More on a Chebyshev-Type Inequality
for Sums of Independent Random Variables

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Mimeograph Series Number 155

April 1968

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MORE ON A CHEBYSHEV-TYPE INEQUALITY FOR SUMS OF INDEPENDENT RANDOM VARIABLES

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1. Introduction and Summary

This paper deals with the same problem and the same conjectured solution to it which we considered in [1].

In section 2, we restate the problem, the conjecture, and, in more concise form, some of the preliminary results of the earlier paper.

In section 3, we give a simpler proof of the conjecture for $n \leq 3$ and, for the first time, a proof for $n = 4$.

In section 4, we give what is essentially a simpler and, we hope, more illuminating proof of Theorem 5.1 of [1].

In section 5, we prove that the conjecture is true for large λ.

2. The Problem

Choose a positive integer n and n positive numbers, $\nu_1 \leq \ldots \leq \nu_n$. Let C be the class of all random variables $S = X_1 + \ldots + X_n$ where the X_i are independent and non-negative, and $E X_i \leq \nu_i$. (It does no harm, and is convenient, to allow $E X_i$ to be less than ν_i.) For each

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$\lambda > \nu_1+\dots+\nu_n$, let $D(\lambda)$ be the sub-class of C in which each X_i has mean ν_i and has not more than two mass points, a_i and b_i, with

$$0 \le a_i \le \nu_i \le b_i \le \lambda - \sum_{j\neq i} a_j,$$

and let $E(\lambda)$ be the subset of $D(\lambda)$ in which each a_i is (or may be taken to be) zero.

The problem is to find

$$\psi(\lambda) = \sup_{S\in C} P(S\ge\lambda). \tag{2.1}$$

From [1], we can state

Lemma 2.1. For each λ, $\psi(\lambda)$ is attained by a member of $D(\lambda)$. If $\psi(\lambda)$ is attained by a unique member of $D(\lambda)$, then there is no other member of C for which it is attained.

Conjecture. For each $\lambda > \nu_1+\dots+\nu_n$,

$$\psi(\lambda) = \max_{0\le k\le n-1} [1-P_k(\lambda)] \tag{2.2}$$

where

$$P_0(\lambda) = \prod_{i=1}^{n}(1-\nu_i/\lambda) \tag{2.3}$$

$$P_k(\lambda) = \prod_{i=k+1}^{n}\left(1-\nu_i/(\lambda-\sum_{j=1}^{k}\nu_j)\right) \quad k=1,\dots,n-1. \tag{2.4}$$

Each of the values $1-P_k(\lambda)$ is attained by a corresponding member of $E(\lambda)$: $1-P_o(\lambda)$ when each $b_i = \lambda$, and, for $k \ge 1$, $1 - P_k(\lambda)$ when $b_i = \nu_i$ $(\lambda-\nu_1-\dots-\nu_k)$ if $i \le (>)\ k$. We call these n members of $E(\lambda)$ the conjectured optimal strategies.

Lemma 2.2. To prove the conjecture for some n, it is sufficient to prove first that it is true for $m < n$, and second that if $S \in E(\lambda)$ with each b_i strictly between ν_i and λ, then there is an $S' = X_1' + \ldots + X_n'$ in C such that either $P(S' \geq \lambda) > P(S \geq \lambda)$, or $P(S' \geq \lambda) = P(S \geq \lambda)$ and $X_i' = \nu_i$ for some i.

This lemma, while not stated in [1], is clearly implied there. Essentially, what we did in [1] was to look for a dominating S' only within $E(\lambda)$, an unnecessary and, as it turns out, unnatural restriction.

3. Solution for $n \leq 4$.

For $n = 1$ the solution is the well-known Markov inequality: $\psi(\lambda) = \nu_i/\lambda = 1 - P_0(\lambda)$.

For $n > 1$ we shall follow the prescription given by Lemma 2.2. We use the following notation

(3.1) $$Z_i = X_i - \nu_i, \; Z_i' = X_i' - \nu_i,$$

(3.2) $$p_i = 1 - q_i = P(X_i = b_i) = \nu_i/b_i,$$

(3.3) $$I_i = 1(0) \quad \text{if} \quad X_i = (\neq)\, b_i$$

(3.4) $$\Delta = P(S' \geq \lambda) - P(S \geq \lambda)$$

Throughout this section we assume that the X_i's are ordered so that

(3.5) $$b_1 \leq b_2 \leq \cdots \leq b_n.$$

Solution for $n = 2$:

$$Z_1' = 0, \; Z_2' = Z_2 + Z_1 I_2,$$
$$\Delta = 0.$$

Solution for $n=3$ when $b_1+b_2<\lambda$:

$$Z_1' = 0,\ Z_2' = Z_2,\ Z_3' = Z_3 + Z_1 I_3,$$
$$\Delta = 0.$$

Solution for $n=3$ when $b_1+b_2 \geq \lambda$: This case can only be handled by breaking it down into at least two sub-cases. One solution is contained in Lemma 4.4 in the next section. We present here a modified solution.

If we use the S' defined in Case 1 of Lemma 4.4, we find, from (4.19),

$$\Delta = \left(p_2q_3+q_2p_3\right)\left[1/(2-p_1)-p_1\right] - p_2p_3q_1{}^2/(2-p_1)^2$$
$$= \left[q_1^2/(2-p_1)^2\right]\left[(1+q_1)\ (p_2q_3+q_2p_3)-p_2p_3\right].$$

If $p_2q_3+q_2p_3 \geq p_2p_3$, this is positive. If not -- which implies $q_2q_3 < p_2q_3 + q_2p_3$ -- Δ is still positive if $P(S<\lambda) = q_2q_3 + q_1(p_2q_3+q_2p_3) \geq p_2p_3$. If the reverse inequality holds, a solution is

$$Z_1' = Z_2' = 0$$

(3.6)
$$Z_3' = Z_3 + Z_2I_3 + Z_1(1-I_3I_2) + Z_2(1-I_3)\,I_1$$
$$-(b_2-b_1+\nu_1)\left[I_3I_2-(1-I_2)(1-I_3)-(1-I_1)(I_2+I_3-I_2I_3)\right].$$

$$\Delta \geq 0\ (=0 \text{ except, e.g., if } b_3+b_2-b_1-\nu_2 \geq \lambda)$$

Here $Z_3' \geq -\nu_3$ and $E\,Z_3' = -(b_2-b_1+\nu_1)\left[p_2p_3 - P(S<\lambda)\right] < 0$.

Before proceeding to $n=4$, we mention a fact which sounds very promising but which we have so far been unable to exploit: For <u>any</u>

S meeting the specifications of Lemma 2.2 and satisfying (3.5), if, in addition,

$$\prod_{i=k}^{n} p_i \geq P(S<\lambda),$$

where k satisfies

$$\sum_{i=k+1}^{n} b_i < \lambda \leq \sum_{i=k}^{n} b_i,$$

then there is a dominating S' with $Z_i' = 0$ for $i < n$, and Z_n' defined analogously to (3.6).

Solution for $n = 4$: From (3.5), the following nine cases are exhaustive:

(a) $b_1 + b_2 \geq \lambda$

(b) $b_1 + b_2 < \lambda \leq b_1 + b_3$

(c) $b_1 + b_3 < \lambda \leq b_1 + b_4,\ b_2 + b_3$

(d) $b_1 + b_4 < \lambda \leq b_2 + b_3$

(e) $b_2 + b_3 < \lambda \leq b_1 + b_4,\ b_1 + b_2 + b_3$

(f) $b_1 + b_4,\ b_2 + b_3 < \lambda \leq b_2 + b_4,\ b_1 + b_2 + b_3$

(g) $b_2 + b_4 < \lambda \leq b_3 + b_4,\ b_1 + b_2 + b_3$

(h) $b_3 + b_4 < \lambda \leq b_1 + b_2 + b_3$

(i) $b_1 + b_2 + b_3 < \lambda.$

Fortunately, cases (c), (d), (f), (g) and (i) can be easily solved as follows:

(c) $Z_1' = 0,\ Z_2' = Z_2,\ Z_3' = Z_3,\ Z_4' = Z_4 + Z_1 I_4$

(d) $Z_1' = 0,\ Z_2' = Z_2,\ Z_3' = Z_3,\ Z_4' = Z_4$

(f) $Z_1' = 0,\ Z_2' = Z_2,\ Z_3' = Z_3 + Z_1 I_3,\ Z_4' = Z_4$

(g) $Z_1' = 0,\ Z_2' = Z_2 + Z_1 I_2,\ Z_3' = Z_3,\ Z_4' = Z_4$

(i) $Z_1' = 0,\ Z_2' = Z_2,\ Z_3' = Z_3,\ Z_4' = Z_4 + Z_1 I_4.$

In all five of these cases $\Delta = 0$.

Cases (a) and (h) are covered by Lemma 4.4, leaving only cases (b) and (e) to consider. Each of these can be solved by a method virtually the same as Lemma 4.4 but we prefer to present simpler solutions here.

Case (b): If $b_1 \le \nu_1 + \nu_2$, a solution is

$$Z_1' = Z_2' = 0,\ Z_3' = Z_3,\ Z_4' = Z_4$$

$$\Delta = q_1 q_2 [p_3 q_4 + q_3 p_4].$$

Henceforth we assume $b_1 > \nu_1 + \nu_2$ which, by (3.5), implies $p_1 + p_2 < 1$. Hence we may use the S' of Lemma 4.4, Case 2:

$$Z_1' = Z_2' = 0,\ Z_3' = Z_3 + Z_3^{(12)} I_3,\ Z_4' = Z_4 + Z_4^{(12)} I_4$$

where the distribution of $Z_3^{(12)}$ and $Z_4^{(12)}$ is given by (4.17). Then

$$\Delta = [p_3 q_4 + q_3 p_4][1/(2-p_1-p_2) - (1-q_1 q_2)] - p_3 p_4 (1-p_1-p_2)^2/(2-p_1-p_2)^2]$$

$$= (1-p_1-p_2)^2/(2-p_1-p_2)^2[(1+\ q_1 q_2)(p_3 q_4 + q_3 p_4) - p_3 p_4]$$

$$+[1-(1-p_1-p_2)^2/(2-p_1-p_2)^2](p_3 q_4 + q_3 p_4).$$

If $p_3q_4 + q_3p_4 \geq p_3p_4$, this is positive. If not -- which implies $q_3q_4 < p_3q_4 + q_3p_4$ -- Δ is still positive if $P(S<\lambda) = q_3q_4 + q_1q_2(p_3q_4+q_3p_4) \geq p_3p_4$.
If the reverse inequality holds, a solution is

$$Z_1' = Z_2' = Z_3' = 0$$

$$Z_4' = Z_4 + Z_3I_4 + (Z_1+Z_2)(1-I_3I_4)+Z_3(1-I_4)(I_1+I_2-I_1I_2)$$

$$-(b_3-b_1+\nu_1+\nu_2)\left[I_3I_4-(1-I_3)(1-I_4)-(I_3+I_4-I_3I_4)(1-I_1)(1-I_2)\right]$$

$$\Delta \geq 0 \text{ (=0 except, e.g., when } b_4+b_3-b_1-\nu_3 \geq \lambda).$$

Here $Z_4' \geq -\nu_4$ and $E\,Z_4' = -(b_3-b_1+\nu_1+\nu_2)\left[p_3p_4-P(S<\lambda)\right] < 0$.

Case (e): If $b_1 \leq \nu_1+\nu_2$, a solution is

$$Z_1' = Z_2' = 0,\ Z_3' = Z_3 + Z_2I_3 + Z_1I_3I_2,\ Z_4' = Z_4$$

$$\Delta = p_4q_1q_2q_3.$$

Henceforth we assume $b_1 > \nu_1+\nu_2$ which, by (3.5), implies $p_1+p_2 < 1$. If $p_4 \leq p_3$, we borrow the random variable $Z_2^{(1)}$ from Lemma 4.4 -- its distribution is given by (4.16). A solution is

$$Z_1' = 0,\ Z_2' = Z_2 + Z_2^{(1)}I_2, Z_3' = Z_3,\ Z_4' = Z_4 + Z_1I_4$$

$$\Delta = (p_3-p_4)\ p_2q_1^{\ 2}/(2-p_1) \geq 0.$$

If $p_4 > p_3$, we first introduce the random variable I^*, independent of all others, with

$$P(I^*=1) = p_1p_2/q_1q_2 = 1-P(I^*=0).$$

I* is well-defined since $p_1+p_2<1$ is equivalent to $p_1p_2<q_1q_2$. A solution is

$$Z_1' = Z_2' = 0, \ Z_3' = Z_3$$

$$Z_4' = Z_4 + I_4\left[Z_1+Z_2-b_1I_1I_2+b_1I^*(1-I_1)(1-I_2)\right]$$

$$\Delta = (p_4-p_3)p_1p_2 > 0.$$

Here $Z_4' \geq -\nu_4$ and $E\,Z_4' = 0$.

4. Independent Trials

The crucial result of this section is Lemma 4.4, whose proof is made possible by the "tricks" of Lemmas 4.1, 4.2, and 4.3. The latter lemma may be of special interest to the reader. From Lemma 4.4, Theorem 4.1 follows easily. It is virtually the same as Theorem 5.1 of [1], though the proof we present here is, we feel, much more satisfactory than the earlier one.

Suppose $S \in D(\lambda)$ is of the following form:

(4.1) $$a_i < \nu_i < b_i \text{ for each } i$$

(4.2) $$b_1-a_1 \leq b_2-a_2 \leq \ldots \leq b_n-a_n$$

(4.3) $$\sum_{i=n-k+1}^{n}(b_i-a_i) < \lambda - \sum_{i=1}^{n}a_i \leq \sum_{i=1}^{k+1}(b_i-a_i)$$

for some k = 1,2,..., or n-1.

We define

$$(4.4)\qquad p_i = 1-q_i = P(X_i=b_i) = (\nu_i-a_i)/(b_i-a_i)$$

$$(4.5)\qquad f(r) = P\ (\text{exactly } r \text{ of } X_i\text{'s} = b_i).$$

Then

$$(4.6)\qquad P(S\geq\lambda) = \sum_{r=k+1}^{n} f\ (r).$$

Lemma 4.1. Since $\lambda > \nu_1+\ldots+\nu_n$, conditions (4.1)-(4.3) imply that $p_1+\ldots+p_n < k+1$.

Proof: From (4.3)

$$\sum_{i=1}^{k+1} (b_i-\nu_i) > \sum_{i=k+2}^{n} (\nu_i-a_i)$$

Hence, by (4.2) and (4.4),

$$\sum_{i=1}^{k+1} (1-p_i) = \sum_{i=1}^{k+1} (b_i-\nu_i)/(b_i-a_i)$$

$$\geq \sum_{i=1}^{k+1} (b_i-\nu_i)/(b_{k+1}-a_{k+1})$$

$$> \sum_{i=k+2}^{n} (\nu_i-a_i)/(b_{k+1}-a_{k+1})$$

$$\geq \sum_{i=k+2}^{n} (\nu_i-a_i)/(b_i-a_i) = \sum_{i=k+2}^{n} p_i .$$

We now re-order the X_i's so that

(4.7) $$b_1 - a_1 = \min_{1 \le i \le n} (b_i - a_i)$$

(4.8) $$p_2 = \max_{2 \le i \le n} p_i .$$

We also define

(4.9) $$f_1(r) = P(\text{exactly } r \text{ of } X_i\text{'s, } i \ge 2, = b_i)$$

(4.10) $$f_{12}(r) = P(\text{exactly } r \text{ of } X_i\text{'s, } i \ge 3, = b_i).$$

We have then, e.g.,

(4.11) $$f_1(r) = p_2 f_{12}(r-1) + q_2 f_{12}(r).$$

Lemma 4.2. The functions f, f_1, and f_{12} are unimodal, first increasing, then decreasing. The modes of f and f_1 are at most $k+1$ while the mode of f_{12} is at most k.

Proof: Since f, f_1, and f_{12} are distributions of numbers of successes in independent trials, their unimodality is well known. The second part of the Lemma follows from Lemma 4.1, equation (4.8), and Theorem 1 and its corollaries in [2].

Corollary 4.1. The following three cases are exhaustive:

Case 1: $f_1(k) \ge f_1(k+1) > \ldots > f_1(n-1)$;

Case 2: $f_{12}(k-1) < f_{12}(k) > f_{12}(k+1) > \ldots > f_{12}(n-2)$ and $p_1 + p_2 \le 1$;

Case 3: $f_{12}(k-1) < f_{12}(k) > f_{12}(k+1) > \ldots > f_{12}(n-2)$ and $p_1 + p_2 > 1$.

Proof: From Lemmá 4.2 and equation (4.11).

We use the usual notation for binomial probabilities:

(4.12) $$B(r;m,p) = \sum_{j=0}^{r} \binom{m}{j} p^j (1-p)^{m-j}.$$

It follows that

(4.13) $$\sum_{m=r+1}^{\infty} B(r;m,p) = (r+1)(1-p)/p,$$

since both sides of (4.13) represent the mean of the same negative binomial distribution. By substituting $r=k-1$ and, respectively, $p=k/(k+1-p_1)$, and $p = k/(k+1-p_1-p_2)$, we obtain

Lemma 4.3. In Case 1 of Corollary 4.1,

(4.14) $$\sum_{j=1}^{n-k-1} f_1(k+j)\, B\Big(k-1;k+j,\ k/(k+1-p_1)\Big) < f_1(k)\left[\Big(k/(k+1-p_1)\Big)^k - p_1\right],$$

while, in Case 2,

(4.15) $$\sum_{j=1}^{n-k-2} f_{12}(k+j) B\Big(k-1;k+j,k/(k+1-p_1-p_2)\Big) < f_{12}(k)\left[\Big(k/(k+1-p_1-p_2)\Big)^k - (1-q_1q_2)\right] - f_{12}(k-1)p_1p_2.$$

Lemma 4.4. Let $S \in D(\lambda)$ and satisfy (4.1). Suppose that (4.3) is also satisfied when the X_i's are ordered to satisfy (4.2). Then there is an $S' = X_1'+\ldots+X_n' = (\nu_1+Z_1') +\ldots+(\nu_n+Z_n')$ in C such that $\Delta = P(S'\geq\lambda)-P(S\geq\lambda) > 0$.

Proof: We henceforth assume that the X_i's are ordered to satisfy

(4.7) and (4.8) rather than (4.2). We introduce the random variables $Z_i^{(1)}$, $Z_i^{(12)}$, and I** which are assumed to be mutually independent and independent of all other random variables, with the following distributions:

$$P\left(Z_i^{(1)}=(b_1-\nu_1)/k\right)=k/(k+1-p_1)=1-P\left(Z_i^{(1)}=-(b_1-a_1)\right), \tag{4.16}$$

$$P\left(Z_i^{(12)}=(b_1+a_2-\nu_1-\nu_2)/k\right)=k/(k+1-p_1-p_2) = 1-P\left(Z_i^{(12)}=-(b_1-a_1)\right), \tag{4.17}$$

$$P(I^{**}=1) = q_1q_2/p_1p_2 = 1-P(I^{**}=0). \tag{4.18}$$

$Z_i^{(1)}$ is always well-defined and has mean zero. $Z_i^{(12)}$ is well-defined when $p_1+p_2 \le 1$ and, by (4.7), $EZ_i^{(12)} \le 0$. We shall use $Z_i^{(12)}$ only when the stronger condition, $b_1+a_2 > \nu_1+\nu_2$, is satisfied. I** is well-defined when $p_1+p_2 \ge 1$, which is equivalent to $q_1q_2 \le p_1p_2$.

We now proceed by considering separately each of the three cases of Corollary 4.1. (In each Case there are "bizarre" choices of the ν_i's, a_i's, and b_i's for which Δ is in fact greater than stated.)

Solution for Case 1:

$$Z_1' = 0,\ Z_i' = Z_i+Z_i^{(1)}I_i \text{ for } i \ge 2,$$

$$\Delta = f_1(k)\left[\left(k/(k+1-p_1)\right)^k - p_1\right] \tag{4.19}$$

$$-\sum_{j=1}^{n-k-1} f_1(k+j)B\left(k-1;k+j,k/(k+1-p_1)\right) > 0 \text{ by } (4.14)$$

Solution for Case 2 when $b_1+a_2 \le \nu_1+\nu_2$:

$$Z_1' = Z_2' = 0, \quad Z_i' = Z_i \text{ for } i \ge 3,$$

(4.20) $$\Delta = f_{12}(k)q_1q_2 - f_{12}(k-1)p_1p_2 > 0.$$

Solution for Case 2 when $b_1+a_2 > \nu_1+\nu_2$:

$$Z_1' = Z_2' = 0, \quad Z_i' = Z_i + Z_i^{(12)}I_i \text{ for } i \ge 3$$

(4.21) $$\Delta = f_{12}(k)\left[\left(k/(k+1-p_1-p_2)\right)^k - (1-q_1q_2)\right] - f_{12}(k-1)p_1p_2 - \sum_{j=1}^{n-k-2} f_{12}(k+j)B\left(k-1;k+j,k/(k+1-p_1-p_2)\right) > 0 \text{ by (4.15)}$$

Solution for Case 3:

$$Z_1' = 0, \quad Z_2' = Z_1+Z_2+b_1(1-I_1)(1-I_2)-b_1I^{**}I_1I_2$$

(4.22) $$\Delta = q_1q_2\left[f_{12}(k)-f_{12}(k-1)\right] > 0.$$

We now define what we call the <u>independent trials case</u>, denoted by $B(\lambda)$. If we agree that, for any member of $D(\lambda)$, $b_i = \nu_i$ whenever $P(X_i=\nu_i) = 1$, then $B(\lambda)$ is the subset of $D(\lambda)$ in which, for some $k = 0,1,\ldots,$ or $n-1$, the events $\{S \ge \lambda\}$ and $\{X_i=b_i$ for at least $k+1$ values of $i\}$ are equal almost surely.

An equivalent definition of $B(\lambda)$ is that it consists of those members of $D(\lambda)$ which, when the X_i's are ordered so that

$$P(X_i=\nu_i) = 1\ (0) \text{ for } i \le (>) r \text{ and}$$

$$b_{r+1} - a_{r+1} \le b_{r+2} - a_{r+2} \le \ldots \le b_n - a_n,$$

satisfy, for some $s=0,1,\ldots,$ or $n-1-r$,

$$\sum_{i=n-s+1}^{n} (b_i - a_i) < \lambda - \sum_{i=1}^{r} \nu_i - \sum_{i=r+1}^{n} a_i \le \sum_{i=r+1}^{r+s+1} (b_i - a_i).$$

It should be noted that $B(\lambda)$ contains all of the conjectured optimal strategies.

Theorem 4.1. $\max_{S\in B(\lambda)} P(S\ge\lambda) = \max_{k=0,1,\ldots,n-1}\left[1-P_k(\lambda)\right].$

We can summarize certain results from [1] to state

Lemma 4.5. To prove Theorem 4.1 it is sufficient to prove that whenever $S \in B(\lambda)$ satisfies the hypotheses of Lemma 4.4, there is an $S' \in B(\lambda)$ such that $X_i' = \nu_i$ for some i and $P(S'\ge\lambda) \ge P(S\ge\lambda)$.

Although the S''s we constructed in Lemma 4.4 are not members of $B(\lambda)$ they can easily be modified to be so in such a way that their probabilities of being at least λ will, at worst, not fall below the values implied by formulas (4.19) - (4.22), respectively. For example, in Case 1, first lower all mass of X_i' from $b_i - (b_1 - a_1)$ to a_i; then transfer enough mass from a_i to $b_i + (b_1-\nu_1)/k$ to restore the mean of X_i' to ν_i. The modifications for the other cases are similar.

There are two differences between this theorem and Theorem 5.1 of [1]. First, in the earlier theorem we restricted ourselves to $a_i = 0$, which we need not have done. Second, the earlier theorem is false as stated (this was pointed out by Martin Fox). To correct it one need only redefine more carefully what we called "$B(\nu_1,\ldots,\nu_n;\lambda)$". The correct definition is exactly that given in this section for $B(\lambda)$.

5. Large λ.

We shall prove the following

Theorem 5.1: There is a λ_0 such that $\lambda \geq \lambda_0$ implies

$$\psi(\lambda) = 1 - \prod_{i=1}^{n} (1- \nu_i/\lambda) = 1-P_0(\lambda) \tag{5.1}$$

which is attained only when $P(X_i = \lambda) = \nu_i/\lambda = 1-P(X_i=0)$ for each i.

The theorem will follow from three lemmas, the first of which was stated without proof in [1].

Lemma 5.1: $$\lim_{\lambda\to\infty} \lambda\ \psi(\lambda) = \sum_{i=1}^{n} \nu_i$$

Proof: For each $\lambda > \nu_1 +\ldots+ \nu_n$,

$$\lambda[1-P_0(\lambda)] \leq \lambda\psi(\lambda) < \sum_{i=1}^{n} \nu_i,$$

since the left side of the inequalities is attained by a member of C, and the second inequality is simply the Markov inequality. The lemma then follows, since

$$\lim_{\lambda\to\infty} \lambda[1-P_0(\lambda)] = \lim_{\lambda\to\infty} \lambda[1- \prod_{i=1}^{n} (1-\nu_i/\lambda)] = \sum_{i=1}^{n} \nu_i.$$

For each $\lambda > \nu_1 + \ldots + \nu_n$, we know from Lemma 2.1 that $\psi(\lambda)$ is attained by some member of $D(\lambda)$. Let $S(\lambda) = X_1(\lambda) + \ldots + X_n(\lambda)$ be such a member; let $a_i(\lambda)$ and $b_i(\lambda)$ be the lower and upper mass points, respectively, of $X_i(\lambda)$, and let

$$(5.2) \qquad p_i(\lambda) = P\big(X_i(\lambda) = b_i(\lambda)\big) = \big(\nu_i - a_i(\lambda)\big)/\big(b_i(\lambda) - a_i(\lambda)\big)$$

<u>Lemma 5.2</u>: Let λ_m, $m=1,2,\ldots$ be any sequence, increasing to ∞, such that the limits

$$\alpha_i = \lim_{m\to\infty} a_i(\lambda_m)$$

$$\beta_i = \lim_{m\to\infty} b_i(\lambda_m)/\lambda_m$$

exist for each i. Then $\alpha_i=0$ and $\beta_i=1$ for each i.

<u>Proof</u>: By Lemma 2.1, $b_i(\lambda_m) \le \lambda_m$, so $\beta_i \le 1$. Of course $0 \le \alpha_i \le \nu_i$. For m sufficiently large,

$$P\big(S(\lambda_m) \ge \lambda_m\big) \le \sum_{\{i:\beta_i=1\}} p_i(\lambda_m)$$

$$+ \sum_{\{j,k:0<\beta_j,\beta_k<1\}} p_j(\lambda_m)\, p_k(\lambda_m).$$

Substituting from (5.2) and applying Lemma 5.1, we obtain

$$\sum_{i=1}^{n} \nu_i = \lim_{m\to\infty} \lambda_m\, P\big(S(\lambda_m) \ge \lambda_m\big) \le \sum_{\{i:\beta_i=1\}} (\nu_i - \alpha_i),$$

which is true if and only if, for each i, $\alpha_i=0$ and $\beta_i=1$.

Lemma 5.3. Under the conditions of Lemma 5.2, for all sufficiently large m, $a_i(\lambda_m)=0$ and $b_i(\lambda_m)=\lambda_m$ for all i.

Proof: From Lemma 5.2 we know that, for sufficiently large m, and for each i,

$$(5.3) \qquad (\lambda_m - \sum_{j=1}^{n} a_j(\lambda_m))/2 < b_i(\lambda_m)-a_i(\lambda_m) \leq \lambda_m - \sum_{j=1}^{n} a_j(\lambda_m),$$

$$(5.4) \qquad a_i(\lambda_m) < \nu_i \; .$$

But Lemma 2.1 of [1] states that, for each i,

$$P(S(\lambda_m) = \lambda_m | X_i(\lambda_m) = b_i(\lambda_m)) > 0.$$

Hence, whenever (5.3) holds for each i, we must have, for each i,

$$(5.5) \qquad b_i(\lambda_m) - a_i(\lambda_m) = \lambda_m - \sum_{j=1}^{n} a_j(\lambda_m).$$

The final step is to apply Lemma 2.3 of [1] which states that if (5.5) holds for each i, we must have each $a_i(\lambda_m)$ equal to 0 or ν_i. But (5.4) holds for each i. Thus $a_i(\lambda_m) = 0$ for each i so, by (5.5), $b_i(\lambda_m) = \lambda_m$ for each i.

From Lemmas 5.2 and 5.3 we conclude that, for all λ sufficiently large, the only S in $D(\lambda)$ which attains $\psi(\lambda)$ -- hence the only S in C which does so -- is the one asserted in the Theorem.

If the means are equal, this S is a sum of independent, identically distributed (IID) random variables. Hence we have an immediate

Corollary 5.1. Among all $S = X_1 + \ldots + X_n$, with the X_i's IID, ≥ 0, with mean ν, and for all λ sufficiently large,

$$P(S \geq \lambda) \leq 1-(1- \nu/\lambda)^n$$

with equality holding if and only if $P(X_i = \lambda) = \nu/\lambda = 1-P(X_i=0)$.

How large is ''sufficiently large''? We don't know yet.

REFERENCES

[1] Samuels, S. M. (1966). On a Chebyshev-type inequality for sums of independent random variables. Ann. Math. Statist. 37 248-259.

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Unclassified
Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Purdue University	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE
More on a Chebyshev-Type Inequality for Sums of Independent Random Variables

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*
Technical Report, April, 1968

5. AUTHOR(S) *(Last name, first name, initial)*
Samuels, S. M.

6. REPORT DATE April, 1968	7a. TOTAL NO. OF PAGES 18	7b. NO. OF REFS 2

8a. CONTRACT OR GRANT NO. NONR 1100(26) b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Mimeograph Series 155 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. AVAILABILITY/LIMITATION NOTICES
Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Office of Naval Research Washington, D.C.

13. ABSTRACT

This paper deals with the same problem and the same conjectured solution to it which we considered in Samuels (Ann. of Math. Statistics, 1966).

In section 2, we restate the problem, the conjecture, and, in more concise form, some of the preliminary results of the earlier paper.

In section 3, we give a simpler proof of the conjecture for $n \leq 3$ and, for the first time, a proof for $n = 4$.

In section 4, we give what is essentially a simpler and, we hope, more illuminating proof of Theorem 5.1 of Samuels (Ann. of Math. Statistics, 1966).

In section 5, we prove that the conjecture is true for large λ.

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Unclassified
Security Classification