of matrix argument (James [7]) and $\Gamma_p(\cdot)$ is the multivariate gamma function defined in [7] . Now define $W_2^{(p)}$ as the second esf in $\frac{1}{2}w_1,\ldots, \frac{1}{2}w_p$. Then from Gupta [5] we have

$$E[W_2^{(p)}]^3 = \frac{7}{64}\,L_{(3^2)} + \frac{1}{12}\,L_{(321)} + \frac{57}{320}\,L_{(2^21^2)} + \frac{3}{40}\,L_{(31^3)} + \frac{1}{8}\,L_{(2^3)} + \frac{9}{40}\,L_{(21^4)} + \frac{27}{64}\,L_{(1^6)} \tag{5.3}$$

where L_κ represents $L_\kappa^\gamma(-\frac{1}{2}\underset{\sim}{\Omega})$, which is the generalized Laguerre polynomial of the form [4], [7]

$$L_\kappa^\gamma(-\tfrac{1}{2}\underset{\sim}{\Omega}) = (\gamma + \tfrac{1}{2}(p+1))_\kappa C_\kappa(\underset{\sim}{I}) \sum_{n=0}^{k} \sum_{\nu} (-1)^n \left[\frac{a_{\kappa,\nu}}{(\gamma+\frac{1}{2}(p+1))_\nu}\right] \frac{C_\nu(-\frac{1}{2}\underset{\sim}{\Omega})}{C_\nu(\underset{\sim}{I})} \quad .$$

Pillai and Gupta [18] have evaluated the first two moments of $W_2^{(p)}$ using $a_{\kappa,\nu}$ coefficients for $k = 2, 4$ available in [4].

Here we evaluate the third moment in (5.3) using the table of $a_{\kappa,\eta}$ coefficients presented in the next section.

$$E[W_2^{(p)}]^3 = \mu_3'^{(\ell)}\{W_2^{(p)}\} + \sum_{i=1}^{4} {}' \sum_{\substack{j=1\\ i\neq j}}^{3} \sum_{\substack{k=1\\ k\neq \ell}}^{3} \sum_{\ell=0}^{2} a_{ijk\ell}\, b_i^k\, b_j^\ell \tag{5.4}$$

where

$a_{1210}= 12\mu_3'\{W_2^{(p)}\}/c_o$, $a_{1220}= c_{-1}[c_{-2}\{c_1(342c_4+70c_o)+c_{-3}(3d_{41}+175c_4)\}$

$+4d_3]/13440$, $a_{2110}= [c_{-2}\{4c_{-3}(175c_4d_{21}+27c_{-4}d_{52}-3c_{-1}d_{41}+627c_1c_2)$

$+8c_1(35c_od_{21}+9c_4\{13c_2-19c_1\})\} +d_3\{7c_2-16c_{-1}\}]/40320$, $a_{2120}= [c_{-2}\{4d_{40}$

$+290c_{-3}-504c_1\}+7c_3\{7c_4-8c_1\}]/840$, $a_{3110}= [c_{-3}\{5c_2(912c_1+245c_4)$

$+135c_{-4}(39c_2+20c_{-5})-c_{-2}d_9+1120c_{-1}c_2\}+c_1\{6c_4(150c_2-666c_{-2}-49c_3)$

$+14(16c_{-1}d_{32}+25c_od_{22})\}]/16800$, $a_{1230}= c_{-1}[c_{-2}d_{13}+c_1c_3]/120$,

$a_{4110}= [10c_{-2}\{2d_{40}+397c_{-3}\}+c_1\{1120d_{32}-400d_{05}+23712c_{-3}+7182c_2\}$

$+35c_{-4}\{7c_3-184c_{-3}+54c_2\}+243c_{-4}\{66c_2+25c_{-5}+56c_{-3}\}/12600$, $a_{2311}= 18$,

$a_{1212}= (3/2)c_1+6$, $a_{1221}= a_{1230}/(6c_{-1})$, $a_{1211}= [c_{-2}\{c_{-3}(d_9-2520c_{-1})$

$+6c_1(666c_4-756c_{-1}+175c_o)\}+42c_1c_3\{7c_4-12c_1\}]/25200$, $a_{1311}= [c_{-3}\{805c_4$

$+1701c_{-4}+2964c_1-2980c_{-2}\}+10c_1\{5d_{05}-378c_{-2}-14c_3\}-16c_{-2}d_{40}]/2520$,

and all other $a_{ijk\ell}= 0$, $c_\alpha= (f+\alpha)(p+\alpha)$, $\mu_3'\{W_2^{(p)}\}$ is the third moment in the central case [5] with $2m = f-p-1$ and

$d_3=7c_1c_3c_4$, $d_{52}= 19c_2-7c_{-5}$, $d_{21}=c_2-c_{-1}$, $d_{40}= 35c_o+99c_4$, $d_{32}= c_3-9c_{-2}$

$d_9=6840c_1+1995c_4+2835c_{-4}$, $d_{22}=c_2-3c_{-2}$, $d_{05}=7c_o+18c_4$, $d_{41}=152c_1+63c_4$.

6. Results for $a_{\kappa,\tau}$. The $a_{\kappa,\tau}$'s are constants [4] satisfying the equality

$$C_\kappa(\underset{\sim}{A} + \underset{\sim}{I}) \;/\; C_\kappa(\underset{\sim}{I}) = \sum_{t=0}^{k} \sum_{\tau} a_{\kappa,\tau} C_\tau(\underset{\sim}{A})/C_\tau(\underset{\sim}{I}) \tag{6.1}$$

where τ is a partition of t. The following are suggested based on the available results. For $\kappa = k-j,1^j$

$$a_{\kappa,\tau} = \begin{cases} j(2k-(j+1))/(2k-(j+2)) & \text{if } \tau = k-j,1^{j-1} \\ (2k-j)(k-(j+1))/(2k-(j+2)) & \text{if } \tau = k-j-1,1^j \end{cases} . \tag{6.2}$$

Also for $\kappa = k-j,j$

$$a_{\kappa,\tau} = \begin{cases} j(2k-(4j-2)) \;/\; (2k-(4j-1)) & \text{if } \tau = k-j,j-1 \\ (k-2j)(2k-(2j-1))/(2k-(4j-1)) & \text{if } \tau = k-j-1,j \end{cases} . \tag{6.3}$$

For $\kappa = k$, $\tau = k-j$

$$a_{\kappa,\tau} = k! \;/\; (j!(k-j)!) \quad . \tag{6.4}$$

As previously stated the $a_{\kappa,\tau}$ for $k = 1,2,3,4$ are available in [4] and for $k = 5,6$ now follow.

Table 1. $a_{\kappa,\tau}$ Coefficients for $k = 5$

$\kappa \backslash \tau$	0	1	2	1^2	3	21	1^3	4	31	2^2	21^2	1^4	5	41	32	31^2	2^21	21^3	1^5
5	1	5	10		10			5					1						
41	1	5	7	3	$\frac{23}{5}$	$\frac{27}{5}$		$\frac{8}{7}$	$\frac{27}{7}$					1					
32	1	5	$\frac{16}{3}$	$\frac{14}{3}$	$\frac{8}{5}$	$\frac{42}{5}$			$\frac{8}{3}$	$\frac{7}{3}$					1				
31^2	1	5	$\frac{13}{3}$	$\frac{17}{3}$	$\frac{7}{5}$	$\frac{33}{5}$	2		$\frac{7}{3}$		$\frac{8}{3}$					1			
2^21	1	5	$\frac{10}{3}$	$\frac{20}{3}$		$\frac{15}{2}$	$\frac{5}{2}$			$\frac{5}{3}$	$\frac{10}{3}$						1		
21^3	1	5	2	8		$\frac{9}{2}$	$\frac{11}{2}$				$\frac{18}{5}$	$\frac{7}{5}$						1	
1^5	1	5		10			10					5							1

Table 2. $a_{\kappa,\tau}$ Coefficients for $k = 6$

$\kappa \backslash \tau$	0	1	2	1^2	3	21	1^3	4	31	2^2	21^2	1^4	5	41	32	31^2	2^21	21^3	1^5	6	51	42	41^2	3^2	321	31^3	2^3	2^21^2	21^4	1^6
6	1	6	15		20			15					6							1										
51	1	6	$\frac{34}{3}$	$\frac{11}{3}$	$\frac{56}{5}$	$\frac{44}{5}$		$\frac{39}{7}$	$\frac{66}{7}$				$\frac{10}{9}$	$\frac{44}{9}$							1									
42	1	6	9	6	$\frac{28}{5}$	$\frac{72}{5}$		$\frac{48}{5}$	$\frac{66}{7}$	$\frac{147}{35}$				$\frac{12}{5}$	$\frac{18}{5}$							1								
41^2	1	6	8	7	$\frac{26}{5}$	$\frac{123}{10}$	$\frac{5}{2}$	$\frac{9}{7}$	$\frac{61}{7}$		5			$\frac{9}{4}$		$\frac{15}{4}$							1							
3^2	1	6	8	7	$\frac{16}{5}$	$\frac{84}{5}$			8	7					6									1						
321	1	6	$\frac{19}{3}$	$\frac{26}{3}$	$\frac{28}{15}$	$\frac{74}{5}$	$\frac{10}{3}$		$\frac{14}{3}$	$\frac{11}{3}$	$\frac{20}{3}$				$\frac{14}{9}$	$\frac{20}{9}$	$\frac{20}{9}$								1					
31^3	1	6	5	10	$\frac{8}{5}$	$\frac{57}{5}$	7		4		$\frac{46}{5}$	$\frac{9}{5}$				$\frac{24}{7}$		$\frac{18}{7}$								1				
2^3	1	6	5	10		15	5			5	10						6										1			
2^21^2	1	6	4	11		12	8			$\frac{5}{2}$	$\frac{52}{5}$	$\frac{21}{10}$					3	3										1		
21^4	1	6	$\frac{7}{3}$	$\frac{38}{3}$		7	13				$\frac{42}{5}$	$\frac{33}{5}$						$\frac{14}{3}$	$\frac{4}{3}$										1	
1^6	1	6		15			20					15							6											1

Table 3. $a_{\kappa,\tau}$ Coefficients* for k = 7

τ κ	0	1	2	1^2	3	21	1^3	4	31	2^2	21^2	1^4	5	41	32	31^2	2^21	21^3	1^5	6	51	42	41^2	3^2	321	31^3	2^3	2^21^2	21^4	1^6
7	1	7	21		35			35					21							7										
61	1	7	$\frac{50}{3}$	$\frac{13}{3}$	22	13		$\frac{115}{7}$	$\frac{130}{7}$				$\frac{59}{9}$	$\frac{130}{9}$						$\frac{12}{11}$	$\frac{65}{11}$									
52	1	7	$\frac{41}{3}$	$\frac{22}{3}$	13	22		$\frac{32}{5}$	22	$\frac{33}{5}$			$\frac{80}{63}$	$\frac{506}{45}$	$\frac{297}{35}$						$\frac{16}{7}$	$\frac{33}{7}$								
51^2	1	7	$\frac{38}{3}$	$\frac{25}{3}$	$\frac{62}{5}$	$\frac{98}{5}$	3	$\frac{43}{7}$	$\frac{146}{7}$		8		$\frac{11}{9}$	$\frac{97}{9}$		9					$\frac{11}{5}$		$\frac{24}{5}$							
43	1	7	12	9	8	27		$\frac{64}{35}$	$\frac{144}{7}$	$\frac{63}{5}$				$\frac{24}{5}$	$\frac{81}{5}$							4		3						
421	1	7	$\frac{31}{3}$	$\frac{32}{3}$	$\frac{19}{3}$	$\frac{49}{2}$	$\frac{25}{6}$	$\frac{54}{35}$	$\frac{1006}{63}$	$\frac{287}{45}$	$\frac{100}{9}$			$\frac{81}{20}$	$\frac{244}{45}$	$\frac{275}{36}$	$\frac{35}{9}$					$\frac{3}{2}$	2		$\frac{7}{2}$					
41^3	1	7	9	12	$\frac{29}{5}$	$\frac{207}{10}$	$\frac{17}{2}$	$\frac{10}{7}$	$\frac{102}{7}$		$\frac{84}{5}$	$\frac{11}{5}$		$\frac{15}{4}$		$\frac{351}{28}$		$\frac{33}{7}$					$\frac{10}{3}$			$\frac{11}{3}$				
3^21	1	7	$\frac{28}{3}$	$\frac{35}{3}$	$\frac{56}{15}$	$\frac{133}{5}$	$\frac{14}{3}$		$\frac{112}{9}$	$\frac{91}{9}$	$\frac{112}{9}$				$\frac{77}{9}$	$\frac{56}{9}$	$\frac{56}{9}$							$\frac{7}{5}$	$\frac{28}{5}$					
32^2	1	7	$\frac{25}{3}$	$\frac{38}{3}$	$\frac{7}{3}$	26	$\frac{20}{3}$		$\frac{70}{9}$	$\frac{85}{9}$	$\frac{160}{9}$				$\frac{35}{9}$	$\frac{50}{9}$	$\frac{104}{9}$								5		2			
321^2	1	7	$\frac{22}{3}$	$\frac{41}{3}$	$\frac{32}{15}$	$\frac{227}{10}$	$\frac{61}{6}$		$\frac{64}{9}$	$\frac{95}{18}$	$\frac{896}{45}$	$\frac{27}{10}$			$\frac{20}{9}$	$\frac{416}{63}$	$\frac{115}{18}$	$\frac{81}{14}$							$\frac{20}{7}$	2		$\frac{15}{7}$		
31^4	1	7	$\frac{17}{3}$	$\frac{46}{3}$	$\frac{9}{5}$	$\frac{86}{5}$	16		6		$\frac{104}{5}$	$\frac{41}{5}$				$\frac{54}{7}$		$\frac{244}{21}$	$\frac{5}{3}$							$\frac{9}{2}$			$\frac{5}{2}$	
2^31	1	7	6	15		$\frac{45}{2}$	$\frac{25}{2}$			$\frac{15}{2}$	24	$\frac{7}{2}$					$\frac{27}{2}$	$\frac{15}{2}$									2	5		
2^21^3	1	7	$\frac{14}{3}$	$\frac{49}{3}$		$\frac{35}{2}$	$\frac{35}{2}$			$\frac{7}{2}$	$\frac{112}{5}$	$\frac{91}{10}$					$\frac{63}{10}$	$\frac{77}{6}$	$\frac{28}{15}$									$\frac{21}{5}$	$\frac{14}{5}$	
21^5	1	7	$\frac{8}{3}$	$\frac{55}{3}$		10	25				16	19						$\frac{40}{3}$	$\frac{23}{3}$										$\frac{40}{7}$	$\frac{9}{7}$
1^7	1	7		21			35					35							21											7

* $a_{\kappa,\tau} = 1$ when $\tau = \kappa$

Table 4. $a_{\kappa,\tau}$ Coefficients* for k=8

κ \ τ	0	1	2	1^2	3	21	1^3	4	31	2^2	21^2	1^4	5	41	32	31^2	2^21	21^3	1^5
8	1	8	28		56			70					56						
71	1	8	23	5	38	18		$\frac{265}{7}$	$\frac{225}{7}$				$\frac{68}{3}$	$\frac{100}{3}$					
62	1	8	$\frac{58}{3}$	$\frac{26}{3}$	$\frac{124}{5}$	$\frac{156}{5}$		$\frac{643}{35}$	$\frac{884}{21}$	$\frac{143}{15}$			$\frac{460}{63}$	$\frac{1456}{45}$	$\frac{572}{35}$				
61^2	1	8	$\frac{55}{3}$	$\frac{29}{3}$	24	$\frac{57}{2}$	$\frac{7}{2}$	$\frac{125}{7}$	$\frac{850}{21}$		$\frac{35}{3}$		$\frac{64}{9}$	$\frac{565}{18}$		$\frac{35}{2}$			
53	1	8	17	11	$\frac{82}{5}$	$\frac{198}{5}$		$\frac{272}{35}$	$\frac{297}{7}$	$\frac{99}{5}$			$\frac{32}{21}$	$\frac{308}{15}$	$\frac{1188}{35}$				
521	1	8	$\frac{46}{3}$	$\frac{38}{3}$	$\frac{72}{5}$	$\frac{183}{5}$	5	$\frac{247}{35}$	$\frac{766}{21}$	$\frac{49}{5}$	$\frac{50}{3}$		$\frac{88}{63}$	$\frac{1673}{90}$	$\frac{438}{35}$	$\frac{35}{2}$	6		
51^3	1	8	14	14	$\frac{68}{5}$	$\frac{162}{5}$	10	$\frac{47}{7}$	$\frac{240}{7}$		$\frac{132}{5}$	$\frac{13}{5}$	$\frac{4}{3}$	$\frac{53}{3}$		$\frac{207}{7}$		$\frac{52}{7}$	
4^2	1	8	16	12	$\frac{64}{5}$	$\frac{216}{5}$		$\frac{128}{35}$	$\frac{288}{7}$	$\frac{126}{5}$				$\frac{64}{5}$	$\frac{216}{5}$				
431	1	8	$\frac{41}{3}$	$\frac{43}{3}$	$\frac{136}{15}$	$\frac{411}{10}$	$\frac{35}{6}$	$\frac{72}{35}$	$\frac{1948}{63}$	$\frac{791}{45}$	$\frac{175}{9}$			$\frac{36}{5}$	$\frac{1006}{45}$	$\frac{140}{9}$	$\frac{98}{9}$		
42^2	1	8	$\frac{38}{3}$	$\frac{46}{3}$	$\frac{112}{15}$	$\frac{201}{5}$	$\frac{25}{3}$	$\frac{9}{5}$	$\frac{226}{9}$	$\frac{689}{45}$	$\frac{250}{9}$			$\frac{63}{10}$	$\frac{574}{45}$	$\frac{325}{18}$	$\frac{170}{9}$		
421^2	1	8	$\frac{35}{3}$	$\frac{49}{3}$	$\frac{106}{15}$	$\frac{183}{5}$	$\frac{37}{3}$	$\frac{12}{7}$	$\frac{1495}{63}$	$\frac{161}{18}$	$\frac{1454}{45}$	$\frac{33}{10}$		6	$\frac{68}{9}$	$\frac{1394}{63}$	$\frac{98}{9}$	$\frac{66}{7}$	
41^4	1	8	10	18	$\frac{32}{5}$	$\frac{153}{5}$	19	$\frac{11}{7}$	$\frac{150}{7}$		$\frac{186}{5}$	$\frac{49}{5}$		$\frac{11}{2}$		$\frac{387}{14}$		$\frac{146}{7}$	2
3^22	1	8	$\frac{35}{3}$	$\frac{49}{3}$	$\frac{14}{3}$	42	$\frac{28}{3}$		$\frac{175}{9}$	$\frac{175}{9}$	$\frac{280}{9}$				$\frac{140}{9}$	$\frac{140}{9}$	$\frac{224}{9}$		
3^21^2	1	8	$\frac{32}{3}$	$\frac{52}{3}$	$\frac{64}{15}$	$\frac{192}{5}$	$\frac{40}{3}$		$\frac{160}{9}$	$\frac{124}{9}$	$\frac{1568}{45}$	$\frac{18}{5}$			$\frac{104}{9}$	$\frac{1088}{63}$	$\frac{152}{9}$	$\frac{72}{7}$	
32^21	1	8	$\frac{29}{3}$	$\frac{55}{3}$	$\frac{8}{3}$	$\frac{75}{2}$	$\frac{95}{6}$		$\frac{100}{9}$	$\frac{245}{18}$	$\frac{367}{9}$	$\frac{9}{2}$			$\frac{50}{9}$	$\frac{800}{63}$	$\frac{224}{9}$	$\frac{90}{7}$	
321^3	1	8	$\frac{25}{3}$	$\frac{59}{3}$	$\frac{12}{5}$	$\frac{321}{10}$	$\frac{43}{2}$		10	$\frac{43}{6}$	$\frac{623}{15}$	$\frac{113}{10}$			3	$\frac{96}{7}$	13	$\frac{503}{21}$	$\frac{7}{3}$
31^5	1	8	$\frac{19}{3}$	$\frac{65}{3}$	2	24	30		$\frac{25}{3}$		$\frac{116}{3}$	23				$\frac{100}{7}$		$\frac{680}{21}$	$\frac{28}{3}$
2^4	1	8	8	20		36	20			15	48	7					36	20	
2^31^2	1	8	7	21		$\frac{63}{2}$	$\frac{49}{2}$			$\frac{21}{2}$	$\frac{231}{5}$	$\frac{133}{10}$					$\frac{126}{5}$	28	$\frac{14}{5}$
2^21^4	1	8	$\frac{16}{3}$	$\frac{68}{3}$		24	32			$\frac{14}{3}$	$\frac{608}{15}$	$\frac{124}{5}$					$\frac{56}{5}$	$\frac{104}{3}$	$\frac{152}{15}$
21^6	1	8	3	25		$\frac{27}{2}$	$\frac{85}{2}$				27	43						30	26
1^8	1	8		28			56					70							56

*$a_{\kappa,\tau} = 1$ when $\tau = \kappa$

Table 4. $a_{\kappa,\tau}$ Coefficients* for k=8 (cont'd.)

$\kappa \backslash \tau$	6	51	42	41^2	3^2	321	31^3	2^3	2^21^2	21^4	1^6	7	61	52	51^2	43	421	41^3	3^21	32^2	321^2	31^4	2^31	2^21^3	21^5	1^7
8	28											8														
71	$\frac{83}{11}$	$\frac{225}{11}$										$\frac{14}{13}$	$\frac{90}{13}$													
62	$\frac{40}{33}$	$\frac{1014}{77}$	$\frac{286}{21}$										$\frac{20}{9}$	$\frac{52}{9}$												
61^2	$\frac{13}{11}$	$\frac{141}{11}$		14									$\frac{13}{6}$		$\frac{35}{6}$											
53	$\frac{144}{35}$	$\frac{121}{7}$		$\frac{33}{5}$										$\frac{18}{5}$		$\frac{22}{5}$										
521		$\frac{132}{35}$	$\frac{97}{14}$	$\frac{46}{5}$		$\frac{81}{10}$								$\frac{22}{15}$	$\frac{40}{21}$		$\frac{162}{35}$									
51^3		$\frac{18}{5}$		$\frac{236}{15}$		$\frac{26}{3}$									$\frac{36}{11}$			$\frac{52}{11}$								
4^2			16		12											8										
431			$\frac{11}{2}$	$\frac{56}{15}$	$\frac{61}{15}$	$\frac{147}{10}$										$\frac{27}{20}$	$\frac{56}{15}$		$\frac{35}{12}$							
42^2			$\frac{7}{2}$	$\frac{14}{3}$		$\frac{33}{2}$		$\frac{10}{3}$									$\frac{14}{3}$			$\frac{10}{3}$						
421^2			$\frac{25}{12}$	$\frac{52}{9}$		$\frac{39}{4}$	$\frac{121}{18}$		$\frac{11}{3}$								$\frac{25}{9}$	$\frac{9}{5}$			$\frac{154}{45}$					
41^4				$\frac{22}{3}$			$\frac{97}{6}$			$\frac{9}{2}$								$\frac{22}{5}$				$\frac{18}{5}$				
3^22					$\frac{7}{3}$	21		$\frac{14}{3}$											$\frac{10}{3}$	$\frac{14}{3}$						
3^21^2					$\frac{28}{15}$	$\frac{528}{35}$	$\frac{16}{3}$		$\frac{40}{7}$										$\frac{8}{3}$		$\frac{16}{3}$					
32^21						$\frac{75}{7}$	$\frac{25}{6}$	$\frac{23}{6}$	$\frac{65}{7}$											$\frac{40}{21}$	$\frac{25}{6}$		$\frac{27}{14}$			
321^3						$\frac{81}{14}$	$\frac{33}{4}$		$\frac{61}{7}$	$\frac{21}{4}$											$\frac{81}{20}$	$\frac{28}{15}$		$\frac{25}{12}$		
31^5							$\frac{25}{2}$			$\frac{195}{14}$	$\frac{11}{7}$											$\frac{50}{9}$			$\frac{22}{9}$	
2^4								8	20														8			
2^31^2								$\frac{7}{2}$	$\frac{91}{5}$	$\frac{63}{10}$													$\frac{7}{2}$	$\frac{9}{2}$		
2^21^4									$\frac{56}{5}$	$\frac{528}{35}$	$\frac{12}{7}$													$\frac{16}{3}$	$\frac{8}{3}$	
21^6										$\frac{135}{7}$	$\frac{61}{7}$														$\frac{27}{4}$	$\frac{5}{4}$
1^8											28															8

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* $a_{\kappa,\tau} = 1$ when $\tau = \kappa$

7. Further uses of $a_{\kappa,\tau}$. Pillai [17] has shown that

$$(7.1) \qquad E[e^{t\,\mathrm{tr}\underset{\sim}{L}}] = e^{-\frac{1}{2}\mathrm{tr}\underset{\sim}{\Omega}} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{n=0}^{k} \sum_{\eta} \frac{(\frac{1}{2}f_2)_\kappa (\frac{1}{2}\nu)_\eta \, a_{\kappa,\eta} t^{k-n} C_\kappa(\underset{\sim}{I}) \, C_\eta(\frac{1}{2}\underset{\sim}{\Omega})}{(\frac{1}{2}\nu)_\kappa \, (\frac{1}{2}f_2)_\eta \, k! \, C_\eta(\underset{\sim}{I})} \quad .$$

From (7.1) we get the moments of $\mathrm{tr}\underset{\sim}{L}$ by differentiation with respect to t and letting $t = 0$. Thus

$$\frac{\partial^r}{\partial t^r} E[e^{t\,\mathrm{tr}\underset{\sim}{L}}] =$$

$$e^{-\frac{1}{2}\mathrm{tr}\underset{\sim}{\Omega}} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{n=0}^{k} \sum_{\eta} \frac{(\frac{1}{2}f_2)_\kappa (\frac{1}{2}\nu)_\eta \, a_{k,\eta}(k-n)(k-n-1)\ldots(k-n-r+1)t^{k-n-r} C_\kappa(\underset{\sim}{I}) C_\eta(\frac{1}{2}\underset{\sim}{\Omega})}{(\frac{1}{2}\nu)_\kappa (\frac{1}{2}f_2)_\eta k! \, C_\eta(\underset{\sim}{I})}$$

and hence

$$(7.2) \qquad E[\mathrm{tr}\underset{\sim}{L}]^r = e^{-\frac{1}{2}\mathrm{tr}\underset{\sim}{\Omega}} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{\eta} \frac{(\frac{1}{2}f_2)_\kappa (\frac{1}{2}\nu)_\eta \, a_{\kappa,\eta} \, r! \, C_\kappa(\underset{\sim}{I}) \, C_\eta(\frac{1}{2}\underset{\sim}{\Omega})}{(\frac{1}{2}\nu)_\kappa \, (\frac{1}{2}f_2)_\eta \, k! \, C_\eta(\underset{\sim}{I})}$$

where η is a partition of $n = k-r$.

Pillai [17] also gives

$$(7.3) \qquad E[e^{t\,\mathrm{tr}\underset{\sim}{R}^2}] = |\underset{\sim}{I}-\underset{\sim}{P}^2|^{\frac{\nu}{2}} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{n=0}^{k} \sum_{\eta} \frac{(\frac{1}{2}f_2)_\kappa ((\frac{1}{2}\nu)_\eta)^2 \, a_{\kappa,\eta} \, t^{k-n} C_\kappa(\underset{\sim}{I}) \, C_\eta(\underset{\sim}{P}^2)}{(\frac{1}{2}\nu)_\kappa (\frac{1}{2}f_2)_\eta \, k! \, C_\eta(\underset{\sim}{I})} \quad ,$$

from which as before we obtain the rth moments

$$(7.4) \qquad E[\mathrm{tr}\underset{\sim}{R}^2]^r = |\underset{\sim}{I}-\underset{\sim}{P}^2|^{\frac{\nu}{2}} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{\eta} \frac{(\frac{1}{2}f_2)_\kappa((\frac{1}{2}\nu)_\eta)^2 a_{\kappa,\eta} r! \; C_\kappa(\underset{\sim}{I}) \; C_\eta(\underset{\sim}{P}^2)}{(\frac{1}{2}\nu)_\kappa(\frac{1}{2}f_2)_\eta k! \; C_\eta(\underset{\sim}{I})} ,$$

where η is as above.

Further, Khatri [10] has obtained the moment generating function of $V^{(p)}$ associated with the test $\lambda \underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ as

$$(7.5) \qquad E[e^{tV^{(p)}}] = |\lambda\underset{\sim}{\Lambda}|^{-\frac{1}{2}f_1} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{n=0}^{k} \sum_{\eta} \frac{(\frac{1}{2}f_1)_\kappa(\frac{1}{2}\nu)_\eta a_{\kappa,\eta} t^{k-n} C_\kappa(\underset{\sim}{I}) \; C_\eta(\underset{\sim}{I}-(\lambda\underset{\sim}{\Lambda})^{-1})}{(\frac{1}{2}\nu)_\kappa \; k! \; C_\eta(\underset{\sim}{I})} .$$

We get the rth moment of $V^{(p)}$ on this case as

$$(7.6) \qquad E[V^{(p)}]^r = |\lambda\underset{\sim}{\Lambda}|^{-\frac{1}{2}f_1} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{\eta} \frac{(\frac{1}{2}f_1)_\kappa(\frac{1}{2}\nu)_\eta \; a_{\kappa,\eta} r! \; C_\kappa(\underset{\sim}{I}) \; C_\eta(\underset{\sim}{I}-(\lambda\underset{\sim}{\Lambda})^{-1})}{(\frac{1}{2}\nu)_\kappa \; k! \; C_\eta(\underset{\sim}{I})} ,$$

where η is a partition of $n = k-p$.

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