# On the moments of traces of two matrices in three situations for complex multivariate normal populations*

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# On the moments of traces of two matrices in three situations for complex multivariate normal populations*

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1. Introduction and Summary. In the case of complex multivariate normal distributions (see Goodman [1]), the classical problems concerning MANOVA model, canonical correlation coefficients and covariance model were studied by Khatri [4] and James [3]. The distribution of the i-th maximum characteristic (ch.) root for these situations are given by Khatri [6]. Moreover, we may note that the three types of problems can be summarised in the following way:

Let $\underset{\sim}{X}$: mxn and $\underset{\sim}{S}$: mxm be jointly distributed as

$$(1)\quad \{\pi^{mn}|\underset{\sim}{\Sigma}_1|^n|\underset{\sim}{\Sigma}_2|^r \tilde{\Gamma}_m(r)\}^{-1} |\underset{\sim}{L}|^m |\underset{\sim}{S}|^{r-m} \operatorname{etr}[-\underset{\sim}{\Sigma}_2^{-1} \underset{\sim}{S} - \underset{\sim}{\Sigma}_1^{-1} (\underset{\sim}{X}-\underset{\sim}{\mu}) \underset{\sim}{L} \overline{(\underset{\sim}{X}-\underset{\sim}{\mu})}']$$

where $\underset{\sim}{\Sigma}_1$ : mxm, $\underset{\sim}{\Sigma}_2$ : mxm and $\underset{\sim}{L}$ : nxn are hermitian positive definite, $\underset{\sim}{\mu}$ : mxn is a complex matrix, $r > m$ and $\tilde{\Gamma}_m(r) = \pi^{\frac{1}{2}m(m-1)} \prod_{i=1}^{m} \Gamma(r-i+1)$.

The usual MANOVA model is when $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$, $\underset{\sim}{L}$ is a fixed matrix and the null hypothesis is $H_0(\underset{\sim}{\mu}=\underset{\sim}{\mu}_0$ given), the model corresponding to cannonical correlation coefficients is obtained when $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$, $H_0(\underset{\sim}{\mu}=\underset{\sim}{\mu}_0$ (given)) and $\underset{\sim}{L}$ is distributed as

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$$(2) \qquad \{\tilde{\Gamma}_n(p)\}^{-1} \, |\Sigma_3|^{-p} \, |L|^{p-n} \, \mathrm{etr}(-\Sigma_3^{-1} L),$$

if $p \geq n$ and $\Sigma_3$ is hermitian positive definite; and the covariance model is obtained when $\mu = 0$ and $H_0$ $(\Sigma_1 = \Sigma_2)$. In all the above cases, the test procedures depend on the ch. roots of $(XL\bar{X}'S^{-1})$ and in terms of zonal polynomials, the distributions of the ch. roots of $(XL\bar{X}'S^{-1})$ have been given by James [3] and in the integral forms by Khatri [4]. Here, we establish lemma 3 (which was conjuctured by Khatri [6] in two particular cases) and this lemma helps us in writing the noncentral distributions of the ch. roots in alternative forms. This is not done here explicitly, but we derive the moments of $T = \mathrm{tr}(XL\bar{X}'S^{-1})$, $T_1 = \mathrm{tr}(X\bar{X}'S^{-1})$ and $V = \mathrm{tr}\, XL\bar{X}'(S + XL\bar{X}')^{-1}$ for the three situations mentioned above.

## 2. Notations and preliminary results.

If $A$ and $B$ are two hermitian matrices such that $A - B$ is positive definite, then we shall write it as $A > B$. Let $A$ be a $m \times m$ hermitian matrix and corresponding to each partition $\kappa = (k_1, \ldots, k_m)$, $k_1 \geq k_2 \geq \ldots \geq k_m \geq 0$, of integer $k$ into not more than $m$ parts, zonal polynomial $\tilde{C}_\kappa(A)$ as defined by James [3] is given by

$$(3) \qquad \tilde{C}_\kappa(S) = \chi_{[\kappa]}(1)\, \chi_{\{\kappa\}}(A)$$

where $\chi_{[\kappa]}(1)$ is the dimension of representation of the symmetric group and is given by

(4a) $$\chi_{[\kappa]}(1) = k! \prod_{i<j}^{m} (k_i - k_j - i + j) / \prod_{i=1}^{m} (k_i + m - i)!$$

and $\chi_{\{\kappa\}}(\underset{\sim}{A})$ is the character of representation $\{\kappa\}$ of the linear group and is given as a symmetric function of the latent roots $a_1, a_2, \ldots, a_m$ of $\underset{\sim}{A}$ as

(4b) $$\chi_{\{\kappa\}}(\underset{\sim}{A}) = |(a_i^{k_j+m-j})| / |(a_i^{m-j})|,$$

$|\underset{\sim}{P}|$ being a determinant of a square matrix $\underset{\sim}{P} = (p_{ij}), i,j=1,2,\ldots,m$ = order of $\underset{\sim}{P}$.

Let $d\underset{\sim}{U}$ be the invariant measure on the unitary group $U(n)$ normalized to make the measure unity. Let $\underset{\sim}{S}$ be a hermitian positive definite and let us use the transformation $\underset{\sim}{S} = \underset{\sim}{U}\underset{\sim}{W}\bar{\underset{\sim}{U}}'$ where $\underset{\sim}{W} = \mathrm{diag}(w_1, \ldots, w_m), w_1 > \ldots > w_m > 0$ $\underset{\sim}{U}$ is an unitary matrix such that the total number of random variables are $m(m-1)$. Then the jacobian of the transformation as given by Khatri [4] is

(5) $$J(\underset{\sim}{S}; \underset{\sim}{W}, \underset{\sim}{U}) = \prod_{i<j}^{m} (w_i - w_j)^2 \, h(\underset{\sim}{U})$$

where $h(\underset{\sim}{U})$ is a function of the elements of $\underset{\sim}{U}$. Noting one to one correspondence between the integration over the elements of $\underset{\sim}{U}$ subject to $\underset{\sim}{U}\bar{\underset{\sim}{U}}' = \underset{\sim}{I}_m$ and over unitary group $U(m)$, we write

$$h(\underset{\sim}{U}) = \pi^{m(m-1)} \{\Gamma_m(m)\}^{-1} d\underset{\sim}{U}. \tag{6}$$

Hence, the jacobian of the transformation (5) is written as

$$J(\underset{\sim}{S};\underset{\sim}{W},\underset{\sim}{U}) = \pi^{m(m-1)} \{\tilde{\Gamma}_m(m)\}^{-1} \prod_{i<j}^{m} (w_i - w_j)^2 \; d\underset{\sim}{U}. \tag{7}$$

If the unitary matrix $\underset{\sim}{U}$ has the total random elements $p(m-p)$ (as for example in the transformation $\underset{\sim}{X} = (\underset{\sim}{T}\ \underset{\sim}{0})\underset{\sim}{U}$ where $X$: pxm is a complex random matrix, $\underset{\sim}{T}$: pxp is a lower triangular matrix with $t_{ii} > 0$ and $\underset{\sim}{U}$: mxm is a unitary matrix, $p < m$), then $h(\underset{\sim}{U})$ in place of (6) will be denoted as

$$h(\underset{\sim}{U}) = \pi^{mp} \{\tilde{\Gamma}_p(m)\}^{-1} d\underset{\sim}{U}. \tag{8}$$

(Note that $h(\underset{\sim}{U})$ obtained by (5) and that by $\underset{\sim}{X} = (\underset{\sim}{T}\ \underset{\sim}{0})\underset{\sim}{U}$ are different, see Khatri [4]).

From James [3], we have the following results over an unitary space:

$$\int_{U(n)} \mathrm{etr}(\underset{\sim}{X_1}\underset{\sim}{U} + \overline{\underset{\sim}{X_1}\underset{\sim}{U}'}) \, dU = \sum_{k=0}^{\infty} \sum_{\kappa} \tilde{C}_\kappa (\underset{\sim}{X}\overline{\underset{\sim}{X}}')/k! \, (n)_\kappa \tag{9}$$

where $\underset{\sim}{X}_1' = (\underset{\sim}{X}'\ \underset{\sim}{0})$: nxn, $\underset{\sim}{X}$: mxn, $n>m$ and $(n)_\kappa = \prod_{i=1}^{m} (n-i+1)_{k_i}$, $(x)_k = x(x+1)\ldots(x+k-1)$

with $\kappa = (k_1,\ldots,k_m)$, $k_1 \geq \ldots \geq k_m \geq 0$ and $\sum_{i=1}^{m} k_i = k$;

$$(10)\qquad \int_{U(n)} \tilde{C}_\kappa\,(\underset{\sim}{A}\underset{\sim}{U}\underset{\sim}{B}\bar{\underset{\sim}{U}}')\; d\underset{\sim}{U} = \tilde{C}_\kappa\,(\underset{\sim}{A})\;\tilde{C}_\kappa\,(\underset{\sim}{B})/\tilde{C}_\kappa(\underset{\sim}{I}_m)$$

where $\underset{\sim}{A}$: nxn and $\underset{\sim}{B}$: nxn are hermitian matrices, and

$$(11)\qquad \int_{\underset{\sim}{S}>\underset{\sim}{0}} \mathrm{etr}(-\underset{\sim}{\Sigma}^{-1}\underset{\sim}{S})|\underset{\sim}{S}|^{r-m}\,\tilde{C}_\kappa(\underset{\sim}{A}\underset{\sim}{S})\;d\underset{\sim}{S} = \tilde{\Gamma}_m(r,\kappa)\;|\underset{\sim}{\Sigma}|^{r}\;\tilde{C}_\kappa(\underset{\sim}{\Sigma}\underset{\sim}{A})$$

where $\underset{\sim}{\Sigma}$: mxm is hermitian positive definite, $\underset{\sim}{A}$: mxm is hermitian and $\tilde{\Gamma}_m(r,\kappa) = \tilde{\Gamma}_m(r)\;(r)_\kappa = \pi^{\frac{1}{2}m(m-1)}\prod_{i=1}^{m}\Gamma(r+k_i-i+1)$. Moreover, let us define the hypergeometric functions as

$$(12)\qquad {}_p\tilde{F}_q = {}_p\tilde{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;\underset{\sim}{A}) = \sum_{k=0}^{\infty}\Sigma_\kappa\Big[\prod_{i=1}^{p}(a_i)_\kappa\Big]\Big[k!\prod_{j=1}^{q}(b_j)_\kappa\Big]^{-1}\tilde{C}_\kappa(\underset{\sim}{A})$$

and

$$(13)\qquad {}_p\tilde{F}_q^{(m)} = {}_p\tilde{F}_q^{(m)}(a_1,\ldots,a_p;b_1,\ldots,b_q;\underset{\sim}{A},\underset{\sim}{B}) = \sum_{k=o}^{\infty}\sum_\kappa \frac{\{\prod_{i=1}^{p}(a_i)_\kappa\}\;\tilde{C}_\kappa(\underset{\sim}{A})\;\tilde{C}_\kappa\;(\underset{\sim}{B})}{\{\prod_{j=1}^{q}(b_j)_\kappa\}\;k!\;\tilde{C}_\kappa(\underset{\sim}{I}_m)}\,.$$

Lemma 1. $\chi_{\{\kappa\}}\;(\underset{\sim}{I}_m) = \prod_{i<j}^{m}(k_i - k_j - i + j)/\prod_{i=1}^{m}\Gamma(m-i+1)$ or

$$\tilde{C}_\kappa(\underset{\sim}{I}_m) = \{\chi_{[\kappa]}(1)\}^2\, \tilde{\Gamma}_m(m,\kappa)/k!\, \Gamma_m(m).$$

Proof. Let $\underset{\sim}{W} = \mathrm{diag}\,(w_1,\ldots,w_m)$ with $w_1>\ldots>w_m$. Then by (4),

$$\chi_{\{\kappa\}}(\underset{\sim}{W}) = |\underset{\sim}{B}_0| / \prod_{i<j}^{m} (w_i - w_j),\ \underset{\sim}{B}_0 = (b_{o,ij}),\ b_{o,ij} = w_i^{k_j+m-j}.$$

Let us write $w_2 = w_1 - y$ and take limit as $y \to 0$. Then, we get

(14)
$$\chi_{\{\kappa\}}(\underset{\sim}{W})]_{w_1=w_2} = |\underset{\sim}{B}_1| / \{\prod_{i=3}^{m} (w_1 - w_i)^2 \prod_{3=j<j'}^{m} (w_j - w_{j'})\}$$

where $\underset{\sim}{B}_1 = (b_{1,ij})$, $b_{1,ij} = b_{0,ij}$ for $i = 1,3,\ldots,m$ and $b_{1,2j} = \frac{d}{dw_1} b_{o,1j}$ for $j = 1,2,\ldots,m$. Let $w_3 = w_1 - y$ in (14) and then take limit as $y \to 0$. Then, we get

(15)
$$\chi_{\{\kappa\}}(\underset{\sim}{W})]_{w_1=w_2=w_3} = |\underset{\sim}{B}_2| / \{\prod_{i=4}^{m} (w_1 - w_i)^3 \prod_{4=j<j'}^{m} (w_j - w_{j'})\}(2!)$$

where $\underset{\sim}{B}_2 = (b_{2,ij}) = b_{2,ij} = b_{1,ij}$ for $i = 1.2.4,\ldots,m$, $b_{1,2j} = (\frac{d}{dw_1})^2 b_{0,1j} = \frac{d}{dw_1} b_{1,2j}$ for $j = 1,2,\ldots,m$. Thus, proceeding, we get finally

(16)
$$\chi_{\{\kappa\}}(\underset{\sim}{I}_m) = |\underset{\sim}{B}| / \prod_{i=1}^{m} \Gamma(m - i + 1)$$

where $\underset{\sim}{B} = (b_{ij})$, $b_{1j} = 1$, $b_{ij} = (k_j + m - j)(k_j + m - j - 1)\ldots(k_j + m - j - i + 2)$ for $i = 2,3,\ldots,m$ and $j = 1,2,\ldots,m$. This establishes lemma 1.

<u>Lemma 2</u>. Let $\underset{\sim}{A}(\underset{\sim}{w}) = (a_j(w_i))$ and $\underset{\sim}{B}(\underset{\sim}{w}) = (b_j(w_i))$ for $i,j = 1,2,\ldots,m$ and let $\mathcal{D}$ be a domain given by $\mathcal{D} = \mathcal{D}\{0<w_m<\ldots< w_t< x < w_{t-1}<\ldots< w_1< \infty\}$.

$$(17) \qquad \int_{\mathcal{D}} |\underset{\sim}{A}(\underset{\sim}{w})|\ |\underset{\sim}{B}(\underset{\sim}{w})|\ dw_1\ldots dw_m = \Sigma_1\ |(c_{\delta_i,j})|$$

where $\Sigma_1$ indicates the summation over the combinations $(\delta_1<\ldots< \delta_{t-1})$ and $(\delta_t< \delta_{t+1}<\ldots< \delta_m)$, $(\delta_1,\ldots,\delta_m)$ being a permutation of $(1,2,\ldots,m)$ and

$$(18) \qquad c_{\delta_i,j} = \int_x^\infty a_j(y)\ b_{\delta_i}(y)\ dy \quad \text{for } i = 1,2,\ldots,t-1 \text{ and } j=1,2,\ldots,m$$

$$= \int_0^x a_j(y)\ b_{\delta_i}(y)\ dy \quad \text{for } i = t,\ldots,m \text{ and } j=1,2,\ldots,m$$

which are assumed to exist for all combinations. When $t = 1$, we rewrite (17) as

$$(19) \qquad \int_{\mathcal{D}} |\underset{\sim}{A}(\underset{\sim}{w})|\ |\underset{\sim}{B}(\underset{\sim}{w})|\ dw_1\ldots dw_m = |\underset{\sim}{C}|$$

where $\underset{\sim}{C} = (c_{ij})$, $c_{ij} = \int_0^x a_j(y)b_i(y)\ dy$ for $i,j=1,2,\ldots,m$ and $\mathcal{D} = \mathcal{D}\ \{0 < w_m<\ldots< w_1< x\}$.

This is a generalized result of lemma 4 given by Khatri [6] and the proof is exactly parallel to that given by lemma 4 after noting the expansion

$$|\underset{\sim}{A}(\underset{\sim}{w})|\ |\underset{\sim}{B}(\underset{\sim}{w})| = |\sum_{\delta=1}^{m} a_i(w_\delta)b_j(w_\delta)| = \Sigma_{\underset{\sim}{\delta}}|(a_i(w_{\delta_j})\ b_j(w_{\delta_j})|$$

where $\Sigma_{\underset{\sim}{\delta}}$ indicates the summation over $\underset{\sim}{\delta} = (\delta_1,\dots,\delta_m)$, the permutations of $(1,2,\dots,m)$. Hence, the proof of this lemma is omitted.

<u>Lemma 3</u>. Let $\underset{\sim}{\Lambda} = \text{diag}\ (\lambda_1,\dots,\lambda_m)$ and $\underset{\sim}{W} = \text{diag}\ (w_1,\dots,w_m)$ with $\lambda_1>\dots>\lambda_m$ and $w_1>\dots>w_m$. Then

$$(20)\qquad {}_p\tilde{F}_q^{(m)}(a_1,\dots,a_p;\ b_1,\dots,b_q;\ \underset{\sim}{\Lambda},\underset{\sim}{W}) = c|\underset{\sim}{G}|\{\prod_{i<j}^{m}(\lambda_i-\lambda_j)(w_i-w_j)\}^{-1}$$

where

$\underset{\sim}{G} = (g_{ij})$, $g_{ij} = {}_pF_q(a_1-m+1,\dots,\ a_p-m+1;\ b_1-m+1,\dots,b_q-m+1;\lambda_i w_j)$ for $i,j=1,2,\dots,m$

and

$$c = \prod_{i=1}^{m}\{\Gamma(m-i+1)\prod_{j=1}^{q}(b_j-i+1)^{i-1}/\prod_{t=1}^{p}(a_t-i+1)^{i-1}\}.$$

In particular, we have

$$(21)\qquad {}_0\tilde{F}_0^{(m)}(\underset{\sim}{\Lambda},\underset{\sim}{W}) = \{\prod_{i=1}^{m}\Gamma(m-i+1)\}\{\prod_{i<j}^{m}(\lambda_i-\lambda_j)(w_i-w_j)\}^{-1}|(\exp(\lambda_i w_j))|$$

and

$$(22)\qquad {}_1\tilde{F}_0^{(m)}(r;\underset{\sim}{\Lambda},\underset{\sim}{W}) = \{\prod_{i=1}^{m} \Gamma(m-i+1)(r-i+1)^{-i+1}\}\{\prod_{i<j}^{m}(\lambda_i-\lambda_j)(w_i-w_j)\}^{-1}|(\{1-\lambda_i w_j\}^{m-r-1})|.$$

We note that when some of the $\lambda_i$'s or $w_i$'s are equal, we obtain the results as limiting cases on the right side of (20) - (22). The results (21) and (22) were conjuctured by Khatri [6].

Proof. It is easy to prove the following result

$$(23)\qquad |\underset{\sim}{G}| = \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} [\prod_{i=1}^{m}\{\prod_{t=1}^{p}(a_t-m+1)_{k_i} / \prod_{j=1}^{q}(b_j-m+1)_{k_i}\}/k_i!]|(\lambda_i w_j)^{k_i})|.$$

Note that

$$|(\lambda_i w_j)^{k_i}| = 0$$

if any two of $k_1,\ldots,k_m$ are equal. Hence (23) can be rewritten as

$$(24)\quad |\underset{\sim}{G}| = \sum_{k=\frac{1}{2}m(m-1)}^{\infty} \sum_{k_1>k_2>\cdots>k_m} [\prod_{i=1}^{m}\{\prod_{t=1}^{p}(a_t-m+1)_{k_i}/k_i!\prod_{j=1}^{q}(b_j-m+1)_{k_i}\}]\sum_{\underset{\sim}{\alpha}}|((\lambda_i w_j)^{k_{\alpha_i}})|$$

$$\text{with} \qquad \sum_{i=1}^{m} k_i = k$$

where $\sum_{\underset{\sim}{\alpha}}$ indicates the summation over $\underset{\sim}{\alpha} = (\alpha_1,\ldots,\alpha_m)$, the permutations of $(1,2,\ldots,m)$. It is easy to verify that

$$(25)\qquad \sum_{\underset{\sim}{\alpha}} \left|\left((\lambda_i w_j)^{k_{\alpha_i}}\right)\right| = \left|\left(\sum_{\alpha=1}^{m} (\lambda_i w_j)^{k_\alpha}\right)\right| = \left|(\lambda_i^{k_j})\right| \left|(w_i^{k_j})\right|,$$

and

$$(26)\qquad \prod_{i=1}^{m}\prod_{t=1}^{s} (v_t - m + 1)_{k_i + m - i} = \prod_{i=1}^{m}\prod_{t=1}^{s} (v_t - i + 1)^{i-1} (v)_\kappa .$$

Obviously changing $k_1,\ldots,k_m$ from inequality $k_1 > .. > k_m \geq 0$ to $k_1 \geq k_2 \geq \ldots \geq k_m \geq 0$ (i.e. $k_i \to k_i + m - i$) and consequently $k \to k + \frac{1}{2}m(m-1)$ and then substituting in (25) and (26), we get

$$(27)\qquad |\underset{\sim}{G}| = \left[\prod_{i=1}^{m}\left\{\prod_{t=1}^{p}(a_t - i+1)^{i-1} \Big/ \prod_{j=1}^{q}(b_j - i+1)^{i-1}\right\}\right] \sum_{k=0}^{\infty} \sum_{\kappa} \left\{\prod_{t=1}^{p}(a_t)_\kappa \Big/ \prod_{j=1}^{q}(b_j)_\kappa\right\}$$

$$\left[\prod_{i=1}^{m}(k_i + m - i)!\right]^{-1} \left|(\lambda_i^{k_j + m - j})\right| \left|(w_i^{k_j + m - j})\right| .$$

Using (27) in the definition of ${}_p\tilde{F}_q^{(m)}(a_1,\ldots,a_p;\, b_1,\ldots,b_q;\, \underset{\sim}{\Lambda},\, \underset{\sim}{W})$ given by (13), we get the required result (20).

The generalised Laguerre polynomials in a hermitian matrix, $\tilde{L}_\kappa^r(\underset{\sim}{S})$, and the generalised Hermite polynomials in a complex matrix, $\tilde{H}_\kappa(\underset{\sim}{T})$, are respectively defined by Hayakawa [2] as under:

$$(28)\quad \mathrm{etr}(-\underset{\sim}{S})\ \tilde{L}_{\kappa}^{r}(\underset{\sim}{S}) = \int_{\underset{\sim}{R}>\underset{\sim}{O}} \{\tilde{\Gamma}_m(r+m)\}^{-1}\ {}_0\tilde{F}_1(r+m;-\underset{\sim}{R}\ \underset{\sim}{S})\ \mathrm{etr}(-\underset{\sim}{R})|\underset{\sim}{R}|^{r}\ \tilde{C}_{\kappa}(\underset{\sim}{R})\ d\underset{\sim}{R}$$

and

$$(29)\quad \mathrm{etr}(-\underset{\sim}{T}\ \overline{\underset{\sim}{T}}')\ \tilde{H}_{\kappa}(\underset{\sim}{T}) = (-1)^k\ \pi^{-mn} \int_{\underset{\sim}{X}} \mathrm{etr}[-\sqrt{-1}(\underset{\sim}{T}\ \overline{\underset{\sim}{X}}'+\underset{\sim}{X}\ \overline{\underset{\sim}{T}}')-\underset{\sim}{X}\ \overline{\underset{\sim}{X}}']\tilde{C}_{\kappa}(\underset{\sim}{X}\ \overline{\underset{\sim}{X}}')\ d\underset{\sim}{X}$$

where $\underset{\sim}{S}$: mxm is a hermitian matrix and $\underset{\sim}{T}$: mxn is a complex matrix.

The following results were established by Hayakawa [2].

$$(30)\qquad \tilde{H}_{\kappa}(\underset{\sim}{T}) = (-1)^k\ \tilde{L}_{\kappa}^{n-m}(\underset{\sim}{T}\ \overline{\underset{\sim}{T}}') \quad \text{if}\quad n > m \ \text{and}\ \underset{\sim}{T}: \text{mxn}.$$

$$(31)\qquad \tilde{L}_{\kappa}^{r}(\underset{\sim}{O}) = (r+m)_{\kappa}\ \tilde{C}_{\kappa}(\underset{\sim}{I}_m),\ \tilde{H}_{\kappa}(0) = (-1)^k\ (n)_{\kappa}\ \tilde{C}_{\kappa}(\underset{\sim}{I}_m).$$

The left side of the equality (32) was proved by Hayakawa [2] in an indirect way. We prove it directly from definition (28).

<u>Lemma 4</u>. Let $\underset{\sim}{Z}$:mxm be a hermitian matrix with the ch. roots $z_1 \geq z_2 \geq \cdots \geq z_m$ such that the absolute values of $z_i$ $(i=1,2,\ldots,m)$ are less than or equal to 1. Then

$$(32)\quad \sum_{k=0}^{\infty}\sum_{\kappa} \tilde{L}_{\kappa}^{r}(\underset{\sim}{S})\ \tilde{C}_{\kappa}(\underset{\sim}{Z})/k!\ \tilde{C}_{\kappa}(\underset{\sim}{I}_m) = |\underset{\sim}{I}-\underset{\sim}{Z}|^{-r-m}\int_{U(m)} \mathrm{etr}[-\underset{\sim}{S}\ \underset{\sim}{U}\ \underset{\sim}{Z}(\underset{\sim}{I}-\underset{\sim}{Z})^{-1}\ \overline{\underset{\sim}{U}}']\ dU$$

$$= \{\prod_{i<j}^{m}(z_i-z_j)(s_i-s_j)\}^{-1}\ |\underset{\sim}{G}(\underset{\sim}{S},\underset{\sim}{Z})|\{\prod_{i=1}^{m}\Gamma(m-i+1)\}$$

where the ch. roots of $\underset{\sim}{S}$ are $s_1 \geq s_2 \geq \ldots \geq s_m$ and $\underset{\sim}{G}(\underset{\sim}{S},\underset{\sim}{Z}) = (g(z_i,s_j))$ with

$$(33) \qquad g(z_i,s_j) = (1-z_i)^{-r-1}\exp(-s_j z_i/(1-z_i)) = \sum_{k=0}^{\infty} L_k^r(s_j)z_i^k/k! \quad \text{for } |z_i| \leq 1,$$

$L_k^r(s)$ being Laguerre polynomial in s.

Proof. Let us write the left side of (32) as L. Then, by definition (28), we have

$$L = \text{etr}\,(\underset{\sim}{S}) \sum_{k=0}^{\infty} \sum_{\kappa} \int_{\underset{\sim}{R}>\underset{\sim}{0}} \{\tilde{\Gamma}_m(r+m)\}^{-1} {}_0\tilde{F}_1(r+m,-\underset{\sim}{R}\underset{\sim}{S})\ \text{etr}(-\underset{\sim}{R})|\underset{\sim}{R}|^r \tilde{C}_\kappa(\underset{\sim}{R})\tilde{C}_\kappa(\underset{\sim}{Z})\{k!\tilde{C}_\kappa(\underset{\sim}{I}_m)\}^{-1} d\underset{\sim}{R}.$$

Interchanging integral and summation sign and using

$${}_0\tilde{F}_0\ (\underset{\sim}{Z},\underset{\sim}{R}) = \int_{U(m)} \text{etr}\ (\underset{\sim}{U}\ \underset{\sim}{Z}\ \bar{\underset{\sim}{U}}'\underset{\sim}{R})\ d\ \underset{\sim}{U},$$

we get

$$(34)\ L = \text{etr}(\underset{\sim}{S})\{\tilde{\Gamma}_m(r+m)\}^{-1} \int_{\underset{\sim}{R}>\underset{\sim}{0}} \int_{U(m)} {}_0\tilde{F}_1(r+m,-\underset{\sim}{R}\underset{\sim}{S})\text{etr}(-\underset{\sim}{U}(\underset{\sim}{I}-\underset{\sim}{Z})\bar{\underset{\sim}{U}}'\underset{\sim}{R})|\underset{\sim}{R}|^r\ d\ \underset{\sim}{R}\ d\ \underset{\sim}{U}\,.$$

Interchanging the two integrals and then integrating over $\underset{\sim}{R}$, we get

$$(35) \qquad L = \text{etr}(\underset{\sim}{S})\ |\underset{\sim}{I}-\underset{\sim}{Z}|^{-r-m} \int_{U(m)} \text{etr}\ [-\underset{\sim}{U}(\underset{\sim}{I}-\underset{\sim}{Z})^{-1}\ \bar{\underset{\sim}{U}}'\underset{\sim}{S}]\ d\ \underset{\sim}{U}$$

because $\operatorname{etr}(\underset{\sim}{X}) = {}_0\tilde{F}_0(\underset{\sim}{X})$. (35) proves the first part of (32). Moreover, (35) can be written as

$$(36) \qquad L = |\underset{\sim}{I}-\underset{\sim}{Z}|^{-r-m} \; {}_0\tilde{F}_0^{(m)}(-\underset{\sim}{S}, \underset{\sim}{Z}(\underset{\sim}{I}-\underset{\sim}{Z})^{-1}) .$$

Using lemma 3, we get

$$(37) \qquad {}_0\tilde{F}_0^{(m)}(-\underset{\sim}{S},\underset{\sim}{Z}(\underset{\sim}{I}-\underset{\sim}{Z})^{-1}) = \prod_{i<j}^{m}[(s_i - s_j)(\frac{z_i}{1-z_i} - \frac{z_j}{1-z_j})]^{-1} \; |\underset{\sim}{A}| \{\prod_{i=1}^{m} \Gamma(m-i+1)\}$$

where $\underset{\sim}{A} = (a_{ij})$, $a_{ij} = \exp\{-s_i z_j/(1-z_j)\}$. Using (37) in (36), we get the second part of (32). Thus, lemma 4 is established.

<u>Corollary 1</u>. The generalised Laguerre polynomials (28) can be calculated from

$$(38) \qquad \tilde{L}_\kappa^r(\underset{\sim}{S}) = \chi_{[\kappa]}(1) \; |(L^r_{k_j+m-j}(s_i))| / |(s_i^{m-j})| .$$

This follows from lemma 4 using the second equality.

<u>Corollary 2</u>. (i) $\sum_{k=0}^{\infty} \sum_{\kappa} \tilde{L}_\kappa^r(\underset{\sim}{S})(-1)^k/k! = 2^{-m(r+m)} \operatorname{etr}(\frac{1}{2}\underset{\sim}{S})$ and

$$(ii) \qquad \sum_{\kappa} \tilde{L}_\kappa^r(\underset{\sim}{S}) = L_k^{m(r+m)-1}(\operatorname{tr}\underset{\sim}{S}) .$$

The proof of (i) is obtained from (32) by putting $\underset{\sim}{Z} = -\underset{\sim}{I}_m$ while that of (ii) is obtained from (32) by putting $Z = z\, I_m$ and then collecting the coefficient of $z^k/k!$ .

Lemma 5. Let $\Sigma$: mxm and $S$: mxm be hermitian positive definite. Then

$$(39) \qquad \int_{S>0} \operatorname{etr}(-\Sigma S)\, \tilde{L}_\kappa^r(S)\, |S|^r\, dS = \tilde{\Gamma}_m(r+m,\kappa)\, |\Sigma|^{-r-m}\, \tilde{C}_\kappa(I-\Sigma^{-1})$$

and

$$(40) \qquad \int_{S>0} \operatorname{etr}(-S)\, \tilde{L}_\eta^r(S)\, \tilde{L}_\kappa^r(S) |S|^r\, dS = 0 \qquad \text{if } \eta \neq \kappa$$

$$= k!\, \tilde{\Gamma}_m(r+m,\kappa)\, \tilde{C}_\kappa(I_m) \quad \text{if } \eta = \kappa.$$

Proof. Let us define $L(S,Z)$ be the left side of (32). Then using the first of the equality of (32), we get

$$\int_{S>0} \operatorname{etr}(-\Sigma\, S)\, |S|^r\, L(S,Z)\, d\, S$$

$$= |I_m-Z|^{-r-m} \int_{S>0} \int_{U(m)} \operatorname{etr}[-(\Sigma+UZ(I-Z)^{-1}\, \bar{U}')S]\, |S|^r\, dU\, dS$$

$$= |I_m-Z|^{-r-m} \int_{U(m)} \int_{S>0} \operatorname{etr}[-(\Sigma-I+U(I-Z)^{-1}\, \bar{U}')S]\, |S|^r\, dS\, dU$$

$$= \tilde{\Gamma}_m(r+m)\, |\Sigma|^{-r-m}\, {}_1\tilde{F}_0^{(m)}(r+m;\, I-\Sigma^{-1},\, Z).$$

Collecting the coefficient of $\tilde{C}_\kappa(Z)$, we get the required result (39). For (40), we have as before

$$\int_{S>0} \operatorname{etr}(-S)\,|S|^r\,\tilde{L}_\eta^r(S)\,L(S,Z)\,dS$$

$$= |I-Z|^{-r-m}\int_{U(m)}\int_{S>0} \operatorname{etr}[-U(I-Z)^{-1}\bar{U}'S]\,|S|^r\,\tilde{L}_\eta^r(S)\,dS\,dU$$

$$= \tilde{\Gamma}_m(r+m,\eta)\,\tilde{C}_\eta(Z) \qquad \text{using (39).}$$

Collecting the coefficient of $\tilde{C}_\kappa(Z)$ from the above, we get (40). Thus, lemma 5 is established.

<u>Corollary 3</u>. Let $T: m\times n$ be a random complex matrix. Then if $\Sigma: m\times m$ is hermitian positive definite,

$$(41) \qquad \int_T \operatorname{etr}(-\Sigma T\bar{T}')\,\tilde{H}_\kappa(T)\,dT = (-1)^k\,\pi^{mn}\,(n)_\kappa\,|\Sigma|^{-n}\,\tilde{C}_\kappa(I-\Sigma^{-1})$$

and

$$(42) \qquad \int_T \operatorname{etr}(-T\bar{T}')\,\tilde{H}_\kappa(T)\,\tilde{H}_\eta(T)\,dT = 0 \qquad \text{if } \kappa \neq \eta$$

$$= k!\,\pi^{mn}(n)_\kappa\,\tilde{C}_\kappa(I_m) \qquad \text{if } \kappa = \eta.$$

This follows from lemma 5 by noting (30) and

$$\tilde{\Gamma}_m(n) \int_{T} \operatorname{etr}(-\Sigma T \bar{T}') f(T \bar{T}') \, dT = \pi^{nm} \int_{S>0} \operatorname{etr}(-\Sigma S) f(S) \, dS.$$

<u>Lemma 6.</u> Let $Z$ and $T$ be arbitrary $m \times n$ $(n \geq m)$ complex matrices. Then,

$$(43) \quad \int_{U(m)} \int_{U(n)} \operatorname{etr}(-Z \bar{Z}' + U_1 T U_2 \bar{Z}' + Z \bar{U}_2' \bar{T}' \bar{U}_1') \, dU_2 \, dU_1$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa} \tilde{H}_\kappa(T) \tilde{C}_\kappa(Z \bar{Z}') \{k! \, (n)_\kappa \tilde{C}_\kappa(I_m)\}^{-1}$$

where $U_1 \in U(m)$ and $U_2 \in U(n)$.

This can be proved in a similar way as that of lemma 4 by noting (9) and (29). This is also given by Hayakawa [2].

<u>Corollary 4.</u>

$$\sum_{k=0}^{\infty} (x^{2k}/k!) \sum_{\kappa} \tilde{H}_\kappa(T) \{(n)_\kappa\}^{-1} = \exp(-mx^2) \int_{U(n)} \operatorname{etr}[x(T_1 U + \overline{T_1 U'})] \, dU$$

$$= \exp(-mx^2) \sum_{k=0}^{\infty} \sum_{\kappa} \tilde{C}_\kappa(T \bar{T}') \, x^{2k} \, \{k! (n)_\kappa\}^{-1}$$

where $T_1' = (T' 0)$: $n \times n$, $T$: $m \times n$, $n > m$.

This follows from (43) by taking $Z = (xI_m \; 0)$. The following lemma can be established in the same way as that of lemma 3 by using corollary 1.

$$(44)\quad \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\{\prod_{i=1}^{p}(a_i)_\kappa\} \tilde{L}_\kappa^r(\Lambda)\, \tilde{C}_\kappa(W)}{\{\prod_{j=1}^{q}(b_j)_\kappa\}\, k!\, \tilde{C}_\kappa(I_m)} = c\, |G_1|\, \{\prod_{i<j}^{m}(\lambda_i - \lambda_j)(w_i - w_j)\}^{-1}$$

where $c$ is the same as defined in (20) and $G_1 = (g_{ij,1})$, $g_{ij,1} = \sum_{k=0}^{\infty}\{\prod_{\alpha=1}^{p}(a_\alpha - m+1)_k\} L_k^r(\lambda_i)\, w_j^k\, \{k!\prod_{\alpha=1}^{q}(b_\alpha - m+1)_k\}^{-1}$ for $i,j = 1,2,\ldots,m$. In particular, when $p = 0$ and $q = 0$, we get (32).

Now, let us consider $E\,\tilde{L}_\kappa^r(S)$ where $S$: mxm $= \nu L \bar{\nu}'$, $\nu$:mxn is a fixed complex matrix and $L$:nxn is distributed as

$$\{\tilde{\Gamma}_n(p)\}^{-1}\, |\Sigma|^{-p}\, |L|^{p-n}\, \mathrm{etr}\,(-\Sigma^{-1}L) \quad \text{for } L > 0,\ p \geq n.$$

Then, using lemma 4, the generating function of $E\,\tilde{L}_\kappa^r(S)$ is given by

$$(45)\qquad \sum_{k=0}^{\infty}\sum_{\kappa} \{E\,\tilde{L}_\kappa^r(S)\}\, \tilde{C}_\kappa(Z)/k!\, \tilde{C}_\kappa(I_m)$$

$$= |I_m - Z|^{-r-m} \int_{U(m)} |I_m + \nu\,\Sigma\,\bar{\nu}' U\, Z(I_m - Z)^{-1}\bar{U}'|^{-p}\, dU\,.$$

Hence, let us write

$$(46)\qquad E\,\tilde{L}_\kappa^r(\nu\, L\, \bar{\nu}') = \xi_\kappa^{(p,r)}(\Lambda),\ \Lambda = \mathrm{diag}\,(\lambda_1,\ldots,\lambda_m),$$

$\lambda \geq \ldots \geq \lambda_m$ are the characteristic roots of $\nu\,\Sigma\,\bar{\nu}'$. Note that the right hand side of (45) can be written as

(47) $$|\underset{\sim}{I}_m - \underset{\sim}{Z}|^{-r-m}\, {}_1\tilde{F}_0^{(m)}(p;\, -\underset{\sim}{\Lambda},\, \underset{\sim}{Z}(\underset{\sim}{I}_m - \underset{\sim}{Z})^{-1})$$

and then using lemma 3, (47) is equal to

(48) $$\{\prod_{i=1}^{m} \Gamma(m-i+1)\}\{\prod_{i=1}^{m} (p-i+1)^{i-1} |(\lambda_i^{m-j})||(z_i^{m-j})|\}^{-1} |\underset{\sim}{G}|$$

where $\underset{\sim}{G} = (g_{ij})$, $g_{ij} = (1-z_i)^{p-r-m}[1 - z_i(1-\lambda_j)]^{-p+m-1}$, $z_1 \geq z_2 \geq \ldots \geq z_m$ are the ch. roots of $\underset{\sim}{Z}$. When $m = 1$ in (46) and (45), we get the univariate result. Hence, we can write

(49) $$g_{ij} = (1-z_i)^{p-r-m}[1-z_i(1-\lambda_j)]^{-p+m-1} = \sum_{k=0}^{\infty} \xi_k^{(p-m+1,r)}(\lambda_j)\, z_i^k/k!.$$

Using this in (48) and equating the coefficient of $\tilde{C}_\kappa(\underset{\sim}{Z})$, we have the following lemma:

<u>Lemma 6</u>. $\xi_\kappa^{(p,r)}(\underset{\sim}{\Lambda})$ defined in (46) satisfies the following relations:

(50) $$\sum_{k=0}^{\infty} \xi_\kappa^{(p,r)}(\underset{\sim}{\Lambda})\, \tilde{C}_\kappa(\underset{\sim}{Z})/k!\, \tilde{C}_\kappa(\underset{\sim}{I}_m)$$

$$= |\underset{\sim}{I}_m - \underset{\sim}{Z}|^{p-n-m} \int_{U(m)} |\underset{\sim}{I}_m - (\underset{\sim}{I}_m - \underset{\sim}{\Lambda})\,\underset{\sim}{U}\,\underset{\sim}{Z}\,\underset{\sim}{U}'|^{-p}\, d\underset{\sim}{U}$$

$$= \prod_{i=1}^{m} \{\Gamma(m-i+1)/(p-i+1)^{i-1}\}\{|(z_i^{m-j})||(\lambda_i^{m-j})|\}^{-1} |(\sum_{k=0}^{\infty} \xi^{(p-m+1,r)}(\lambda_j) z_i^k/k!)|$$

and

$$\xi_{\kappa}^{(p,r)}(\underset{\sim}{\Lambda}) = \chi_{[\kappa]}(1)\ |(\xi_{k_j+m-j}^{(p-m+1,r)}(\lambda_i))|\{\prod_{i=1}^{m}(p-i+1)^{i-1}|(\lambda_i^{m-j})|\}^{-1} \tag{51}$$

where

$$\xi_k^{(p,r)}(\lambda) = \sum_{j=0}^{k} (-1)^j \binom{k}{j}\ \Gamma(p+j)\ \Gamma(r+k+1)\ \{\Gamma(p)\ \Gamma(r+j+1)\}^{-1}\ \lambda^j, \tag{52}$$

or

$$\rho^r(1+\rho)^{-p}\ \xi_k^{(p,r)}(\rho/(1+\rho)) = (\tfrac{d}{d\rho})^k\ [\rho^{r+k}(1+\rho)^{-p}]. \tag{52'}$$

3. Moments of $T = \mathrm{tr}(\underset{\sim}{S}^{-1}\underset{\sim}{X}\ \underset{\sim}{L}\ \underset{\sim}{\bar{X}}')$ and $T_1 = \mathrm{tr}(\underset{\sim}{S}^{-1}\underset{\sim}{X}\ \underset{\sim}{\bar{X}}')$.

(3.1). Moments of T. We have

$$E\ T^k = \Sigma_\kappa\ E\ \tilde{C}_\kappa(\underset{\sim}{S}^{-1}\underset{\sim}{X}\ \underset{\sim}{L}\ \underset{\sim}{\bar{X}}'). \tag{53}$$

Using the following result given by Khatri [5],

$$\int_{\underset{\sim}{S}>\underset{\sim}{0}} \mathrm{etr}(-\underset{\sim}{\Sigma}^{-1}\underset{\sim}{S})|\underset{\sim}{S}|^{r-m}\ \tilde{C}_\kappa\ (\underset{\sim}{A}\ \underset{\sim}{S}^{-1})\ d\underset{\sim}{S} = \tilde{\Gamma}_m(r,-\kappa)\ |\underset{\sim}{\Sigma}|^r\ \tilde{C}_\kappa(\underset{\sim}{\Sigma}^{-1}\underset{\sim}{A})$$

where $\tilde{\Gamma}_m(r,-\kappa) = \pi^{\frac{1}{2}m(m-1)}\prod_{i=1}^{m}\Gamma(r-m-k_i+1) = \tilde{\Gamma}_m(r)/(r)_{(\kappa)}$, $r \geq m + k$,

$(r)_{(\kappa)} = \prod_{i=1}^{m}(r-m+i-1)_{(k_i)}$ and $(r-m+i-1)_{(k_i)} = (r-m+i-1)(r-m+i-2)\ldots(r-m+i-k_i)$, we get

(54) $$E(T^k) = \sum_{\kappa} E\, \tilde{C}_{\kappa}\, (\Sigma_2^{-1} X L \bar{X}')/\ (r)_{(\kappa)}.$$

(3.1.1) Let us assume $\Sigma = \Sigma_1 = \Sigma_2$ and $L$ is fixed. In this case, we shall assume without loss of generality $n \geq m$, because if $n \leq m$, we consider the distribution of $L^{\frac{1}{2}} \bar{X}' \Sigma^{-1} X L^{\frac{1}{2}}$ instead of $\Sigma^{-\frac{1}{2}} X L X' \Sigma^{-\frac{1}{2}} = R$ and $\tilde{C}_{\kappa}(R) = \tilde{C}_{\kappa}(L \bar{X}' \Sigma^{-1} X)$ if $k_1 \geq k_2 \geq \ldots \geq k_n \geq 0$, $n \geq m$ and $\tilde{C}_{\kappa}(R) = 0$, otherwise. Hence, under the condition $\Sigma_1 = \Sigma_2 = \Sigma$ and $n \geq m$, the density function of $R$ is given by

(55) $$\mathrm{etr}(-\Lambda - R)\ \{\tilde{\Gamma}_m(n)\}^{-1}\ |R|^{n-m}\ {}_0\tilde{F}_1(n;\ \Lambda\, R)$$

where $\Lambda = \Sigma^{-\frac{1}{2}} \mu\, L\, \bar{\mu}' \Sigma^{-\frac{1}{2}}$. Then, using the definition (28) in (54), we get

(56) $$E(T^k) = (-1)^k \sum_{\kappa} \tilde{L}_{\kappa}^{n-m}\ (-\Lambda)/(r)_{(\kappa)} \qquad \text{if } r \geq m+k,\ n \geq m \text{ and } L \text{ is fixed.}$$

(3.1.2) Let us assume that $\Sigma_1 = \Sigma_2 = \Sigma$ and $L$ is distributed as Wishart whose density function is given by

(57) $$\{\tilde{\Gamma}_n\,(p)\}^{-1}\ |\Sigma_3|^{-p}\ |L|^{p-n}\ \mathrm{etr}\,(-\Sigma_3^{-1}\, L), \quad \text{for } p \geq n.$$

Then, using (46), we get

$$(58)\qquad E(T^k) = (-1)^k \Sigma_\kappa\, \xi_\kappa^{(p,n-m)}(-\underset{\sim}{\Lambda}_1)/(r)_{(\kappa)} \quad \text{if} \quad r \geq m+k,\ p \geq n \geq m$$

and $\underset{\sim}{\Lambda}_1$ is the diagonal matrix with diagonal elements as the ch. roots of $\underset{\sim}{\Sigma}^{-1}\, \underset{\sim}{\mu}\, \underset{\sim}{\Sigma}_3\, \underset{\sim}{\bar{\mu}}'$.

(3.1.3) Let us assume $\underset{\sim}{\mu} = \underset{\sim}{0}$. Then, using (11) in (54), we get

$$(59)\qquad E(T^k) = \sum_\kappa \tilde{C}_\kappa\, (\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1})\, (n)_\kappa/(r)_{(\kappa)} \quad \text{if} \quad r \geq m+k,\ n \geq m$$

$$= \sum_\kappa \tilde{C}_\kappa\, (\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1})\, (m)_\kappa/(r)_{(\kappa)} \quad \text{if} \quad r \geq m+k,\ n \leq m.$$

(3.2) Moments of $T_1$.

Let us assume $\underset{\sim}{\mu} = \underset{\sim}{0}$. Then using Khatri's result [5, (58) on p. 477], we get

$$(60)\qquad E(T_1^k) = \sum_\kappa \tilde{C}_\kappa\, (\underset{\sim}{L}^{-1})\, \tilde{C}_\kappa(\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1})\, (n)_\kappa/(r)_{(\kappa)}\, \tilde{C}_\kappa(\underset{\sim}{I}_n) \quad \text{if} \quad r \geq m+k,\ n \geq m$$

$$= \sum_\kappa \tilde{C}_\kappa\, (\underset{\sim}{L}^{-1})\, \tilde{C}_\kappa\, (\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1})\, (m)_\kappa/(r)_{(\kappa)}\, \tilde{C}_\kappa(\underset{\sim}{I}_n) \quad \text{if} \quad r \geq m+k,\ n \leq m$$

when $\underset{\sim}{L}$ is fixed, while

$$(61)\qquad E(T_1^k) = \sum_\kappa \tilde{C}_\kappa\, (\underset{\sim}{\Sigma}_3^{-1})\, \tilde{C}_\kappa\, (\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1})\, (n)_\kappa/\{(r)_{(\kappa)}(p)_{(\kappa)}\, \tilde{C}_\kappa(\underset{\sim}{I}_n)\}.$$

where $\underset{\sim}{A} = (a_{ij})$ and

(64) $a_{ij} = E\{w^{m+k_j-j}(1+w)^{-k_j+j-1}\}$ when the density function $w$ is given by

(65) $\Gamma(r+n-m+1)\ \{\Gamma(n-m+1)\ \Gamma(r)\ \theta_i^{n-m+1}\}^{-1}\ w^{n-m+1}/(1+w/\theta_i)^{r+n-m+1}$ for $0 < w < \infty$.

To obtain the convergent expression for $a_{ij}$, we shall write it as

$$(66)\quad a_{ij} = \{\Gamma(n-m+1)\Gamma(r)(1+\theta_i)^{r+n-m+1}\}^{-1}\theta_i^r\Gamma(r+n-m+1)\int_0^1 x^{n+k_j-j}(1-x)^{r-m}\left[1 - \frac{1-x+x\ \theta_i}{1+\theta_i}\right]^{-r-n+m-1} dx.$$

Hence (63) can be rewritten as

$$(63')\quad E\ \tilde{C}_\kappa(\underset{\sim}{W}^{-1}+\underset{\sim}{I}_m)^{-1} = \pi^{m(m-1)}\{\Gamma(r-m+1)\Gamma(n)\}^m\{\tilde{\Gamma}_m(r)\tilde{\Gamma}_m(n)|(\theta_i^{m-j})|\}^{-1}\ |\underset{\sim}{\theta}|^r\ |\underset{\sim}{I}_m+\underset{\sim}{\theta}|^{-r-n+m+1}\ |\underset{\sim}{B}|$$

where $\underset{\sim}{B} = (b_{ij})$,

$$(64')\quad b_{ij} = \{\Gamma(n)\Gamma(r-m+1)\}^{-1}\Gamma(r+n-m+1)\int_0^1 x^{n+k_j-j}(1-x)^{r-m}\left[1-\frac{1-x+x\theta_i}{1+\theta_i}\right]^{-r-n+m-1} dx.$$

in which $\underset{\sim}{B} = (b_{ij})$ and

(75) $b_{ij} = E\, x^{m+k_j-j}$ when the density function of $x$ is given by

(77) $$(1-\rho_i)^{p-m+1}\, \Gamma(r+n-m+1)\, \{\Gamma(r)\Gamma(n-m+1)\}^{-1}\, x^{n-m}\, (1-x)^{r-1}$$

$$_2F_1\,(r+n-m+1,\ p-m+1;\ n-m+1;\ \rho_i x).$$

Noting (69) and (74), we can rewrite these expressions in terms of moment generating function as under:

(78) $$E(\exp(\varphi V)) = \{\Gamma(r)\}^m \{\prod_{i=1}^{m} \Gamma(r-i+1)\ |(\lambda_i^{m-j})|\}^{-1}\ |\underset{\sim}{G}(\varphi,\underset{\sim}{\Lambda})|$$

where $\underset{\sim}{G}(\varphi,\underset{\sim}{\Lambda}) = (g_{ij}(\varphi,\lambda_i))$ and $\underset{\sim}{L}$ is fixed,

(79) $g_{ij}(\varphi,\lambda_i) = E[x^{m-j} \exp(\varphi\, x)]$, the density of $x$ is given by (71),

while

(80) $$E(\exp(\varphi V)) = \{\Gamma(r)\}^m \{\prod_{i=1}^{m} (p-i+1)^{i-1}\ \Gamma(r-i+1)\}\ |(\rho_i^{m-j})|^{-1} |\underset{\sim}{I}-\underset{\sim}{\rho}|^{m-1} |\underset{\sim}{G}_1(\varphi,\underset{\sim}{\rho})|,$$

where $\underset{\sim}{G}_1(\varphi,\underset{\sim}{\rho}) = (g_{ij}^{(1)}(\varphi,\rho_i))$ and the density of $\underset{\sim}{L}$ is given by (57),

(81) $g_{ij}^{(1)}(\varphi,\rho_i) = E[x^{m-j}\exp(\varphi x)]$, the density of $x$ is given by (77).

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13. ABSTRACT

Let the joint density function of complex random variables of $\underset{\sim}{X}$: mxn and $\underset{\sim}{S}$: mxm be constant $|\underset{\sim}{S}|^{r-m}$ etr$[-\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{S} - \underset{\sim}{\Sigma}_1^{-1}(\underset{\sim}{X}-\underset{\sim}{\mu})\underset{\sim}{L}(\overline{\underset{\sim}{X}-\underset{\sim}{\mu}})']$ where $\underset{\sim}{\Sigma}_1$, $\underset{\sim}{\Sigma}_2$ and $\underset{\sim}{L}$ are hermitian positive definite, $\underset{\sim}{\mu}$: pxn is complex and fixed, and L be fixed or random. In this paper, the moments of $T = tr(\underset{\sim}{X}\,\underset{\sim}{L}\,\overline{\underset{\sim}{X}}'\underset{\sim}{S}^{-1})$, $T_1 = tr(\underset{\sim}{X}\,\overline{\underset{\sim}{X}}'\underset{\sim}{S}^{-1})$ and

$V = tr\,\underset{\sim}{X}\,\underset{\sim}{L}\,\overline{\underset{\sim}{X}}'(\underset{\sim}{S} + \underset{\sim}{X}\,\underset{\sim}{L}\,\overline{\underset{\sim}{X}}')^{-1}$ are established under various situations in terms of generalized Laguerre polynomials and zonal polynomials in complex arguments. Some of the properties of Laguerre polynomials are studied.

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