A Discussion on Brownian Motion of 1-Dimensional, Continuous Mechanical System in a Viscous Medium*

by

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I. INTRODUCTION

In 1954, Baxter [1] considered certain continuous, 1-dimensional mechanical systems undergoing Brownian motion. The physical systems were assumed to have the kinetic and potential energies in the forms

$$T = \frac{1}{2}\int_a^b r(x)\left(\frac{\partial\xi}{\partial t}\right)^2 dx$$

(1) $$V = \frac{\alpha}{2}\xi^2(a) + \frac{1}{2}\int_a^b \left\{p(x)\left(\frac{\partial\xi}{\partial x}\right)^2 + q(x)\xi^2\right\}dx + \frac{\beta}{2}\xi^2(b)$$

where $\alpha, \beta \geq 0$ are constants, $p(x)$ is a positive continuous differentiable function on $[a,b]$, $q(x)$ is a continuous non-negative function on $[a,b]$, and $r(x)$ is a positive continuous function on $[a,b]$. The system was assumed also to be immersed in a viscous medium and was subjected to certain type of random forces which will be given later. He did not publish his results and we will give a description of the methods he used and some of his results here. First, he gave a lemma which is in some sense a proper extension of the Kolmogorov consistency theory. The statement of the lemma is:

Let T be a bounded linear set and suppose that for $n = 1, 2, 3, \ldots$ and any choice of n points $t_1, t_2, \ldots, t_n$ of T there is given an n-variate distribution function $F_{t_1,\ldots,t_n}(\lambda_1,\ldots,\lambda_n)$. For any fixed integer k and points $s_1, s_2, \ldots, s_k$ of T let $G_{s_1,\ldots,s_k}(\lambda_1,\ldots,\lambda_k)$

be a (possibly) multiple-valued function whose values for any $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the values of k-variate distribution functions which are implied by some n-variate $(n \geq k)$ distribution function $F_{t_1,\ldots,t_n}(\lambda_1,\ldots,\lambda_n)$ whose parameter values include $s_1, s_2, \ldots, s_k$. If, for every positive integer k and every choice of k points $s_1, s_2, \ldots, s_k$ of T, the limit

$$\lim_{\substack{\max\limits_{t\in T}\min\limits_{t_i} |t-t_i| \to 0}} G_{s_1,\ldots,s_k}(\lambda_1,\ldots,\lambda_k)$$

exists uniformly in $\lambda_1, \lambda_2, \ldots, \lambda_k$, then the class of the limiting functions gives rise to a stochastic process.

Then he considered four types, and discussed only one type because of similarity, of density functions which are, crudely speaking, formed by using a Riemann sum approximation to V as the exponent of a Gaussian density function. The type he discussed is

$$(2) \quad \sqrt{\frac{|A_{n+1}|}{(2\pi)^{n+1}}} \int_{-\infty}^{\infty} \exp\left\{-\tfrac{1}{2}\left[\sum_{k=1}^{n+1} \{p(t_k)\frac{(\xi_k-\xi_{k-1})^2}{t_k-t_{k-1}} + q(t_k)\xi_k^2(t_k-t_{k-1})\} + \beta\,\xi_{n+1}^2\right]\right\} d\xi_{n+1}$$

where $\xi_0 = 0$ and A_{n+1} is the matrix in the quadratic form appeared in the integral. This corresponds to the case that the boundary condition $\xi(a) = 0$ for the system. There is a restriction to be placed so that the differential system corresponding to (2)

$$(3) \qquad \begin{cases} \frac{d}{dx}\left\{p(x)\,\frac{d\xi}{dx}\right\} - q(x)\xi = 0 \\ \text{B.C's (The corresponding boundary contitions for the physical system)} \end{cases}$$

has a Green's function $K(x,y)$ which can be separated into $u(x)v(y)(x \leq y)$ on account of the boundary conditions (which will be given later). Let k fixed points $(s_1, s_2, \ldots, s_k)$ on (a,b) be subset of $(t_1, t_2, \ldots t_n)$, where $t_j = a + j/(n+1)$, and let A_k^* be the matrix in the quadratic form appearing in the exponent of the implied density function, which is implied from (2), associated with parameters $s_1, s_2, \ldots, s_k$. If we set $(A_k^*)^{-1} = (b_{ij})$, then he proved that $b_{ij} = K(s_i, s_j) + O(1)$ as $n \to \infty$. Hence b_{ij} tends to $K(s_i, s_j)$ as n tends to infinity. This implies the convergence of the characteristic function of the implied distribution associated with $s_1, s_2, \ldots, s_k$. Hence the limit of the implied distribution associated with $s_1, s_2, \ldots, s_k$ exists as $n \to \infty$. And, by the lemma, the class of limiting distribution functions gives rise to a stochastic process which, obviously, has $K(x,y)$ as covariance function. But the stochastic process is defined only on the set of rational numbers in (0,1). The continuity of $K(x,y)$ in square $[a,b] \times [a,b]$ implies that the process can be defined on every parameter in $[a,b]$ in the sense of mean square convergent. The process is Markovian since $K(x,y) = u(x)v(y)(x \leq y)$ on account of the boundary conditions considered. (See [2]).

When applying the preceding result to the Brownian motion of continuous, 1-dimensional systems, he discretized the system so that it became a finite system of coupled harmonic oscillators type and satisfied the conditions of Wang and Uhlenbeck [3] which will be given later, then he applied the result of Wang and Uhlenbeck, the covariance function of the stationary state was then $D\ A_{n+1}^{-1}$ where D is a constant, hence the density function for $t_1, t_2, \ldots, t_n$ was the same type as (2), preceding result implies that the class of limiting functions, as $n \to \infty$, gives rise to a Markov process which has, now, $D\ K(x,y)$ as its covariance function. Hence the covariance function of the stationary state is D times the Green's function of (3).

In this paper, we will give a different approach to analyse the Brownian motion for the same continuous 1-dimensional mechanical system. First use the same method as Baxter used to analyse string vibration with fixed ends, the result justifies, in some sense, the main result of this paper. And we will point out the difficulties for higher dimensional case.

II. STRING VIBRATION WITH FIXED ENDS

Let us consider a uniformly flexible string with fixed end points 0 and 1. Assume its kinetic and potential energies are of the form

$$T = \frac{1}{2}\int_0^1 \left(\frac{\partial \xi}{\partial t}\right)^2 dx \tag{4}$$

$$V = \frac{1}{2}\int_0^1 \left(\frac{\partial \xi}{\partial x}\right)^2 dx \quad . \tag{5}$$

We assume also that the string is immersed in a viscous medium and is subjected to Gaussian random forces $F(x,t)$, $t \geq 0$, with

$$\begin{cases} \langle F(x,t)\rangle_{Av} = 0 \\ \langle F(x,t)\, F(y,s)\rangle_{Av} = 2(2\omega)\kappa\, T\, \delta(x-y)\, \delta(t-s) \end{cases} \tag{6}$$

where 2ω is the damping coefficient of the system, κ and T are constants which can be interpreted as Boltzman's constant and the temperature of the gas if the string is placed in a rarefield gas and the random forces are induced from the colisions of the gas molecules with the string.

Before we start to discuss this mechanical system, we like to mention some results for the coupled harmonic oscillator system treated by Wang and Uhlenbeck. In the electrical notation, the couple system is

$$\sum_{j=1}^{n} \left\{ L_{ij} \frac{d^2 y_j}{dt^2} + R_{ij} \frac{dy_j}{dt} + G_{ij} \, y_j \right\} = \sum_{j=1}^{n} E_{ij} \qquad i = 1,2,\ldots, n \quad .$$

where the random forces $\{E_{ij}\}$ form a Gaussian process satisfying

$$(7) \qquad \begin{cases} \langle E_{ij} \rangle_{Av} = 0 \\ \langle E_{ij}(t) \, E_{ji}(s) \rangle_{Av} = 2R_{ij} \, \kappa \, T \, \delta(t-s) \\ \langle E_{ij}(t) \, E_{ij}(s) \rangle_{Av} = 2|R_{ij}| \kappa \, T \, \delta(t-s) \\ \langle E_{ij} \, E_{k\ell} \rangle_{Av} = 0 \text{ if either } k \neq i, \, k \neq j \text{ or } \ell \neq i, \, \ell \neq j \quad . \end{cases}$$

where (L_{ij}) is a non-singular matrix and the roots of

$$\text{Det } (\{L_{ij}\lambda^2 + R_{ij}\lambda + G_{ij}\}) = 0$$

have negative real parts. Then the $2n$ variables $(y_1,\ldots,y_n;\dot{y}_1,\ldots,\dot{y}_n)$ form a Gaussian Markov process whose transition probability density function is the solution of the Fokker-Planck equation.

$$\frac{\partial P}{\partial t} = - \sum_{j=1}^{2n} \frac{\partial}{\partial \xi_j} (A_j P) + \frac{1}{2} \sum_{j,k=1}^{2n} \frac{\partial^2}{\partial \xi_j \partial \xi_k} (D_{jk} P)$$

$$\xi_j = y_j, \quad \xi_{n+j} = \dot{y}_j = \frac{dy_j}{dt} \qquad 1 \leq j \leq n \quad .$$

in which

$$A_j = \sum_{k=1}^{2n} a_{jk}\, \xi_k \qquad 1 \leq j \leq 2n \quad .$$

$$A = (a_{ij}) = \begin{pmatrix} 0 & I \\ -L^{-1}G & -L^{-1}R \end{pmatrix} \qquad (2n \times 2n)$$

$$D = (D_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & 2\kappa TL^{-1}\, RL^{-1} \end{pmatrix} \qquad (2n \times 2n) \quad .$$

Let $\overline{\xi}_i(t)$ denote the mean of $\xi_i(t)$, $1 \leq i \leq 2n$ and C be the matrix which diagonalizes A. Then the mean vectors and covariance matrix are given by

$$\text{(8)} \qquad \begin{cases} X(t) = e^{At}\, X_o \\ B(t) = (b_{ij}(t)) = C^{-1} \cdot \mu \cdot (C^T)^{-1} \end{cases}$$

where $X(t) = \begin{pmatrix} \xi_1(t) \\ \vdots \\ \xi_{2n}(t) \end{pmatrix}$ and $X_o = \begin{pmatrix} \xi_1(o) \\ \vdots \\ \xi_{2n}(o) \end{pmatrix}$ is the vector of initial values.

$$b_{ij}(t) = \left\langle [\xi_i(t) - \overline{\xi}_i(t)]\, [\xi_j(t) - \overline{\xi}_j(t)] \right\rangle_{Av} \qquad 1 \leq i,j \leq 2n$$

and

$$\mu_{ij} = -\frac{\sigma_{ij}}{\lambda_i + \lambda_j}\, [1 - \exp(\lambda_i + \lambda_j)t] \qquad 1 \leq i,j \leq 2n \quad ,$$

in which $\sigma = (\sigma_{ij}) = C \cdot D \cdot C^T$ and λ_i's are the $2n$ eigenvalues for the matrix A.

Armed with above results, we are able to analyze the Brownian motion of the string. Partition first the interval $[0,1]$ into n equal subintervals $\{x_o = 0 < x_1 < \ldots < x_n = 1\}$. Then the "Riemann sum" approximation of (4) and (5) are given by

$$T' = \tfrac{1}{2} \sum_{j=1}^{n} \dot{\xi}_j^2 (x_j - x_{j-1}) \tag{9}$$

$$V' = \tfrac{1}{2} \sum_{j=1}^{n} \frac{(\xi_j - \xi_{j-1})^2}{x_j - x_{j-1}} \qquad \xi_o = \xi_n = 0 \ . \tag{10}$$

Apply the Hamilton's principle (see Appendix II), then

$\frac{\partial}{\partial \tau}\frac{\partial T'}{\partial \dot{\xi}_i} - \frac{\partial}{\partial \xi_i} (T' - V') = 0$, to (9) and (10), and by considering the damping and the external forces, we get the Lagrange's general equation of motion.

$$(x_i - x_{i-1})\ddot{\xi}_i + 2\omega(x_i - x_{i-1})\dot{\xi}_i + \frac{\xi_i - \xi_{i-1}}{x_i - x_{i-1}} + \frac{\xi_i - \xi_{i+1}}{x_{i+1} - x_i} = F_i(t)$$

or

$$\frac{1}{n}\ddot{\xi}_i + \frac{2\omega}{n}\dot{\xi}_i + n(2\xi_i - \xi_{i-1} - \xi_{i+1}) = F_i(t) \qquad 1 \le i \le n-1$$

where
$$F_i(t) = \int_{x_{i-1}}^{x_i} F(x,t)\, dt \quad .$$

From the properties of $F(x,t)$, we see that $\{F_i(t)\}_i$ are Gaussian random process satisfying (7) with $E_{ij}(t) = \delta_{ij} F_i(t)$, since

$$L = (L_{ij}) = \frac{1}{n} (\delta_{ij}) = \frac{1}{n} I \quad .$$

$$R = (R_{ij}) = \frac{2\omega}{n} (\delta_{ij}) = \frac{2\omega}{n} \cdot I$$

$$G = (G_{ij}) = \begin{pmatrix} 2n, & -n, & 0, \ldots & & \\ -n, & 2n, & -n, & 0, \ldots & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \ldots & 0, & -n, & 2n \end{pmatrix}$$

and

$$\langle F_i(t) \rangle_{Av} = \int_{x_{i-1}}^{x_i} \langle F(x,t) \rangle_{Av} dx = 0$$

$$\langle F_i(t)\, F_k(s) \rangle_{Av} = \int_{x_{i-1}}^{x_i} \int_{x_{k-1}}^{x_k} \langle F(x,t)\; F(y,s) \rangle_{Av}\, dx\, dy$$

$$= 4\omega\kappa T\; \delta(t-s) \int_{x_{i-1}}^{x_i} \int_{x_{k-1}}^{x_k} \delta(x-y)\; dx\; dy = 4\omega\kappa T \delta(t-s) \delta_{ik} \int_{x_{k-1}}^{x_k} dx$$

$$= 2\; (\frac{2\omega}{n})\; \kappa T\; \delta(t-s)\; \delta_{ik} \quad .$$

Hence, the method of Wang and Uhlenbeck applies, then $(\xi_1, \ldots, \xi_{n-1}; \dot{\xi}_1, \ldots \dot{\xi}_{n-1})$ forms a $2(n-1)$ dimensional Gaussian Markov process with mean vectors and covariance matrix in form of (8).

The eigenvalues for A are the roots of $\mathrm{Det}(a_{ij} - \lambda\delta_{ij}) = 0$, or of

$$\mathrm{Det}\ (L_{ij}\,\lambda^2 + R_{ij}\,\lambda + G_{ij}) = 0$$

$$= \begin{bmatrix} \frac{\lambda^2+2\omega\lambda+2n^2}{n}, & -n, & 0, & \cdots & \cdots \\ -n, & \frac{\lambda^2+2\omega\lambda+2n^2}{n}, & -n, & 0, & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0, & -n, & \frac{\lambda^2+2\omega\lambda+2n^2}{n} \end{bmatrix} .$$

Let $(\lambda^2+2\omega\lambda+2n^2) / n^2 = a$, then they become the roots of

$$D_{n-1} = \begin{vmatrix} a, & -1, & 0 & \ldots & \\ -1, & a, & -1, & 0, & \ldots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & \ldots & 0, & -1, & a \end{vmatrix} = 0 .$$

If we set $D_o = 1$, the it is easily checked that $\{D_k\}$ satisfies the equation $D_k = aD_{k-1} - D_{k-2}$ with $D_o = 1$, $D_1 = a$. In order to solve this equation, let $E\,D_k = D_{k+1}$, $k = 0, 1, 2, 3, \ldots$. Then we have the following difference equation.

$$(11) \qquad \begin{cases} (E^2 - aE + 1)\,D_k = 0 \\ D_o = 1,\ D_1 = a \end{cases} .$$

The solution of (11) is given by

$$D_k = p_1 r_1^k + p_2 r_2^k = \frac{r_1^{k+1} - r_2^{k+1}}{r_1 - r_2}$$

where r_1, r_2 are the solution of the quadratic equation $y^2 - ay + 1 = 0$ and p_1, p_2 satisfy the initial value

$$\begin{cases} p_1 + p_2 = 1 \\ p_1 r_1 + p_2 r_2 = a \end{cases} .$$

If r_1, r_2 are not real, then $r_1 \neq r_2$, this implies that $D_{n-1} = 0$ if and only if $r_1^n = r_2^n$, or the same, if and only if

$$\frac{a + \sqrt{a^2-4}}{a - \sqrt{a^2-4}} = e^{\frac{2k\pi}{n} i} \qquad 1 \leq k \leq n-1 \quad .$$

We find that $a = 2 \cos \frac{k\pi}{n}$ for $k = 1, 2, \ldots, n-1$. From the definition of a, the eigenvalues are then given by

$$\lambda = -\omega \pm \sqrt{\omega^2 - 2n^2(1 - \cos \frac{k\pi}{n})} \qquad 1 \leq k \leq n-1 \quad .$$

Let

$$\begin{cases} \lambda_j = \tilde{\lambda}_{n-1+j} = -\omega + \sqrt{\omega^2 - 2n^2(1-\cos \frac{j\pi}{n})} \\ \lambda_{n-1+j} = \tilde{\lambda}_j = -\omega - \sqrt{\omega^2 - 2n^2(1-\cos \frac{j\pi}{n})} \qquad 1 \leq j \leq n-1 \quad . \end{cases}$$

The best way to find the matrix C which diagonalizes A is to find the matrix which is formed by using the components of the eigenvectors corresponds to A as its row vectors. Now, to find the eigenvectors for λ_j's, we start from the equation

$$0 = X(A - \lambda_j I) = (x_1, \ldots, x_{2n-2}) \left[\begin{array}{c|c} -\lambda_j I & I \\ \hline \begin{array}{l} -2n^2, n^2, 0, \; \ldots\ldots \\ n^2, -2n^2, n^2, 0 \; \ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots \; 0, n^2, -2n^2 \end{array} & -(2\omega+\lambda_j)I \end{array}\right]$$

Multiply by a non-singular matrix $\begin{bmatrix} I & 0 \\ \lambda_j I & I \end{bmatrix}$ from the right, and note that $\lambda_j^2 + 2\omega\lambda_j + 2n^2 = 2n^2 \cos(j\pi/n)$, $\lambda_j + \widetilde{\lambda}_j = -2\omega$, then we have

$$0 = (x_1, \ldots, x_{2n-2}) \left[\begin{array}{c|c} 0 & I \\ \hline \begin{array}{l} -2n^2 \cos(j\pi/n), n^2, 0, \; \ldots \\ n^2, -2n^2\cos(j\pi/n), n^2, 0, \; \ldots \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots \; 0, n^2, -2n^2 \cos\frac{j\pi}{n} \end{array} & \widetilde{\lambda}_j I \end{array}\right].$$

This is equivalent to the following equations

$$(12) \qquad \begin{cases} x_{n+1} = 2\cos(j\pi/n)\, x_n \\ x_{n+k} = 2\cos(j\pi/n)\, x_{n-1+k} - x_{n-2+k} \qquad k = 2,3,\ldots,n-2. \end{cases}$$

$$(13) \qquad x_{2n-3} = 2\cos(j\pi/n)\, x_{2n-2}$$

$$(14) \qquad x_j = -\tilde{\lambda}_j\, x_{n-1+j} \qquad j = 1, 2, \ldots, n-1 .$$

If we set $x_n = \sin\frac{j\pi}{n}$, then (12) becomes the difference equations

$$(15) \begin{cases} (D^2 - 2\cos\frac{j\pi}{n} D + 1)\, x_{n+k} = 0 & k = 2, 3, \ldots, n-2 \\ \text{initial values} \quad x_n = \sin\frac{j\pi}{n},\; x_{n+1} = \sin\frac{j\pi}{n}\, 2\cos\frac{j\pi}{n} \end{cases}$$

With $a = 2\cos\frac{j\pi}{n}$, (15) is the same as (11) except that both the initial values of (15) have a factor $\sin(j\pi/n)$. So, if we let $q_1, q_2 = \cos\frac{j\pi}{n} \pm i\sin\frac{j\pi}{n}$ be the roots of $y^2 - 2\cos(j\pi/n)\, y + 1 = 0$. Then the solutions are given by

$$x_{n+k} = \frac{q_1^{k+1} - q_2^{k+1}}{q_1 - q_2}\sin\frac{j\pi}{n} = \frac{e^{\frac{(k+1)j\pi}{n}i} - e^{-\frac{(k+1)j\pi}{n}i}}{e^{\frac{j\pi i}{n}} - e^{-\frac{j\pi}{n}i}} \cdot \sin\frac{j\pi}{n}$$

$$= \sin(k+1)\, j\pi/n \qquad k = 2, 3, \ldots, n-2 .$$

One can check easily that the above solutions satisfy also (13) as it should be. The eigenvector corresponding to λ_j is then, by (14)

$$\left(-\tilde{\lambda}_j \sin\frac{j\pi}{n}, \ldots, -\tilde{\lambda}_j \sin\frac{(n-1)j\pi}{n}, \sin\frac{j\pi}{n}, \ldots, \sin\frac{(n-1)j\pi}{n}\right) .$$

The matrix C which diagonalizes A is

$$C = (c_{ij}) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

where

$$C_{12} = C_{22} = (\sin\frac{ij\pi}{n})$$

$$= \begin{bmatrix} \sin\frac{\pi}{n} & \sin\frac{2\pi}{n} & \cdots & \sin\frac{(n-1)\pi}{n} \\ \sin\frac{2\pi}{n} & \sin\frac{4\pi}{n} & \cdots & \sin\frac{2(n-1)\pi}{n} \\ \cdots & \cdots & \cdots & \cdots \\ \sin\frac{(n-1)\pi}{n} & \sin\frac{2(n-1)\pi}{n} & \cdots & \sin\frac{(n-1)^2\pi}{n} \end{bmatrix}$$

and

$$C_{11} = (-\tilde{\lambda}_i \delta_{ij})\, C_{22}, \quad C_{21} = (-\lambda_i \delta_{ij})\, C_{22} \quad .$$

In order to find the inverse of C, we use the formula $\Sigma_{k-1}^{m} \cos kx = \sin(\frac{mx}{2}) \cos(\frac{m+1}{2}x) / \sin\frac{x}{2}$ to get

$$\sum_{k=1}^{n-1} \sin\frac{kj\pi}{n} \sin\frac{ki\pi}{n} = \frac{n}{2}\,\delta_{ij} \quad . \tag{16}$$

Then, C^{-1} is easily to find to be

$$C^{-1} = (c_{ij}^{-1}) = \frac{2}{n} \begin{bmatrix} C'_{11} & C'_{12} \\ C'_{21} & C'_{22} \end{bmatrix}$$

where

$$C'_{11} = C_{22}((\lambda_i - \tilde{\lambda}_i)^{-1}\delta_{ij})$$

$$C'_{12} = C_{22}((\tilde{\lambda}_i - \lambda_i)^{-1}\delta_{ij})$$

$$C'_{21} = C_{22}((\frac{\lambda_i}{\lambda_i - \tilde{\lambda}_i})\delta_{ij})$$

$$C'_{22} = C_{22}((\frac{\tilde{\lambda}_i}{\tilde{\lambda}_i - \lambda_i})\delta_{ij}) \quad .$$

The matrix $\sigma = (\sigma_{ij})$ is, by using (16)

$$\sigma = C\ D\ C^T = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4\omega\kappa Tn\ I \end{bmatrix} \begin{bmatrix} C_{11}^T & C_{21}^T \\ C_{12}^T & C_{22}^T \end{bmatrix}$$

$$= 4\omega\kappa Tn \begin{bmatrix} C_{12} \cdot C_{12}^T & C_{12} \cdot C_{22}^T \\ C_{22} \cdot C_{12}^T & C_{22} \cdot C_{22}^T \end{bmatrix} = 2\omega\kappa T\ n^2 \begin{bmatrix} I & I \\ I & I \end{bmatrix} \quad .$$

Hence $\sigma_{ij} \neq 0$ iff $i = j$ or $i = n-1+j$ or $i+n-1=j$ iff $\mu_{ij} \neq 0$. From (8), we have

$$b_{ij}(t) = \sum_{r,s=1}^{2n-2} c_{ir}^{-1}\, \mu_{rs} (c^{-1})_{sj}^T = \sum_{r,s=1}^{2n-2} c_{ir}^{-1}\, \mu_{rs}\, c_{js}^{-1}$$

where c_{rs}^{-1} is the elements in C^{-1} .

For $1 \leq i, j \leq n-1$, note that $x_i = i/n$, $x_j = j/n$.

$$(17)\qquad b_{ij}(t) = 8\omega\kappa T \sum_{r=1}^{n-1} \left\{ \frac{1}{(\lambda_r-\tilde{\lambda}_r)^2}\,\frac{1}{2\lambda_r}\,(1-e^{2\lambda_r t}) + \frac{1}{(\lambda_r-\tilde{\lambda}_r)^2}\,\frac{-1}{2\tilde{\lambda}_r}\,(1-e^{2\tilde{\lambda}_r t}) \right.$$

$$\left. + 2\,\frac{-1}{(\lambda_r-\tilde{\lambda}_r)^2}\,\frac{-1}{\lambda_r+\tilde{\lambda}_r}\,[1-e^{(\lambda_r+\tilde{\lambda}_r)t}] \right\} \sin\frac{ri\pi}{n}\,\sin\frac{rj\pi}{n}$$

$$= 8\omega\kappa T \sum_{r=1}^{n-1} \left\{ \frac{1}{2(\lambda_r-\tilde{\lambda}_r)^2\lambda_r\tilde{\lambda}_r}(\tilde{\lambda}_r\,e^{2\lambda_r t} + \lambda_r\,e^{2\tilde{\lambda}_r t} - \lambda_r - \tilde{\lambda}_r) \right.$$

$$\left. - \frac{1}{\omega(\lambda_r-\tilde{\lambda}_r)^2}(1-e^{-2\omega t}) \right\} \sin rx_i\pi \quad \sin rx_j\pi$$

$$= \omega\kappa T \sum_{r=1}^{n-1} \left\{ \frac{1}{4n^2(1-\cos\frac{r\pi}{n})\,[n^2-2n^2(1-\cos\frac{r\pi}{n})]} \Big[2\omega \right.$$

$$- \omega\, e^{-2\omega t}\left(e^{2\sqrt{\omega^2-2n^2[1-\cos(\frac{r\pi}{n})]}\,t} + e^{-2\sqrt{\omega^2-2n^2[1-\cos(\frac{r\pi}{n})]}\,t}\right)$$

$$- \sqrt{\omega^2-2n^2(1-\cos\tfrac{r\pi}{n})}\; e^{-2\omega t}\;\Big(e^{2\sqrt{\omega^2-2n^2(1-\cos\frac{r\pi}{n})}\;t}$$

$$\left. - e^{-2\sqrt{\omega^2-2n^2(1-\cos\frac{r\pi}{n})}\,t}\Big)\Big] - \frac{1-e^{-2\omega t}}{\omega[\omega^2-2n^2(1-\cos\frac{r\pi}{n})]} \right\}$$

$$\cdot\quad 2\sin rx_i\pi\,\sin rx_j\pi \quad .$$

$$(18)\qquad b_{i,n-1+j}(t) = \omega\kappa T \sum_{r=1}^{n-1} \left\{ \frac{e^{2\sqrt{\omega^2-2n^2(1-\cos\frac{r\pi}{n})}\,t} - 2 + e^{-2\sqrt{\omega^2-2n^2(1-\cos\frac{r\pi}{n})}\,t}}{2[\omega^2-2n^2(1-\cos\frac{r\pi}{n})]}\, e^{-2\omega t} \right\}$$

$$\times\ 2\sin rx_i\pi\,\sin rx_j\pi$$

$$(19)\qquad b_{n-1+i,n-1+j}(t) = \omega\kappa T \sum_{r=1}^{n-1} \left\{ \frac{1}{\omega} + \frac{2n^2(1-\cos\frac{r\pi}{n})\, e^{-2\omega t}}{[\omega^2-2n^2(1-\cos\frac{r\pi}{n})]\omega} \right.$$

$$- \frac{\omega\, e^{-2\omega t}\left(e^{2\sqrt{\omega^2-2n^2(1-\cos\frac{r\pi}{n})}\,t} + e^{-2\sqrt{\omega^2-2n^2(1-\cos\frac{r\pi}{n})}\,t}\right)}{2[\omega^2-2n^2(1-\cos\frac{r\pi}{n})]}$$

$$\left. + \frac{\sqrt{\omega^2-2n^2(1-\cos\frac{r\pi}{n})}\; e^{-2\omega t}\left(e^{2\sqrt{\omega^2-2n^2(1-\cos\frac{r\pi}{n})}\,t} - e^{-2\sqrt{\omega^2-2n^2(1-\cos\frac{r\pi}{n})}\,t}\right)}{2[\omega^2-2n^2(1-\cos\frac{r\pi}{n})]} \right\}$$

$$\times\ 2 \sin rx_i\pi \ \sin rx_j\pi \quad .$$

The above covariances are corresponding to $x_i = i/n$, $x_j = j/n$. Since we want to pass n to infinity, let us fix $i/n = x$, $j/n = y$. We shall discuss (17) first. For convenience, we assume $\omega^2 < \pi^2$ and set $c_n = \sqrt{n^2\pi^2 - \omega^2}$, then we claim that the limit of $b_{ij}(t)$, as $n \to \infty$, has the form

$$(20)\qquad b(x,y;t) = \kappa T \sum_{r=1}^{\infty} \left\{ \frac{1}{r^2\pi^2} - \frac{e^{-2\omega t}}{c_r^2} + \frac{\omega^2\, e^{-2\omega t}\cos 2c_r t}{c_r^2\, r^2\pi^2} \right.$$

$$\left. - \frac{\omega\, c_r\, e^{-2\omega t} \sin 2\, c_r t}{c_r^2\, r^2\pi^2} \right\}\ 2 \sin rx\pi \ \sin ry\pi \quad .$$

To show this, let N be any large integer, $n \geq N$. It is trivial that for $r \leq N$, the r-th term in (17) converges to rth term in (20). We need only to show that the summation of terms for $r \geq N$ in (17) has order $o(1)$ uniformly in n. If this is the case, then (17) tends to (20) as n tends to infinity. We note that each term appearing in (17) is bounded, say by M, uniformly in n and that

$$\sum_{r=N}^{\infty} \frac{1}{r^2\pi^2} \sim \sum_{r=N}^{\infty} \frac{1}{c_r^2} \sim o(1) \quad ;$$

hence the summation over $N \leq r \leq n$ has order $o(1)$ uniformly in n. This implies that (17) tends to (20) as $n \to \infty$. From the above procedure, we see that for each pair of rational parameters (x,y), the corresponding limiting covariance $b(x,y;t)$ is independent of n except that x,y are integral multiple of $1/n$ so that the covariance can be defined at (x,y) for each n, as $n \to \infty$. Hence the limiting covariance $b(x,y;t)$ is well defined at each pair of rational parameters. We note also that the series (20) is dominated by some absolutely convergent series which has order equivalent to $\Sigma\frac{1}{n^2}$. Now since each term in (20) is continuous with respect to x,y, series (20) is continuous for $(x,y) \in [0,1] \times [0,1]$, hence we can extend the limiting covariance function to each pair of $(x,y) \in [0,1] \times [0,1]$, ([4]).

If we let $t \to \infty$, then the limiting covariance becomes the covariance of the stationary state. It appears to be

$$\kappa T \sum_{r=1}^{\infty} \frac{2 \sin rx\pi \sin ry\pi}{r^2 \pi^2} \quad . \tag{21}$$

Let us consider now the Strum-Liouville Boundary Valued Problem

$$(22)\qquad \begin{cases} \xi_{xx} + \lambda^2 \xi = 0 \\ \xi(o) = \xi(1) = 0 \end{cases}$$

which is normally associated with the classical problem of the vibrating string. We see that the eigenvalues and eigenfunctions are

$$\begin{cases} \lambda_n^2 = n^2 \pi^2 \\ \varphi_n = \sqrt{2} \sin n\pi x \qquad n = 1, 2, \ldots \end{cases} .$$

Applying Mercer's theory (See Appendix I) [5], we have then

$$\sum_{r=1}^{\infty} \frac{2 \sin r\pi x \sin r\pi y}{r^2 \pi^2} = \sum_{r=1}^{\infty} \frac{\varphi_n(x)\, \varphi_n(y)}{\lambda_n^2} = K(x,y)$$

where $K(x,y)$ is the Green's function for (22). Hence the covariance function of the stationary state is κT times the Green's function of the differential system corresponding to (4) and (5). This coincides with the result of Baxter.

Now let $d_{i,j}(t) = b_{i,n-1+j}(t), e_{ij}(t)=b_{n-1+i,n-1+j}(t)$, then, as $n \to \infty$ $d_{ij}(t)$, $e_{ij}(t)$ tend to, say, $d(x,y;t)$, $e(x,y;t)$ respectively, where

$$(23)\qquad d(x,y;t) = \kappa T \sum_{r=1}^{\infty} \frac{1-\cos 2 c_r t}{c_r^2}\, \omega\, e^{-2\omega t}\, 2 \sin rx\pi \sin ry\pi$$

$$(24)\qquad e(x,y;t) = \kappa T \sum_{r=1}^{\infty} \left\{ \frac{\omega^2 \cos 2c_r t + \omega c_r \sin 2c_r t - \omega^2}{c_r^2} e^{-2\omega t} + (1-e^{-2\omega t}) \right\}$$

$$\times\ 2 \sin rx\pi \sin ry\pi$$

$$= \kappa T \sum_{r=1}^{\infty} \frac{\omega^2 \cos 2c_r t + \omega c_r \sin 2c_r t - \omega^2}{c_r^2} e^{-2\omega t} 2 \sin rx\pi \sin ry\pi$$

$$+ \kappa T (1-e^{-2\omega t})\ \delta(x-y) \quad .$$

As we treated for $b(x,y;t)$, we can extend the definition of $d(x,y;t)$ from pair of rational numbers to pair of real numbers in $[0,1] \times [0,1]$. We see that $d(x,y;t) \to 0$ when $t \to \infty$, this means that the correlation between velocity at point x and displacement at y falls off exponentially. Hence the velocity and the displacement become uncorrelated as $t \to \infty$. As to $e(x,y;t)$, we know immediately that it does not converge, but we can regard it as a generalized function and then extend it in the natural way to all pair of real numbers $(x,y) \in [0,1] \times [0,1]$.

The initial values $\xi_1(o), \ldots, \xi_{2n-2}(o)$ in (8) are defined by $\xi_j(o) = \xi(x_j, o)$ and $\xi_{n-1+j}(o) = \xi_t(x_j, o)$ for $1 \le j \le n-1$, where $\xi(x,o)$ and $\xi_t(x,o)$ are the initial displacement and initial velocity of the string. The vector $X(t) = e^{At} X_o$ is a vector whose components are function of t which are not random functions, hence $X(t)$ coincides with its mean vector $\overline{X}(t)$. We note that the mean vector $X(t)$ depends only on the systematic part of the system and tends to 0 exponentially as $t \to \infty$ due to the damping of the system. (In fact, all the eigenvalues of A has negative real parts implies the conclusion).

III. GENERAL APPROACH

In this section we shall give a different approach, apply it to the general problem, and the precise form for the position and velocity functions will be given. We restrict ourself to the physical system which has constant density and it is no loss of generality to assume that the density equals 1. The kinetic and potential energies are then

$$T = \frac{1}{2}\int_0^{\ell} \left(\frac{\partial \xi}{\partial t}\right)^2 dx \tag{25}$$

$$V = \frac{\alpha}{2}\xi^2(o) + \frac{1}{2}\int_0^{\ell} \{p(x)\left(\frac{\partial \xi}{\partial x}\right)^2 + q(x)\xi^2\}\, dx + \frac{\beta}{2}\xi^2(\ell) \tag{26}$$

where α, β, $p(x)$, $q(x)$ are the same as in introduction, and the system is assumed to be immersed in a viscous medium with damping coefficient 2ω and is subjected to random force $F(x,t)$ which is Gaussian and satisfies condition(6).

If we apply variation principle to (25), (26) and considering the damping together with the external force, then we have the general equation of the motion.

$$\begin{cases} \xi_{tt} = \{p(x)\xi_x\}_x - q(x)\xi - 2\omega\xi_t + F(x,t) \\ \text{B.C. and I.C.} \end{cases} \tag{27}$$

where B.C. is the boundary conditions from one of (28), and I.C. is the initial condition for $\xi(x,t)$, say $\xi(x,o)$ and $\xi_t(x,o)$.

$$(28)\quad \begin{cases} \text{(a)}\quad \xi(o,t) = 0 \quad \xi(\ell,t) = 0 \\ \text{(b)}\quad \xi(o,t) = 0 \quad \beta\xi(\ell,t) + p(\ell)\xi_x(\ell,t) = 0 \\ \text{(c)}\quad \alpha\xi(o,t) - p(o)\xi_x(o,t) = 0, \quad \xi(\ell,t) = 0 \\ \text{(d)}\quad \alpha\xi(o,t) - p(o)\xi_x(o,t) = 0, \quad \beta\xi(\ell,t) + p(\ell)\xi_x(\ell,t) = 0 \quad . \end{cases}$$

Let us consider first the problem

$$(29)\quad \begin{cases} u_{tt} = \{p(x)\ u_x\}_x - q(x)\ u - 2\omega\ u_t \\ \text{B. C.} \end{cases}$$

Separate the time and coorinate variables $u(x,t) = q(t)\ v(x)$, then (29) becomes

$$\begin{cases} \dfrac{\ddot{q}(t) - 2\omega\dot{q}(t)}{q(t)} = \dfrac{\{p(x)\ v'(x)\}' - q(x)\ v(x)}{v(x)} \\ (\text{B. C.})' \quad . \end{cases}$$

The ratio is known to be non-positive, say $-\lambda^2$ (see [5]). We have

$$(30)\quad \begin{cases} \{p(x)\ v'(x)\} - q(x)\ v(x) + \lambda^2 v(x) = 0 \\ (\text{B. C.})' \end{cases}$$

where $(B.C.)'$ is the corresponding boundary condition from one of the following

$$(a') \quad v(o) = 0, \quad v(\ell) = 0$$

$$(b') \quad v(o) = 0, \quad \beta v(\ell) + p(\ell)v'(\ell) = 0$$

$$(c') \quad \alpha v(o) - p(o)\, v'(o) = 0, \quad v(\ell) = 0$$

$$(d') \quad \alpha v(o) - p(o)\, v'(o) = 0, \quad \beta v(\ell) + p(\ell)v'(\ell) = 0 \quad .$$

This is one of the well known type of Strum-Liouville boundary problem. Of course, we are going to assume that $\lambda = 0$ is not an eigenvalue so that the Green's function exists. We note that the assumption on $p(x)$, $q(x)$ are so nice that the Green's function is square summerable and the Mercer's Theory applies, also Green's function can be separated into $u(x)\, v(y)\ (x \leq y)$ on account of the boundary conditions considered. Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues and $\varphi_1, \varphi_2, \ldots$ be the corresponding complete normalized eigenfunctions.

Consider sequence of independent Gaussian random functions $\{A_n(t)\}_n$, which are identically distributed for each fixed t, with zero means and

$$E\{A_n(t)\ A_m(s)\} = 2(2\omega)\kappa T\ \delta_{nm}\ \delta(t-s) \quad .$$

Then the covariance function of $\sum A_n(t)\ \varphi_n(x)$ has the form

$$E\left\{\left[\sum_{n=1}^{\infty} A_n(t)\ \varphi_n(x)\right]\left[\sum_{n=1}^{\infty} A_m(s)\ \varphi_m(y)\right]\right\} = \sum_{n=1}^{\infty} \varphi_n(x)\varphi_n(y)\ E\{A_n(t)\ A_n(s)\}$$

$$= 4\omega\kappa T\ \delta(t-s) \sum_{n=1}^{\infty} \varphi_n(x)\ \varphi_n(y) = 4\omega\kappa T\ \delta(t-s)\delta(x-y) = E\{F(x,t)F(y,s)\}$$

$$E\left(\sum_{n=1}^{\infty} A_n(t)\ \varphi_n(x) \right) = 0 \quad .$$

Hence $\Sigma A_n(t)\ \varphi_n(x)$ satisfies (3), and since it is Gaussian, we can express

$$F(x,t) = \sum_{1}^{\infty} A_n(t)\ \varphi_n(x) \quad .$$

In order to solve (21), substitute $u(x,t) = \Sigma\, b_n(t)\ \varphi_n(x)$ into (29), note that since we are considering continuous physical system, $\xi(x,o)$, $\xi_t(x,o)$ are assumed to be continuous, we can express $\xi(x,o) = \Sigma S(o)\varphi_n(x)$ and $\xi_t(x,o) = \Sigma\dot{S}(o)\ \varphi_n(x)$. We have

$$\sum_{n=1}^{\infty} \ddot{b}_n(t)\ \varphi_n(x) = \sum_{n=1}^{\infty} b_n(t)\ \{p(x)\ \varphi_n'(x)\}' - \sum_{n=1}^{\infty} b_n(t)\ q(x)\ \varphi_n(x)$$

$$- 2\omega \sum_{n=1}^{\infty} \dot{b}_n(t)\ \varphi_n(x) + \sum_{n=1}^{\infty} A_n(t)\ \varphi_n(x) \quad .$$

Then we get a sequence of differential equations

$$\ddot{b}_n(t)\ \varphi_n(x) = b_n(t)\{p(x)\varphi_n'(x)\}' - b_n(t)q(x)\varphi_n(x) - 2\omega\dot{b}_n(t)\varphi_n(x) + A_n(t)\varphi_n(x)$$

or

$$\frac{\ddot{b}_n(t)+2\omega\dot{b}_n(t)-A_n(t)}{b_n(t)} = \frac{\{p(x)\varphi_n'(x)\}'-q(x)\varphi_n(x)}{\varphi_n(x)} \qquad n = 1,2,\ldots \quad .$$

Since $\{\varphi_n(x)\}$ are solutions of (30). The right hand side is equal to $-\lambda_n^2$, so we have

$$(31) \qquad \begin{cases} \ddot{b}_n(t) + 2\omega\dot{b}_n(t) + \lambda_n^2\, b_n(t) = A_n(t) \\ b_n(o) = s(o), \quad \dot{b}_n(o) = \dot{s}_n(o) \end{cases} \quad .$$

Then $(b_n(t), \dot{b}_n(t))$ forms a Markov Gaussian process with (see [3])

$$(32) \quad \begin{cases} \left\langle \dot{b}_n(t) \right\rangle_{Av} = \dfrac{\dot{s}_n(o)}{c_n}\, e^{-\omega t}\, (c_n \cos c_n t - \omega \sin c_n t) - \dfrac{\lambda_n^2}{c_n}\, s_n(o) e^{-\omega t} \sin c_n t \\ \left\langle b_n(t) \right\rangle_{Av} = \dfrac{\dot{s}_n(o)}{c_n}\, e^{-\omega t} \sin c_n t + \dfrac{s_n(o)}{c_n}\, e^{-\omega t}(c_n \cos c_n t + \omega \sin c_n t) \\ \left\langle (\dot{b}_n(t)-\overline{\dot{b}_n(t)})^2 \right\rangle_{A_v} = \kappa T\{1 - \dfrac{e^{-2\omega t}}{c_n^2}\, (c_n^2+2\omega^2 \sin^2 c_n t - 2\omega c_n \sin c_n t \cos c_n t)\} \\ \left\langle (b_n(t)-\overline{b_n(t)})^2 \right\rangle_{Av} = \dfrac{\kappa T}{\lambda_n^2}\, \{1 - \dfrac{e^{-2\omega t}}{c_n^2}(c_n^2+2\omega^2 \sin^2 c_n t + 2\omega\, c_n \sin c_n t \cos c_n t)\} \\ \left\langle (\dot{b}_n(t)-\overline{\dot{b}_n(t)})(b_n(t)-\overline{b_n(t)}) \right\rangle_{Av} = \dfrac{2\omega\kappa T}{c_n^2}\, e^{-2\omega t} \sin^2 c_n t \end{cases}$$

where $c_n = \sqrt{\lambda_n^2-\omega^2}$, and $\overline{b_n(t)}$, $\overline{\dot{b}_n(t)}$ denote the means of $b_n(t)$ and $\dot{b}_n(t)$. The formula is given only for the under damped case; for the aperiodic case, let $c_n \to 0$, and for the over damped case, put $c_n = ic_n'$.

Under Wiener's assumption, some people like to rewrite (31) in the form of

$$d\,\dot{b}_n(t) + 2\omega \dot{b}_n(t)\,dt + \lambda_n^2\, b_n(t)\,dt = dB_n(t)$$

where, for each n, $B_n(t)$ is a Wiener process with variance $4\omega\kappa T$, and for different n, they are independent. The solution is given by the following stochastic integrals.

$$(33) \qquad b_n(t) = s_n(t) + \begin{cases} \dfrac{1}{c_n'} \displaystyle\int_0^t e^{-\omega(t-\eta)} \sinh c_n'(t-\eta)\,dB_n(\eta) & \text{if } \omega > \lambda_n \\ \displaystyle\int_0^t e^{-\omega(t-\eta)} (t-\eta)\, dB_n(\eta) & \omega = \lambda_n \\ \dfrac{1}{c_n} \displaystyle\int_0^t e^{-\omega(t-\eta)} \sin c_n(t-\eta)\, dB_n(\eta) & \omega < \lambda_n \end{cases}$$

$$\dot{b}_n(t) = \dot{s}_n(t) + \begin{cases} \displaystyle\int_0^t e^{-\omega(t-\eta)} \cosh c_n'(t-\eta)\,dB_n(\eta) - \frac{\omega}{c_n'} \int_0^t e^{-\omega(t-\eta)} \sinh c_n'(t-\eta)\,dB_n(\eta) \\ \displaystyle\int_0^t e^{-\omega(t-\eta)}\, dB_n(\eta) - \omega \int_0^t e^{-\omega(t-\eta)} (t-\eta)\, dB_n(\eta) \\ \displaystyle\int_0^t e^{-\omega(t-\eta)} \cos c_n(t-\eta)\,dB_n(\eta) - \frac{\omega}{c_n} \int_0^t e^{-\omega(t-\eta)} \sin c_n(t-\eta)\,dB_n(\eta) \end{cases}$$

for $\omega > \lambda_n$ or $\omega = \lambda_n$ or $\omega < \lambda_n$

where $c_n = \sqrt{\lambda_n^2 - \omega^2}$, $c_n' = ic_n$ and $s_n(t)$, $\dot{s}_n(t) = \frac{d}{dt} s_n(t)$ are solutions of

$$\ddot{y}(t) + 2\omega \dot{y}(t) + \lambda_n^2\, y(t) = 0$$

with the same initial values as in (31). Hence $s_n(t)$, $\dot{s}_n(t)$ are the systematic part and not random functions. It is well known that $s_n(t)$, $\dot{s}_n(t)$ tend to 0 in order $e^{-\beta t}$ as $t \to \infty$ for sufficiently large n, and for small n, they tend also to zero exponentially as $t \to \infty$. They will play no important roles.

Let us now investigate the process $\{b_n(t), \dot{b}_n(t)\}$. We will make calculation only for the case $\omega < \lambda_n$. Assume $t \leq s$, then,

(34) $$E\left\{\int_0^t e^{-\omega(t-\eta)} \sin c_n(t-\eta)dB_n(\eta) \int_0^s e^{-\omega(s-\zeta)}\sin c_n(s-\zeta)dB_n(\zeta)\right\}$$

$$= \int_0^t \int_0^s e^{-\omega(t+s-\eta-\zeta)}\sin c_n(t-\eta)\sin c_n(s-\zeta)dd'E(B_n(\eta)B_n(\zeta))$$

$$= 4\omega\kappa T \int_0^t e^{-\omega(t+s-2\eta)} \sin c_n(t-\eta) \sin c_n(s-\eta)d\eta$$

$$= 2\omega\kappa T \int_0^t e^{-\omega(t+s-2\eta)} \{\cos c_n(s-t) - \cos c_n(t+s-2\eta)\}d\eta$$

$$= \kappa T \cos c_n(s-t)(e^{-\omega(s-t)}-e^{-\omega(s+t)})-\omega\kappa T\frac{e^{-\omega\eta}(-\omega \cos c_n\eta+c_n\sin c_n\eta)\Big|_{s-t}^{s+t}}{\omega^2+c_n^2}$$

$$= \kappa T \cos c_n(s-t) \{e^{-\omega(s-t)}-e^{-\omega(s+t)}\}$$

$$+ \frac{\omega\kappa T}{\lambda_n^2} e^{-\omega(s+t)} \{\omega \cos c_n(s+t) - c_n \sin c_n(s+t)\}$$

$$- \frac{\omega\kappa T}{\lambda_n^2} e^{-\omega(s-t)} \{\omega \cos c_n(s-t) - c_n\sin c_n(s-t)\}$$

$$= \kappa T \frac{c_n^2}{\lambda_n^2} - \kappa T e^{-2\omega t}+ \frac{\omega\kappa T}{\lambda_n^2} e^{-2\omega t} \{\omega \cos 2c_nt-c_n\sin 2c_nt\} \quad \text{if} \quad s = t .$$

Also,

$$(35) \qquad E\left\{\int_0^t e^{-\omega(t-\eta)}\cos c_n(t-\eta)dB_n(\eta) \int_0^s e^{-\omega(s-\zeta)}\cos c_n(s-\zeta)dB_n(\zeta)\right\}$$

$$= \kappa T \cos c_n(s-t)\{e^{-\omega(s-t)}-e^{-\omega(s+t)}\}$$

$$+ \frac{\omega\kappa T}{\lambda_n^2} e^{-\omega(s-t)}\{\omega \cos c_n(s-t) - c_n \sin c_n(s-t)\}$$

$$- \frac{\omega\kappa T}{\lambda_n^2} e^{-\omega(s+t)}\{\omega \cos c_n(s+t) - c_n \sin c_n(s+t)\}$$

$$= \kappa T\frac{\lambda_n^2+\omega^2}{\lambda_n^2} - \kappa T e^{-2\omega t} - \frac{\omega\kappa T}{\lambda_n^2} e^{-2\omega t}\{\omega \cos 2c_n t - c_n \sin 2c_n t\} \quad \text{if} \quad s = t.$$

The following computations are similarly made.

$$(36) \qquad E\left\{\int_0^t e^{-\omega(t-\eta)}\sin c_n(t-\eta)dB_n(\eta) \int_0^s e^{-\omega(s-\zeta)}\cos c_n(s-\zeta)dB_n(\zeta)\right\}$$

$$= \frac{\omega\kappa T}{\lambda_n^2} e^{-\omega(s-t)}\{\omega \sin c_n(s-t) + c_n \cos c_n(s-t)\}$$

$$- \kappa T \sin c_n(s-t)\{e^{-\omega(s-t)}-e^{-\omega(s+t)}\}$$

$$- \frac{\omega\kappa T}{\lambda_n^2} e^{-\omega(s+t)}\{\omega \sin c_n(s+t) + c_n \cos c_n(s+t)\}$$

(37) $$E\left\{\int_o^t e^{-\omega(t-\eta)} \cos c_n(t-\eta)dB_n(\eta)\int_o^s e^{-\omega(s-\zeta)}\sin c_n(s-\zeta)dB_n(\zeta)\right\}$$

$$= \frac{\omega\kappa T}{\lambda_n^2} e^{-\omega(s-t)}\{\omega \sin c_n(s-t)+c_n\cos c_n(s-t)\}$$

$$+ \kappa T \sin c_n(s-t)\{e^{-\omega(s-t)}-e^{-\omega(s+t)}\}$$

$$- \frac{\omega\kappa T}{\lambda_n^2} e^{-\omega(s+t)}\{\omega \sin c_n(s+t) + c_n \cos c_n(s+t)\}$$

$$= \frac{\omega\, c_n\kappa T}{\lambda_n^2} - \frac{\omega\kappa T}{\lambda_n^2} e^{-2\omega t}(\omega \sin 2c_nt + c_n \cos 2c_nt) \quad \text{if} \quad s = t$$

(38) $$E\left\{\int_o^t e^{-\omega(t-\eta)} (t-\eta)dB_n(\eta) \int_o^s e^{-\omega(s-\zeta)}(s-\zeta)dB_n(\zeta)\right\}$$

$$= \frac{\kappa T}{\omega} e^{-\omega(s-t)}[(s-t)+ \frac{1}{\omega}] -2\kappa T e^{-\omega(s+t)}(st + \frac{s+t}{2\omega} + \frac{1}{2\omega^2})$$

(39) $$E\left\{\int_o^t e^{-\omega(t-\eta)}(t-\eta)dB_n(\eta) \int_o^s e^{-\omega(s-\zeta)} dB_n(\zeta)\right\}$$

$$= \frac{\kappa T}{\omega} e^{-\omega(s-t)} - 2\kappa T e^{-\omega(s+t)}(t + \frac{1}{2\omega})$$

(40) $$E\left\{\int_o^t e^{-\omega(t-\eta)}dB_n(\eta) \int_o^s e^{-\omega(s-\zeta)}(s-\zeta)dB_n(\zeta)\right\}$$

$$= 2\kappa T e^{-\omega(s-t)}[(s-t) + \frac{1}{2\omega}] - 2\kappa T e^{-\omega(s+t)}(s + \frac{1}{2\omega})$$

(41) $$E\left\{\int_0^t e^{-\omega(t-\eta)}dB_n(\eta)\int_0^s e^{-\omega(s-\zeta)}dB_n(\zeta)\right\}$$

$$= 2\kappa T\{e^{-\omega(s-t)}-e^{-\omega(s+t)}\}$$

(42) $$E\left\{\int_0^t e^{-\omega(t-\eta)}\sinh c_n'(t-\eta)dB_n(\eta)\int_0^s e^{-\omega(s-\zeta)}\sinh c_n'(s-\zeta)dB_n(\zeta)\right\}$$

$$= \frac{\omega\kappa T[e^{(s+t)(c_n'-\omega)}-e^{(s-t)(c_n'-\omega)}]}{2(c_n'-\omega)} + \frac{\omega\kappa T[e^{-(s-t)(c_n'+\omega)}-e^{-(s+t)(c_n'+\omega)}]}{2(c_n'+\omega)}$$

$$+ \kappa T\cosh c_n'(s-t)[e^{-\omega(s+t)}-e^{-\omega(s-t)}]$$

(43) $$E\left\{\int_0^t e^{-\omega(t-\eta)}\sinh c_n'(t-\eta)dB_n(\eta)\int_0^s e^{-\omega(s-\zeta)}\cosh c_n'(s-\zeta)dB_n(\zeta)\right\}$$

$$= \frac{\omega\kappa T[e^{(s+t)(c_n'-\omega)}-e^{(s-t)(c_n'-\omega)}]}{2(c_n'-\omega)} + \frac{\omega\kappa T[e^{-(s+t)(c_n'+\omega)}-e^{-(s-t)(c_n'+\omega)}]}{2(c_n'+\omega)}$$

$$+ \kappa T\sinh c_n(s-t)[e^{-\omega(s+t)}-e^{-\omega(s-t)}]$$

(44) $$E\left\{\int_0^s e^{-\omega(s-\eta)}\sinh c_n'(s-\eta)dB_n(\eta)\int_0^t e^{-\omega(t-\zeta)}\cosh c_n'(t-\zeta)dB_n(\zeta)\right\}$$

$$= \frac{\omega\kappa T}{2(c_n'-\omega)}\left[e^{(s+t)(c_n'-\omega)}-e^{(s-t)(c_n'-\omega)}\right]$$

$$+ \frac{\omega\kappa T}{2(c_n'+\omega)}\left[e^{(s+t)(c_n'+\omega)}-e^{(s-t)(c_n'+\omega)}\right]$$

$$- \kappa T\sinh c_n'(s-t)[e^{-\omega(s+t)}-e^{-\omega(s-t)}]$$

$$(45)\qquad E\left\{\int_0^t e^{-\omega(t-\eta)}\cosh c_n'(t-\eta)dB_n(\eta)\int_0^s e^{-\omega(s-\zeta)}\cosh c_n'(s-\zeta)dB_n(\zeta)\right\}$$

$$= \frac{\omega\kappa T}{2(c_n'-\omega)}[e^{(s+t)(c_n'-\omega)} - e^{(s-t)(c_n'-\omega)}]$$

$$+ \frac{\omega\kappa T}{2(c_n'+\omega)}[e^{-(s-t)(c_n'+\omega)} - e^{-(s+t)(c_n'+\omega)}]$$

$$+ \kappa T \cosh c_n'(s-t)\ [e^{-\omega(s-t)} - e^{-\omega(s+t)}] \quad .$$

We can check that the covariance function of $b_n(t)$, $\dot{b}_n(t)$ are the same as (32) if we use formulas (34) - (45) by putting $t = s$.

Consider now the solution of (27) $\xi(x,t) = \Sigma_{n=1}^{\infty} b_n(t)\ \varphi_n(x)$. Let $N_1 = \{n: \lambda_n^2 < \omega^2\}$, $N_2 = \{n: \lambda_n^2 = \omega\}$, $N_3 = \{n: \lambda_n^2 > \omega^2\}$ note that for $n \in N_3$,

$$s_n(t) = e^{-\omega t}[s_n(o)\ \cos c_n t + \frac{\dot{s}_n(o) + \omega s_n(o)}{c_n} \sin c_n(t)] \quad .$$

Hence $\Sigma_{N_3}\ s_n^2(t)$ converges and in fact continuous in t since $\Sigma\ s_n^2(o) < \infty$ $\Sigma\ \dot{s}_n^2(o) < \infty$. Also, from (34), we find that $\Sigma_{N_3}\ E(b_n(t) - s_n(t))^2 < \infty$ for all t, and in fact continuous in t.

Hence $\Sigma\ b_n^2(t) < \infty$, in probability term, almost everywhere. And $\xi(x,t)$ is then well defined by the series $\Sigma\ b_n(t)\ \varphi_n(x)$. The covariance function of $\xi(x,t)$

$$(46) \qquad E\{[\xi(x,t) - \overline{\xi(x,t)}]\,[\xi(y,t) - \overline{\xi(y,t)}]\}$$

$$= \sum_{n=1}^{\infty} E\{(b_n(t) - s_n(t))^2\}\ \varphi_n(x)\ \varphi_n(y)$$

$$= \sum_{n=1}^{\infty} \frac{\kappa T}{\lambda_n^2}\ \varphi_n(x)\ \varphi_n(y)$$

$$+ \sum_{N_1} \frac{\omega \kappa T}{c_n'^2} \left\{ \frac{e^{-2\omega t}}{\omega} - \frac{e^{-2(\omega - c_n')t}}{2(\omega - c_n')} - \frac{e^{-2(\omega + c_n')t}}{2(\omega + c_n')} \right\} \varphi_n(x)\ \varphi_n(y)$$

$$- \sum_{N_2} 2\kappa T\ e^{-2\omega t}\ \left(t^2 + \frac{t}{\omega} + \frac{1}{2\omega^2}\right) \varphi_n(x)\ \varphi_n(y)$$

$$- \sum_{N_3} \frac{\omega \kappa T\ e^{-2\omega t}}{c_n^2} \left\{ \frac{1}{\omega} - \frac{\omega \cos 2c_n t}{\lambda_n^2} + \frac{c_n \sin 2c_n t}{\lambda_n^2} \right\} \varphi_n(x)\ \varphi_n(y) \quad .$$

It is clear that the series converges uniformly in t. The covariance function of stationary state is then, by letting $t \to \infty$, and applying Mercer's theory,

$$\sum_{n=1}^{\infty} \frac{\kappa T}{\lambda_n^2}\ \varphi_n(x)\ \varphi_n(y) = \kappa T\ K(x,y)$$

where $K(x,y)$ is the Green's function of (30). Thus we give a different approach to the Baxter result. The Markov property of the process will be discussed in Appendix III.

Let us consider here some examples:

Example 1.

In section II, the covariance functions $b(x,y;t)$, $d(x,y;t)$ and $e(x,y;t)$ are defined by extending the definition from a pair of rational numbers to a pair of real numbers in $[0,1]$. In that case

$$\varphi_n(x) = \sqrt{2}\sin n\pi x\ , \quad \lambda_n^2 = n^2\pi^2 \qquad \text{for} \quad n = 1, 2, \ldots$$

hence

$$\xi(x,t) = \sum_{n=1}^{\infty} b_n(t)\ \sqrt{2}\sin n\pi x$$

where $b_n(t)$ is defined by (33). Using formulae (34) - (37), we see that $(\omega^2 < \pi^2)$

$$E\{[\xi(x,t) - \overline{\xi(x,t)}]\,[\xi(y,t) - \overline{\xi(y,t)}]\}$$

$$= \sum_{n=1}^{\infty} 2\sin n\pi x \sin n\pi y\, \frac{1}{c_n^2}\{\kappa T\, \frac{c_n^2}{n^2\pi^2} - \kappa T\, e^{-2\omega t}$$

$$+ \frac{\omega\kappa T}{n^2\pi^2}\, e^{-2\omega t}(\omega\cos 2c_n t - c_n \sin 2c_n t)\} = b(x,y;t)$$

$$E\{[\xi(x,t) - \overline{\xi(x,t)}]\,[\xi_t(y,t) - \overline{\xi_t(y,t)}]\}$$

$$= \kappa T \sum_{n=1}^{\infty} \{\frac{1}{c_n}[\frac{\omega\, c_n}{n^2\pi^2} - \frac{\omega}{n^2\pi^2}\, e^{-2\omega t}(\omega\sin 2c_n t + c_n\cos 2c_n t)$$

$$- \frac{\omega}{c_n^2}[\frac{c_n^2}{n^2\pi^2} - e^{-2\omega t} + \frac{\omega}{n^2\pi^2}\, e^{-2\omega t}(\omega\cos 2c_n t - c_n \sin 2c_n t)]\}$$

$$2\sin n\pi x \sin n\pi y \;=\; d(x,y;t)$$

and

$$E\{[\xi_t(x,t) - \overline{\xi_t(x,t)}]\,[\xi_t(y,t) - \overline{\xi_t(y,t)}]\}$$

$$= \kappa T \sum_{n=1}^{\infty} \Big\{ \frac{\omega^2}{c_n^2} \Big[\frac{c_n^2}{n^2\pi^2} - e^{-2\omega t} + \frac{\omega}{n^2\pi^2} e^{-2\omega t}(\omega \cos 2c_n t - c_n \sin 2c_n t) \Big]$$

$$+ \frac{n^2\pi^2+\omega^2}{n^2\pi^2} - e^{-2\omega t} - \frac{\omega}{n^2\pi^2} e^{-2\omega t}(\omega \cos 2c_n t - c_n \sin 2c_n t)$$

$$- \frac{2\omega}{c_n} \Big[\frac{\omega\, c_n}{n^2\pi^2} - \frac{\omega}{n^2\pi^2} e^{-2\omega t}(c_n \cos 2c_n t + \omega \sin 2c_n t) \Big] \Big\} 2 \sin n\pi x \sin n\pi y$$

$$= e(x,y;t) \quad .$$

Example 2:

Suppose a uniformly flexible string with constant mass density ρ having a constant tension τ is immersed in a viscous medium with damping coefficient 2ω and is subjected to the random forces as considered in the last section. Assume also that one end is fixed and one end is free. The equation of the motion is then

$$\begin{cases} \rho U_{tt} = \tau U_{xx} - 2\omega\, U_t(x,t) + F(x,t) & 0 \le x \le \ell \\ U(o,t) = 0, \quad U_x(\ell,t) = 0 & \end{cases}$$

and the corresponding Strum-Liouville boundary valued problem is

$$(47) \qquad \begin{cases} \frac{\tau}{\rho} v''(x) + \lambda^2 v(x) = 0 \\ v(o) = 0 \quad v_x(\ell) = 0 \end{cases} .$$

The eigenvalues and the corresponding eigenfunctions are

$$\lambda_n^2 = \frac{\tau}{\rho} \frac{(2n+1)^2\pi^2}{4\ell^2} , \qquad \varphi_n(x) = \sqrt{\frac{2}{\ell}} \sin \frac{(2n+1)\pi x}{2\ell} \qquad n = 1, 2, \ldots$$

$$U(x,t) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{\ell}}\, b_n(t) \sin \frac{(2n+1)\pi x}{2\ell} .$$

The covariance function of the stationary state is then

$$E\{[U(x,\infty) - \overline{U(x,\infty)}]\,[U(y,\infty) - \overline{U(y,\infty)}]\} = E\{U(x,\infty)\, U(y,\infty)\}$$

$$= \sum_{n=1}^{\infty} \kappa T \frac{\rho\, 4\ell^2}{\tau(2n+1)^2\pi^2} \frac{2}{\ell} \sin \frac{(2n+1)\pi x}{2\ell} \sin \frac{(2n+1)\pi y}{2\ell}$$

$$= \frac{4\kappa T\rho}{\tau\, \pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \{\cos \frac{(2n+1)\pi(x-y)}{2\ell} - \cos \frac{(2n+1)\pi(x+y)}{2\ell}\}$$

$$= \frac{4\kappa T\rho}{8\tau\, \pi^2} [\pi^2 - 2\pi \frac{\pi(x-y)}{2\ell} - (\pi^2 - 2\pi \frac{\pi(x+y)}{2\ell})] \qquad \text{if} \quad y \le x$$

$$= \frac{\kappa T\rho y}{\tau} \qquad \text{if} \quad y \le x$$

$$= \frac{\kappa T\rho x}{\tau} \qquad \text{if} \quad x \le y$$

$$= \kappa T\, K(x,y)$$

where $K(x,y)$ is the Green's function of (47) .

IV. THE DIFFICULTIES IN HIGHER DIMENSIONAL CASE

One may consider higher dimensional mechanical system undergoing Brownian motion analogous to what is discussed before, but the result turns out to be unbelievable. If we apply the method in section III, the equation of motion will be

$$\begin{cases} \xi_{tt} = p\{\xi_{xx} + \xi_{yy}\} + p_x \xi_x + p_y \xi_y - q\,\xi - 2\omega\xi_t + F(x,y;t) \\ \text{B. C.} \end{cases}$$

where $F(x,y;t)$ is Gaussian process with zero mean and

$$\langle F(x,y;t)\, F(\eta,\zeta;s)\rangle_{Av} = 2(2\omega)\kappa T\delta(x-\eta)\delta(y-\zeta)\delta(t-s) \quad . \tag{48}$$

By separating the time and coordinate variables for ξ, one gets the solution

$$\xi(x,y;t) = \sum_{n=1}^{\infty} b_n(t)\, \varphi_n(x,y)$$

where $b_n(t)$ is defined in (33) and $\{\varphi_n(x,y)\}$ is the complete orthonormal system which satisfies the equation

$$\begin{cases} p(U_{xx} + U_{yy}) + p_x U_x + p_y U_y - qU + \lambda^2 U = 0 \\ (\text{B. C.})' \end{cases} \tag{49}$$

Then the covariance function of $\xi(x,y;t)$, say $\Gamma(x,y;\eta,\zeta;t)$, is the same as (46) except that we have to replace $\varphi_n(x),\varphi_n(y)$ by $\varphi_n(x,y)$, $\varphi_n(\eta,\zeta)$. It is divergent when $(x,y)=(\xi,\eta)$, since the Green's function of (49) diverges when $(x,y) = (\xi,\eta)$. This gives the trouble.

If we apply the method of Wang and Uhlenbeck, a similiar disappointing result occurs. Let us consider a simple case; a continuous 2-dimensional mechanical system for $0 \leq x, y \leq 1$. $\xi(x,y;t) = 0$ if either $x = 0$ or $x = 1$ or $y = 0$ or $y = 1$. The kinetic and potential energies are assumed to be

$$(50) \qquad \begin{cases} T = \frac{1}{2}\int_0^1 \int_0^1 (\xi_t)^2 \, dx \, dy \\ V = \frac{1}{2}\int_0^1 \int_0^1 (\xi_x^2 + \xi_y^2) \, dx \, dy \end{cases} .$$

Set $x_k = y_k = \frac{k}{n+1}$ for $k = 0, 1, 2, \ldots, n+1$, then

$$\begin{cases} T' = \frac{1}{2}\sum_{i,j=1}^{n+1} \dot{\xi}_{ij}^2 \frac{1}{(n+1)^2} \\ V' = \frac{1}{2}\sum_{i,j=1}^{n+1} \{(\xi_{ij} - \xi_{i-1,j})^2 + (\xi_{i,j} - \xi_{i,j-1})^2\} \end{cases}$$

where $\xi_{ij} = 0$ if either $i = 0$ or $i = n+1$ or $j = 0$ or $j = n+1$. The Lagrange equation is then

$$(51)\qquad \frac{1}{(n+1)^2}\ddot{\xi}_{ij} + \frac{2\omega}{(n+1)^2}\dot{\xi}_{ij} + \{(\xi_{ij} - \xi_{i-1,j}) + (\xi_{i,j} - \xi_{i+1,j})\}$$

$$+ \{(\xi_{i,j} - \xi_{i,j-1}) + (\xi_{i,j} - \xi_{i,j+1})\} = F_{ij}(t)(= \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} F(x,y;t)\,dx\,dy) \quad .$$

By (48), we have

$$\left\langle F_{ij}(t)\, F_{k\ell}(s)\right\rangle_{Av} = 2\left(\frac{2\omega}{(n+1)^2}\right) KT\delta(t-s)\,\delta_{ik}\,\delta_{j\ell} \quad .$$

We note that $\{F_{ij}(t)\}$ should be independently identically distributed so that $F(t) = \Sigma_{i,j} F_{ij}(t)$ is the random force acting on the whole drum (say) with covariance function

$$\left\langle F(t)\, F(s)\right\rangle_{Av} = c \cdot \delta(t-s)$$

and then

$$\left\langle F_{ij}(t)\, F_{k\ell}(s)\right\rangle_{Av} = \frac{c}{(n+1)^2}\,\delta_{ik}\,\delta_{j\ell}\,\delta(t-s)$$

is a reasonable result.

Thus applying to Wang and Uhlenbeck model;then the covariance function $\Gamma_{n+1}(i,j;k,\ell)$ of the stationary state is equal to the inverse of $\frac{1}{KT}G$ where G is, from (51), of the form

$$G = \begin{bmatrix} A & B & O & \cdot & \cdot & \cdot & \cdot \\ B & A & B & O & \cdot & \cdot & \cdot \\ O & B & A & B & O & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & O & B & A \end{bmatrix} \qquad (n^2 \times n^2)$$

where $B = -I$ is $n \times n$ matrix and

$$A = \begin{bmatrix} 4 & -1 & 0 & . & . & . & . & . \\ -1 & 4 & -1 & 0 & . & . & . & . \\ 0 & -1 & 4 & -1 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & 0 & -1 & 4 \end{bmatrix} \quad (n \times n) \quad .$$

Note that the system (50) is continuous, and equals to zero at boundary, it is reasonable to assume that the covariance funtion tends to zero if any variable tends to 0 or 1. Hence $\Gamma_{n+1}(1,1;\ 1,1)$, $\Gamma_{n+1}(1,1;\ 1,2)$, $\Gamma_{n+1}(1,1;\ 2,1)$ all tend to zero as $n \to \infty$. But

$$(\Gamma_{n+1}(i,j;\ k,\ell))\ \frac{1}{KT}\ G = I \quad .$$

Hence

$$4\Gamma_{n+1}(1,1;\ 1.1) - \Gamma_{n+1}(1.1;\ 1,2) - \Gamma_{n+1}(1,1;\ 2,1) = (KT) \neq 0 \ .$$

The right hand side of the above equality remains constant, Hence at least one of the Γ_{n+1}'s in the left hand side does not tend to zero. Thus one gets a contradiction. The method of Wang and Uhlenbeck still doesn't apply.

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APPENDIX I

We want to show Mercer's theorem here, but we give some definition and properties first. A (complex-valued) continuous kernel $K(x,t)$ on $a \leq s,\ t \leq b$ is positive definite if

$$(K\, f,f) \geq 0 \quad \forall \quad f \in L^2(a,b) \quad \text{where} \quad K(f)(s) = \int_a^b K(s,t)\, f(t)\, dt$$

then $K(s,s) \geq 0 \ \forall \ s \in [a,b]$ and $K(s,t)$ is Hermitian symmetric, i.e., $K(s,t) = \overline{K(t,s)}$.

Mercer's theorem: A positive definite kernel $K(s,t)$ can be expanded in a series

$$K(s,t) = \sum_{j=1}^{\infty} \lambda_j^{-2} \varphi_j(s)\, \overline{\varphi_j(t)}$$

which converges absolutely and uniformly on $a \leq s,\ t \leq b$, where $\{\lambda_j^{-2}\}$ $\{\varphi_n\}$ are the eigenvalues and the corresponding orthonormal eigenfunction, $\lambda_n > 0$.

Proof:

The Fourier expansion of $(Kf)(s)$ is

$$\text{(a)} \qquad \sum_{j=1}^{\infty} (Kf,\varphi_j)\varphi_j(s) = \sum_{j=1}^{\infty} (f,K\varphi_j)\varphi_j(s) = \sum_{j=1}^{\infty} \lambda_j^{-2}(f,\varphi_j)\varphi_j(s) \quad .$$

Set, for each n ,

$$K_n(s,t) = K(s,t) - \sum_{j=1}^{n} \lambda_j^{-2}\varphi_j(s)\,\overline{\varphi_j(t)}$$

then

$$(K_n f)(s) = \sum_{j=n+1}^{\infty} \lambda_j^{-2}\varphi_j(s)(f,\varphi_j) \quad .$$

Since λ_j^2 is positive, we have

$$(K_n f,f) = \sum_{j=n+1}^{\infty} \lambda_j^{-2}\,(f,\varphi_j)(\overline{f,\varphi_j}) \geq 0$$

which shows that $K_n(s,t)$ is positive definite. Hence

$$K_n(s,s) = K(s,s) - \sum_{j=1}^{n} \lambda_j^{-2}\,\varphi_j(s)\,\overline{\varphi_j(s)} \geq 0$$

implies that $\Sigma_{j=1}^{\infty} \lambda_j^{-2}\,|\varphi_j(s)|^2$ converges and is not greater than $K(s,s)$. Now set

$$S(s,t) = \sum_{j=1}^{\infty} \lambda_j^{-2}\,\varphi_j(s)\,\overline{\varphi_j(t)} \quad .$$

Then we obtain, by Schwarz and Bessel inequalities, that

$$\left(\sum_{j=n}^{m} |\lambda_j^{-2} \varphi_j(s) \overline{\varphi_j(t)}| \right)^2 \leq \left(\sum_{j=n}^{m} \lambda_j^{-2} |\varphi_j(s)|^2 \right) \left(\sum_{j=n}^{m} \lambda_j^{-2} |\varphi_j(t)|^2 \right)$$

$$\leq \sum_{j=n}^{m} \lambda_j^{-2} |\varphi_j(s)|^2 K(t,t) \quad .$$

Hence, the series $S(s,t)$ converges absolutely and uniformly with respect to t for fixed s and also with respect to s for fixed t.

We set now

$$R(s,t) = K(s,t) - S(s,t) \quad .$$

Then for any continuous function $f(s)$, we have

$$\text{(b)} \qquad \int_a^b K(s,t)\, f(t)\, dt = \int_a^b S(s,t)\, f(t)\, dt + \int_a^b R(s,t)\, f(t)\, dt$$

the left hand side is (a). On the other hand, since $S(s,t)$ converges uniformly with respect to t, we obtain, by term by term integration, that the first term on the right hand side of (b) is equal to $\Sigma_{j=1}^{\infty} \lambda_j^{-2}(f,\varphi)\varphi_j(s)$. Hence

$$\text{(c)} \qquad \int_a^b R(s,t)\, f(t)\, dt \equiv 0 \quad .$$

Since the series $S(s,t)$ uniformly converges with respect to t for fixed s, $R(s,t)$ is continuous function of t for any fixed s. Hence, by setting $f(t) = \overline{R(s,t)}$ in (c), we have $R(s,t) \equiv 0$ for fixed s; hence $R(s,t) \equiv 0$. Thus we have

$$K(s,t) = \sum_{j=1}^{\infty} \lambda_j^{-2} \varphi_j(s) \overline{\varphi_j(t)} \quad .$$

To show that the series converges uniformly, we recall that $K(s,s) = \Sigma_{j=1}^{\infty} \lambda_j^{-2}|\varphi_j(s)|^2$ is positive continuous in s, $g_n(s)=K(s,s) - \Sigma_{j=1}^{n} \lambda_j^{-2}|(\varphi_j(s)|^2$ is positive continuous and $\{g_n(s)\}$ is a sequence of monotone decreasing positive continuous function converges to 0 everywhere, hence the convergence is uniformly, i.e. for any $\epsilon > 0$. $\exists$ $N = N(\epsilon)$ s.t. for any $s \in [a,b]$

$$\epsilon \geq K(s,s) - \sum_{j=1}^{N} \lambda_j^{-2} |\varphi_j(s)|^2 \quad .$$

Now by Schwarz inequality, for all $a \leq s, t \leq b$.

$$\Big(\sum_{j=n}^{m} \lambda_j^{-2}|\varphi_j(s) \overline{\varphi_j(t)}\Big)^2 \leq \sum_{j=n}^{m} \lambda_j^{-2}|\varphi_j(s)|^2 \sum_{j=n}^{m} \lambda_j^{-2}|\varphi_j(t)|^2 \leq \epsilon^2 \quad .$$

Thus, we prove the uniformly and absolutely convergent for all $m \geq n \geq N$ .with respect with respect to t,s of the series $\Sigma_{j=1}^{\infty} \lambda_j^{-2}\varphi_j(t) \overline{\varphi_j(s)}$.

APPENDIX II

Variation principle and Hamilton's principle: Consider

$$\text{(a)} \qquad J[y] = \int_{x_0}^{x_1} F(x,y,y')\,dx$$

where the values x_0, x, $y(x_0)$, $y(x_1)$ are given. The function F is twice differentiable w.r.t. its three arguments x, y, y'. y'' is assumed also to be continuous. The variation principle is to determine the minimum of $J[y]$. Suppose $y = y(x) = f(x)$ is the desire extremal function yielding the minimum. Let $\eta(x) \in C^2[x_0, x_1]$ s.t. $\eta(x_0) = 0$, $\eta(x_1) = 0$. We construct $\bar{y} = y + \epsilon\,\eta(x) = y + \delta y$, where ϵ is a parameter. The quantity $\delta y = \epsilon\,\eta(s)$ is known as the variation of the function $y = f(x)$. If ϵ is small then, $\bar{y}$ lies in a small neighborhood of y. Therefore the integral $J[\bar{y}] = J[y + \epsilon\,\eta]$, which may be regarded as a function $\Phi(\epsilon)$ of ϵ, must have a minimum at $\epsilon = 0$ relative to all values of ϵ in a sufficient small neighborhood of 0, hence $\Phi'(o) = 0$. Now if we differentiate the integral $\Phi(\epsilon) = \int_{x_0}^{x_1} F(x, y+\epsilon\eta, y' + \epsilon\eta')\,dx$ w.r.t. ϵ under the integral sign we obtain as a necessary condition the equation

$$\text{(b)} \qquad \Phi'(o) = \int_{x_0}^{x_1} (F_y \eta + F_{y'} \eta')\,dx = 0 \quad ,$$

which must hold for all $\eta(x)$ which satisfies the above requirement. Use partial integration, note that $\eta(x_0) = \eta(x_1) = 0$, then (b) becomes

$$\int_{x_0}^{x_1} \eta(F_y - \frac{d}{dx} F_{y'})\, dx = 0$$

for any twice differentiable η with $\eta(x_0) = \eta(x_1) = 0$. This implies $-[F]_y = \frac{d}{dx} F_{y'} - F_y = 0$.

We called this the fundamental differential equation of Euler. Now we state the Hamilton's principle: Between two instants of time to a t_1 the motion proceeds in such a way that for the functions $q_i(t)$ the integral

$$J = \int_{t_0}^{t_1} (T - U)\, dt$$

is stationary (minimum) in comparison with the neighboring functions $\bar{q}_i(t)$ for which $\bar{q}_i(t_0) = q_i(t_0)$ and $\bar{q}_i(t_1) = q_i(t_1)$. Where T is kinetic energy and is assumed to be $T = \Sigma_{i,k=1}^{n} p_{ik}(q_1, q_2, \ldots, q_{n,t})\, \dot{q}_i \dot{q}_k$ with constant p_{ik}, and $U = U(q_1 \ldots q_{n,} t)$ is the potential energy. Hence variation principle implies the Lagrange's general equation of motion

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial}{\partial q_i} (T - U) = 0 \qquad i = 1, 2, \ldots, n \quad .$$

If we consider (a) but no longer impose conditions on $y(x)$ at $x = x_0, x_1$. Then the necessary condition for J to be stationary is that

$$\delta J = F_{y'}\, \delta y \Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} [F]_y\, \delta y\, dy$$

vanish. It is clear that Euler equation $[F]_y = 0$ must be satisfied. For, if J is stationary w.r.t. variations which do not have prescribed boundary values, then it is certainly stationary w.r.t. the smaller class of variations for which $\delta y = 0$ on the boundary, hence Euler's equation holds. Because of the arbitrariness of δy at the boundary, we obtain as a necessary condition the "natural boundary condition

$$F_{y'} = 0 \quad \text{for} \quad x = x_0 \quad \text{and} \quad x = x_1 \quad .$$

We discuss now the variation problem for two independent variables, for more variables the discussion is similar. Consider the integer

$$\text{(c)} \qquad J = \int \int_G F(x,y,u,u_x,u_y)\, dx\, dy$$

to find $u(x,y) \in C^2(G)$ satisfies the prescribed values on boundary G. Introduce arbitrary function $\eta(x,y) \in C^2(G)$, $\eta(x,y) = 0$ on boundary of G. As a necessary condition for an extremum, then

$$\delta J = \epsilon \left(\frac{d}{d\epsilon}\, \Phi(\epsilon)\right)_{\epsilon=0} = \epsilon\left(\frac{d}{d\epsilon}\, J[u + \epsilon\eta]\right)_{\epsilon=0}$$

i.e.

$$\delta J = \epsilon \int \int_G (F_u \eta + F_{u_x} \eta_x + F_{u_y} \eta_y) dx\, dy = 0 \quad .$$

Use partial integration, assuming that the boundary case Γ of G has a tangent which turns piecewise continuous. Then according to Gauss' integral theorem, we have

APPENDIX III

We will give here an informal prove that $\{\xi(x,t), \xi_t(x,t)\}$ is a Markov process. Note that $\xi(x,t) = \Sigma_{n=1}^{\infty} b_n(t)\, \varphi_n(x)$, $\xi_t(x,t) = \Sigma_{n=1}^{\infty} \dot{b}_n(t)\, \varphi_n(x)$ where $\{b_n(t), \dot{b}_n(t)\}$ forms a Markov process and for different n, they are independent. Also $\{\varphi_n(x)\}$ is an complete orthonormal basis for the Hilbert space $L^2[0,1]$, hence we consider $\xi(x,t)$, $\xi_t(x,t)$ as elements of infinite dimensional space with components $b_n(t)$, $\dot{b}_n(t)$ respectively. Let $t \leq s \leq u$, $E_n^{(v)}$, $F_n^{(v)}$, for $n=1,2,\ldots$, $v = t,s,u$, be measurable sets of real numbers and except for finite number of n, they equal to the whole set of real numbers. Then

$$P\left\{\begin{matrix} b_n(u) \in E_n^{(u)} \\ \dot{b}_n(u) \in F_n^{(u)},\ n = 1,2,\ldots \end{matrix}\,\middle|\,\begin{matrix} b_n(t) \in E_n^{(t)}\ \ b_n(s) \in E_n^{(s)} \\ \dot{b}_n(t) \in F_n^{(t)},\ \dot{b}_n(s) \in F_n^{(s)},\ n=1,2,\ldots \end{matrix}\right\}$$

$$= \frac{\prod_{n=1}^{\infty} P\left\{\begin{matrix} b_n(t) \in E_n^{(t)},\ b_n(s) \in E_n^{(s)},\ b_n(u) \in E_n^{(u)} \\ \dot{b}_n(t) \in F_n^{(t)},\ \dot{b}_n(s) \in F_n^{(s)},\ \dot{b}_n(u) \in F_n^{(u)} \end{matrix}\right\}}{\prod_{n=1}^{\infty} P\left\{\begin{matrix} b_n(t) \in E_n^{(t)},\ b_n(s) \in E_n^{(s)} \\ \dot{b}_n(t) \in F_n^{(t)},\ \dot{b}_n(s) \in F_n^{(s)} \end{matrix}\right\}}$$

$$= \frac{\prod_{n=1}^{\infty} P\left\{\begin{matrix} b_n(s) \in E_n^{(s)},\ b_n(u) \in E_n^{(u)} \\ \dot{b}_n(s) \in F_n^{(s)},\ \dot{b}_n(u) \in F_n^{(u)} \end{matrix}\right\}}{\prod_{n=1}^{\infty} P\ \left\{b_n(s) \in E_n^{(s)},\ \dot{b}_n(s) \in F_n^{(s)}\right\}}$$

$$= P\left\{\begin{matrix} b_n(u) \in E_n^{(u)} \\ \dot{b}_n(u) \in F_n^{(u)},\ n = 1,2,\ldots \end{matrix}\,\middle|\,\begin{matrix} b_n(s) \in E_n^{(s)} \\ \dot{b}_n(s) \in F_n^{(s)},\ n = 1,2,\ldots \end{matrix}\right\} .$$

The above equality is true for cylinder sets in $\Pi_{n=1}^{\infty}(R_n \times R_n)$, where R_n denotes the real line, then it is true for all measurable sets in the σ-field which is generated by the cylinder sets. Thus $\{\xi(x,t),\ \xi_t(x,t)\}$ is Markovian.

If we know that $\Sigma_{n=1}^{\infty} b_n^2(t) < \infty$, $\Sigma_{n=1}^{\infty} \dot{b}_n(t) < \infty$ a.e. (or, for fixed t, the sample functions $\xi(x,t)$, $\xi_t(x,t)$ are continuous in x), consider the σ-field generated by the open sets of weak star topology in $L^2[a,b]$

(or $C[a,b]$, apply the preceding arguments, then we see that

$$P\left\{\begin{matrix}\xi(x,u)\in K_3 \\ \xi_t(x,u)\in M_3\end{matrix}\;\middle|\;\begin{matrix}\xi(x,t)\in K_1, & \xi(x,s)\in K_2 \\ \xi_t(x,t)\in M_1, & \xi(x,s)\in M_2\end{matrix}\right\}$$

$$= P\left\{\begin{matrix}\xi(x,u)\in K_3 \\ \xi_t(x,u)\in M_3\end{matrix}\;\middle|\;\begin{matrix}\xi(x,s)\in K_2 \\ \xi(x,s)\in M_2\end{matrix}\right\}$$

where K_i, M_j are measurable sets in $L^2[a,b]$ (or $C[a,b]$). The definition of Markov process and the discussion are then valid.