Some Distribution Problems Concerning Characteristic Roots and Vectors in Multivariate Analysis

by

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CHAPTER I

POWER COMPARISONS OF TESTING $\delta \Sigma_1 = \Sigma_2$ BASED ON INDIVIDUAL CHARACTERISTIC ROOTS

1. Introduction and Summary

In this chapter, exact non-central distributions of individual characteristic roots have been obtained first in two and three roots cases in connection with tests of the hypothesis $\delta \Sigma_1 = \Sigma_2$, where Σ_1 and Σ_2 are covariance matrices of two normal populations and $\delta > 0$, known. Powers of tests using individual roots are tabulated for the test of this hypothesis against various one-sided simple alternatives and comparisons of powers made.

2. Non-Central cdf of the Largest Root For Testing $\delta \Sigma_1 = \Sigma_2$

Let $S_i(p \times p)$, $(i = 1, 2)$ be independently distributed as Wishart (n_i, p, Σ_i). Let the characteristic (Ch.) roots of $S_1 S_2^{-1}$ and $\Sigma_1 \Sigma_2^{-1}$ be denoted by c_i and λ_i, $i = 1, \ldots, p$ respectively such that $0 < c_1 < c_2 \ldots < c_p < \infty$ and $0 < \lambda_1 < \ldots < \lambda_p < \infty$. Let $g_i = \delta c_i/(1+\delta c_i)$, $i = 1, \ldots, p$; $\delta > 0$ and $G = \text{diag}\,(g_1, \ldots, g_p)$ and $\Lambda = \text{diag}\,(\lambda_1, \ldots, \lambda_p)$, then the distribution of $g_1, \ldots, g_p$ is given by Khatri [14] in the following form

$$(2.1) \qquad c(p,m,n)|\delta \Lambda|^{-\frac{1}{2}n_1}|G|^m|I-G|^n \prod_{i>j} (g_i - g_j)\ {}_1F_0(\tfrac{1}{2}\nu;\ \Lambda_1, G) \quad ,$$

where

$$c(p,m,n) = [\pi^{p/2} \prod_{i=1}^{p} \Gamma\{\tfrac{1}{2}(2m+2n+p+i+2)\}] \,/$$

$$[\prod_{i=1}^{p} \Gamma\{\tfrac{1}{2}(2m+i+1)\}\, \Gamma\{\tfrac{1}{2}(2n+i+1)\}\, \Gamma(\tfrac{i}{2})] \quad ,$$

$\underset{\sim}{\Lambda}_1 = \underset{\sim}{I} - (\delta\underset{\sim}{\Lambda})^{-1}$, $m = \frac{1}{2}(n_1-p-1)$, $n = \frac{1}{2}(n_2-p-1)$, $n_1 + n_2 = \nu$ and ${}_1F_0$ is the hypergeometric function of matrix argument defined by James [10] as

(2.2) $${}_sF_t(a_1,\ldots, a_s;\, b_1,\ldots, b_t;\, \underset{\sim}{S}, \underset{\sim}{T}) =$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots\cdots (a_s)_\kappa\; C_\kappa(\underset{\sim}{S}) C_\kappa(\underset{\sim}{T})}{(b_1)_\kappa \cdots\cdots (b_t)_\kappa\; C_\kappa(\underset{\sim}{I}_p) k!} \quad ,$$

where $a_1,\ldots, a_s$, $b_1,\ldots, b_t$ are real or complex constants and the multivariate coefficient $(a)_\kappa$ is given by

(2.3) $$(a)_\kappa = \prod_{i=1}^{p} (a-\tfrac{1}{2}(i-1))_{k_i} \quad ,$$

where

(2.4) $$(a)_k = a(a+1) \ldots (a+k-1)$$

and κ is the partition of k such that $\kappa = (k_1,\ldots, k_p)$, $k_1 \geq k_2 \geq \ldots \geq k_p \geq 0$ and the zonal polynomials $C_\kappa(\underset{\sim}{S})$ are expressible in terms of elementary symmetric functions (esf) of the characteristic roots of $\underset{\sim}{S}$[10].

Now define by $V(q_p,n;\ \ldots\ldots;\ x',\ x'',\ q_j,\ n;\ldots\ldots;\ q_1,n)$ the determinant

$$
(2.5)\qquad \left|\begin{array}{l}
\int_{x_{p-1}}^{1} x_p^{q_p}(1-x_p)^n\,dx_p \int_{x_{p-2}}^{1} x_{p-1}^{q_p}(1-x_{p-1})\,dx_{p-1}\cdots\int_{x'}^{x''} x_j^{q_p}(1-x_j)^n dx_j\cdots \\
\qquad\qquad \int_0^{x_3} x_2^{q_p}(1-x_2)^n\,dx_2 \int_0^{x_2} x_1^{q_p}(1-x_1)^n\,dx_1 \\
\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\int_{x_{p-1}}^{1} x_p^{q_1}(1-x_p)^n\,dx_p \int_{x_{p-2}}^{1} x_{p-1}^{q_1}(1-x_{p-1})^n\,dx_{p-1}\cdots \int_{x'}^{x''} x_j^{q_1}(1-x_j)^n dx_j\cdots \\
\qquad\qquad \int_0^{x_3} x_2^{q_1}(1-x_2)^n\,dx_2 \int_0^{x_2} x_1^{q_1}(1-x_1)^n\,dx_1
\end{array}\right| .
$$

It may be observed that the cdf of the largest root from (2.1) under the null hypothesis $\delta\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ can be thrown into the form $V(0,x;q_p,n;\ \ldots,\ q_1;n)$, which for simplicity of notation will be written hereafter $V(0,x;q_p,\ldots,q_1;n)$, multiplied by $C(p,m,n)$ [16], [17], [19]. Further, in view of the fact that the zonal polynomials $C_\kappa(\underset{\sim}{S})$ in (2.2) can be expressed in terms of the esf's of ch-roots of $\underset{\sim}{S}$, by the use of Pillai's lemma on the multiplication of the basic Vandermonde type determinant by powers of esf's, [19], it is easy to see that the non-central distribution of the cdf of g_p in (2.1) can be expressed as a series whose terms are linear compounds of determinants of type $V(0,x;q_p',\ldots,\ q_1';\ n)$, where $(q_p',\ldots,\ q_1')$ may differ from term to term.

Further, it has been shown that [16], [17]

$$
(2.6)\qquad V(0,x;q_s,q_{s-1},\ldots,q_1;n) = (q_s+n+1)^{-1}(A^{(s)}+ B^{(s)}+ q_sC^{(s)}) ,
$$

where

$$A^{(s)} = - I_o(0,x;q_s, n+1)\, V(0,x;q_{s-1},\dots,q_1;n),$$

$$B^{(s)} = 2 \sum_{j=s-1}^{1} (-1)^{s-j-1}\, I(0,x;q_s + q_j; 2n+1)$$

$$V(0,x;q_{s-1},\dots, q_{j+1},q_{j-1},\dots, q_1;n),\ C^{(s)} = V(0,x;q_s-1,q_{s-1},\dots,$$

$$q_1;n),\ I_o(x',x'';q_s,n+1) = x^{q_s}(1-x)^{n+1} \Big|_{x'}^{x''}, \text{ and}$$

$$I(x',x'';q,r) = \int_{x'}^{x''} x^q(1-x)^r\, dx \quad .$$

It may be noted that $C^{(s)}$ vanishes if $q_s = q_{s-1}+1$. Using (2.6) in each of the determinants of the linear compounds involved in the series obtainable from (2.2), after the necessary number of reductions, the cdf of the largest root, g_p, can be ultimately reduced in terms of simple incomplete beta functions.

3. Non-Central cdf's of Individual Roots

In this section we give the non-central cdf's of individual roots, associated power function tabulations and comparisons of powers for testing $\delta\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ against various simple hypotheses.

a) Non-Central cdf of g_2. Now putting $p = 2$ in (2.1) and using the method outlined in the preceding section the cdf of the largest root is obtained in the following form:

$$(3.1) \qquad \Pr\{g_2 \le x\} = K\Big\{-I_o(0,x;m+1,n+1)\Big[\Big(\sum_{i=0}^{6} B_i x^i\Big)I(0,x;m,n)$$

$$+ \Big(\sum_{i=2}^{6} C_i x^{i-1}\Big)I(0,x;m+1,n) + \Big(\sum_{i=4}^{6} D_i x^{i-2}\Big)I(0,x;m+2,n)$$

$$+ E_6 x^3 I(0,x;m+3,n)\Big] + 2\Big[(B_6 + C_6 + D_6 + E_6)$$

$$I(0,x;2m+7,2n+1) + (B_5 + C_5 + D_5)I(0,x;2m+6,2n+1)$$

$$+ (B_4 + C_4 + D_4)I(0,x,2m+5,2n+1)$$

$$+ (B_3 + C_3)I(0,x;2m+4,2n+1) + (B_2 + C_2)I(0,x;2m+3,2n+1)$$

$$+ B_1 I(0,x;2m+2,2n+1) + B_0 I(0,x;2m+1,2n+1)\Big]\Big\}$$

where $K = (\delta^2\lambda_1\lambda_2)^{-\frac{1}{2}n_1} c(2,m,n)$, B's, C's, D's and E_6 are obtained from Pillai [24] by making the following changes:

In the A_{ij} coefficients in [24], delete each linear factor involving n_2 in the denominator, each linear factor involving ν in the numerator should be raised only to a single power instead of two and b_1 and b_2 should be changed to $2 - (1/\lambda_1 + 1/\lambda_2)/\delta$ and $[1-1/(\delta\lambda_1)][1-1/(\delta\lambda_2)]$ respectively.

In obtaining the cdf of g_2 on (3.1), zonal polynomials of degree 1 to 6 were used. The expression for the cdf of g_2 in (3.1) has been used to compute the power of test H_o: $\delta\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$, $\delta > 0$, known, against $\delta\lambda_i \ge 1$, $i = 1,\ldots, p$, $\sum_{i=1}^{p} (\delta\lambda_i) > p$, for various pairs of values

$(\delta\lambda_1, \delta\lambda_2)$ and the results are presented in Table 1.

b) Non-central cdf's of individual roots for $p = 3$.

i) Largest root: Put $p = 3$ in (2.1) and using the method outlined in section (2), the cdf of the largest root is obtained in the following form.

$$\begin{aligned}
(3.2)\qquad \Pr\{g_3 \le x\} = K_1\Big\{-I_o(0,x;m+2,n+1)\Big[&\Big(\sum_{i=0}^{6} B_i^{(0)}x^i\Big)V(0,x;m+1,m;n) \\
&+ \Big(\sum_{i=2}^{6} C_i^{(0)}x^{i-1}\Big)V(0,x;m+2,m;n) \\
&+ \Big(\sum_{i=3}^{6} D_i^{(0)}x^{i-2}\Big)V(0,x;m+2,m+1;n) \\
&+ \Big(\sum_{i=4}^{6} E_i^{(0)}x^{i-2}\Big)V(0,x;m+3,m;n) \\
&+ \Big(\sum_{i=5}^{6} F_i^{(0)}x^{i-3}\Big)V(0,x;m+3,m+1;n) \\
&+ G^{(0)}x^3V(0,x;m+4,m;n) + H^{(0)}x^2V(0,x;m+3,m+2;n)\Big] \\
&+ 2I(0,x;m,n)\sum_{i=0}^{6}\big(B_i^{(1)}I(0,x;2m+3+i,2n+1)\big) \\
&- 2I(0,x;m+1,n)\sum_{i=0}^{6}\big(B_i^{(2)}I(0,x;2m+2+i,2n+1)\big)
\end{aligned}$$

$$- 2I(0,x;m+2,n) \sum_{i=0}^{4} (B_i^{(3)} I(0,x;2m+3+i,2n+1))$$

$$- 2I(0,x;m+3,n) \sum_{i=0}^{2} (B_i^{(4)} I(0,x;2m+4+i,2n+1))$$

$$- 2G^{(0)} I(0,x;m+4,n)\ I(0,x;2m+5,2n+1)\Big\} \quad ,$$

where

$$K_1 = c(3,m,n)\left(\prod_{i=1}^{3} \delta\lambda_i\right)^{-\frac{1}{2}n_1} \quad ,$$

and the $B_i^{(0)}$'s, $C_i^{(0)}$'s, $D_i^{(0)}$'s, $E_i^{(0)}$'s, $F_i^{(0)}$'s, G^0 and the $B_i^{(j)}$ coefficients are obtained from corresponding coefficients in Pillai and Dotson [23] by making changes in the A_{ij} coefficients as described in the preceding section and $b_1 = 3 - \frac{1}{\delta\lambda_1} - \frac{1}{\delta\lambda_2} - \frac{1}{\delta\lambda_3}$,

$b_2 = (1 - \frac{1}{\delta\lambda_1})(1 - \frac{1}{\delta\lambda_2}) + (1 - \frac{1}{\delta\lambda_1})(1 - \frac{1}{\delta\lambda_3}) + (1 - \frac{1}{\delta\lambda_2})(1 - \frac{1}{\delta\lambda_3})$ and

$b_3 = (1 - \frac{1}{\delta\lambda_1})(1 - \frac{1}{\delta\lambda_2})(1 - \frac{1}{\delta\lambda_3})$.

ii) Smallest root: The non-central cdf's of the smallest root for $p = 2, 3$ are obtained from the corresponding non-central cdf's of the largest root by making the following changes.

$$\begin{aligned} (3.3) \qquad & - I_o(0,x;q_p,n+1) \rightarrow (-1)^p\ I_o(x,1;q_p,n+1) \\ & I(0,x;q,r) \rightarrow I(x,1;q,r) \\ & V(0,x;q_p,\ldots,\ q_1;n) \rightarrow V(q_p,n;\ldots;x,1,q_1,n) \quad . \end{aligned}$$

iii) Median root: In obtaining the non-central cdf of the median root for $p = 3$, the following changes may be made in (3.2)

$$- I_o(0,x;m+2,n+1) \rightarrow I_o(0,x;m+2,n+1)$$

$$V(0,x;q_2,q_1;n) \rightarrow I(x,1;q_2,n)I(0,x;q_1,n) - I(x,1;q_1,n)I(0,x;q_2,n)$$

$$I(0,x;q_j,n)I(0,x;q_3+ q_j,2n+1) \rightarrow \beta(q_j+1,n+1)I(x,1;q_3+ q_j,2n+1),$$

$$j = 1,2 \quad .$$

Tabulations of powers of individual roots for test of hypothesis H_o given earlier have been done extensively and in Table 2 are presented powers for selected values of the parameters.

4. Power Comparisons

For tabulating the powers of the tests of H_o based on individual roots for $p = 2$ and $p = 3$ against simple alternatives such that $\delta\lambda_i \geq 1$, $i = 1,\ldots, p$, $\sum_{i=1}^{p} \delta\lambda_i > p$, the upper 5% points for the largest root were taken from Pillai [24] and those of the median and smallest roots from Pillai and Dotson [23]. These were used to compute powers on IBM 7094 for values of $m = 0,1,2,5$ and $n = 5(5)30,40,60$ but in Tables 1 and 2 are presented only the tabulations for $n = 5,15$ and 40.

Now we compare the powers of individual roots for the test of H_o. Cases $p = 2$ and $p = 3$ may be considered separately.

$p = 2$. When $p = 2$, the following observations may be made (Table 1).

1) Although the larger root has generally more power than the smaller root, for small values of n, the smaller root has generally greater power for small deviations (except for $m = 0$).

2) For $\delta(\lambda_1 + \lambda_2)$ = constant and small deviations, the power of the larger root decreases as the two roots tend to be equal while that of the smaller root increases.

3) The individual root possesses monotonicity property of power with respect to individual population roots but not with respect to their sum or product.

4) For larger deviations or larger values of n, the power of the largest root is always greater (and more often considerably so) than that of the smaller root.

<u>$p = 3$</u>. The following observations may be made when $p = 3$.

1') Although the largest root has generally more power than the other roots, for small values of n and small deviations, the median root has greater power and sometimes (for $m = 2$ and 5) even the smallest root. But the power of the smallest root is always less than that of the median root.

2') For $\delta(\lambda_1 + \lambda_2 + \lambda_3)$ = constant, the power of the largest root seems to attain its maximum when $\delta\lambda_1 = \delta\lambda_2 = 1$ (at least for small deviations) while those of the other two roots when $\delta\lambda_1 = \delta\lambda_2 = \delta\lambda_3$. The power of the largest root decreases as the roots tend to be equal (at least for small deviations) while those of the other two increase.

3') is the same as 3) above for $p = 2$.

4') For large n, the power of the largest root is generally greater than those of the others except possibly in the case of the median root when the population roots tend to be equal.

It may be pointed out that the monotonicity property of the power of the individual roots with respect to individual population roots for the above test was shown earlier by Anderson and Das Gupta [3]. A comparative study of powers of four criteria for this test has been carried out by Pillai and Jayachandran [24].

Table 1. Powers of individual roots for $p = 2$ for testing

$\delta\lambda_1 = 1$, $\delta\lambda_2 = 1$ against different simple alternative hypotheses, $\alpha = .05$

$\delta\lambda_1$	$\delta\lambda_2$	g_2	g_1	g_2	g_1	g_2	g_1	g_2	g_1
		m=0,n=5		m=1,n=5		m=2,n=5		m=5,n=5	
1	1.001	.050073	.050067	.050080	.050084	.050084	.050094	.050089	.050112
1	1.1	.05778	.056531	.05854	.05833	.05901	.05945	.05975	.06133
1.05	1.05	.05763	.05683	.05834	.05873	.05877	.05992	.05944	.06192
1	1.5	.0955	.0793	.1022	.0885	.1067	.0946	.1153	.1050
1.25	1.25	.0926	.0864	.0977	.0988	.1009	.1073	.1063	.1225
1	2	.152	.101	.173	.118	.189	.128	.239	.142
1.333	1.5	.126	.112	.137	.136	.144	.152	.160	.183
1	4	.378	.132	.479	.143	.573	.136	.807	
1	5	.467	.	.594		.703		.912	
2	4	.469		.586		.689		.895	
3	3	.472		.587		.692		.902	
1	8	.645		.794		.890		.989	
4.5	4.5	.690		.833		.920		.994	
1	11	.745		.883		.951		.998	
6	6	.810		.929		.977		.999	
		m=0,n=15		m=1,n=15		m=2,n=15		m=5,n=15	
1	1.001	.050093	.050072	.050106	.050094	.050115	.050109	.050133	.050139
1	1.1	.05998	.05698	.06164	.05930	.06284	.06095	.06516	.06419
1.05	1.05	.05973	.05732	.06126	.05980	.06234	.06157	.06440	.06508
1	1.5	.1105	.0811	.1254	.0927	.1371	.1011	.1630	.1181
1.25	1.25	.1058	.0891	.1172	.1054	.1258	.1181	.1434	.1456
1	2	.186	.104	.228	.124	.262	.138	.348	.165
1.333	1.5	.151	.117	.175	.147	.193	.172	.235	.227
1	4	.456	.134	.585	.149	.684	.153	.873	
1	5	.548		.692		.792		.945	
2	4	.557		.692		.787		.939	
3	3	.564		.696		.792		.943	
1	8	.716		.857		.930		.993	
4.5	4.5	.769		.893		.953		.997	
1	11	.803		.922		.970		.999	
6	6	.867		.957		.987		.999	

Table 1. (Continued)

$\delta\lambda_1$	$\delta\lambda_2$	g_2	g_1	g_2	g_1	g_2	g_1	g_2	g_1
		m=0,n=40		m=1,n=40		m=2,n=40		m=5,n=40	
1	1.001	.050102	.050073	.050120	.050098	.050134	.050117	.050162	.050155
1	1.1	.06105	.05716	.06332	.05975	.06509	.06168	.06902	.06583
1.05	1.05	.06075	.05752	.06282	.06029	.06442	.06239	.06786	.06695
1	1.5	.1179	.0818	.1382	.0945	.1556	.1041	.1992	.1252
1.25	1.25	.1124	.0902	.1280	.1085	.1408	.1235	.1709	.1596
1	2	.203	.105	.257	.126	.304	.143	.425	.181
1.333	1.5	.163	.118	.196	.153	.223	.182	.289	.253
1	4	.487	.135	.632	.151	.737	.162	.911	
1	5	.580		.734		.834		.963	
2	4	.593		.737		.832		.961	
3	3	.601		.744		.837		.964	
1	8	.742		.882		.947		.996	
4.5	4.5	.798		.916		.967		.998	
1	11	.823		.937		.978		.999	
6	6	.887		.968		.992		.999	

Table 2. Powers of individual roots for $p = 3$ for testing

$\delta\lambda_i = 1$, $i = 1,2,3$ against different simple alternative hypothesis, $\alpha = .05$

$\delta\lambda_1$	$\delta\lambda_2$	$\delta\lambda_3$	g_3	g_2	g_1	g_3	g_2	g_1	g_3	g_2	g_1
			m = 0, n = 5			m = 0, n = 15			m = 0, n = 40		
1	1	1.001	.050054	.050061	.050046	.050073	.050069	.050048	.050083	.050073	.050049
1	1	1.15	.05891	.05924	.05651	.06239	.06051	.05676	.06432	.06110	.05687
1	1.05	1.1	.05868	.05951	.05691	.06194	.06086	.05720	.06373	.06149	.05733
1.05	1.05	1.05	.05856	.05964	.05713	.06171	.06103	.05744	.06344	.06168	.05757
1	1	1.5	.08482	.08001	.06865	.10097	.08398	.06928	.11012	.08577	.06954
1	1.25	1.25	.08208	.08390	.07291	.09580	.08905	.07385	.10352	.09145	.07426
1	1	2	.132	.107	.080	.172	.112	.081	.194	.115	.082
1	1	3	.248	.137	.090	.329	.147	.091	.370	.151	.091
1	2	2	.224	.198	.128	.300	.222	.131	.341	.232	.132
1	1	4	.366	.146	.100	.464	.157	.101	.511	.161	.101
1	2	3	.343	.240	.140	.445	.271	.144	.496	.286	.145
2	2	2	.319	.310	.199	.424	.360	.207	.477	.385	.210
1	1	6	.554			.650			.692		
1	2	5	.556			.655			.700		
2	2	4	.542			.653			.703		
2	3	3	.543			.656			.708		
1	1	7	.623			.711			.749		
1	2	6	.633			.722			.762		
2	2	5	.628			.727			.771		
3	3	3	.637			.739			.785		
1	1	8	.678			.758			.792		
1	1	10	.760			.824			.851		

Table 2. (Continued)

$\delta\lambda_1$	$\delta\lambda_2$	$\delta\lambda_3$	g_3	g_2	g_1	g_3	g_2	g_1	g_3	g_2	g_1
			m = 1, n = 5			m = 1, n = 15			m = 1, n = 40		
1	1	1.001	.050057	.050070	.050061	.050080	.050083	.050065	.050094	.050089	.050067
1	1	1.15	.05948	.06067	.05858	.06392	.06271	.05919	.06662	.06373	.05947
1	1.05	1.1	.05919	.06096	.05915	.06330	.06311	.05985	.06576	.06420	.06017
1.05	1.05	1.05	.05905	.06109	.05945	.06299	.06329	.06020	.06533	.06441	.06055
1	1	1.5	.08806	.08555	.07486	.10991	.09221	.07640	.12376	.09541	.07708
1	1.25	1.25	.08445	.09012	.08118	.10234	.09878	.08361	.11351	.10321	.08473
1	1	2	.145	.116	.090	.201	.128	.092	.235	.134	.093
1	1	3	.302	.148	.098	.406	.166	.099	.465	.178	.100
1	2	2	.259	.148	.159	.359	.274	.167	.417	.296	.171
1	1	4	.466	.144	.107	.576	.167	.102	.634	.184	.103
1	2	3	.427	.260	.168	.544	.317	.179	.608	.348	.184
2	2	2	.372	.363	.214	.498	.458	.236	.568	.521	.247
1	1	6	.697			.778			.819		
1	2	5	.694			.779			.822		
2	2	4	.661			.762			.813		
2	3	3	.661			.763			.815		
1	1	7	.768			.835			.867		
1	2	6	.774			.842			.875		
2	2	5	.756			.836			.874		
3	3	3	.763			.842			.881		
1	1	8	.820			.874			.900		
1	1	10	.886			.923			.940		

Table 2 (Continued)

$\delta\lambda_1$	$\delta\lambda_2$	$\delta\lambda_3$	g_3	g_2	g_1	g_3	g_2	g_1	g_3	g_2	g_1
			m = 2, n = 5			m = 2, n = 15			m = 2, n = 40		
1	1	1.001	.050059	.050075	.050069	.050086	.050092	.050077	.050102	.050101	.050081
1	1	1.15	.05985	.06162	.05995	.06506	.06435	.06095	.06851	.06582	.06145
1	1.05	1.1	.05952	.06191	.06064	.06429	.06479	.06179	.06739	.06636	.06237
1.05	1.05	1.05	.05936	.06204	.06101	.06392	.06498	.06225	.06685	.06659	.06287
1	1	1.5	.09043	.08930	.07906	.11719	.09855	.08158	.13586	.10336	.08277
1	1.25	1.25	.08604	.09430	.08697	.10741	.10640	.09109	.12197	.11308	.09312
1	1	2	.157	.123	.097	.226	.140	.100	.273	.148	.101
1	1	3	.365	.146	.101	.478	.174	.103	.548	.195	.104
1	2	2	.300	.245	.176	.414	.310	.192	.485	.345	.200
1	1	4	.569	.169		.671	.204		.731	.242	
1	2	3	.516	.250		.632	.329		.700	.377	
2	2	2	.408	.382		.544	.524		.627	.628	
1	1	6	.809			.865			.897		
1	2	5	.802			.865			.899		
2	2	4	.747			.831			.877		
2	3	3	.745			.827			.874		
1	1	7	.869			.910			.932		
1	2	6	.871			.915			.937		
2	2	5	.840			.898			.928		
3	3	3	.840			.894			.925		
1	1	8	.908			.938			.954		
1	1	10	.952			.968			.977		

Table 2 (Continued)

$\delta\lambda_1$	$\delta\lambda_2$	$\delta\lambda_3$	g_3	g_2	g_1	g_3	g_2	g_1	g_3	g_2	g_1
			m = 5, n = 5			m = 5, n = 15			m = 5, n = 40		
1	1	1.001	.050062	.050085	.050085	.050096	.050110	.050101	.050120	.050127	.050109
1	1	1.15	.06046	.06326	.06241	.06738	.06770	.06454	.07288	.07053	.06575
1	1.05	1.1	.06006	.06355	.06333	.06629	.06820	.06580	.07108	.07121	.06724
1.05	1.05	1.05	.05987	.06367	.06381	.06576	.06839	.06649	.07021	.07146	.06806
1	1	1.5	.09616	.09593	.08747	.13431	.11179	.09246	.16730	.12146	.09523
1	1.25	1.25	.08896	.10202	.09758	.11816	.12294	.10703	.14235	.13695	.11244
1	1	2	.208	.129	.138	.300	.158	.123	.377	.176	.124
1	1	3	.577	.108	.010	.671	.143	.162	.744	.180	.149
1	2	2	.446	.236	.139	.566	.348	.168	.656	.426	.191
1	1	4	.817			.868			.905		
1	2	3	.750			.826			.877		
2	2	2	.495			.658			.842		
1	1	6	.965			.976			.984		
1	2	5	.959			.975			.984		
2	2	4	.680			.733			.812		
2	3	3	.523			.487			.513		
1	1	7	.983			.989			.993		
1	2	6	.982			.989			.993		
2	2	5	.876			.907			.941		
3	3	3	.999			.999			.999		
1	1	8	.991			.994			.996		
1	1	10	.997			.998			.999		

CHAPTER II

NON-CENTRAL DISTRIBUTIONS OF THE SECOND LARGEST ROOTS OF THREE MATRICES AND THE VECTORS CORRESPONDING TO THE LARGEST AND SECOND LARGEST ROOTS

1. Introduction and Summary

In this chapter, the non-central distributions of the second largest roots in the MANOVA situation, the canonical correlations, and equality of two covariance matrices are obtained. The central distribution of the second largest (smallest) root following the Fisher-Girshick-Hsu-Roy distribution under certain null-hypothesis comes as a special case of the MANOVA situation. Further, the distribution of the second largest root of the covariance matrix is obtained as limiting case. The largest root and its non-central distributions have been considered by Pillai and Sugiyama [25] for the situations stated above. However, in this chapter, the joint densities of the largest and the second largest roots are derived in all the above cases from which the distributions of the largest roots can be obtained, although in more elaborate forms. In the last section the distribution of the characteristic vectors is obtained corresponding to the largest and second largest root of a sample covariance matrix. The three roots-case is dealt with in more detail.

2. Non-Central Distribution of the Second Largest Root in the MANOVA Case

Let $\underset{\sim}{X}$ be a $p \times n_1$ matrix variate $(p \leq n_1)$ and $\underset{\sim}{Y}$ a $p \times n_2$ matrix variate $(p \leq n_2)$ and the columns be all independently normally distributed with covariance matrix $\underset{\sim}{\Sigma}$, $E\underset{\sim}{X} = \underset{\sim}{M}$ and $E\underset{\sim}{Y} = \underset{\sim}{0}$. Then it is known that $\underset{\sim}{X}\,\underset{\sim}{X}' = \underset{\sim}{S}_1$ is non-central Wishart with n_1 degrees of freedom and $\underset{\sim}{Y}\,\underset{\sim}{Y}' = \underset{\sim}{S}_2$ is central Wishart with n_2 degrees of freedom and the covariance matrix $\underset{\sim}{\Sigma}$, respectively.

Let $0 < \ell_1 < \ell_2 < \ldots < \ell_p < 1$ be the characteristic roots of $\underset{\sim}{S}_1\,\underset{\sim}{S}_2^{-1}$, then the joint density function of $\ell_1, \ldots, \ell_p$ is given by Constantine [4]

$$(2.1) \qquad c(p,m,n)\exp(\mathrm{tr}\,-\underset{\sim}{\Omega})\,|\underset{\sim}{L}|^m\,|\underset{\sim}{I}-\underset{\sim}{L}|^n \prod_{i>j}(\ell_i-\ell_j)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}\nu)_\kappa c_\kappa(\underset{\sim}{\Omega})c_\kappa(\underset{\sim}{L})}{(n_{1/2})_\kappa\; c_\kappa(\underset{\sim}{I})k!}$$

where $\underset{\sim}{\Omega}$ is the non-centrality matrix, $\frac{1}{2}\underset{\sim}{M}'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{M}$, and $\underset{\sim}{L} = \mathrm{diag}(\ell_1, \ldots, \ell_p)$ and $c(p,m,n)$, m, n and ν is defined in (2.1), of Chapter 1, and $c_\kappa(\underset{\sim}{L})$ are zonal polynomials defined in [10]. Consider the transformation $q_i = \ell_i/\ell_{p-1}$, $i = 1, \ldots, p-2$, and decompose $c_\kappa(\underset{\sim}{L}) = \sum_{\tau,\mu} a_{\tau,\mu}\, \ell_p^{k_1} c_\mu(\underset{\sim}{L}_1)$ where $\underset{\sim}{L}_1 = \mathrm{diag}(\ell_1, \ldots, \ell_{p-1})$ and the summation is over the partitions τ of k_1 and μ of k_2 such that $k_1 + k_2 = k$, and κ is the partition of k, and $a_{\tau,\mu}$ are constants defined in [8]. Then the joint distribution of $q_1, \ldots, q_{p-2}$, ℓ_{p-1}, ℓ_p can be written in the form

$$(2.2)\qquad q(\ell_{p-1},\ell_p)|\underset{\sim}{Q}|^m|\underset{\sim}{I}-\underset{\sim}{Q}|\,|\underset{\sim}{I}-\ell_{p-1}\underset{\sim}{Q}_1|^n|\underset{\sim}{I}-(\ell_{p-1}|\ell_p)\underset{\sim}{Q}_1|$$

$$\prod_{i>j}(q_i-q_j)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}\nu)_\kappa c_\kappa(\underset{\sim}{\Omega})}{k!c_\kappa(\underset{\sim}{I})(n_1/2)_\kappa}\sum_{\tau,\mu}a_{\tau,\mu}\,\ell_p^{k_1}\ell_{p-1}^{k_2}\,c_\mu(\underset{\sim}{Q}_1)$$

where $\underset{\sim}{Q} = \mathrm{diag}(q_1,\ldots,q_{p-2})$, $\underset{\sim}{Q}_1 = \mathrm{diag}(q_1,\ldots,q_{p-2},1)$, and $q(\ell_{p-1},\ell_p) = c(p,n_1,n_2)\exp \mathrm{tr}-\underset{\sim}{\Omega}\,.\;\ell_{p-1}^{m(p-1)+\frac{1}{2}(p-2)(p+1)}\ell_p^{m+p-1}(1-\ell_p)^n$. By expanding $|\underset{\sim}{I}-\ell_{p-1}\underset{\sim}{Q}_1|^n$ as well as $|\underset{\sim}{I}-(\ell_{p-1}|\ell_p)\underset{\sim}{Q}_1|$ and the use of the results from Khatri and Pillai [15] for multiplication of zonal polynomials we write (2.2) in the form

$$(2.3)\qquad q(\ell_{p-1},\ell_p)|\underset{\sim}{Q}|^m|\underset{\sim}{I}-\underset{\sim}{Q}|\prod_{i>j}(q_i-q_j)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}\nu)_\kappa c_\kappa(\underset{\sim}{\Omega})}{k!(n_1/2)_\kappa c_\kappa(\underset{\sim}{I})}$$

$$\sum_{s=0}^{\infty}\sum_{\eta}((-n)_\eta\ell_{p-1}^s/s!)\sum_{\ell=0}^{p-2}(c(\ell)\ell_{p-1}^{\ell}/\ell!\ell_p^{\ell})$$

$$\sum_{\tau,\mu}a_{\tau,\mu}\,\ell_p^{k_1}\,\ell_{p-1}^{k_2}\sum_{\delta'}g_{\ell\,(1,\eta,\mu)}^{\delta'}\,c_{\delta'}(\underset{\sim}{Q}_1)\quad ,$$

where η and δ' are the partitions of s and $\ell+s+k_2$ respectively such that $\eta=(\eta_1,\ldots,\eta_p)$ and $\delta'=(\delta_1,\ldots,\delta_p)$ where $s=\sum_{i=1}^{p}\eta_i$, $\ell+s+k_2=\sum_{i=1}^{p}\delta_i$, $g_{\ell\,(1,\eta,\mu)}^{\delta'}$ are constants defined in [15] and

$$c(\ell)=\frac{(-1)^\ell(2\ell)!}{(\ell!)\;2^\ell\;\chi_{[21^\ell]}(1)}\quad ,$$

where $\chi_{[21^\ell]}$ is the degree of the representation $[21^\ell]$ of the symmetric group on 2ℓ symbols, and such that $\chi_{[\kappa]}(1) = k! \prod_{i<j}^{p} (k_i - k_j - i + j) / \prod_{i=1}^{p} (k_i + p - 1)!$ and $\kappa = (k_1 \geq k_2 \geq \dots \geq k_p \geq 0)$. Now integrate (2.3) with respect to $0 \leq q_1 \leq q_2 \leq \dots \leq q_{p-2} < 1$ by the use of the lemma in [29], we get the joint density function of ℓ_{p-1}, ℓ_p in the form

$$(2.4) \qquad (\Gamma_{p-1}((p-1)/2)/\Pi^{(p-1)^2/2})\Gamma_{p-1}(p/2)q(\ell_{p-1}, \ell_p)$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}\nu)_\kappa c_\kappa(\underset{\sim}{\Omega})}{k!(n_1/2)c_\kappa(\underset{\sim}{I})} \sum_{s=0}^{\infty} \sum_{\eta} \{(-n)_\eta / s!\} \sum_{\ell=0}^{p-2} \{c(\ell)/\ell! \ell_p^\ell\}$$

$$\sum_{\tau,\mu} a_{\tau,\mu} \ell_p^{k_1} \ell_{p-1}^{s+\ell+k_2} \sum_{\delta'} g_{\ell\,(1,\eta,\mu)}^{\delta'} c_{\delta'}(\underset{\sim}{I}_{p-1}) \; ((n_1-1)(p-1) /$$

$$2+s+\ell+k_2)(\Gamma_{p-1}((n_1-1)/2,\delta')/\Gamma_{p-1}((n_1+p-1)/2,\delta') \quad .$$

Further, integrate (2.4) with respect to ℓ_p, then the density function of ℓ_{p-1} can be written

$$(2.5) \qquad c_1(p,n_1,n_2)\exp(\mathrm{tr}\,\underset{\sim}{\Omega})\,\ell_{p-1}^{m(p-1)+(p-2)(p+1)/2} \cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}\nu)_\kappa c_\kappa(\underset{\sim}{\Omega})}{k!(n_1/2)_\kappa c_\kappa(\underset{\sim}{I}_p)}$$

$$\sum_{s=0}^{\infty} \sum_{\eta} \{(-n)_\eta / s!\} \sum_{\ell=0}^{p-2} (c(\ell)/\ell!) \sum_{\tau,\mu} a_{\tau,\mu}\, \ell_{p-1}^{s+\ell+k_2} \cdot$$

$$I(\ell_{p-1},1;m+p+k_1-\ell-1;n) \sum_{\delta'} g_{\ell\,(1,\eta,\mu)}^{\delta'} \; c_{\delta'}(\underset{\sim}{I}_{p-1})(n_1-1)(p-1)\,/$$

$$2+s+\ell+k_2)(\Gamma_{p-1}((n_1-1)/2,\delta')/\Gamma_{p-1}((n_1+p-1)/2,\delta'))$$

where $c_1(p,n_1,n_2) = \Pi^{p-1}\Gamma_p(\nu/2)\Gamma_{p-1}((p-1)/2)/\Gamma_p(n_1/2)\Gamma_p(n_2/2)$. It may be pointed out that the density function of the largest root can be obtained from (2.4) by integrating it with respect to ℓ_{p-1} over the range $0 < \ell_{p-1} < \ell_p$, however a simpler form has been given in [25] .

Let $\underset{\sim}{\Omega} = \underset{\sim}{0}$ in (2.5) then the central case is of the form

$$(2.6) \qquad c_1(p,n_1,n_2)\,\ell_{p-1}^{m(p-1)+(p-2)(p+1)/2} \sum_{s=0}^{\infty} \sum_{\eta} \{(-n)_\eta/s!\} \sum_{\ell=0}^{p-2} \{c(\ell)/\ell!\} \cdot$$

$$\cdot\; \ell_{p-1}^{s+\ell}\, I(\ell_{p-1},1;m+p-\ell-1;n) \sum_{\delta'(1,\eta)} g_{\ell}^{\delta'} \; c_{\delta'}(\underset{\sim}{I}_{p-1})((n_1-1)(p-1)\,/$$

$$2+s+\ell)\Gamma_{p-1}((n_1-1)/2,\delta')/\Gamma_{p-1}((n_1+p-1)/2,\delta'))$$

where δ' is the partition of $\ell+s$.

3. The Distribution of the Second Largest Root in the Canonical Correlation Case

Let the columns of $\begin{pmatrix} \underset{\sim}{X}_1 \\ \underset{\sim}{X}_2 \end{pmatrix}$ be n independent normal $(p+q)$-dimensional variates $(p \leq q)$ with zero means and covariance matrix

$$\Sigma = \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}'_{12} & \underset{\sim}{\Sigma}_{22} \end{pmatrix} .$$

Let $\underset{\sim}{R} = \text{diag}\,(r_1, r_2, \ldots, r_p)$, where $r_1^2, \ldots, r_p^2$ are the characteristic roots of the equation

$$|\underset{\sim}{X}_1 \underset{\sim}{X}'_2 (\underset{\sim}{X}_2 \underset{\sim}{X}'_2)^{-1} \underset{\sim}{X}_2 \underset{\sim}{X}'_1 - r^2 \underset{\sim}{X}_1 \underset{\sim}{X}'_1| = 0$$

and also $\underset{\sim}{P} = \text{diag}\,(\rho_1, \rho_2, \ldots, \rho_p)$ where $\rho_1^2, \ldots, \rho_p^2$ are the characteristic roots of the equation

$$|\underset{\sim}{\Sigma}_{12} \underset{\sim}{\Sigma}_{22}^{-1} \underset{\sim}{\Sigma}'_{12} - \rho\, \underset{\sim}{\Sigma}_{11}| = 0 .$$

Then, the density function of $r_1^2, \ldots, r_p^2$ is given by Constantine [4] in the following form

$$(3.1) \qquad c(n,p,q)|\underset{\sim}{I}-\underset{\sim}{P}^2|^{n/2}\, |\underset{\sim}{R}^2|^{(q-p-1)/2}\, |\underset{\sim}{I}-\underset{\sim}{R}^2|^{(n-p-q-1)/2} \prod_{i>j} (r_i^2 - r_j^2)$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa (n/2)_\kappa\; c_\kappa(\underset{\sim}{R}^2)\; c_\kappa(\underset{\sim}{P}^2)}{(q/2)_\kappa \quad k! \quad c_\kappa(\underset{\sim}{I}_p)}$$

where

$$c(n,p,q) = \frac{\Gamma_p(n/2) \quad \Pi^{p^2/2}}{\Gamma_p(q/2)\,\Gamma_p((n-q)/2)\Gamma_p(p/2)} \quad .$$

By using the same transformation, namely $q_i = \dfrac{r_i^2}{r_{p-1}^2}$, $i = 1,\ldots, p-2$ and the same method as in section 2, the joint density function of r_{p-1}^2, r_p^2 can be shown to have the following form

$$(3.2) \qquad c_1(n,p,q)|I-P^2|^{n/2}(r_{p-1}^2)^{\{(q-p-1)(p-1)+(p-2)(p+1)\}/2}$$

$$(r_p^2)^{(q+p-3)/2}(1-r_p^2)^{(n-p-q-1)/2}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(n/2)_\kappa(n/2)_\kappa c_\kappa(\underset{\sim}{P}^2)}{(q/2)_\kappa k!\,c_\kappa(\underset{\sim}{I}_p)}$$

$$\sum_{s=0}^{\infty}\sum_{\eta}\frac{((p+q+1-n/2)_\eta}{s!}\sum_{\ell=0}^{p-2}\{c(\ell)/\ell!(r_p^2)^{\ell}\}\sum_{\tau,\mu} a_{\tau,\mu}\,(r_p^2)^{k_1}$$

$$(r_{p-1}^2)^{s+\ell+k_2}\sum_{\delta'} g_{\ell\,(1,\eta,\mu)}^{\delta'} \quad c_{\delta'}(\underset{\sim}{I}_{p-1})((q-1)(p-1)/2+s+\ell+k_2)$$

$$(\Gamma_{p-1}((q-1)/2,\delta'))/\Gamma_{p-1}((q+p-1)/2,\delta') \quad ,$$

where $c_1(n,p,q) = \Pi^{p-1}\,\Gamma_{p-1}((p-1)/2)\,\Gamma_p(n/2)/\,\Gamma_p(q/2)\,\Gamma_p((n-q)/2)$. Now, integrate (3.2) with respect to r_p^2 then the density function of r_{p-1}^2 can be written in the form

$$(3.3)\qquad c_1(n,p,q)|\underset{\sim}{I}-\underset{\sim}{P}^2|^{n/2}\,(r_{p-1}^2)^{\{(q-p-1)(p-1)+(p-2)(p+1)\}/2}$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(n/2)_\kappa(n/2)_\kappa c_\kappa(\underset{\sim}{P}^2)}{(q/2)_\kappa k!\,c_\kappa(\underset{\sim}{I}_p)}\sum_{s=0}^{\infty}\sum_{\eta}\frac{(p+q+1-n)/2)_\eta}{s!}$$

$$\sum_{\ell=0}^{p-2}\{c(\ell)/\ell!\}\sum_{\tau,\mu}a_{\tau,\mu}(r_{p-1}^2)^{s+\ell+k_2}\,I(r_{p-1}^2,1;(q+p-3)/2+k_1-\ell;$$

$$(n-p-q-1)/2)\sum_{\delta'}g_{\ell\,(1,\eta,\nu)}^{\delta'}\;c_{\delta'}(\underset{\sim}{I}_{p-1})((q-1)(p-1)/2+s+\ell+k_2)$$

$$(\Gamma_{p-1}((q-1)/2,\delta')/\Gamma_{p-1}\,((q+p-1)/2,\delta'))\quad .$$

4. Non-Central Distribution of the Second Largest Root of $\underset{\sim}{S}_1\,\underset{\sim}{S}_2^{-1}$

In this section we consider the distribution of the second largest root of $\underset{\sim}{S}_1\,\underset{\sim}{S}_2^{-1}$ as defined in (2.1) of Chapter 1. Then, as before, we can obtain the joint density function of g_{p-1} and g_p in the following form

$$(4.1) \quad c_1(p,n_1,n_2)|\delta\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1} g_{p-1}^{m(p-1)+\frac{1}{2}(p-2)(p+1)}$$

$$g_p^{m+p-1} (1-g_p)^n \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}\nu)_\kappa c_\kappa(\underset{\sim}{I}-(\delta\underset{\sim}{\Lambda})^{-1})}{k!\ c_\kappa(I)} \sum_{s=0} \sum_{\eta} \{(-n)_\eta/s!\}$$

$$\sum_{\ell=0}^{p-2} \{c(\ell)/\ell!\} \sum_{\tau,\mu} a_{\tau,\mu}\, g_p^{k_1-\ell}\, g_{p-1}^{s+\ell+k_2} \sum_{\delta'} g_{\ell\ (1,\eta,\mu)}^{\delta'}\, c_{\delta'}(I_{p-1})$$

$$((n_1-1)(p-1)/2 + s+\ell+k_2)(\Gamma_{p-1}((n_1-1)/2,\delta') /$$

$$\Gamma_{p-1}((n_1+p-1)/2,\delta') \quad .$$

Now, integrate (4.1) with respect to g_p, the density function of g_{p-1} can be written in the following form

$$(4.2) \quad c_1(p,n_1,n_2)|\delta\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1} g_{p-1}^{m(p-1)+(p-2)(p+1)/2} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}\nu)_\kappa c_\kappa(I-(\delta\underset{\sim}{\Lambda})^{-1})}{k!\ c_\kappa\ (I)}$$

$$\sum_{s=0}^{\infty} \sum_{\eta} \{(-n)/s!\} \sum_{\ell=0}^{p-2} \{c(\ell)/\ell!\} \sum_{\tau,\mu} a_{\tau,\mu}\, g_{p-1}^{s+\ell+k_2}$$

$$I(g_{p-1},1;m+p+k_1-\ell-1;n) \sum_{\delta'} g_{\ell\ (1,\eta,\mu)}^{\delta'}\, c_{\delta'}(I_{p-1})((n_1-1)(p-1)/2$$

$$+ s+\ell+k_2)(\Gamma_{p-1}((n_1-1)/2,\delta')\Gamma_{p-1}((n_1+p-1)/2,\delta')) \quad .$$

5. The Distribution of the Second Largest Root of a Covariance Matrix

The distribution of the characteristics roots, $0 < \omega_1 \leq \cdots \leq \omega_p < \infty$, of $\underset{\sim}{X}\,\underset{\sim}{X}'$ depends only upon the characteristic roots of $\underset{\sim}{\Sigma}$ and can be given in the form (James [9])

$$(5.1) \qquad k(p,n)|\Sigma|^{-\frac{1}{2}n}|\underset{\sim}{W}|^{m}\{\exp(-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{W})\} \prod_{i>j}(\omega_i - \omega_j)\ {}_0F_0(\tfrac{1}{2}(\underset{\sim}{I}_p - \underset{\sim}{\Sigma}^{-1}), \underset{\sim}{W})$$

where $k(p,n) = \Pi^{\frac{1}{2}p^2}/2^{\frac{1}{2}pn}\,\Gamma_p(n/2)\,\Gamma_p(p/2)$, $W = \mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_p)$.

It may be pointed out that the form (5.1) can also be viewed as a limiting form of (4.1), when $n_2 \to \infty$.

However, by methods similar to those in the previous sections, the density function of the second largest root $\gamma_{p-1} = \frac{\omega_{p-1}}{2}$ can be written in the form

$$(5.2) \qquad k_1(p,n)|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}\,\gamma_{p-1}^{m(p-1)+\frac{(p-2)(p+1)}{2}}\,e^{-\gamma_{p-1}}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{c_\kappa(\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1})}{k!\,c_\kappa(\underset{\sim}{I})}$$

$$\sum_{\tau,\nu} a_{\tau,\nu}\sum_{s=0}^{\infty}\sum_{\eta}\frac{(-1)^s}{s!}\sum_{\ell=0}^{p-2}\{c(\ell)\gamma_{p-1}^{s+\ell+k_2}/\ell!\}\sum_{i=0}^{k_2}\sum_{\delta} b_{\delta,\nu}\sum_{\mu} g^{\mu}_{(\delta,1^\ell,\eta)}$$

$$c_\mu(I_{p-2})[\Gamma_{p-2}(n-2)/2,\mu)/\Gamma_{p-2}((n+p)/2,\mu)]$$

$$[\gamma(\gamma_{p-1},\infty;m+p+k_1-j) - \gamma_{p-1}\,\gamma(\gamma_{p-1},\infty;m+p+k_1-j-1)] \quad ,$$

where $b_{\delta,\nu}$ are constants defined in [15], δ and μ are the partions of i and $i+\ell+s$ respectively.

$$k_1(p,n) = k(p,n)\ \Gamma_{p-2}((p-2)/2)\Gamma_{p-2}((p+1)/2)/\Pi^{\frac{(p-2)^2}{2}} ,$$

and

$$\gamma(a,b;c) = \int_a^b x^{c-1} e^{-x} dx \quad .$$

It may be noted that the cdf of the second largest root can be obtained by integrating the corresponding densities over the region $0 \leq \ell_{p-1} \leq x$. Hence from (2.6) we obtain

$$(5.3) \qquad \Pr\{\ell_{p-1} \leq x\} = c_1(p,n_1,n_2) \sum_{s=0}^{\infty} \sum_{\eta} \{(-n)_\eta/s!\} \sum_{\ell=0}^{p-2} \{c(\ell)/\ell!\}$$

$$[I(x,1;b,n)\ x^{a+1} + I(0,x;a+b+1;n)] \sum_{\delta'} g^{\delta'}_{\ell\,(1^\ell,\eta)}$$

$$c_{\delta'}(\underset{\sim}{I}_{p-1})((n_1-1)(p-1)/2+s+\ell)\Gamma_{p-1}((n_1-1)/2,\delta') /$$

$$(a+1)\Gamma_{p-1}((n_1+p-1)/2,\delta')) \quad .$$

where $a = m(p-1) + (p-2)(p+1)/2 + s + \ell$, $b = m+p-\ell-1$. The individual characteristic root could be very useful in testing hypotheses, for instance, Anderson [2] in testing the null hypotheses that the rank of $\Omega = r$ against the alternative that it is greater we reject the null hypothesis if the $p - r$ smallest roots are not sufficiently small.

6. The Distribution of the Characteristic Vectors Corresponding to the Largest and Second Largest Roots of a Sample Covariance Matrix

Let $\underset{\sim}{U}$ has the Wishart distribution $W(p,n,\underset{\sim}{\Sigma})$, the probability elements of $\underset{\sim}{U}$ are

$$(6.1) \qquad K_2|\underset{\sim}{U}|^{(n-p-1)/2} \exp\left(-\tfrac{1}{2}\mathrm{tr}\ \underset{\sim}{\Sigma}^{-1}\ \underset{\sim}{U}\right) d\,\underset{\sim}{U} \quad ,$$

where

$$K_2 = |\underset{\sim}{\Sigma}^{-1}|^{-\frac{1}{2}n}/2^{pn/2}\ \Gamma_p(\tfrac{1}{2}n) \quad .$$

Now there exists an orthogonal matrix $\underset{\sim}{L}$ such that $\underset{\sim}{\Sigma} = \underset{\sim}{L}\ \underset{\sim}{D}_{\mu}\ \underset{\sim}{L}'$ where $\underset{\sim}{D}_{\mu} = \mathrm{diag}\ (\mu_1, \ldots, \mu_p)$ and further make the transformation $\underset{\sim}{V} = \underset{\sim}{L}'\ \underset{\sim}{U}\ \underset{\sim}{L}$, then the distribution of $\underset{\sim}{V}$ is given by

$$(6.2) \qquad K_2|\underset{\sim}{V}|^{(n-p-1)/2} \exp\left(-\tfrac{1}{2}\mathrm{tr}\ \underset{\sim}{D}_{\gamma}\ \underset{\sim}{V}\right) d\underset{\sim}{V} \quad ,$$

where $\gamma_i = 1/\mu_i$, $(i = 1, \ldots, p)$.

Now transform $\underset{\sim}{V} = \underset{\sim}{H}\ \underset{\sim}{W}\ \underset{\sim}{H}'$ where the orthogonal matrix $\underset{\sim}{H}$ is represented in terms of rotations angles. The $p \times p$ orthogonal matrix has only $p(p-1)/2$ independent elements and every rotation in the p-dimensional space consists of $p(p-1)/2$ single rotations which is such a rotation in the two dimensional space. Let $\underset{\sim}{R}_p^{\nu}(\theta)$ be a single rotation matrix defined by

$$(6.3) \qquad \underset{\sim}{R}_p^{\nu}(\theta) = \begin{pmatrix} \underset{\sim}{I}_{p-\nu} & o & o & 0 \\ o & \cos\theta & -\sin\theta & o \\ o & \sin\theta & \cos\theta & o \\ 0 & o & o & \underset{\sim}{I}_{\nu-2} \end{pmatrix},$$

where $\underset{\sim}{I}_{\nu}$ is the identity matrix $(\nu \times \nu)$, then H is defined by

$$(6.4) \qquad \underset{\sim}{H} = \underset{\sim}{H}_p^{p}(\theta_{pj})\, \underset{\sim}{H}_p^{p-1}(\theta_{p-1,j}) \cdots \underset{\sim}{H}^{2}(\theta_{22})$$

and

$$H_p^{\nu}(\theta_j) = R_p^2(\theta_2) \cdots R_p^{\nu}(\theta_{\nu});\ 0 \leq \theta_{i2} \leq 2\pi;\ 0 \leq \theta_{ij} \leq \pi, (j \geq 3)$$

and

$$\underset{\sim}{W} = \text{diag}\,(\omega_p, \omega_{p-1}, \ldots, \omega_1),\ 0 < \omega_1 < \omega_2 < \ldots < \omega_p < \infty \ .$$

Then the Jacobion of this transformation as found by Tumura [31] will be

$$(6.5) \qquad \prod_{i>j} (\omega_i - \omega_j) \prod_{i=p}^{3} \prod_{j=i}^{3} \sin^{j-2} \theta_{ij}$$

$$(6.6)\qquad \operatorname{tr} \underset{\sim}{D}_\gamma \underset{\sim}{V} = \operatorname{tr} \underset{\sim}{D}_\gamma \underset{\sim}{H}^p(\theta_{pj}) \begin{pmatrix} 1 & 0 \\ 0 & \underset{\sim}{H}_{p-1} \end{pmatrix} \begin{pmatrix} \omega_p & & 0 \\ & \omega_{p-1} & \\ 0 & & \ddots \omega_1 \end{pmatrix} .$$

$$\begin{pmatrix} 1 & 0 \\ 0 & \underset{\sim}{H}_{p-1} \end{pmatrix}' H^{p'} = h_p' \underset{\sim}{D}_\gamma h_p \omega_p + \operatorname{tr} \underset{\sim}{H}^{p'} \underset{\sim}{D}_\gamma \underset{\sim}{H}^p \begin{pmatrix} 1 & 0 \\ 0 & \underset{\sim}{H}_{p-1} \end{pmatrix}$$

$$\begin{pmatrix} \omega_{p-1} & & 0 \\ & \ddots & \\ 0 & & \omega_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \underset{\sim}{H}_{p-1} \end{pmatrix}'$$

where $0 < \omega_1 < \omega_2 < \ldots < \omega_p < \infty$ and h_p is the first column of $\underset{\sim}{H}^p(\theta_{pj})$, and h_p will be of the form

$$(6.7)\quad h_p' = (h_{pp} h_{p,p-1} \cdots h_{p1}) = (\cos\theta_{pp} \sin\theta_{pp} \cos\theta_{p,p-1} \cdots$$

$$\prod_{\nu=p}^{p-i} \sin\theta_{p\nu} \cos\theta_{pi} \cdots \prod_{\nu=p}^{3} \sin\theta_{p\nu} \cos\theta_{p2} \prod_{\nu=p}^{3} \sin\theta_{p\nu} \sin\theta_{p2}) \quad ,$$

$\underset{\sim}{H}^p(\theta_{pj})$ is an orthogonal matrix with $p - 1$ independent elements θ_{pp}, $\theta_{p,p-1} \cdots \theta_{p2}$, $\underset{\sim}{H}_{p-1}$ is the orthogonal matrix of the $p - 1$ dimensional space with $(p-1)(p-2)/2$ independent elements θ_{ij}, $(i = p-1, \ldots, 2,$ $j = i, \ldots, 2)$ denote $\underset{\sim}{D}_{p-1}$ as the $(p-1) \times (p-1)$ matrix obtained from $\underset{\sim}{H}^{p'} \underset{\sim}{D}_\gamma \underset{\sim}{H}^p$, deleting the first row and column, and $\underset{\sim}{W}_1 = \operatorname{diag}(\omega_{p-1}, \ldots, \omega_1)$ then (6.6) can be written

$$(6.8) \qquad \mathrm{tr}\, \underset{\sim}{D}_{\gamma}\underset{\sim}{V} = h_p' \underset{\sim}{D}_{\gamma} h_p \omega_p + \mathrm{tr}\, \underset{\sim}{D}_{p-1} \underset{\sim}{H}_{p-1} \underset{\sim}{W}_1 \underset{\sim}{H}_{p-1}'$$

$$= h_p' \underset{\sim}{D}_{\gamma} h_p \omega_p + \mathrm{tr}\, \underset{\sim}{D}_{p-1} \underset{\sim}{H}_{p-1}^{p-1}(\theta_{p-1,j})$$

$$\begin{pmatrix} 1 & o \\ o & \underset{\sim}{H}_{p-2} \end{pmatrix} \begin{pmatrix} \omega_{p-1} & & o \\ & \ddots & \\ o & & \omega_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \underset{\sim}{H}_{p-2} \end{pmatrix}' \underset{\sim}{H}_{p-1}^{p-1}(\theta_{p-1,j})$$

$$= h_p' \underset{\sim}{D}_{\gamma} h_p \omega_p + h_{p-1}' \underset{\sim}{D}_{p-1} h_{p-1} \omega_{p-1} + \mathrm{tr} \underset{\sim}{D}_{p-2} \underset{\sim}{H}_{p-2} \underset{\sim}{W}_2 \underset{\sim}{H}_{p-2}' ,$$

where $\underset{\sim}{H}_{p-1}^{p-1}(\theta_{p-1,j})$ is an orthogonal matrix with p-2 independent elements $\theta_{p-1,p-1}, \theta_{p-1,p-2}, \ldots, \theta_{p-1,2}$ and is obtained from $\underset{\sim}{H}_p^{p-1}(\theta_{p-1,j})$ by deleting the 1st row and column H_{p-2} is the orthogonal matrix of the p-2 dimensional space with $\frac{1}{2}(p-2)(p-3)$ independent elements θ_{ij}, $i = p-2,\ldots 2$, $j = i,\ldots, 2$; h_{p-1} is the first vector of $\underset{\sim}{H}_{p-1}^{p-1}(\theta_{p-1,j})$ and is given by

$$(6.9) \qquad h_{p-1} = (h_{p-1,p-1},\ldots,h_{p-1,1}) = (\cos\theta_{p-1,p-1} \sin\theta_{p-1,p-1} \cdots\cdots \prod_{\nu=p-1}^{p-i-1} \sin\theta_{p-1,\nu} \cos\theta_{p-1,i} \cdots\cdots \prod_{\nu=p-1}^{3} \sin\theta_{p-1,\nu}$$

$$\cos\theta_{p-1,2} \prod_{\nu=p-1}^{3} \sin\theta_{p-1,\nu} \sin\theta_{p-1,2})$$

and $\underset{\sim}{W}_2 = \mathrm{diag}\,(\omega_{p-2},\ldots,\omega_1)$, and D_{p-2} is the $(p-2) \times (p-2)$ matrix obtained from $\underset{\sim}{H}_{p-1}^{p-1\prime} \underset{\sim}{D}_{p-1} \underset{\sim}{H}_{p-1}^{p-1}$ by deleting the first row and column. Hence the distribution of $\omega_1,\ldots,\omega_p$, $\theta_{ij}(i = p,\ldots,2,\ j = i,\ldots,2)$ can be written in the form

(6.10) $$K_2(\omega_p\omega_{p-1})^{\frac{1}{2}(n-p-1)}|\underset{\sim}{W}_2|^{\frac{1}{2}(n-p-1)}\exp(-\tfrac{1}{2}h_p' \underset{\sim}{D}_\gamma h_p\omega_p)$$

$$\exp(-\tfrac{1}{2}h_{p-1}'\underset{\sim}{D}_{p-1}h_{p-1}\omega_{p-1})\ \exp(-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{D}_{p-2}\underset{\sim}{H}_{p-2}\underset{\sim}{W}_2\underset{\sim}{H}_{p-2}'$$

$$\cdot \prod_{i=p-2}^{3}\ \prod_{j=i}^{3} \sin^{j-2}\theta_{ij} \cdot \prod_{j=p}^{3} \sin^{j-2}\theta_{pj} \prod_{p-1}^{3} \sin^{j-2}\theta_{p-1,j}$$

$$\prod_{i>j}(\omega_i-\omega_j) \quad .$$

Consider the case, $p = 3$. The joint density of θ_{33}, θ_{32}, θ_{22}, ω_1, ω_2, ω_3 can be deduced from (6.10) and is given by (writing K_2 again for K_2, $p = 3$)

(6.11) $$K_2|\omega_1\omega_2\omega_3|^{\frac{1}{2}(n-4)} \prod_{i>j}^{3} (\omega_i-\omega_j) \sin\theta_{33}\exp(-\tfrac{1}{2}h_3' \underset{\sim}{D}_\gamma h_3\omega_3)$$

$$\exp\left(-\mathrm{tr}\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix}\begin{pmatrix}\cos\theta_{22} & -\sin\theta_{22} \\ \sin\theta_{22} & \cos\theta_{22}\end{pmatrix}\right.$$

$$\left.\begin{pmatrix}\omega_2 & 0 \\ 0 & \omega_1\end{pmatrix}\begin{pmatrix}\cos\theta_{22} & \sin\theta_{22} \\ -\sin\theta_{22} & \cos\theta_{22}\end{pmatrix}\right)$$

where

$$h_3' = (\cos\theta_{33}\ \cos\theta_{32}\ \sin\theta_{33}\ \sin\theta_{32}\ \sin\theta_{33}) \quad ,$$

$$a_{11} = \gamma_1 + (\gamma_2 - \gamma_1)\cos^2\theta_{33} + (\gamma_3 - \gamma_2)\sin^2\theta_{32}\cos^2\theta_{33} \quad ,$$

$$a_{12} = (\gamma_3 - \gamma_1)\cos\theta_{33}\sin 2\theta_{32}/2 \quad ,$$

and

$$a_{22} = \gamma_2 + (\gamma_3 - \gamma_2) \cos^2 \theta_{32} \quad .$$

Now (6.11) can be written in the form

$$(6.12) \qquad K_2(\omega_1\omega_2\omega_3)^{\frac{1}{2}(n-4)} \prod_{i>j} (\omega_i - \omega_j) \sin \theta_{33} \exp(-b_3 \omega_3)$$

$$\left[\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} b_2^i b_1^{k-i} \omega_2^i \omega_1^{k-i} \right]$$

where

$$b_1 = -\tfrac{1}{2}(\sin \theta_{22} - \cos \theta_{22}) \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \sin \theta_{22} \\ -\cos \theta_{22} \end{pmatrix} \quad ,$$

$$b_2 = -\tfrac{1}{2}(\cos \theta_{22} \ \sin \theta_{22}) \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \cos \theta_{22} \\ \sin \theta_{22} \end{pmatrix} \quad ,$$

and

$$b_3 = \tfrac{1}{2} h_3' \underset{\sim}{D}_\gamma h_3 \quad .$$

Let $\ell = \omega_1/\omega_2$, then the distribution of $\theta_{33}, \theta_{32}, \theta_{22}, \ell_1, \omega_2, \omega_3$ is given by

$$(6.13) \qquad k\, \omega_3^{\frac{1}{2}(n-4)}\, \omega_2^{n-2}(\omega_3 - \omega_2)\, \sin\theta_{33}\, \exp(-b_3\omega_3)$$

$$\Big[\sum_{k=0}^{\infty} \frac{\omega_2^k}{k!} \sum_{i=0}^{k} \binom{k}{i} b_2^i\, b^{k-i} (\omega_3 (1-\ell)\ell^{\frac{1}{2}n+k-i-2}$$

$$- \omega_2 (1-\ell)\ell^{\frac{1}{2}n+k-i-1}) \Big] \quad .$$

Integrate (6.5) with respect to ℓ, then

$$(6.14) \qquad K_2\, \omega_2^{n-2}\, \omega_3^{\frac{1}{2}(n-4)}\, (\omega_3 - \omega_2)\, \sin\theta_{33}\, \exp(-b_3\,\omega_3)$$

$$\Big[\sum_{k=0}^{\infty} \frac{\omega_2^k}{k!} \sum_{i=0}^{k} \binom{k}{i} b_2^i\, b_1^{k-i} (\omega_3\, \beta(\tfrac{1}{2}n+k-i-1,2) - \omega_2 \beta(\tfrac{1}{2}n+k-i,2)) \Big] \, .$$

Again make the transformation $t = \omega_2/\omega_3$, integrate with respect to t and then with respect to ω_3, we can write the distribution of θ_{33}, θ_{32}, θ_{22} in the form

$$(6.15) \qquad K_2 \sin\theta_{33} \Big[\sum_{k=0}^{\infty} \frac{\Gamma\{3n/2)+k\}}{k!\; b_3^{(3n/2)+k}} \sum_{i=0}^{k} \binom{k}{i} b_2^i\, b_1^{k-i}$$

$$\beta(n+k-1,2)\; \beta(\tfrac{1}{2}n+k-i-1,2)(1 - \frac{(n+k)(\frac{1}{2}n+k-i)}{(n+k+2)(\frac{1}{2}n+k-i+2)}) \Big] \; .$$

For any p, integrate (6.10) with respect to $\frac{1}{2}(p-2)(p-3)$ independent elements of $\underset{\sim}{H}_{p-2}$ by using Lemma (3.2) of Sugiyama [29], we can write the distribution of $\omega_1,\ldots,\omega_p$, $\theta_{ij}(i=p,p-1;j=i,\ldots 2)$ in the form

$$(6.16)\qquad k\{\pi^{(p-2)^2/2}/\Gamma_{p-2}(\tfrac{1}{2}(p-2))\}(\omega_p-\omega_{p-1})^{\frac{1}{2}(n-p-1)}|\underset{\sim}{W}_2|^{\frac{1}{2}(n-p-1)}$$

$$\exp(-\tfrac{1}{2}h_p'\underset{\sim}{D}_\gamma h_p\omega_p)\ \exp(-\tfrac{1}{2}h_{p-1}'\underset{\sim}{D}_{p-1}h_{p-1}\omega_{p-1})\prod_{j=p}^{3}\sin^{j-2}\theta_{pj}$$

$$\prod_{j=p-1}^{3}\sin^{j-2}\theta_{p-1,j}\prod_{i>j}(\omega_i-\omega_j)\Big[\sum_{k=0}^{\infty}\sum_{\kappa}\{C_\kappa(-\tfrac{1}{2}\underset{\sim}{D}_{p-2})$$

$$C_\kappa(\underset{\sim}{W}_2)/k!\ C_\kappa(\underset{\sim}{I}_{p-1})\}\Big]\ .$$

Now make the transformation $\ell_i' = \omega_i/\omega_{p-1}$, $i = 1,\dots, p-2$, and using James [10], the distribution of $\ell_1', \ell_2',\dots, \ell_{p-2}', \omega_{p-1}, \omega_p$, $\theta_{ij}(i = p,p-1;\ j = i,\dots, 2)$ can be written in the form

$$(6.17)\qquad K_2\{\pi^{(p-2)^2/2}/\Gamma_{p-2}\ (\tfrac{p-2}{2})\}\qquad \omega_p^{\frac{1}{2}(n+p-5)}\omega_{p-1}^{\frac{1}{2}(np-p-n-1)}(\omega_p-\omega_{p-1})$$

$$|\underset{\sim}{L}'|^{\frac{1}{2}(n-p-1)}|\underset{\sim}{I}-\underset{\sim}{L}'|\prod_{i>j}(\ell_i'-\ell_j')\ \exp(-\tfrac{1}{2}h_p'\underset{\sim}{D}_\gamma h_p\omega_p)$$

$$\exp(-\tfrac{1}{2}h_{p-1}'D_{p-1}h_{p-1}\omega_{p-1})\prod_{j=p}^{3}\sin^{j-2}\theta_{pj}\prod_{j=p-1}^{3}\sin^{j-2}$$

$$\theta_{p-1,j}\Big[\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{p-2}\{C_\kappa(\tfrac{1}{2}\underset{\sim}{D}_{p-2})C_\kappa(\underset{\sim}{L}')C_{(1^j)}(\underset{\sim}{L}')(-1)^j(2j)!$$

$$\omega_{p-1}^{j+k}\ /\ \omega_p^j(j!)^2k!\ \chi_{(21^j)}(1)C_\kappa(\underset{\sim}{I})\}\Big]\ .$$

Now by multiplication of two zonal polynomials [15] and integrating (6.17) with respect to $0 < \ell_1' \le \ell_2' \le \dots \le \ell_{p-2}' \le 1$, we get the distribution of $\omega_{p-1}, \omega_p, \theta_{ij}(i = p,\ p-1;\ j = i,\dots, 2)$

$$(6.18) \quad K_2 \, \Gamma_{p-2}\left(\frac{p+1}{2}\right) \omega_p^{\frac{1}{2}(n+p-5)} \exp(-\tfrac{1}{2}h_p' \underset{\sim}{D}_\gamma h_p \omega_p) \prod_{j=p}^{3} \sin^{j-2} \theta_{pj}$$

$$\prod_{j=p-1}^{3} \sin^{j-2} \theta_{p-1,j} \Big[\sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{j=0}^{p-2} \sum_{\tau} \{(-1)^j (2j)!$$

$$g^{\tau}_{(\kappa,1^j)} \, C_\kappa(-\tfrac{1}{2}D_{p-2}) \, \Gamma_{p-2}\{\tfrac{1}{2}(n-2),\tau\} (-\tfrac{1}{2}h_{p-1}' \underset{\sim}{D}_{p-1} h_{p-1})^r$$

$$C_\tau(\underset{\sim}{I})(\omega_p - \omega_{p-1}) \, \omega_{p-1}^{\frac{1}{2}(np-n-p-1)+k+i+r} / \, \omega_p^j (j!)^2 \, k! \, r! \, \chi_{(21^j)}(1)$$

$$C_\kappa(\underset{\sim}{I}) \, \Gamma_{p-2}(\tfrac{1}{2}(n+p-1),\tau)\} \Big]$$

where τ and $g^{\tau}_{(\kappa,1^j)}$ and $\chi_{(21^j)}(1)$ as defined in section 2. Further let $\omega_{p-1} = \ell\omega_p$, integrate ℓ and then ω_p, the distribution of $\theta_{ij}(i = p, p-1; j = i,\ldots, 2)$ in the form:

$$(6.19) \quad K_2 \, \Gamma_{p-2}(\tfrac{1}{2}(p+1)) \prod_{j=p}^{3} \sin^{j-2} \theta_{pj} \prod_{j=p-1}^{3} \sin^{j-2} \theta_{p-1,j}$$

$$\Big[\sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{j=0}^{p-2} \sum_{\tau} \{(-1)^j (2j)! \, g^{\tau}_{(\kappa,1^j)} \, C_\kappa(-\tfrac{1}{2}\underset{\sim}{D}_{p-2}) \, C_\tau(\underset{\sim}{I})$$

$$(-\tfrac{1}{2}h_{p-1}' \underset{\sim}{D}_{p-1} h_{p-1})^r \, \beta(\tfrac{1}{2}(np-n-p+1)+k+j+r,2)\Gamma\{\tfrac{1}{2}(np+k+r)\} \, /$$

$$\chi_{(21^j)}(1) \, C_\kappa(\underset{\sim}{I})\Gamma_{p-2}\{\tfrac{1}{2}(n+p+1),\tau\}(-\tfrac{1}{2}h_p' \underset{\sim}{D}_\gamma h_p)^{\frac{1}{2}(np)+k+r}\} \Big] \ .$$

When $\underset{\sim}{\Sigma} = \underset{\sim}{I}$, we get from (6.19)

$$(6.20) \qquad (\Gamma(p-1)/2^p \, \Pi^{p-1}) \prod_{j=p}^{3} \sin^{j-2} \theta_{pj} \prod_{j=p-1}^{3} \sin^{j-2} \theta_{p-1,j} \quad .$$

CHAPTER III

NON-CENTRAL DISTRIBUTIONS OF THE SMALLEST AND SECOND SMALLEST ROOTS OF MATRICES IN MULTIVARIATE ANALYSIS

1. Introduction and Summary

While the second chapter dealt with non-central distribution of the second largest root, this chapter deals with the non-central distributions of the smallest and (second smallest) root of a covariance matrix and those in the case of MANOVA, canonical correlation and test of equality of covariance matrices.

2. The Distribution of the Smallest Root of a Covariance Matrix

In this section we obtain the distribution of $g_1' = \frac{1}{2}\omega_1$, where $0 < \omega_1 \leq \omega_2 \leq \ldots \leq \omega_p < \infty$, has the joint density defined in (5.1) of the previous chapter.

Now transform $q_i = g_1' / g_i'$, $i = 2,\ldots, p$, then the joint density of g_1' and $q_2,\ldots, q_p$ can be written as

$$(2.1) \qquad K_1(p,n)|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} g_1'^{\frac{1}{2}np-1} e^{-g_1' \operatorname{tr} \underset{\sim}{Q}_1^{-1}} |\underset{\sim}{Q}|^{-m-p-1} |\underset{\sim}{I}-\underset{\sim}{Q}|$$

$$\prod_{i>j} (q_j - q_i)\ {}_0F_0((\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1}), g_1' \underset{\sim}{Q}_1^{-1}) \quad .$$

Now, by using the results of Constantine [5], namely,

$$C_\kappa(\underset{\sim}{L}^{-1}) = |\underset{\sim}{L}|^{-e_1} (C_\kappa(\underset{\sim}{I})/C_{\kappa*}(\underset{\sim}{I}))C_{\kappa*}(\underset{\sim}{L})$$

where e_1 is any integer $\geq k_1$ and $\kappa* = (e_1-k_p,\ldots,e_1-k_1)$, and $\kappa = (k_1,\ldots,k_p)$. Also expand $|\underset{\sim}{Q}_1|^{-m-p-e_1-1}$ as well as $C_\kappa(\underset{\sim}{I}-\underset{\sim}{Q}_1)$. Then using the results of Khatri and Pillai [15] on the multiplication of two zonal polynomials, (2.1) can be written as

$$(2.2) \qquad K_1(p,n)\, g_1'^{\frac{1}{2}np-1} |\underset{\sim}{I}-\underset{\sim}{Q}| \prod_{i>j} (q_j - q_i) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{I} - \underset{\sim}{\Sigma}^{-1})}{k!\, C_\kappa(\underset{\sim}{I})}$$

$$\sum_{s=0}^{\infty} \sum_{\eta} \frac{(-1)^s g_1'^{k+s}}{s!} \sum_{\delta} g^{\delta}_{\eta,\kappa*} \frac{C_\delta(\underset{\sim}{I})}{C_{\delta*}(\underset{\sim}{I})} \sum_{t=0} \sum_{\tau} \frac{(m+p+e_1+1)_\tau}{t!}$$

$$C_\tau(\underset{\sim}{I}) \sum_{d=0}^{t} \sum_{\mu} \frac{(-1)^d a_{\tau,\mu}}{C_\mu(\underset{\sim}{I})} \sum_{\gamma} g^{\gamma}_{\delta*,\mu}\, C_\gamma(\underset{\sim}{Q}_1) \quad ,$$

where δ, γ are the partitions of $k + s$ and $d + pe_1 - s - k$ respectively, and $\delta* = (e_1 - \delta p,\ldots, e_1 - \delta_1)$ where e_1 is any integer $\geq \delta_1$ and $\delta = (\delta_1,\ldots, \delta_p)$. The constants $g^{\delta}_{\eta,\kappa*}$, $g^{\gamma}_{\delta*,\mu}$ are defined in [15], and $a_{\tau,\mu}$ are defined in [8] .

Now, integrate (2.2) with respect to $1 > q_2 \geq \ldots \geq q_p > 0$, the density function of g_1' can be written as

$$(2.3)\qquad \Gamma_p((p+1/2)\,/\,\Gamma_p(\tfrac{1}{2}n)\; g_1'^{\frac{1}{2}np-1} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{C_\kappa(\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1})}{k!\; C_\kappa(\underset{\sim}{I})} \sum_{s=0}^{\infty}\sum_{\eta} \frac{(-1)^s g_1'^{k+s}}{s!}$$

$$\sum_{\delta} g^{\delta}_{\eta,\kappa} \frac{C_\delta(\underset{\sim}{I})}{C_{\delta*}(\underset{\sim}{I})} \sum_{t=0}^{\infty}\sum_{\tau} \frac{(m+p+e_1+1)_\tau}{t!}\; C_\tau(\underset{\sim}{I}) \sum_{d=0}^{t}\sum_{\mu} \frac{(-1)^d a_{\tau,\mu}}{c_\mu(\underset{\sim}{I})}$$

$$\sum_{\gamma} g^{\gamma}_{\delta*,\mu}\; (p(p+1)/2 + d+pe_1-s-k)(\Gamma_p((p+1)/2,\gamma)/\;\Gamma_p(p+1,\gamma)).$$

If $\underset{\sim}{\Sigma} = \underset{\sim}{I}$, in (2.1), then the density of g_1' can be written as

$$(2.4)\qquad K_2(p,n)\; g_1'^{\frac{1}{2}pn-1}\; e^{-g_1'} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(-g_1)^\kappa\; C_\kappa(\underset{\sim}{I})}{k!\; C_{\kappa*}(\underset{\sim}{I})} \sum_{t=0}^{\infty}\sum_{\tau} \frac{(m+p+e_1+1)}{t!}\,\tau$$

$$C_\tau(\underset{\sim}{I}) \sum_{d=0}^{t}\sum_{\mu} \frac{(-1)^d\; a_{\tau,\mu}}{C_\mu(\underset{\sim}{I})} \sum_{\delta} g^{\delta}_{\kappa*,\mu}\; C_\delta(\underset{\sim}{I})\; (\Gamma_{p-1}(p/2,\delta)\;/$$

$$\Gamma_{p-1}(p+1,\delta))\quad ,$$

where $K_2(p,n) = \Pi^{p-\frac{1}{2}}\;\Gamma_{p-1}(p/2+1)\;/\;\Gamma_p(n/2)\;\Gamma(p/2)$.

3. The Distribution of the Second Smallest Root

Let $\underset{\sim}{\Sigma} = \underset{\sim}{I}$ in (2.1) and transform $q_i = g_2'/g_i'$, $i = 3,\ldots, p$ and by the same method as in section (2), the joint density of g_1', g_2' can be written as

$$(3.1) \qquad K_3(p,n)\, g_1'^{m}\, g_2'^{m(p-1)+\frac{1}{2}(p-2)(p+3)}\, e^{-(g_1'+g_2')}(g_2'-g_1')$$

$$\sum_{k=0}^{\infty}\sum_{\kappa} \frac{(-g_2')^k\, C_\kappa(\underset{\sim}{I})}{k!\, C_{\kappa*}(\underset{\sim}{I})} \sum_{t=0}^{\infty}\sum_{\tau} \frac{(m+p+e_1+1)_\tau}{t!}\, C_\tau(\underset{\sim}{I})$$

$$\sum_{d=0}^{t}\sum_{\mu} \frac{(-1)^d\, a_{\tau,\mu}}{C_\mu(\underset{\sim}{I})} \sum_{\ell=0}^{p-2} \frac{c(\ell)\, g_1'^{\ell}}{g_2'^{\ell}} \sum_{\delta} g^{\delta}_{(\kappa*,\mu,1^\ell)}\, C_\delta(\underset{\sim}{I}_{p-2})$$

$$(\Gamma_{p-2}((p-1)/2,\delta)/\Gamma_{p-2}(p,\delta)) \;;$$

where $K_3(p,n) = 2\Pi^{2p-3}\,\Gamma_{p-2}((p+1)/2) \,/\, \Gamma_p(n/2)\,\Gamma_{p-2}((p-2)/2)$. Integrate (3.1) with respect to g_1', then the density of g_2' is given by

$$(3.2) \qquad K_3(p,n)\, g_2'^{\,m(p-1)+\frac{1}{2}(p-2)(p+3)}\, e^{-g_2'} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(-g_2')^{\kappa} C_\kappa(\underset{\sim}{I})}{k!\, C_{\kappa*}(\underset{\sim}{I})}$$

$$\sum_{t=0}\sum_{\tau} \frac{(m+p+e_1+1)_\tau}{t!}\, C_\tau(\underset{\sim}{I}) \sum_{d=0}^{t}\sum_{\mu} \frac{(-1)^d\, a_{\tau,\mu}}{C_\mu(\underset{\sim}{I})} \sum_{\ell=0}^{p-2} c(\ell) g_2'^{-\ell}$$

$$\sum_{\delta} g^{\delta}_{(\kappa*,\mu,1^\ell)}\, C_\delta(\underset{\sim}{I}_{p-2})(\Gamma_{p-2}((p-1)/2,\delta)/\Gamma_{p-2}(p,\delta))$$

$$(g_2'\gamma(0,g_2';\, m+\ell+1) - \gamma(0,g_2';\, m+\ell+2)) \quad .$$

4. Non-Central Distribution of the Smallest and (Second Smallest) Roots in MANOVA Case

In this section we obtain the distributions of the smallest root ℓ_1 and the second smallest ℓ_2, when the distribution of $0 < \ell_1 < \ell_2 < \ldots < \ell_p < 1$

is described in (2.1) of the previous chapter.

Now transform $z_i = 1-\ell_i$ and expand $C_\kappa(\underset{\sim}{I}-\underset{\sim}{Z})$, then the joint density of $1 > z_1 \geq z_2 \geq \dots \geq z_p > 0$ can be written as

$$(4.1) \qquad C(p,n_1,n_2)\ \exp(\mathrm{tr}-\underset{\sim}{\Omega})|\underset{\sim}{Z}|^n|\underset{\sim}{I}-\underset{\sim}{Z}|^m \prod_{i>j}(z_j-z_i) \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(\nu/2)_\kappa C_\kappa(\underset{\sim}{\Omega})}{(n_1/2)_\kappa\ k!}$$

$$\sum_{s=0}^{k}\sum_{\eta} (-1)^s a_{\kappa,\eta}\ C_\eta(\underset{\sim}{Z}) \,/\, C_\eta(\underset{\sim}{I}) \quad .$$

Now, from the results of Pillai and Sugiyama [25], the density of ℓ_1 can be written as

$$(4.2) \qquad C_2(p,n_1,n_2)\ \exp(\mathrm{tr}-\underset{\sim}{\Omega}) \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(\nu/2)_\kappa\ C_\kappa(\underset{\sim}{\Omega})}{(n_1/2)_\kappa\ k!} \sum_{s=0}^{k}\sum_{\eta} \frac{(-1)^s a_{\eta,\kappa}}{C_\eta(\underset{\sim}{I})}$$

$$\sum_{t=0}^{\infty}((pn_2/2+s+t)/t!) \sum_{\delta',\sigma}^{\infty} g_{\eta,\sigma}^{\delta'} \frac{((p+1-n_1)/2)^\sigma (n_2/2)_{\delta'}}{((n_2+p+1)/2)_{\delta'}}\ C_{\delta'}(\underset{\sim}{I})$$

$$(1-\ell_1)^{pn_2/2+s+t-1} \quad ,$$

where σ and δ' are the partitions of t and $s+t$ respectively, and $C_2(p,n_1,n_2) = \Gamma_p((p+1)/2)\ \Gamma_p(\nu/2) \,/\, \Gamma_p(n_1/2)\ \Gamma_p((n_2+p+1)/2)$. Also from the results of Chapter II, the density of ℓ_2 can be written as

$$(4.3) \qquad C_1(p,n_1,n_2)\ \exp(\mathrm{tr}{-}\Omega)(1-\ell_2)^{n(p-1)+(p-2)(p+1)/2} \sum_{k=0}^{\infty}\sum_{\kappa}$$

$$\frac{(\nu/2)_\kappa\ C_\kappa(\Omega)}{k!\ (n_1/2)_\kappa} \sum_{s=0}^{k}\sum_{\eta} \frac{(-1)^s\ a_{\kappa,\eta}}{C_\eta(\underset{\sim}{I})} \sum_{t=0}^{\infty}\sum_{\sigma}\frac{(-m)_\sigma}{t!}\sum_{\ell=0}^{p-2} c(\ell)/\ell!$$

$$\sum_{\tau,\mu} a_{\tau,\mu}(1-\ell_2)^{t+\ell+s_2}\ \ I(1-\ell_2,1;n+p+s_1-\ell-1,m)\ \sum_{\gamma} g^{\gamma}_{(1^\ell,\sigma,\mu)}$$

$$C_\gamma(\underset{\sim}{I}_{p-1})((n_2-1)(p-1)/2+t+\ell+s_2)(\Gamma_{p-1}((n_2-1)/2,\gamma)\ /$$

$$\Gamma_{p-1}((n_2+p-1)/2,\gamma))\quad .$$

5. The Distribution of the Smallest and (Second Smallest) Roots in the Canonical Correlation Case

In this section, we obtain the distributions of r_1^2 and r_2^2 as the joint density of $0 < r_1^2 \leq r_2^2 \leq \ldots \leq r_p^2 < 1$ is defined in (3.1) of the previous chapter.

As before, transform $r_i^{2'} = 1-r_i^2$, $i = 1,\ldots, p$. Then the density of r_1^2 can be written as

$$(5.1) \qquad C_2(n,p,q)\,|\underset{\sim}{I}-\underset{\sim}{P}^2|^{n/2} \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(n/2)_\kappa\ (n/2)_\kappa\ C_\kappa(\underset{\sim}{P}^2)}{(q/2)_\kappa\ k!} \sum_{s=0}^{k}\sum_{\eta}\frac{(-1)^s a_{\kappa,\eta}}{C_\eta(\underset{\sim}{I})}$$

$$\sum_{t=0}^{\infty} ((p(n-q)/2+s+t/t!)\sum_{\sigma,\delta'} g^{\delta'}_{\eta,\sigma}\ ((p-q+1)/2)_\sigma$$

$$\frac{((n-q)/2)\delta'}{((n-q+p+1)/2)_{\delta'}}\ \ C_{\delta'}(\underset{\sim}{I}_p)(1-r_1^2)^{p(n-q)/2+s+t-1} \quad ,$$

where $C_2(n,p,q) = \Gamma_p(n/2)\ \Gamma_p((p+1)/2)\ /\ \Gamma_p(q/2)\ \Gamma_p((n-q+p+1)/2)$.
Also the density of r_2^2 can be written as

$$(5.2)\qquad C_1(n,p,q)|\underset{\sim}{I}-\underset{\sim}{P}^2|^{n/2}(1-r_2^2)^{\alpha} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(n/2)_\kappa (n/2)_\kappa\ C_\kappa(\underset{\sim}{P}^2)}{(q/2)_\kappa\ k!}$$

$$\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{\eta,\sigma} \frac{(-1)^s\ a_{\kappa,\eta}((p-q+1)/2)\sigma}{C_\eta(\underset{\sim}{I})\ t!} \sum_{\ell=0}^{p-2} c(\ell)/\ell!$$

$$\sum_{\tau,\mu} a_{\tau,\mu}(1-r_2^2)^{t+\ell+s_2}\ I((1-r_2^2),1;(n-q+p-3)/2+k_1-\ell;(q-p-1)/2)$$

$$\sum_{\gamma} g^{\gamma}_{(1^\ell,\sigma,\mu)}\ C_\gamma(I_{p-1})((n-q-1)(p-1)/2+t+\ell+s_2)$$

$$(\Gamma_{p-1}((n-q-1)/2,\gamma)\ /\ \Gamma_{p-1}((n-q+p-1)/2,\gamma)) \quad ,$$

where $\alpha = \{(n-q-p-1)(p-1) + (p-2)(p+1)\}/2$.

6. Non-Central Distribution of the Smallest (and Second Smallest) Roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$

In this section we obtain the distribution of g_1 and g_2 where $0 < g_1 \leq g_2 \leq \cdots \leq g_p < 1$ has the joint distribution described in (2.1) of Chapter 1. Then, as before, the density of g_1 can be written as

$$(6.1)\qquad c_2(p,n_1,n_2)|\delta\Lambda|^{-n_1/2} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(\nu/2)_\kappa\, C_\kappa(I-(\delta\Lambda)^{-1})}{k!} \sum_{s=0}^{k}\sum_{\eta}$$

$$\frac{(-1)^s\, a_{\kappa,\eta}}{C_\eta(I)} \sum_{t=0}^{\infty}\sum_{\sigma,\delta'} \{(pn_2/2+s+t)\, g^{\delta'}_{\eta,\sigma}((p+1-n_2)/2)_\sigma$$

$$(n_2/2)_{\delta'}\, C_{\delta'}(I_p)(1-g_1)^{pn_2/2+s+t-1}/t!\,(n_2+p+1)_{\delta'}\} \quad .$$

Also, the density of g_2 can be written as

$$(6.2)\qquad c_1(p,n_1,n_2)|\delta\Lambda|^{-n_1/2}(1-g_2)^{n(p-1)+(p-2)(p+1)/2}$$

$$\sum_{k=0}^{\infty}\sum_{\kappa} \frac{(\nu/2)_\kappa\, C_\kappa(I-(\delta\Lambda))^{-1}}{k!} \sum_{s=0}^{k}\sum_{t=0}^{\infty}\sum_{\eta,\sigma} \frac{(-1)^s\, a_{\kappa,\eta}(-m)_\sigma}{t!\, C_\eta(I)}$$

$$\sum_{\ell=0}^{p-2} c(\ell)/\ell! \sum_{\tau,\mu} a_{\tau,\mu}(1-g_2)^{t+\ell+s_2}\, I(1-g_2,1;n+p+s_1-\ell-1;m)$$

$$\sum_{\gamma} g^{\gamma}_{(1^\ell,\sigma,\mu)}\, C_\gamma(I_{p-1})((n_2-1)(p-1)/2+t+\ell+s_2)$$

$$(\Gamma_{p-1}((n_2-1)/2,\gamma)\ /\ \Gamma_{p-1}((n_2+p-1)/2,\gamma)) \quad .$$

CHAPTER IV

ON THE DISTRIBUTION OF THE iTH LATENT ROOT UNDER NULL HYPOTHESES CONCERNING COMPLEX MULTIVARIATE NORMAL POPULATIONS

1. Introduction and Summary

Khatri [12], has pointed out that one can handle all the classical problems of point estimation and testing hypotheses concerning the parameters of complex multivariate normal populations much as one handles those for multivariate normal populations in real variates. Further Khatri [12], has suggested the maximum latent root statistic for testing the reality of a covariance matrix. The joint distribution of the latent roots under certain null hypotheses can be written as, [11], [12],

$$c_1\{\prod_{j=1}^{q} w_j^m (1-w_j)^n\} \prod_{i>j} (w_i-w_j)^2 \tag{1.1}$$

where $c_1 = \prod_{j=1}^{q} \Gamma(n+m+q+j) / \{\Gamma(n+j)\,\Gamma(m+j)\,\Gamma(j)\}$ and $0 \le w_1 \le w_2 \le \cdots \le w_q \le 1$. We may also note that when n is large, the joint distribution of $nw_j = f_j$, $j = 1,\ldots, q$, $0 \le f_1 \le \cdots \le f_q < \infty$, can be written as

$$c_2 \prod_{j=1}^{q} f_j^m \exp(-\sum_{j=1}^{q} f_j)\{\prod_{i>j} (f_i-f_j)^2\} \tag{1.2}$$

where Σ means summation over all permutations $(j_1, j_2, \ldots, j_q)$ of $(1,2,\ldots, q)$, and $|A|$ means the determinant of A.

For Proof, see Khatri [10].

Lemma 3.

$$\sum \int_{\mathcal{D}'} \prod_{j=1}^{s} [x_j^{m_j'}(1-x_j)^{n_j'} \, dx_j] = \prod_{j=1}^{s} [\int_x^1 x_j^{m_j}(1-x_j)^{n_j} \, dx_j] \quad ,$$

where $\mathcal{D}': (x \leq x_1 \leq x_2 \leq \ldots \leq x_s \leq 1)$, and on the left hand side $(m_s', n_s'), \ldots, (m_1', n_1')$ is any permutation of $(m_s, n_s), \ldots, (m_1, n_1)$ and the summation is taken over all such permutations.

Proof is similar to Lemma 1.

3. The Distribution of w_{q-1}

In this section we obtain first the cdf's of w_{q-1} and f_{q-1} and in the next those of w_i and f_i. Note that

$$(3.1) \qquad \Pr\{w_{q-1} \leq x\} = \Pr\{w_q \leq x\} + \Pr\{w_{q-1} \leq x < w_q \leq 1\}$$

Khatri [11], showed that

$$(3.2) \qquad \Pr\{w_q \leq x\} = c_1 |(\beta_{i+j-2})| = c_1 \begin{vmatrix} \beta_0 & \beta_1 & \ldots & \beta_{q-1} \\ \beta_1 & \beta_2 & & \beta_q \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \beta_{q-1} & \beta_q & & \beta_{2q-2} \end{vmatrix} \quad ,$$

where c_1 is defined in (1.1), $\beta_{i+j-2} = \int_0^x w^{m+i+j-2}(1-w)^n \, dw$ for $i,j = 1, 2, \ldots, q$ and (β_{i+j-2}) is a $q \times q$ matrix. Now the determinant in Lemma 2, can be written as

$$(3.3) \qquad \sum_1 \text{sign}\,(t_1,\ldots,t_q)\, w_{j1}^{q-1+t_1}\, w_{j2}^{q-2+t_2} \cdots\cdots w_{jq}^{tq} \quad ,$$

where $(t_1,\ldots,t_q)$ is a permutation of $(0,1,\ldots,q-1)$, sign $(t_1,\ldots,t_q)$ is positive if the permutation is even and negative if the permutation is odd, and Σ_1 means the summation over all such permutations. Then (1.1) can be written as

$$(3.4) \qquad c_1\{\prod_{j=1}^{q} w_j^m(1-w_j)^n \sum_{j_1,\ldots,j_{q-1}} \sum_1 \text{sign}(t_1,\ldots,t_q)$$

$$[w_q^{q-1+t_1}\, w_{j1}^{q-2+t_2}\, w_{j2}^{q-3+t_3} \cdots w_{jq-1}^{t_q} + w_q^{q-2+t_2}\, w_{j1}^{q-1+t_1}$$

$$w_{j2}^{q-3+t_3} \cdots w_{jq-1}^{t_q} + \cdots + w_q^{t_q}\, w_{j1}^{q-1+t_1}\, w_{j2}^{q-2+t_2} \cdots w_{jq-1}^{1+t_q}] \quad .$$

First taking summation over $(j_1,\ldots, j_{q-1})$, the permutation of $(1,2,\ldots,q-1)$ and integrate w_q over $x < w_q < 1$, and apply lemma 1, we get

$$(3.5) \qquad \Pr(w_{q-1} \leq x \leq w_q < 1) = c_1 \sum_1 \text{sign}\,(t_1,\ldots,t_q)\,[\beta'_{q-1+t_1}\beta_{q-2+t_2}$$

$$\cdots \beta_{t_q} + \beta_{q-1+t_1}\beta'_{q-2+t_2}\cdots\beta_{t_q} + \cdots \beta_{q-1+t_1}\beta_{q-2+t_2}\cdots\beta'_{t_q}\,] \quad ,$$

where $\beta'_{i+j-2} = \int_x^1 w^{m+i+j-2}(1-w)^n \, dw$, then (3.5) can be written as

$$(3.6) \qquad c_1 \sum_{k=1}^{q} |(\beta^{(k)}_{i+j-2})| \quad ,$$

where $|(\beta^{(k)}_{i+j-2})|$ is the determinant obtained from $|(\beta_{i+j-2})|$ by replacing, the kth column of $|(\beta_{i+j-2})|$, β_α, by the corresponding β'_α's. So we proved the following theorem.

__Theorem 1.__ If the joint distribution of $w_1, \ldots, w_q$ is given by (1.1), then

$$(3.7) \qquad Pr\{w_{q-1} \leq x\} = c_1 \sum_{k=0}^{q} |(\beta^{(k)}_{i+j-2})|$$

where $|(\beta^{(0)}_{i+j-2})| = |(\beta_{i+j-2})|$, and $|(\beta^{(k)}_{i+j-2})|$ is defined in (3.6), and c_1 is defined in (1.1).

__Theorem 2.__ If the distribution of $f_1, \ldots, f_q$ is given by (1.2) then

$$(3.8) \qquad Pr\{f_{q-1} \leq x\} = c_2 \sum_{k=0}^{q} |(\gamma^{(k)}_{i+j-2})| \quad ,$$

where $\gamma_{i+j-2} = \int_0^x w^{m+i+j-2} \exp(-w)dw$, (γ_{i+j-2}) is a q x q matrix and $(\gamma^{(k)}_{i+j-2})$ is defined similar to that of (3.7) and c_2 is defined in (1.2). Proof is similar to that of Theorem 1.

4. The Distribution of w_i

It may be noted here that

$$(4.1) \qquad \Pr\{w_i \leq x\} = \Pr\{w_{i+1} \leq x\} + \Pr\{w_i \leq x < w_{i+1}\}, i = 1,\ldots,q-1 .$$

To evaluate the second term of (4.1), we may write

$$(4.2) \qquad \prod_{i>j} (w_i - w_j)^2 = \sum_1 \text{sign } (t_1,\ldots,t_q) \sum_2 \sum_{i_1,\ldots,i_{q-i}} w_{i_1}^{\alpha_1} w_{i_2}^{\alpha_2} \cdots$$

$$\cdots w_{i_{q-i}}^{\alpha_{q-i}} \sum_{j_1,\ldots,j_i} w_{j1}^{\alpha_{q-i+1}} w_{j2}^{\alpha_{q-i+2}} \cdots w_{ji}^{\alpha_q}$$

where $(i_1,\ldots, i_{q-i})$ is permutation of $(i+1,\ldots,q)$ and $\sum_{i_1,\ldots,i_{q-i}}$ runs over all such permutations; $(j_1,\ldots,j_i)$ is a permutation of $(1,\ldots, i)$, and $\sum_{j1,\ldots,ji}$ runs over all such permutations; Σ_2 is the summation over the terms $\binom{q}{q-i}$ terms of obtained by taking $q-i$, $(\alpha_1,\ldots,\alpha_{q-i})$, at a time of $q-1+t_1$, $q-2+t_2,\ldots, t_q$.

Substituting (4.2) in (1.1) and using Lemma 1, and Lemma 3, and as in Section (3), we get

$$(4.3) \qquad \Pr(w_i \leq x < w_{i+1}) = c_1 \sum_2 |(\beta_{i+j-2}^{(i_0)})| ,$$

where $(\beta_{i+j-2}^{(i_0)})$ is a $q \times q$ matrix obtained from (β_{i+j-2}) by replacing i columns of (β_{i+j-2}) by the corresponding β'_α 's. Therefore by (4.1), (4.3) and Theorem 1 and reduction process, we can get the distribution of w_i.

It may be pointed out that, [23],

$$(4.4) \qquad \Pr\{w_i \leq x;\ m,n) = 1 - \Pr(w_{q-i+1} \leq 1 - x;\ n,m)$$

where on the right side of (4.4) the parameters m and n are interchanged, hence the distribution of w_1, [11], can be written as

$$(4.5) \qquad \Pr\{w_1 \leq x\} = 1 - c_1 |(\delta_{i+j-2})| \quad ,$$

where $\delta_{i+j-2} = \int_0^{1-x} z^{n+i+j-2} (1-z)^m \, dz$, and (δ_{i+j-2}) is a $q \times q$ matrix, similarly, if we define $\delta'_{i+j-2} = \int_{1-x}^{1} z^{n+i+j-2} (1-z)^m \, dz$, the distribution of w_2 can be written as

$$(4.6) \qquad \Pr\{w_2 \leq x\} = 1 - c_1 \sum_{k=0}^{q} |(\delta^{(k)}_{i+j-2})| \quad ,$$

where, as before, $|(\delta^{(k)}_{i+j-2})|$ is the determinant obtained from $|(\delta_{i+j-2})|$ by replacing the kth column of $|(\delta_{i+j-2})|$ by the corresponding δ'_α 's, and $(\delta^{(0)}_{i+j-2}) = (\delta_{i+j-2})$. A similar method gives

$$(4.7) \qquad \Pr\{f_i \leq x\} = \Pr\{f_{i+1} \leq x\} + \Pr\{f_i \leq x < f_{i+1}\} \quad ,$$

$i = 1,2,\ldots, q-1$, and

CHAPTER V

ON THE DISTRIBUTION OF THE SUM OF THE TWO SMALLEST ROOTS OF A COVARIANCE MATRIX AND NON-CENTRAL WILKS' Λ

1. Introduction and Summary

In this chapter, the distribution of the sum of the two smallest roots of a covariance matrix, studied for $p = 3$, 4 and 5 when $\underset{\sim}{\Sigma} = \underset{\sim}{I}_p$. This criterion is useful for various tests of hypotheses, for example, those regarding the number of independent linear equations satisfied by the means, μ_{it}, $i = 1,\ldots, p$, $t = 1,\ldots, N$ in N-p variate normal populations with a common covariance matrix ([1], [25]). The distribution of the sum of the two smallest, largest and the sum of the roots are considered for $p = 4$. In the last section, the non-central distribution of Wilks' Λ criterion has been obtained for $p = 2$, 3 and 4. In this connection a lemma has been proved using some results on Mellin transform.

2. The Distribution of the Sum of the Two Smallest Roots

Let $\underset{\sim}{\Sigma} = \underset{\sim}{I}_p$ in (5.1) and transform $g_i' = \frac{1}{2}\omega_i$, $i = 1,\ldots, p$, we get the joint density of $g_1',\ldots, g_p'$ in the form

$$(2.1) \qquad K_1(p,n) \prod_{i=1}^{p} (g_i'^{m} e^{-g_i'}) \prod_{i>j} (g_i' - g_j') \ ,$$

$$0 < g_1' \leq g_2' \leq \cdots \leq g_p' < \infty \quad .$$

In this section we will derive the distribution of $M_1 = g_1' + g_2'$ for $p = 3$, 4, and 5.

<u>Case i</u>. Put $p = 3$ in (2.1) and let $M = \ell_1' + \ell_2'$, $G = \ell_1'\ell_2'$, where $\ell_i' = g_i'/g_3'$, $i = 1, 2$. Then the joint distribution of M and g_3' can be written in the form

$$(2.2) \qquad K_1(3,n)\, e^{-g_3'(1+M)}\, g_3'^{\,3m+5} \int_0^{M^2/4} G^m (1-M+G)\,dG, \quad 0 < M \leq 1 \quad .$$

Further, transform $M_1 = g_3'M$ and we get

$$(2.3) \qquad K_2(3,n)\, g_3'^{\,m}\, M_1^{2m+2} \{(g_3' - M_1/2)^2 - M_1^2/(4(m+2))\}\, e^{-(g_3'+M_1)} \quad ,$$

where

$$K_2(p,n) = K_1(p,n) \,/\, \{(m+1)2^{2m+2}\} \quad .$$

Now integrating g_3' from M_1 to ∞ we get for $0 < M \leq 1$

$$(2.3') \quad K_2(3,n)\, e^{-M_1}\, M_1^{2m+2} [a_0 \gamma(M_1,\infty;\, m+3) + a_1\, M_1 \gamma(M_1,\infty;\, m+2) + a_2\, M_1^2 \gamma(M_1,\infty;\, m+1)] \quad ;$$

where $a_0 = 1$, $a_1 = -1$, $a_2 = (m+1)/\{4(m+2)\}$.

Now we consider the case when $1 \leq M \leq 2$. Let $\ell_i'' - 1 - \ell_i'$, $i = 1, 2$ such that $M' = 2-M$, $G' = (1-M+G)$, then the distribution of g_3' and M' can be written in the form

$$(2.4) \qquad K_1(3,n)\ e^{-g_3'(3-M')}\ g_3'^{3m+5}\left[\frac{(1-M'/2)^{2m+2}}{(m+1)}\left(\frac{M'^2}{4}-\frac{(1-M'/2)^2}{m+2}\right)+\frac{(1-M')^{m+2}}{(m+1)(m+2)}\right] .$$

Integrate (2.4) with respect to g_3', from $M_{1/2}$ to M_1 and combine the result with (2.3), then the distribution of M_1 can be written in the form

$$(2.5) \qquad K_2(3,n)\ e^{-M_1}\left[M_1^{2m+2}\sum_{i=0}^{2} a_i\ M^i\ \gamma(M_{1/2},\infty;\ m+3-i) + 2^{2m+2}(m+2)^{-1}\int_{M_{1/2}}^{M_1} g_3^{2m+2}(M_1-g_3)^{m+2}\ e^{-g_3}\ dg_3\right] ,$$

$$0 < M_1 < \infty .$$

Case ii. Put $p = 4$ in (2.1) and integrate g_4', then the distribution of g_3' and M is given by

$$(2.6) \qquad K_2(4,n)\ e^{-g_3'(2+M)}\ M^{2m+2}\sum_{r=0}^{m+2}(r+1)\ g_3'^{4m+7-r}\left[(a-bM)\{(1-M/2)^2 - M^2/4(m+2)\} + a_2CM^2\ \{(1-M/2)^2 - M^2/4(m+3)\}\right] ,$$

where $a = (m+2)!\ /\ (m+2-r)!$, $b = (m+1)!/(m+1-r)!$ and $0 < M \leq 1$, $C = m!\ /\ (m-r)!$. As before transform $M_1 = g_3'M$, and integrate g_3', then the distribution of M_1, for $0 < M \leq 1$, takes the form

$$(2.7) \qquad 2^{-(2m+5)} K_2(4,n)\, e^{-M_1} \sum_{r=0}^{m+2} (r+1) \{M_1^{2m+2} \sum_{i=0}^{3} 2^{r+i} M_1^i$$

$$a_i' \gamma(2M_1,\infty;\ 2m+5-r-i) + a_2 C M_1^{2m+4} \sum_{i=0}^{2} 2^{r+i+2} M_1^i\, b_i \gamma(2M_1,\infty;$$

$$2m+3-r-i)\},\ 0 \leq M \leq 1 \quad ,$$

where $a_0' = a$, $a_1' = -(a+b)$, $a_2' = a(m+1)/\{(m+2)4\}+b$, $a_3' = -b(m+1)/4(m+2)$, $b_0 = 1$, $b_1 = -1$ and $b_2 = (m+2)/4(m+3)$. Now, when $1 \leq M \leq 2$, as before, transform to M' and G' and integrate out G', and further transform to $M = 2-M'$ and $M_1 = g_3' M$ and integrate out g_3' between $M_1/2$ and M_1 and combining the result with (2.7) we get

$$(2.8) \qquad 2^{-m} K_2(4,n)\, e^{-M_1} \sum_{r=0}^{m+2} (r+1) \Big[M_1^{2m+2} \sum_{i=0}^{4} 2^{r+i-m-7} c_i$$

$$M_1^i \gamma(M_1,\infty;\ 2m-r-i+5) + (m+2)^{-1}\{(a-c) \sum_{i=0}^{m+2} \binom{m+2}{i} (-1)^i$$

$$g(r,i+1) + (c-b) \sum_{i=0}^{m+2} \binom{m+2}{i}(-1)^i g(r,i) - c(m+3)^{-1}$$

$$\sum_{i=0}^{m+3} \binom{m+3}{i} (-1)^i\, 2g(r,i), \qquad 0 < M_1 < \infty \quad ,$$

where

$$g(r,i) = 2^{r-i-2} M_1^{m+3-i} \gamma(M_1, 2M_1; 3m+4+i-r) ,$$

$$c_0 = 4a, \quad c_1 = -4(a+b), \quad c_2 = (C+a)(m+1)(m+2)^{-1} + 4b$$

$$c_3 = -(C+b)(m+1)(m+2)^{-1}, \text{ and } c_4 = C(m+1)/\{4(m+3)\} .$$

Case iii. Put $p = 5$ in (2.1) and integrate g_5' and g_4', then the distribution of g_3' and M is given by

$$\text{(2.9)} \qquad K_2(5,n)\, e^{-g_3'(3+M)}\, g_3'^{3m+5}\, M^{2m+2} \sum_{r=0}^{6} \eta_r M^r\, g_3'^{2m+7-i-j}$$

where $\eta_0 = K_{o,i,j}/(m+1)$, $\eta_1 = (K_{1,i,j} - K_{o,i,j}) / (m+1)$,

$$\eta_2 = (K_{o,i,j} + K_{3,i,j})/4(m+2) + (K_{2,i,j} - K_{1,i,j})/(m+1) ,$$

$$\eta_3 = (K_{1,i,j} - K_{3,i,j} + K_{4,i,j})/4(m+2) - K_{2,i,j}/(m+1) ,$$

$$\eta_4 = (K_{2,i,j} - K_{4,i,j})/4(m+2) + (K_{3,i,j} + K_{5,i,j})/2^4(m+3) ,$$

$$\eta_5 = (K_{4,i,j} - K_{5,i,j})2^4(m+3), \text{ and } \eta_6 = K_{5,i,j}/2^6(m+4)$$

and the $K_{\ell,i,j}$ are defined by

$$\text{(2.10)} \qquad K_{\ell,i,j} = \sum_{j=o}^{2m+7-i-\ell_\delta} \sum_{i=o}^{m+k} \frac{i}{2^{j+1}} \Big[a_\ell^{(1)} (2m+7-i-\ell_\delta)_{-j}$$

$$- a_\ell^{(2)} (2m+6-i-\ell_\delta)_{-j} + a_\ell^{(3)} (2m+5-i-\ell_\delta)_{-j} \Big] ,$$

where

$$\ell_\delta = \begin{cases} \ell\ , & \text{for } \ell = 0,\ 1,\ \text{and } 2, \\ \ell-1 & \text{for } \ell = 3,\ 4,\ \text{and } 5\ , \end{cases}$$

and

$$K = \begin{cases} 4 & \text{for } \ell = 0,\ 1,\ 3 \\ 3 & \text{for } \ell = 2,\ 4 \\ 2 & \text{for } \ell = 5 \end{cases}$$

and

$$(2.11) \qquad \begin{array}{lll} a_o^{(1)} = (m+3)_{-i+1}\ , & a_o^{(2)} = -a_i(m+2)_{-i+2}, & a_o^{(3)} = (m+2)_{-i+1} \\ a_1^{(1)} = -a_o^{(2)}\ , & a_1^{(2)} = b_i(m+1)_{-i+3}\ , & a_1^{(3)} = -c_i(m+1)_{-i+1} \\ a_2^{(1)} = a_o^{(3)}\ , & a_2^{(2)} = -c_i(m+1)_{-i+2}, & a_2^{(3)} = (m+1)_{-i+1} \\ a_3^{(1)} = d_i(m+1)_{-i+3}, & a_3^{(2)} = -c_i(m)_{-i+4}\ , & a_3^{(3)} = g_i(m)_{-i+3} \\ a_4^{(1)} = a_2^{(2)}\ , & a_4^{(2)} = k_i(m)_{-i+3}\ , & a_4^{(3)} = -\ell_i(m)_{-i+2} \\ a_5^{(1)} = a_2^{(3)}\ , & a_5^{(2)} = a_4^{(3)}\ , & a_5^{(3)} = (m)_{-i+1} \end{array}$$

and $(a)_{-i+b} = a(a-1)$ ----$(a-i+b+1)$; $a_1 = 2$, $a_i = 2m+7-i$, $i \geq 2$; $b_1 = 4$, $b_2 = 4m+8$ and $b_i = (2m+7-i)(2m+5-i) + i-1$ for $i \geq 3$; $c_1 = 2$, $c_i = 2m+5-i$ for $i \geq 2$; $d_1 = 2$, $d_2 = 2m+4$ and $d_i = (m+2)_2 + (m+3-i)_2$ for $i \geq 3$; $e_1 = 4$, $e_2 = 4m+6$, $e_3 = \sum_{i=o}^{3} (m+i)_{-2}$ and $e_i = \sum_{K=o}^{3} (m+2-i+K)_{3-K}(m+1)_K$ for $i \geq 4$; $g_1 = 2$, $g_2 = 2m+2$, $g_i = (m+1)_2 + (m+2-i)_2$ for $i \geq 3$; $\ell_1 = 2$, $\ell_i = 2m-i+3$, $i \geq 2$; $k_1 = 4$, $k_2 = 4m+4$,

$k_i = 4m^2 + 16m = 4im + i^2 - 7i + 14$ for $i \geq 3$.

As before transform $M_1 = g_3'M$, and integrate g_3', then the distribution of M_1, for $0 < M \leq 1$, takes the form

$$(2.12) \qquad \bar{K}_2(5,n)\, M_1^{2m+2}\, e^{-M_1} \sum_{r=o}^{6} \eta_r M_1^r \gamma(3M_1,\infty;\, 3m+10-i-j-r) \,/\, 3^{3m+10-i-j-r} \quad .$$

Now, when $1 \leq M \leq 2$, proceeding as before, and combining the result with (2.12) we get

$$(2.13) \qquad K_3(5,n)\, M_1^{m+2}\, e^{-M_1}\Big[(3M_1)^m \sum_{r=o}^{6} 3^{i+j+r}\, \eta_r M_1^r \cdot \gamma(3M_1/2,\infty;\, 3m+10-i-j-r) + 2^{2m+2} \sum_{s=o}^{m+2} \binom{m+2}{s}(-1)^s \sum_{r=o}^{2} p_r M_1^{r-s}\, 3^{s+i+j+r}\, \gamma(3M_1/2, 3M_1; 4m+10+s-i-j-r)\Big]$$

where $K_3(5,n) = K_2(5,n)/3^{4m+10}$,

$p_o = K_{o,i,j}/(m+1)(m+2) - K_{3,i,j}/(m+2)(m+3) + K_{5,i,j}/(m+3)(m+4)$,

$p_1 = K_{1,i,j}/(m+1)(m+2) + (K_{3,i,j} - K_{4,i,j})/(m+2)(m+3) - 2K_{5,i,j}/(m+3)(m+4)$

$p_2 = K_{2,i,j}/(m+1)(m+2) + K_{4,i,j}/(m+2)(m+3) + K_{5,i,j}/(m+3)(m+4)$.

3. The Distribution of the Sum of the Two Smallest and (Largest) Roots and Their Sum and Ratio of a Covariance Matrix

Transform $M_1 = g_1' + g_2'$, $M_2 = g_3' + g_4'$ in (2.1) and integrate g_1' and g_3' over the region $0 \le g_1' \le M_1/2$ and $M_1/2 \le g_3' \le M_2/2$ respectively, then the joint density of M_1 and M_2 can be written as

$$(3.1) \qquad K_1'(4,n)\, e^{-(M_1+M_2)}\, M_1^{2m+1} \sum_{k=0}^{m} \binom{m}{k}(-2)^{-k} M_2^{m-k} \sum_{i=0}^{m} \binom{m}{i}(-2)^{-i}$$

$$\sum_{j=1}^{s} M_1^{j} M_2^{5-j} \left(a_j M_2^{m+k+2} - b_j M_1^{m+k+2}\right) \quad ,$$

where

$$K_1'(4,n) = K_1(4,n)/2^{2m+3} \quad ,$$

and

$$a_1 = \{(m+1)^2+15(m+k+4)\}/8(m+i+1)_2(m+k+3)_4 \quad ,$$

$$a_2 = -(m+k+6)/2(m+i+1)_2(m+k\_2)_3,$$

$$a_3 = -(m+k+6)(m+i)/2(m+i+1)_2(m+k+2)_3+[(m+i+2)(m+k+1)_2]^{-1}-(3m+3i+13)$$

$$\{4(m+k+4)+(m+k+1)_2\}/(m+i+3)_2(m+k+1)_4$$

$$a_4 = -(m+i+6)/2(m+k+1)_2(m+i+2)_3,$$

$$a_5 = \{(4m+4i+25)(m+3+i)_2-8(m+i+1)(m+i+5)_2\}/(m+i+3)_4(m+k+1)_2$$

$$b_1 = 0, \quad b_2 = \{(m+k+2)_3(m+i+6)(m+i+1) - (m+i+3)_2(m+k+4)$$

$$(m+k+1)/2(m+i+1)_4(m+k+1)_3\} \quad ,$$

$$b_3 = (m+k+6)/2(m+i+1)_2(m+k+3)_2 - (2m+2i+1)/(m+i+1)_2(m+k+2)+(m+i)/$$

$$2(m+i+2)_2(m+k+1)+3/(m+i+3)(m+k+2)+(m+k)/2(m+i+4)(m+k+1)_2 \quad ,$$

$$b_4 = \sum_{i_1,i_2}^{6} c_{i_1,i_2} \,/\, (m+i+i_1)(m+k+i_2)$$

where

$$c_{1,1} = c_{1,2} = c_{2,1} = c_{3,5} = c_{4,5} = c_{5,i_2} = c_{b,i_2} = 0, \quad \forall\, i_2 > 1$$

$$c_{1,3} = -3/2, \quad c_{1,4} = -\tfrac{1}{2}, \quad c_{1,5} = 5/8, \quad c_{2,2} = 1, \quad c_{2,3} = 3, \quad c_{2,4} = \tfrac{1}{2}$$

$$c_{2,5} = -5/8, \quad c_{3,1} = \tfrac{1}{2}, \quad c_{3,2} = -3/2, \quad c_{3,4} = -\tfrac{1}{4}, \quad c_{4,1} = -\tfrac{1}{2},$$

$$c_{4,2} = \tfrac{1}{2}, \quad c_{4,3} = 3/4, \quad c_{5,1} = 5/8, \quad c_{6,1} = -1/8 \quad ,$$

$$b_5 = (3m+3k+20)/(m+i+1)_2(m+k+4)(m+k+6)+(m+i+2)/2(m+i+3)_2(m+k+2)$$

$$+ (2m+2i+9)/4(m+i+3)_2(m+k+1) - (4m+4i+25)/8(m+i+5)_2(m+k+2) \quad .$$

Integrate (3.1) with respect to M_1, then the density of M_2 can be written as

$$(3.2) \qquad K_1'(4,n)\, e^{-M_2} \sum_{k=0}^{m} \binom{m}{k}(-2)^{-k} M_2^{m-k} \sum_{i=0}^{m} \binom{m}{i}(-2)^{-i}$$

$$\sum_{j=1}^{5} M_2^{5-j}(a_j M_2^{m+k+2}\, \gamma(0,M_2;2m+j+1)-b_j(0,M_2;3m+k+j+3)).$$

Iy may be pointed out that the density of M_1 can be found from (3.1) by integrating M_2. Now, let $T = M_1 + M_2$ in (3.1) and integrate M_1 then the density function of T can be written as

$$(3.3) \qquad \frac{1}{\Gamma(4m+10)} T^{4m+9} e^{-T} .$$

Further, transform $R_1 = M_1/M_2$ in (3.1) and integrate M_2, then the density of R_1 can be written as

$$(3.4) \quad K_3(4,n)(1+R_1)^{-(4m+10)} R_1^{2m+1} \sum_{k=0}^{m} \binom{m}{k}(-2)^{-k} \sum_{i=0}^{m} \binom{m}{i}(-2)^{-i} \sum_{j=1}^{s} R_1^j(a_j - b_j M_1^{m+k+2})$$

where

$$K_3(4,n) = \Gamma(4m+10)\, K_1'(4,n) \quad .$$

4. The Non-Central Distribution of Wilks' Criterion

In this section we shall derive the non-central distribution of Wilks' criterion, namely $\Lambda = W^{(p)} = \prod_{i=1}^{p} (1-r_i)$ where $r_1, \ldots, r_p$ are the characteristic roots of the equation

$$|\underset{\sim}{S}_1 - r(\underset{\sim}{S}_1 + \underset{\sim}{S}_2)| = 0 \quad ,$$

where $\underset{\sim}{S}_1$ is a $(p \times p)$ matrix distributed non-central Wishart with s degrees of freedom and a matrix of non-centrality parameters $\underset{\sim}{\Omega}$ and $\underset{\sim}{S}_2$

has the Wishart distribution with t degrees of freedom, the covariance matrix in each case being $\underset{\sim}{\Sigma}$. For this, first we state below a few results on Mellin transform and then prove a lemma.

Theorem 1. If s is any complex variate and $f(x)$ is a function of a real variable x, such that

$$(4.1) \qquad F(x) = \int_0^\infty x^{s-1} f(x)\, dx$$

exists. Then, under certain conditions [6]

$$(4.2) \qquad f(x) = \frac{1}{2\Pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s)\, ds \quad .$$

$F(s)$ in (4.1) is called the Mellin transform of $f(x)$ and $f(x)$ in (4.2) is called the inverse Mellin transform of $F(s)$. Now we state another theorem [6] .

Theorem 2. If $f_1(x)$ and $f_2(x)$ are the inverse Mellin transform of $F_1(s)$ and $F_2(s)$ respectively, then the inverse Mellin transform of $F_1(s)\, F_2(s)$ is given by

$$(4.3) \qquad \frac{1}{2\Pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} F_1(s)\, F_2(s)\, ds = \int_0^\infty f_1(u)\, f_2(x/u) \cdot \frac{du}{u} \quad .$$

Further we use theorem 2 to prove the following lemma.

Lemma 1. If s is a complex variabe, a, b, c, d, m, n, p and ℓ are reals then

$$(4.4) \qquad I = \frac{1}{2\Pi i} \int_{c-i\infty}^{c+i\infty} X^{-s} \frac{\Gamma(s+a)\,\Gamma(s+b)\,\Gamma(s+c)\,\Gamma(s+d)}{\Gamma(s+a+m)\,\Gamma(s+b+n)\,\Gamma(s+c+p)\,\Gamma(s+d+\ell)}\, ds$$

$$= \frac{X^d(1-X)^{m+n+p+\ell-1}}{\Gamma(m+n+p)} \sum_{k=o}^{\infty} \frac{(d+\ell-a)_k}{k!} \sum_{r=o}^{\infty} \frac{(p)_r(b+n-c)_r}{r!\,(m+n+p)_r} (1-X)^{k+r}$$

$$\frac{\Gamma(m+n+p+k+r)}{\Gamma(m+n+p+\ell+k+r)}\; {}_3F_2(a+m-b,n+p+r,m+n+p+k+r;$$

$$m+n+p+r,m+n+p+\ell+k+r;1-X) \quad .$$

<u>Proof</u>: Let $F_1(s) = \{\Gamma(s+a)\,\Gamma(s+b)\,\Gamma(s+c)/\Gamma(s+a+m)\,\Gamma(s+b+n)\,\Gamma(s+c+p)\}$, $F_2(s) = \Gamma(s+d)/\Gamma(s+d+\ell)$, then

$$(4.5) \qquad f_1(X) = X^a(1-X)^{m+n+p-1}[\Gamma(m+n+p)]^{-1} \sum_{r=0}^{\infty} \frac{(p)_r(b+n-c)_r}{r!\,(m+n+p)_r} (1-X)^r$$

$${}_2F_1(a+m-b,n+p+r;\; m+n+p+r;\; 1-X) \quad ,$$

and

$$f_2(X) = \frac{X^d(1-X)^{\ell-1}}{\Gamma(\ell)} \quad , \quad 0 < X < 1, \qquad [7] \quad .$$

Now by the use of Theorem 2, we get

$$(4.6) \qquad I = \frac{X^d}{\Gamma(\ell)\,\Gamma(m+n+p)} \int_X^1 u^{a-d-\ell}(1-U)^{m+n+p-1} \sum_{r=0}^{\infty} \frac{(p)_r(b+n-c)_r}{r!(m+n+p)_r}$$

$$(1-U)^r\; {}_2F_1(a+m-b,n+p+r;\; m+n+p+r;\; 1-U)(U-X)^{\ell-1}\, du \quad .$$

Further, put $u = 1 - (1-X)t$ in the above and by simplifying, we have

$$(4.7) \qquad I = \frac{X^d(1-X)^{m+n+p+\ell-1}}{\Gamma(\ell)\,\Gamma(m+p+n)} \int_0^1 \sum_{k=0}^{\infty} \frac{(d+\ell-a)_k}{k!} \sum_{r=0}^{\infty} \frac{(p)_r(b+n-c)_r}{r!\,(m+n+p)_r}$$

$$\sum_{i=0}^{\infty} \frac{(a+m-b)_i(m+p+r)_i}{i(m+n+p+r)_i} (1-X)^{k+i+r} t^{m+n+p+k+i+r-1}(1-t)^{\ell-1}\, dt.$$

Now integrate (4.7) with respect to t, then the lemma follows immediately.

The moments of the Wilks' Criterion has been given [4] in the following form.

$$(4.8) \qquad E\{W^{(p)}\}^h = \left[\Gamma_p(h+\tfrac{1}{2}t)\,\Gamma_p(\nu)/\Gamma_p(t/2)\,\Gamma_p(h+\nu)\right] {}_1F_1(h;h+\nu;-\underset{\sim}{\Omega})\right] \;;$$

where $\nu = \frac{1}{2}(s+t)$, and $\Gamma_p(u) = \Pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma(u-\frac{1}{2}(i-1))$.

By using Kummar transformation, (4.8) can be written in the following form

$$(4.9) \qquad E\{W^{(p)}\}^h = \left[\Gamma_p(h+\tfrac{1}{2}t)\,\Gamma_p(\nu)/\Gamma_p(t/2)\Gamma_p(h+\nu)\right] e^{-tr\underset{\sim}{\Omega}} {}_1F_1(\nu;h+\nu;\underset{\sim}{\Omega}) \;.$$

__Case i.__ Put $p = 2$ in (4.9), then

$$(4.10) \qquad E\{W^{(2)}\}^h = \frac{\Gamma(2\nu-1)}{2^s\Gamma(t-1)}\, e^{-tr\underset{\sim}{\Omega}} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\nu)_\kappa C_\kappa(\underset{\sim}{\Omega})}{k!} \;.$$

$$\frac{\Gamma(r)\;\Gamma(r+\frac{1}{2})}{\Gamma(r+\frac{1}{2}s+k_1+\frac{1}{2})\Gamma(r+\frac{1}{2}s+k_2)} \;,$$

where $r = h+\frac{1}{2}t-\frac{1}{2}$ and $k_1 \geq k_2 \geq 0$, $k_1 + k_2 = k$, then

$$(4.11) \qquad f(W^{(2)}) = \frac{\Gamma(2\nu-1)}{2^s\Gamma(t-1)} \exp(\mathrm{tr}-\underset{\sim}{\Omega}) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\nu)_\kappa C_\kappa(\underset{\sim}{\Omega})}{k!} \; .$$

$$\frac{1}{2\Pi i} \int_{c-i\infty}^{c+i\infty} \{W^{(2)}\}^{-h-1} \left[\Gamma(r)\Gamma(r+\tfrac{1}{2})/\Gamma(r+\tfrac{1}{2}s+k_2)\Gamma(r+\tfrac{1}{2}s+\tfrac{1}{2}+k_1)\right] dr \; .$$

Now, by the use of the results of Consul [7], we get the density function of $W^{(2)}$ in the following form

$$(4.12) \qquad f(W^{(2)}) = \frac{\Gamma(2\nu-1)}{2^s\Gamma(t-1)} \{W^{(2)}\}^{\frac{1}{2}(t-3)} \exp(\mathrm{tr}-\underset{\sim}{\Omega}) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\nu)_\kappa C_\kappa(\underset{\sim}{\Omega})}{k!\Gamma(s+k)}$$

$$(1-W^{(2)})^{s+k-1} \; {}_2F_1(\tfrac{1}{2}s+k_1, \tfrac{1}{2}s+k_2-\tfrac{1}{2}; s+k; 1-W^{(2)}) \; .$$

Putting $\underset{\sim}{\Omega} = \underset{\sim}{0}$, then the central case can be written in the following form

$$(4.13) \qquad f(W^{(2)}) = \frac{\Gamma(2\nu-1)}{2^s\Gamma(t-1)\Gamma(s)} \{W^{(2)}\}^{\frac{1}{2}(t-3)} (1-W^{(2)})^{s-1} \, {}_2F_1(s/2, (s-1)/2; s; 1-W^{(2)}) \; .$$

It may be pointed out that (4.13) can be reduced to

$$(4.14) \qquad \frac{\Gamma(2\nu-1)}{2\Gamma(t-1)\Gamma(s)} \{W^{(2)}\}^{\frac{1}{2}(t-3)} \left(1-\sqrt{W^{(2)}}\right)^{s-1} \; ,$$

by observing that

$$(4.15) \qquad {}_2F_1(s/2,(s-1)/2;s;1-U) = 2^{s-1}/(1+\sqrt{U})^{s-1} \qquad ([28]) \ .$$

Also the density function of $W^{(2)}$ can be written in the following form by the use of the results in [6].

$$(4.16) \qquad f(W^{(2)}) = \frac{\Gamma(2\nu-1)}{2\Gamma(t-1)} \{W^{(2)}\}^{\frac{1}{2}(t-3)} \exp(\mathrm{tr}\,\underset{\sim}{\Omega}) \cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\nu)_\kappa C_\kappa(\underset{\sim}{\Omega})}{k!\Gamma(s+2k_2)}$$

$$4^{k_2}(1-W^{(2)})^{k_1-k_2} \sum_{r=0}^{s+2k_2-1} \binom{s+2k_2-1}{r} (-1)^r \{W^{(2)}\}^{r/2} \ .$$

$${}_2F_1(k_1-k_2;\ (r+1-s)/2 - k_2;\ k_1-k_2+1;\ 1-W^{(2)}) \ .$$

Setting $\underset{\sim}{r} = \underset{\sim}{0}$, then (4.16) reduces to (4.14).

Case ii. Put $p = 3$ in (4.9), and by the use of (4.2) the density function of $W^{(3)}$ can be written in the following form

$$(4.17) \qquad f(W^{(3)}) = \frac{\Gamma_3(\nu)}{\Gamma_3(t/2)} \exp(\mathrm{tr}\,\Omega)\{W^{(2)}\}^{\frac{1}{2}(t-4)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\nu)_\kappa C_\kappa(\underset{\sim}{\Omega})}{k!}$$

$$\frac{1}{2\Pi i} \int_{c-i\infty}^{c+i\infty} \frac{\{W^{(3)}\}^{-r}\ \Gamma(r)\ \Gamma(r+\frac{1}{2})\ \Gamma(r+1)\ dr}{\Gamma(r+\frac{1}{2}s+k_3)\ \Gamma(r+\frac{1}{2}s+k_2+\frac{1}{2})\ \Gamma(r+\frac{1}{2}s+k_1+1)}$$

where $k_1 \geq k_2 \geq k_3 \geq 0$, $k_1 + k_2 + k_3 = k$.

By (4.5), the density function of $W^{(3)}$ can be written in the form

$$(4.18)\qquad f(W^{(3)}) = \frac{\Gamma_3(\nu)}{\Gamma_3(t/2)} \exp(\mathrm{tr}\,-\underset{\sim}{\Omega})\ \{W^{(3)}\}^{\frac{1}{2}(t-1)}(1-W^{(3)})^{\frac{3}{2}s-1}\ .$$

$$\sum_{k=0}^{\infty}\sum_{\kappa} \frac{(\nu)_\kappa C_\kappa(\underset{\sim}{\Omega})}{k!\Gamma(3s/2+k)} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}s+k_1)_r(\frac{1}{2}(s-1)+k_2)_r}{r!(3s/2+k)_r}$$

$$(1-W^{(3)})^{r+k}\ {}_2F_1(\tfrac{1}{2}(s-1)+k_3,s+k_1+k_2+r;3s/2+k+r;1-W^{(3)})\ .$$

<u>Case</u> <u>iii</u>. Put $p = 4$ in (4.9) and by the use of (4.2) the density function of $W^{(4)}$ can be written in the form

$$(4.19)\qquad f(W^{(4)}) = \frac{\Gamma_4(\nu)}{\Gamma_4(\frac{1}{2}t)} \exp(\mathrm{tr}\,-\underset{\sim}{\Omega})\{W^{(4)}\}^{\frac{1}{2}(t-5)} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(\nu)_\kappa C_\kappa(\underset{\sim}{\Omega})}{k!}\ .$$

$$\frac{1}{2\Pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(r)\ \Gamma(r+\frac{1}{2})\ \Gamma(r+1)\ \Gamma(r+\frac{3}{2})\ \{W^{(4)}\}^{-r}\ dr}{\Gamma(r+\frac{1}{2}s+k_4)\Gamma(r+\frac{1}{2}s+\frac{1}{2}+k_3)\Gamma(r+\frac{1}{2}s+1+k_2)\Gamma(r+\frac{1}{2}s+\frac{3}{2}+k_1)}$$

where $k_1 \geq k_2 \geq k_3 \geq k_4 \geq 0$, and $\sum_{i=1}^{4} k_i = k$.

By using Lemma 1, the density function of $W^{(4)}$ can be written in the form

$$(4.20) \qquad f(W^{(4)}) = \frac{\Gamma_4(\nu)}{\Gamma_4(t/2)} \exp(\mathrm{tr}\,\underset{\sim}{\Omega}) \{W^{(4)}\}^{\frac{1}{2}(t-2)} (1-W^{(4)})^{2s-1}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\nu)_\kappa C_\kappa(\underset{\sim}{\Omega})}{k!} \sum_{k=0}^{\infty} \frac{(\frac{1}{2}(s+3)+k_1)_j}{j!} \sum_{r=0}^{\infty}$$

$$\frac{(\frac{1}{2}(s+k_2)_r(\frac{1}{2}(s-1)+k_3)_r}{\Gamma(3s/2+k-k_1+r)r!} (1-W^{(4)})^{k+j+r} \frac{\Gamma(3s/2+k+j-k_1+r)}{\Gamma(2s+k+j+r)}$$

$${}_3F_2(\tfrac{1}{2}(s-1)+k_4, s+k_2+k_3+r, 3s/2+k-k_1+j+r;$$

$$3s/2+k-k_1+r, 2s+j+k+r; 1-W^{(4)}) \; .$$

It may be pointed out that the non-central distribution of Wilks' criterion could be found for more than $p = 4$ by extending Lemma 1. However the distribution would be complicated.

CHAPTER VI

DISTRIBUTION OF RATIOS AND DIFFERENCES OF THE ROOTS OF A COVARIANCE MATRIX

1. Introduction and Summary

While the earlier chapters deal with the studies of individual roots of some matrices in multivariate analysis, this chapter presents first the distribution of differences and ratios respectively of characteristic roots which follow the Fisher-Hsu-Girshick-Roy distribution. In regard to differences, the study has been carried out up to (including) the four roots case while for the ratios, results have been obtained up to five roots. The last section deals with the non-central distribution of the ratios of a covariance matrix which follow (5.1) of Chapter 2. The study has been carried out up to (including) the four roots. The distributions of such ratios are useful in testing the hypothesis $\delta \underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$, $\delta > 0$ unknown, has been pointed out where $\underset{\sim}{\Sigma}_1$ and $\underset{\sim}{\Sigma}_2$ are the covariance matrices of two p-variate normal populations.

2. The Distribution of the Differences of the Characteristic Roots

In this section we find the joint and the marginal distributions of of the differences $\theta_i - \theta_j$, $i > j$ when $p = 2, 3, 4$. The joint density of a p non-null roots of a matrix derived from sample observations under certain null hypotheses including that of Chapter 1, can be expressed in the form

$$(2.1)\qquad C(p,m,n)\prod_{i=1}^{p}\{\theta_i^m(1-\theta_i)^n\}\prod_{i>j}(\theta_i-\theta_j)\quad ,$$

$0 < \theta_1 \leq \theta_2 \leq \ldots \leq \theta_p < 1$, and parameters m and n are differently for various situations described in [22].

Transform $q_i = \theta_i/\theta_p$, $i = 1,\ldots, p-1$ then the distribution of $q_1,\ldots, q_{p-1}, \theta_p$ can be written as

$$(2.2)\qquad C(p,m,n)\theta_p^{mp+(p-1)(1+\frac{p}{2})}(1-\theta_p)^n\prod_{i=1}^{p-1}\{q_i^m(1-q_i\theta_p)^n(1-q_i)\}$$

$$\prod_{i>j}(q_i-q_j),\ 0 < q_1\leq\ldots\leq q_{p-1} < 1.$$

Now consider the transformation $d_i = \theta_p(1-q_i)$, $i = 1,\ldots, p-1$. Then $d_1,\ldots, d_{p-1}, \theta_p$ will be distributed as

$$(2.3)\qquad C(p,m,n)|\underset{\sim}{D}|\prod_{i<j}(d_i-d_j)\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)_\kappa}{k!}C_\delta(\underset{\sim}{D})\,C_\kappa(\underset{\sim}{D})\,\theta_p^{mp-d}\,(1-\theta_p)^{np-k}\ ,$$

where κ,δ are the partitions of k and d respectively and $\underset{\sim}{D} = \mathrm{diag}(d_1\ldots, d_{p-1})$. Now integrate (2.3) with respect to θ_p, then $d_1,\ldots, d_{p-1}$ are distributed in the form

$$(2.4)\qquad c(p,m,n)|D| \prod_{i<j} (d_i - d_j) \Big[\sum_{d=0}^{\infty} \sum_{\delta} \frac{(-m)_\delta}{d!} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-1)^k(-n)_\kappa}{k!} \quad C_\delta(\underset{\sim}{D})$$

$$C_\kappa(\underset{\sim}{D})\ I(d_1,1;mp-d,np-k)\Big],\quad 0 < d_{p-1} \leq \dots \leq d_1 < 1 \ .$$

For $p = 2$, (2.4) reduces to

$$(2.5)\qquad f(d_1) = c(2,m,n) \Big[\sum_{j=0}^{m} \binom{m}{j}(-1)^j \sum_{i=0}^{n} \binom{n}{i}\ d_1^{m+n+1-(i+j)}$$

$$I(d_1,1;\ m+j,n+i)\Big] \ .$$

For $p = 3$, the joint density of d_1, d_2 can be written in the form

$$(2.6)\qquad c(3,m,n) \Big[\sum_{d=0}^{\infty} \sum_{\delta} \frac{(-m)_\delta}{d!} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-1)^k(-n)_\kappa}{k!} \sum_{\tau} g^{\tau}_{(\delta,\kappa)} \sum_{i+j=t}$$

$$h^{T}_{ij}\{(d_1^{i+2}\ d_2^{j+1} - d_1^{i+1}\ d_2^{j+2})\ I(d_1,1;3m-d,3n-k)\}\Big] \ ,$$

where $g^{T}_{\delta,\kappa}$ is as defined in the previous sections and h^{T}_{ij} are such that $C_\tau\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = \sum_{i+j=t} h^{T}_{ij}\ d_1^i\ d_2^j$, τ is the partition of t and $t = k+d$.

Integrate (2.6) with respect to d_2, then the density of d_1 is of the form

$$(2.7) \qquad c(3,m,n)\Big[\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)_\kappa}{k!}$$

$$\sum_{T} g^T_{(\delta,\kappa)} \sum_{i+j=t} h^T_{ij}\{\frac{d_1^{t+4}}{(j+2)_2}\ I(d_1,1;3m-d,3n-k)\}\Big] .$$

Again, integrate (2.6) with respect to d_1, by parts, then the density of d_2 is given by

$$(2.8) \qquad c(3,m,n)\Big[\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)_\kappa}{k!}$$

$$\sum_{T} g^T_{(\delta,\kappa)} \sum_{i+j=t} h^T_{ij}\ \frac{1}{(i+2)_2}\ \{d_2^{t+4} I(d_2,1;3m=d,3n-k)$$

$$+ d_2^{j+1}((i+2)I(d_2,1;3m-d+i+3,3n-k)$$

$$- (i+3)\ d_2^{j+2}\ I(d_2,1;\ 3m-d+i+2,3n-k))\}\Big] .$$

Now let $\delta_{12} = d_1 - d_2 = \theta_2 - \theta_1$, then the distribution δ_{12} and d_1 can be written in the form

$$(2.9) \qquad c(3,m,n)\Big[\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)_\kappa}{k!}\sum_{T} g^T_{(\delta,\kappa)}\sum_{i+j=t} h^T_{ij}$$

$$\{\sum_{r=0}^{j+1} (-1)^r \binom{j+1}{r} \delta_{12}^{r+1} d_1^{t+2-r} I(d_1,1;3m-d,3n-k)\}\Big] ,$$

$$0 < \delta_{12} \le d_1 < 1 .$$

Integrating (2.9) with respect to d_1, we get the density of δ_{12} in the form

$$(2.10) \qquad c(3,m,n)\Big[\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)_\kappa}{k!}\sum_{T} g^T_{(\delta,\kappa)}\sum_{i+j=t} h^T_{ij}$$

$$\{\sum_{r=0}^{j+1}[(-1)^r\binom{j+1}{r}/t+r-3](-\delta_{12}^{t+4}I(\delta_{12},1;3m-d,3n-k)$$

$$+\ \delta_1^{r+1}I(\delta_{12},1;3m-d+t+3-r,3n-k))\}\Big] \ .$$

For $p = 4$, the joint density of d_1, d_2, d_3 can be written in the form

$$(2.11) \qquad c(4,m,n)\Big[\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)_\kappa}{k!}\sum_{T} g^T_{(\delta,\kappa)}$$

$$\sum_{i_1+i_2+i_3=t} h^T_{i_1,i_2,i_3}\ c(d_2-d_3)(d_1^2-(d_2+d_3)d_1+d_2d_3)$$

$$I(d_1,1;\ a,b)\Big] \ ,$$

where

$$a = 4m-d, \quad b = 4n-k, \quad c = d_1^{i_1+1}\ d_2^{i_2+1}\ d_3^{i_3+1} \ .$$

Integrating (2.11) with respect to d_1, by parts, and further with respect to d_2, we get the density of d_3 in the form

$$(2.12) \qquad c(4,m,n) \sum_{d=0}^{\infty} \sum_{\delta} \frac{(-m)_\delta}{d!} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-1)^k(-n)_\kappa}{k!} \sum_{T} g^T_{(\delta,\kappa)}$$

$$\sum_{i_1+i_2+i_3=t} h^T_{i_1,i_2,i_3} d_3^{i_3+1} \Big[- \frac{2d_3^{i_1+i_2+7}}{(i_1+2)_3(i_1+i_2+5)_3}$$

$$I(d_3,1;a,b) + \frac{I(d_3,1;e+3,b)}{(i_2+3)_2(i_1+i_2+7)} - \frac{2d_3 I(d_3,1;e+2,b)}{(i_2+2)(i_2+4)(i_1+i_2+6)}$$

$$+ \frac{d_3^2 I(d_3,1;e+1,b)}{(i_2+2)_2(i_1+i_2+5)} - \frac{d^{i_2+3} \; I(d_3,1;e_1+2,b)}{(i_2+2)_2(i_1+4)} \Big] \quad ,$$

where

$$e = i_1 + i_2 + 4 + a, \quad e_1 = a + i_1 + 2 \quad .$$

Similarly starting with (2.11) we can obtain the density of d_1 as

$$(2.13) \qquad c(4,m,n) \sum_{d=0}^{\infty} \sum_{\delta} \frac{(-m)_\delta}{d!} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-1)^k(-n)_\kappa}{k!} \sum_{T} g^T_{(\delta,\kappa)}$$

$$\sum_{i_1+i_2+i_3=t} h^T_{i_1,i_2,i_3} \frac{2(i_2+2i_3+9)}{(i_3+2)_3(i_2+i_3+5)_3} \; I(d_1,1;a,b) \quad ,$$

and the density of d_2 as

$$(2.14)\qquad c(4,m,n) \sum_{d=0}^{\infty} \sum_{\delta} \frac{(-m)_\delta}{d!} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-1)^k(-n)_\kappa}{k!} \sum_{T} g^T_{(\delta,\kappa)}$$

$$\sum_{i_1+i_2+i_3=t} h^T_{i_1,i_2,i_3}\, d_2^{i_2+i_3+4}\Big[\frac{2(i_1-i_3)d_2^{i_1+4}}{(i_1+2)_3(i_3+2)_3} I(d_2,1;a,b)$$

$$\frac{I(d_2,1;e_1+2,b)}{(i_1+4)(i_3+2)_2} - \frac{2d_2 I(d_2,1;e_1+1,b)}{(i_1+3)(i_3+2)(i_3+4)} + \frac{d_2^2 I(d_2,1;e_1,b)}{(i_1+2)(i_3+3)_2}\Big] .$$

Now make the transformation

$$(2.15)\qquad d_1 = \delta_1 + \delta_2 + \delta_3,\quad d_2 = \delta_2 + \delta_3,\quad d_3 = \delta_3,\quad \delta_{13} = \theta_3 - \theta_1 .$$

Using (2.15), then from the joint distribution of δ_1, d_2 can be obtained in the form:

$$(2.16)\qquad f(\delta_1,d_2) = c(4,m,n) \sum_{d=0}^{\infty} \sum_{\delta} \frac{(-m)_\delta}{d!} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-1)^k(-n)_\kappa}{k!} \sum_{T} g^T_{(\delta,\kappa)}$$

$$\sum_{i_1+i_2+i_3=t} h^T_{i_1,i_2,i_3} \Big[\sum_{r=0}^{i_1+1} \binom{i_1+1}{r} \delta_1^{r+1} d_2^{t+5-r}\Big(\frac{\delta_1}{(i_3+2)_2}$$

$$+ \frac{2d_2}{(i_3+2)_3}\Big) I(d_2+\delta_1,1;a,b)\Big] .$$

Further, integrate d_2 over $0 \le d_2 \le 1 - \delta_1$ then the distribution of δ_1 can be written in the form

$$(2.17)\qquad c(4,m,n)\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{K}\frac{(-1)^k(-n)_K}{k!}\sum_{T}g^T_{(\delta,K)}$$

$$\sum_{i_1+i_2+i_3=t}h^T_{i_1,i_2,i_3}\left[\frac{\delta_1}{(i_3+2)_2}\int_0^{1-\delta_1}\right.$$

$$\left\{\sum_{r=0}^{i_1+1}\binom{i_1+1}{r}\delta_1^{r+1}d_2^{t+6-r}(d_2+\delta_1)^a(1-d_2-\delta_1)^b\right.$$

$$\left.\left.\left(\frac{\delta_1}{t+6-r}+\frac{2}{(i_3+4)(t+7-r)}\right)/t+6-r\right\}dd_2\right].$$

Similarly the density of δ_2 can be written in the form

$$(2.18)\qquad c(4,m,n)\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{K}\frac{(-1)^k(-n)_K}{k!}\sum_{T}g^T_{(\delta,K)}\sum_{i_1+i_2+i_3=t}h^T_{i_1,i_2,i_3}$$

$$\left[\int_0^{1-\delta_2}\left\{\frac{\sum_{r=0}^{i_1+i_2+6}\binom{i_1+i_2+6}{r}}{(i_1+3)_2}-\frac{2\sum_{r=0}^{i_1+i_2+5}\binom{i_1+i_2+5}{r}}{(i_1+2)(i_1+4)}+\frac{\sum_{r=0}^{i_1+i_2+4}\binom{i_1+i_2+4}{r}}{(i_1+2)_2}\right\}\right.$$

$$\frac{\delta_2^r\,\delta_3^{t+8-r}}{t+8-r}(\delta_2+\delta_3)^a(1-\delta_2-\delta_3)^b d\delta_3+\left(\frac{\sum_{r=0}^{i_2+2}\binom{i_2+2}{r}-\sum_{r=0}^{i_2+1}\binom{i_2+1}{r}}{i_1+4}\right)$$

$$q(\delta_2,r,0)-\left(\frac{\sum_{r=0}^{i_2+3}\binom{i_2+3}{r}-\sum_{r=0}^{i_2+1}\binom{i_2+1}{2}}{i_1+3}\right)q(\delta_2,r,1)$$

$$\left.+\left(\frac{\sum_{r=0}^{i_2+3}\binom{i_2+3}{r}-\sum_{r=0}^{i_2+2}\binom{i_2+2}{2}}{i_1+2}\right)q(\delta_2,r,2)\right],$$

where

$$q(\delta_2,r,j) = \delta_2^2 \int_0^{1-\delta_2} \frac{\delta_3^{i_2+i_3+4-r+j}}{i_2+i_3+4-r+j} \; (\delta_2+\delta_3)^{e_1+2-j} (1-\delta_2-\delta_3)^b \, d\delta_3 \; ,$$

$$j = 0,\, 1,\, 2 \; .$$

Similarly the distribution of δ_{13} can be written in the form

$$\text{(2.19)} \qquad c(4,m,n) \sum_{d=0}^{\infty} \sum_{\delta} \frac{(-m)_\delta}{d!} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-n)_\kappa (-1)^k}{k!} \sum_{T} g^T_{(\delta,\kappa)} \sum_{i_1+i_2+i_3=t}$$

$$h^T_{i_1,i_2,i_3} \, \delta_{13} \Big[\Big\{ A(r) \, \delta_{13}^r \, I(\delta_{13},1;a+7+t-r,b)$$

$$- A(r) \, \delta_{13}^{t+7} \, I(\delta_{13},1;a,b) \Big\} / t+7-r \Big] \; ,$$

where

$$A(r) = \Big[\sum_{r=0}^{i_3+1} \binom{i_3+1}{r} (-1)^r - \sum_{r=0}^{i_2+i_3+5} \binom{i_2+i_3+5}{r} (-1)^r \Big] \Big/ (i_2+3)_2$$

$$+ \sum_{r=0}^{i_2+i_3+4} \binom{i_2+i_3+4}{r} (-1)^r - \sum_{r=0}^{i_3+2} \binom{i_3+2}{r} (-1)^r \Big] \Big/ (i_2+2)_2 \; .$$

3. The Distribution of the Ratios of the Characteristic Roots

The ratios of the characteristic roots are useful in various respects, but one immediate use can be seen from Chapter (1), for tests of hypotheses when δ is not known.

$$(3.6)\qquad c(4,m,n)(m_1m_2)^m(1-m_1)(1-m_2)(m_2-m_1)\Big[\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-n)_\kappa}{k!}C_\kappa(\underset{\sim}{M}_1)\beta(c_1,n+1)$$

$$\{\beta(c_2,2)-(m_1+m_2)\beta(c_2+1,2)+m_1m_2\beta(c_2+2,2)\}\Big] ,$$

where

$$\underset{\sim}{M}_1 = \mathrm{diag}(m_1,m_2,1),\quad c_1 = 4m+k+10,\quad c_2 = 3m+k+6 \quad .$$

Now let $n_1 = m_1/m_2$ and integrate with respect to m_2 then the distribution of n_1 can be obtained in the form

$$(3.7)\qquad c(4,m,n)n_1^m(1-n_1)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-n)_\kappa}{k!}\beta(c_1,n+1)\sum_{i=0}^{k}\sum_{\delta} b_{(\kappa,\delta)}\, c_\delta\begin{pmatrix}1 & 0\\ 0 & n_1\end{pmatrix}$$

$$\{\beta(c_2,2)\beta(s_1,2) - \beta(s_1+1,2)((n_1+1)\beta(c_2+1,2)+n_1\beta(c_2,2))$$

$$+ \beta(s_1+2,2)(n_1\beta(c_2+2,2) + n_1(n_1+1)\beta(c_2+1,2)-n_1^2\beta(s_1+3,2))\} ,$$

where

$$s_1 = 2m+i+3 \quad .$$

We may note that the distribution of q_1 can be found from (2.1) as the distribution of the smallest root as in Chapter (1) and that of m_2 by integrating (3.6) with respect to m_1 .

For $p = 5$, integrate (3.2) with respect to q_4, the joint density of m_1, m_2, m_3 can be written in the form

$$(3.8) \qquad c(5,m,n)|\underset{\sim}{M}|^m|\underset{\sim}{I}-\underset{\sim}{M}| \prod_{i>j}(m_i-m_j)\Big[\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-n)_\kappa}{k!}\beta(c_3,n+1)$$

$$\sum_{j=0}^{3}\frac{(-1)^j\,(2j)!}{(j!)^2 2^j \chi_{[21^j]}(1)}\beta(s_2,2)\Big(\sum_{i=0}^{k}\sum_{\delta} b_{(\delta,\kappa)}\sum_{\tau} g^{\tau} C_\tau(\underset{\sim}{M})\Big)\Big] ,$$

where

$$c_3 = 5m+k+15 \quad \text{and} \quad s_2 = 4m+10+j+k \quad .$$

Now consider the transformation $n_i = m_i/m_3$, $i = 1,2$ and integrate with respect to m_3, then the joint density of n_1, n_2 can be written in the form

$$(3.9) \qquad c(5,m,n)(n_1n_2)^m(1-n_1)(1-n_2)(n_2-n_1)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-n)_\kappa}{k!}\beta(c_3,n+1)$$

$$\sum_{j=0}^{3}\frac{(-1)^j\,(2j)!}{(j!)^2 2^j \chi_{[21^j]}(1)}\beta(s_2,2)\sum_{i=0}^{k}\sum_{\delta} b_{(\delta,\kappa}\sum_{\tau} g^{\tau}_{(\delta,1^j)}$$

$$C_\tau(\underset{\sim}{N}_1)\{\beta(t_1,2)-(n_1+n_2)\beta(t_1+1,2) + n_1n_2\beta(t_1+2,2)\} \quad ,$$

where

$$t_1 = 3m+i+j+6 \quad \text{and} \quad \underset{\sim}{N}_1 = \text{diag}\,(1,n_1,n_2) \quad .$$

Further, let $x = \frac{n_1}{n_2}$ and integrate with respect to n_2, we get the density of x as

$$(3.10) \qquad c(5,m,n)\, x^m(1-x)\Big[\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-n)_\kappa}{k!}\beta(c_3,n+1)\sum_{j=0}^{3}\frac{(-1)^j(2j)!}{(j!)^2 2^j \chi_{(21^j)}(1)}$$

$$\beta(s_2,2)\sum_{i=0}^{k}\sum_{\delta} b_{\delta,k}\sum_{T} g^{T}_{(\delta,1^j)}\sum_{r=0}^{i+j}\sum_{\eta} b_\eta C_\eta\begin{pmatrix}1 & 0\\ 0 & x\end{pmatrix}$$

$$\{(1-x)\beta(t_1,2)\beta(s_3,2)-(1-x^2)\beta(t_1+1,2)\beta(s_3+1,2)$$

$$+\, x(1-x)\beta(t_1+2,2)\beta(s_3+2,2)\}\Big] \quad ,$$

where $s_3 = 2m+r+3$, b_η are constants and η denote the partition of $i + j$.

We may note that the distribution of q_1 and q_4 can be found from (3.1) as the smallest and the largest roots respectively and m_3 can be found from (3.8) as its largest root.

4. The Distribution of the Ratios of the Roots of a Covariance Matrix

In this section we consider the distribution of the latent roots as in (5.1) of Chapter 2, which can be viewed as a limiting form of the non-central distribution of the latent roots Khatri [13] associated with test of hypothesis $\delta\Sigma_1=\Sigma_2$, where Σ_1 and Σ_2 are the covariance matrices of two p-variate normal populations, when $n_2\to\infty$, where n_2 is the size of the sample from the second population. Now if we wish to test instead the null hypothesis $\delta\Sigma_1=\Sigma_2$, $\delta>0$ unknown, the ratios of the latent roots would be of interest as test criteria. In this context, in the limiting form (5.1) of Chapter II, Σ should be replaced by $\delta\Sigma_1\Sigma_2^{-1}$.

$$(4.3)\, q \quad K_1(3,n)|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}(q_1\ q_2)^{\frac{1}{2}(n-4)}(q_2 - q_1)(1-q_1)(1-q_2)$$

$$\left[\sum_{k=0}^{\infty} \frac{\Gamma(a_k)}{k!} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{I}_3 - \underset{\sim}{\Sigma}^{-1})}{C_\kappa(I_3)} \sum_{i=0}^{k} \sum_{\eta} b_{\eta,k}\, C_\eta\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}\right.$$

$$\left.\sum_{r=0}^{\infty} \binom{-a_k}{r} q_1^r\, (1+q_2)^{-r-a_k}\right] \quad ,$$

where $a_k = (3n/2) + k$, $b_{\eta,\kappa}$ are the constants defined [14], and η is the partition of i into not more than p elements.

It may be noted that the distribution of q_1 and of q_2 can be found by writing $C_\eta\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} = \sum_{i_1+i_2=i} a_{i_1,i_2}\, q_1^{i_1}\, q_2^{i_2}$ and expanding $(1+q_2)^{-r-a_k}$ then integrating q_2 and q_1 respectively.

Let $r_1 = q_1/q_2$ so the distribution of r_1, q_2 can be written in the form

$$(4.4) \qquad K_1(3,n)|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}\, r_1^{\frac{1}{2}(n-4)}\, (1-r_1) \left[\sum_{k=0}^{\infty} \frac{\Gamma(a_k)}{k!} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{I}_3 - \underset{\sim}{\Sigma}^{-1})}{C_\kappa(\underset{\sim}{I}_3)}\right. \quad .$$

$$\sum_{i=0}^{k} \sum_{\eta} b_{\eta,\kappa}\, C_\eta\begin{pmatrix} r_1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{r=0}^{\infty} \binom{-a_k}{r} r_1^r \sum_{h=0}^{\infty} \binom{-r-a_k}{h}$$

$$\left. q_2^{n-2+i+r+h}\, (1-q_2)(1-r_1 q_2)\right] \quad .$$

Integrating (4.4) with respect to q_2, the distribution of r_1 can be written in the form where $b = \frac{3}{2}(n-1)+i+h+r$ and $\underset{\sim}{R}_1 = \mathrm{diag}(r_1, r_2, 1)$. Now, we can find the distribution of r_1 or r_2 by expressing $(r_1+r_2)^r$ in terms of zonal polynomials of $R = \mathrm{diag}(r_1, r_2)$ and using the method outlined in Section (2) and integrating with respect to r_2 or r_1 such that $0 < r_1 \leq r_2 < 1$.

Now, let $r_1' = r_1/r_2$, then the distribution of r_1' can be written in the form

$$(4.8) \qquad K_1(4,n)|\Sigma|^{-\frac{1}{2}n} \, r_1'^{\frac{1}{2}(n-5)}(1-r_1') \sum_{k=0}^{\infty} \frac{\Gamma(2n+k)}{k!} \sum_{\kappa} \frac{C_\kappa(I-\Sigma^{-1})}{C_\kappa(I_4)} \sum_{i=0}^{k} \sum_{\eta}$$

$$b_{\kappa,\eta} \sum_{t=0}^{i} \sum_{\tau} b'_{i,\tau} \, C_\tau\begin{pmatrix} r_1' & 0 \\ 0 & 1 \end{pmatrix} \sum_{r=0}^{\infty} \binom{-2n-k}{r}(1+r_1')^r$$

$$\sum_{h=0}^{\infty} \binom{-2n-k-r}{h} \{\beta(b,2)\beta(C,2)+r_1'[\beta(C+2,2)\beta(b+2,2)$$

$$-\beta(C+1,2)\beta(b,2)]+(1+r_1')\beta(b+1,2)(r_1'\beta(C+2,2)$$

$$-\beta(C+1,2))-r_1'^2\beta(b+2,2)\beta(C+3,2)\}$$

where $C = n-2+t+r$ and the constants $b'_{i,\tau}$ are defined in [13] .

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