On the non-central distributions of Wilks' - Λ*

for tests of three hypotheses

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1. Introduction and Summary. In multivariate analysis we are interested in testing three hypotheses, namely

1) that of equality of the dispersion matrices of two p-variate normal populations,

2) that of equality of the p-dimensional mean vectors for k p-variate normal populations having a common covariance matrix and

3) that of independence between a p-set and a q-set of variates in a (p+q)-variate normal population, with $p \leq q$. We obtain the non-central distribution of Wilks' criterion $\Lambda = W^{(p)} = \prod_{i=1}^{p} (1-c_i)$ in each of the above cases, where the c_i's are functions of the characteristic roots of the appropriate matrices. The density functions for case 2 have been obtained by Pillai and Al-Ani [8] for $p = 2,3,4$ and here we obtain the density functions for all three cases for general p in terms of Meijer's G-function [7] with special cases being explicitly evaluated. In this connection a theorem has been proved using some results on Mellin transforms [2,3,4]. Also the cumulative distribution function (c.d.f.) of $W^{(p)}$ is obtained for $p = 2$ in the above three cases. The densities in all cases may be put in a single general form given by

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$$(1.1)\quad f(W^{(p)}) = \frac{\Gamma_p(\delta)}{\Gamma_p(\frac{1}{2}\gamma)}\ \alpha\ \{W^{(p)}\}^{\frac{1}{2}(\gamma-p-1)} \sum_{k=o}^{\infty} \sum_{\kappa} \frac{(\delta)_\kappa\ \beta}{k!} C_\kappa(\underset{\sim}{M})\ G^{p,o}_{p,p}(W^{(p)}\Big|^{a_1,a_2,\ldots,a_p}_{b_1,b_2,\ldots,b_p})$$

where

$$a_i = \tfrac{1}{2}(2\delta-\gamma) + k_{p-i+1}+b_i \quad \text{and} \quad b_i = (i-1)/2$$

and

	Case 1	Case 2	Case 3
$\gamma =$	n_2	t	$n-q$
$\delta =$	$\frac{1}{2}n$	ν	$\frac{1}{2}n$
$\beta =$	$(\frac{1}{2}n_1)_\kappa$	1	$(\frac{1}{2}n)_\kappa$
$\alpha =$	$\lvert\lambda\ \underset{\sim}{\Lambda}\rvert^{-\frac{1}{2}n_1}$	$e^{-tr\underset{\sim}{\Omega}}$	$\lvert\underset{\sim}{I}_p - \underset{\sim}{P}^2\rvert^{\frac{1}{2}n}$
$\underset{\sim}{M} =$	$\underset{\sim}{I}_p - (\lambda\ \underset{\sim}{\Lambda})^{-1}$	$\underset{\sim}{\Omega}$	$\underset{\sim}{P}^2$

See the following sections for definitions of the parameters as well as the G-function.

2. Preliminary Results. Some results on Mellin transforms [2,3,4] and Meijer's G-function [7] useful in proving the theorem below will now be given.

Lemma 1. If s is any complex variate and $f(x)$ is a function of a real variable x, such that

$$(2.1)\qquad F(s) = \int_o^\infty x^{s-1} f(x)\, dx$$

exists, then under certain regularity conditions

(2.2) $$f(x) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s)\, ds.$$

$F(s)$ is called the Mellin transform of $f(x)$ and $f(x)$ is the inverse Mellin transform of $F(s)$.

<u>Lemma 2</u>. If $f_1(x)$ and $f_2(x)$ are the inverse Mellin transforms of $F_1(s)$ and $F_2(s)$ respectively, then the inverse Mellin transform of $F_1(s)F_2(s)$ is

(2.3) $$(2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s} F_1(s)F_2(s)\, ds = \int_0^\infty f_1(u)f_2(x/u)\, du/u.$$

Meijer [7] defined the G-function by

(2.4) $$G_{p,q}^{m,n}\left(x\middle|\begin{matrix}a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_p\end{matrix}\right) = (2\pi i)^{-1} \int_C \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j - s)}\, x^s ds.$$

where C is a curve separating the singularities of $\prod_{j=1}^{m}\Gamma(b_j - s)$ from those of $\prod_{j=1}^{n}\Gamma(1-a_j+s)$, $q \geq 1$, $0 \leq n \leq p \leq q$, $0 \leq m \leq q$; $x \neq 0$ and $|x| < 1$ if $q = p$; $x \neq 0$ if $q > p$. It is easily verified that

$$(2.5)\quad G^{2,0}_{2,2}\left(x\middle|{a_1,a_2 \atop b_1,b_2}\right) = \frac{x^{b_1}(1-x)^{a_1+a_2-b_1-b_2-1}}{\Gamma(a_1+a_2-b_1-b_2)}\, {}_2F_1(a_2-b_2, a_1-b_2; a_1+a_2-b_1-b_2; 1-x) \quad 0<x<1$$

where the generalized hypergeometric function ${}_2F_1$ is given by James [5]. The G-function of (2.4) can be expressed as a finite number of generalized hypergeometric functions as follows.

$$G^{m,n}_{p,q}\left(x\middle|{a_1,\ldots,a_p \atop b_1,\ldots,b_q}\right) = \sum_{h=1}^{m} \frac{\prod_{\substack{j=1\\ j\neq h}}^{m} \Gamma(b_j-b_h) \prod_{j=1}^{n} \Gamma(1+b_h-a_j)}{\prod_{j=m+1}^{q} \Gamma(1+b_h-b_j) \prod_{j=n+1}^{p} \Gamma(a_j-b_h)}\, x^{b_h}$$

$$\cdot\ {}_pF_{q-1}(1+b_h-a_1,\ldots,1+b_h-a_p;\ 1+b_h-b_1,\ldots*\ldots,1+b_h-b_q;\ (-1)^{p-m-n}x)$$

where the asterisk denotes that the number $1+b_h-b_h$ is omitted in the sequence $1+b_h-b_1,\ldots,1+b_h-b_q$. Although the following theorem gives a more complicated form for expressing the G-function, it is useful in that expression (2.4) of Consul [4] and Lemma 1 of Pillai and Al-Ani [8] are special cases.

Theorem 1. If s is a complex variate, a_i, b_i, $i=1,2,\ldots,p$ are reals, then for $p \geq 3$

$$(2.6)\quad G^{p,o}_{p,p}\left(x\middle|{a_1,a_2,\ldots,a_p \atop b_1,b_2,\ldots,b_p}\right) = \frac{x^{b_p}(1-x)^{c-1}}{\Gamma(c_1+c_2+c_3)} \prod_{i=1}^{p-3} \left(\sum_{j_i=o}^{\infty} \frac{(b_{p-i+1}+c_{p-i+1}-b_{p-i})_{j_i}}{(j_i)!}\right)$$

$$\cdot \sum_{j=o}^{\infty} \frac{(c_1)_j(b_2+c_2-b_1)_j}{(c_1+c_2+c_3)_j\ j!}(1-x)^{j+\sum_{i=1}^{p-3} j_i} \prod_{\ell=1}^{p-3} \left[\frac{\Gamma(g_\ell+j_\ell)}{\Gamma(h_\ell)}\right]$$

$$\cdot\ {}_{p-1}F_{p-2}\ (b_3+c_3-b_2,f_1,f_2,\ldots,f_{p-2};\ g_1,g_2,\ldots,g_{p-2};1-x) \qquad 0<x<1$$

where for notational convenience $c_i = a_i - b_i$, $c = \sum_{i=1}^{p} c_i$, $f_\ell = \sum_{i=1}^{\ell+1} c_i + \sum_{i=1}^{\ell-1} j_i + j$, $g_\ell = \sum_{i=1}^{\ell+2} c_i + \sum_{i=1}^{\ell-1} j_i + j$, $h_\ell = \sum_{i=1}^{\ell+3} c_i + \sum_{i=1}^{\ell} j_i + j$ and $(a)_k = a(a+1)\ldots(a+k-1)$.

<u>Proof</u>. Using mathematical induction starting with $p=3$, we see making the substitution $(a,b,c,m,n,p) \to (b_3,b_2,b_1,c_3,c_2,c_1)$ in (2.4) of Consul [3] that

$$(2.7)\qquad G^{3,0}_{3,3}\left(x\middle|{a_1,a_2,b_3 \atop b_1,b_2,b_3}\right) = \frac{x^{b_3}(1-x)^{c_1+c_2+c_3-1}}{\Gamma(c_1+c_2+c_3)} \sum_{j=o}^{\infty} \frac{(c_1)_j(b_2+c_2-b_1)_j}{j!(c_1+c_2+c_3)_j}\ (1-x)^j$$

$$\cdot\ {}_2F_1\ (b_3+c_3-b_2,\ c_1+c_2+j;\ c_1+c_2+c_3+j;\ 1-x) \qquad 0<x<1$$

which is (2.6) with $p=3$. Now assuming (2.6) is true for $p=n$, we show it holds for $p=n+1$. Applying Lemma 2 with

$$F_1(s) = \frac{\prod_{i=1}^{n} \Gamma(s+b_i)}{\prod_{i=1}^{n} \Gamma(s+a_i)} \quad \text{and} \quad F_2(s) = \frac{\Gamma(s+b_{n+1})}{\Gamma(s+a_{n+1})}$$

we have $f_1(x)$ is (2.6) with $p=n$ and $f_2(x) = \dfrac{x^{b_{n+1}}(1-x)^{c_{n+1}-1}}{\Gamma(c_{n+1})}$

and it follows that

$$(2.8) \qquad G^{n+1,o}_{n+1,n+1}\left(x\middle|{a_1,a_2,\ldots,a_{n+1} \atop b_1,b_2,\ldots,b_{n+1}}\right) = \frac{x^{b_{n+1}}}{\Gamma(c_1+c_2+c_3)\,\Gamma(c_{n+1})}\int_x^1 u^{b_n-b_{n+1}-c_{n+1}}(1-u)^{\sum_{i=1}^{n} c_i+1}$$

$$\cdot \prod_{i=1}^{n-3}\left(\sum_{j_i=o}^{\infty}\frac{(b_{n-i+1}+c_{n-i+1}-b_{n-i})_{j_i}}{(j_i)!}\right)\sum_{j=o}^{\infty}\frac{(c_1)_j(b_2+c_2-b_1)_j}{(c_1+c_2+c_3)_j\, j!}(1-u)^{j+\sum_{i=1}^{n-3} j_i}\prod_{\ell=1}^{p-3}\frac{\Gamma(g_\ell+j_\ell)}{\Gamma(h_\ell)}$$

$$\cdot \ {}_{n-1}F_{n-2}(b_3+c_3-b_2,f_1,f_2,\ldots,f_{p-2};g_1,g_2,\ldots,g_{p-2};\ 1-u)(u-x)^{c_{n+1}-1}\ du.$$

Expanding $u^{b_n-b_{n+1}-c_{n+1}}$ in powers of $1-u$ when $b_{n+1}+c_{n+1}> b_n$, letting $u = x+(1-x)t$ and integrating with respect to t, the result is the same as (2.6) with $p=n+1$.

It is easily verified that Lemma 1 of Pillai and Al-Ani [8] is a special case of (2.6) with $p=4$ by making the following substitution

$$(b_1,b_2,b_3,b_4,c_1,c_2,c_3,c_4) \to (c,b,a,d,p,n,m,\ell).$$

It should be mentioned that this theorem doesn't apply when $p=1,2$. This is due to the fact that a simplification in the form of the G-function for $p=3$ reduces the hypergeometric function involved from ${}_3F_2$ to ${}_2F_1$. A general form for all p can be given as below, but we see it is more cumbersome to use because we have ${}_pF_{p-1}$ rather than ${}_{p-1}F_{p-2}$ as in (2.6)

$$G^{p,o}_{p,p}\left(x\left|{a_1,a_2,\ldots,a_p \atop b_1,b_2,\ldots,b_p}\right.\right) = \frac{x^{b_1}(1-x)^{c-1}}{\Gamma(c)} \prod_{i=1}^{p-3}\left[\sum_{\ell_{p-i-2}}^{\infty} \frac{(b_i+c_i-b_{i+1})_{\ell_{p-i-2}}}{(\ell_{p-i-2})!}\right] \tag{2.9}$$

$$\cdot \sum_{r=o}^{\infty} \frac{(b_{p-2}+c_{p-2}-b_{p-1})_r(c_{p-1}+c_p)_r \prod_{i=1}^{p-3}\left(\sum_{j=i+2}^{p} c_j + \sum_{j=1}^{p-4} \ell_j+r\right)_{\ell_i}}{r!\ (c)_{\ell+r}} (1-x)^{\ell+r}$$

$$\cdot\ {}_pF_{p-1}(c_p,b_{p-1}+c_{p-1}-b_p,f_1,\ldots,f_{p-2};\ c_{p-1}+c_p,g_1,\ldots,g_{p-2};\ 1-x) \quad 0<x<1$$

where

$$\ell = \sum_{i=1}^{p-3} \ell_i,\ f_i = \sum_{j=p-i}^{p} c_i + \sum_{j=1}^{p+i-6} \ell_j+r, \quad g_i = \sum_{j=p-i-1}^{p} c_j + \sum_{j=1}^{p+i-6} \ell_j+r, \quad c = \sum_{i=1}^{p} c_i.$$

It follows that letting $p=2$ we get (2.5) and $p=1$ gives

$$G^{1,0}_{1,1}\left(x\left|{a_1 \atop b_1}\right.\right) = x^b(1-x)^{c_1-1}/\Gamma(c_1).$$

3. The Non-Central Distribution of $W^{(p)}$ in Case 1. Let $\underset{\sim}{X}(pxn_1)$ and $\underset{\sim}{Y}(pxn_2)$ $p \leq n_i$, i=1,2, be independent matrix variates with the columns of $\underset{\sim}{X}$ independently distributed as $N(0,\underset{\sim}{\Sigma}_1)$ and those of $\underset{\sim}{Y}$ independently distributed as $N(0,\underset{\sim}{\Sigma}_2)$. Hence $\underset{\sim}{S}_1 = \underset{\sim}{X}\underset{\sim}{X}'$ and $\underset{\sim}{S}_2 = \underset{\sim}{Y}\underset{\sim}{Y}'$ are independently distributed as Wishart $(n_i,p,\underset{\sim}{\Sigma}_i)$, i=1,2. Let $0 < f_1 < f_2 < \ldots < f_p < \infty$ be the characteristic (ch.) roots of the determinantal equation

$$(3.1) \qquad |\underset{\sim}{S}_1 - f\,\underset{\sim}{S}_2| = 0$$

and $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p < \infty$ be the ch. roots of

$$(3.2) \qquad |\underset{\sim}{\Sigma}_1 - \gamma\,\underset{\sim}{\Sigma}_2| = 0.$$

For testing the hypothesis H_o: $\lambda\,\underset{\sim}{\Lambda} = \underset{\sim}{I}_p$, $\lambda > 0$ being given, we will use

$$(3.3) \qquad W^{(p)} = \prod_{i=1}^{p} (1-w_i)$$

where

$$\underset{\sim}{\Lambda} = \text{diag}\,(\lambda_1,\lambda_2,\ldots,\lambda_p), \quad w_i = \lambda f_i/(1-\lambda f_i) \qquad i=1,2,\ldots,p.$$

Khatri [6] has shown that

$$(3.4)\quad f(w_1,w_2,\ldots,w_p) = C|\lambda\Lambda|^{-\frac{1}{2}n_1}|W|^{\frac{1}{2}(n_1-p-1)}|I_p-W|^{\frac{1}{2}(n_2-p-1)}\prod_{i<j}(w_i-w_j)\,{}_1F_0(\tfrac{1}{2}n;I_p-(\lambda\Lambda)^{-1},W)$$

where

$$W = \mathrm{diag}(w_1,w_2,\ldots,w_p),\quad n=n_1+n_2,\quad \Gamma_p(t) = \pi^{p(p-1)/4}\prod_{j=1}^{p}\Gamma(t-\tfrac{1}{2}j+\tfrac{1}{2}),$$

$$C = \pi^{\frac{1}{2}p^2}\,\Gamma_p(\tfrac{1}{2}n)\,[\Gamma_p(\tfrac{1}{2}p)\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)]^{-1}.$$

To find $E[W^{(p)}]^h$ we multiply (3.4) by $|I_p - W|^h = [\prod_{i=1}^{p}(1-w_i)]^h$, transform $W \to H\,V\,H'$, where H is an orthogonal and V is a symmetric matrix, integrate out H and V using (44) and (22) of Constantine [1] and we find

$$(3.5)\quad E[W^{(p)}]^h = \frac{\Gamma_p(\frac{1}{2}n)\Gamma(\frac{1}{2}n_2+h)}{\Gamma(\frac{1}{2}n_2)\Gamma_p(\frac{1}{2}n+h)}\,|\lambda\Lambda|^{-\frac{1}{2}n_1}\,{}_2F_1(\tfrac{1}{2}n,\tfrac{1}{2}n_1;\tfrac{1}{2}n+h;I_p-(\lambda\Lambda)^{-1}).$$

Using Lemma 1, the density of $f(W^{(p)})$ has the form

$$(3.6)\qquad f(W^{(p)}) = C_p\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}n)_\kappa(\frac{1}{2}n_1)_\kappa}{k!}\,C_\kappa(I_p-(\lambda\Lambda)^{-1})\,\{W^{(p)}\}^{\frac{1}{2}(n_2-p-1)}$$

$$\cdot\,(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\{W^{(p)}\}^{-r}\,\frac{\prod_{i=1}^{p}\Gamma(r+b_i)}{\prod_{i=1}^{p}\Gamma(r+a_i)}\,dr$$

where

$$r = \tfrac{1}{2}n_2 + h - \tfrac{1}{2}(p-1), \quad b_i = \tfrac{1}{2}(i-1), \quad a_i = \tfrac{1}{2}n_1 + k_{p-i+1} + b_i,$$

$$C_p = \frac{\Gamma_p(\frac{1}{2}n)}{\Gamma_p(\frac{1}{2}n_2)} |\lambda\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1}, \quad (a)_\kappa = \prod_{i=1}^{p} \Gamma(a-\tfrac{1}{2}(i-1))_{k_i}, \quad (a)_k = a(a+1)..(a+k-1),$$

$\sum_\kappa$ is the sum over all partitions κ of the integer k where

$$\kappa = (k_1,k_2,\dots,k_p), \; k_1 \geq k_2 \geq \dots \geq k_p > 0, \; \sum_{i=1}^{p} k_1 = k, \quad \text{and}$$

$C_\kappa(\underset{\sim}{S})$ is a zonal polynomial; see James [5].

Noting that the integral in (3.6) is in the form of Meijer's G-function we can write the density of $W^{(p)}$ as

$$(3.7)\quad f(W^{(p)}) = C_p\{W^{(p)}\}^{\frac{1}{2}(n_2-p-1)} \sum_{k=o}^{\infty} \sum_\kappa \frac{(\frac{1}{2}n)_k(\frac{1}{2}n_1)_\kappa}{k!} C_k(\underset{\sim}{I}_p-(\lambda\underset{\sim}{\Lambda})^{-1}) \, G^{p,o}_{p,p}\left(W^{(p)}\middle|{}^{a_1,a_2,\dots,a_p}_{b_1,b_2,\dots,b_p}\right).$$

Letting $p=2$ in (3.7) and using (2.5) we obtain

$$(3.8)\quad f(W^{(2)}) = C_2\{W^{(2)}\}^{\frac{1}{2}(n_2-3)} \sum_{k=o}^{\infty} \sum_\kappa \frac{(\frac{1}{2}n)_\kappa(\frac{1}{2}n_1)_\kappa}{k!} C_\kappa(\underset{\sim}{I}_2-(\lambda\underset{\sim}{\Lambda})^{-1}) \frac{\{1-W^{(2)}\}^{a_1+a_2-b_1-b_2-1}}{\Gamma(a_1+a_2-b_1-b_2)}$$

$$\cdot \; {}_2F_1(a_2-b_2,\, a_1-b_2,\, a_1+a_2-b_1-b_2;\, 1-W^{(2)}).$$

The probability that $W^{(2)} \leq w(\leq 1)$ can be obtained by integrating (3.8) by parts a_1 times when n_1 is even. Using the relation [3]

$$(3.9) \qquad (d^n/dz^n)[z^{c-1}{}_2F_1(a,b;c;z)] = (c-n)_n\, z^{c-n-1}{}_2F_1(a,b;\, c-n;\, z),$$

and recalling that $\kappa=(k_1,k_2)$, we obtain the c.d.f. of $W^{(2)}$ in terms of a_i's and b_i's as

$$(3.10) \quad \Pr\{W^{(2)} \leq w\} = |\underset{\sim}{\lambda\Lambda}|^{-\frac{1}{2}n_1} \sum_{k=o}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}n_1)_\kappa C_\kappa(\underset{\sim}{I_2}-(\underset{\sim}{\lambda\Lambda})^{-1})\, w^{\frac{1}{2}(n_2-1)}}{k!}$$

$$\cdot \left\{\frac{\Gamma_2(\frac{1}{2}n)(\frac{1}{2}n)_\kappa}{\Gamma_2(\frac{1}{2}n_2)\Gamma(a_1+a_2-b_1-b_2)} \sum_{r=o}^{a} \frac{(a_1+a_2-b_1-b_2-r)_r}{\{\frac{1}{2}(n_2-1)\}_{r+1}}\; w^r(1-w)^{a_1+a_2-b_1-b_2-r-1}\right.$$

$$\left. \cdot\ {}_2F_1(a_2-b_2, a_1-b_2,\, a_1+a_2-b_1-b_2-r;\, 1-W) + I_W(\tfrac{1}{2}n_2,\, b)\right\}$$

where a_i, b_i are defined in (3.6), $a=a_1-1$ and $b=a_2-b_2$. When n_1 is odd, after integrating (3.8) by parts a_2 times, the c.d.f. of $W^{(2)}$ is (3.10) with $a=a_2-1$ and $b=a_1-b_2$.

Letting $p=3$ in (3.7) we have

$$(3.11) \qquad f(W^{(3)}) = \frac{\Gamma_3(\frac{1}{2}n)}{\Gamma_3(\frac{1}{2}n_2)} |\lambda\Lambda|^{-\frac{1}{2}n_1} \{W^{(3)}\}^{\frac{1}{2}(n_2-4)}$$

$$\cdot \sum_{k=o}^{\infty} \sum_{K} \frac{(\frac{1}{2}n)_K(\frac{1}{2}n_1)_K}{k!} C_K(I_3-(\lambda\Lambda)^{-1})\, G^{3,0}_{3,3}\left(W^{(3)}\middle|{a_1,a_2,a_3 \atop b_1,b_2,b_3}\right)$$

where a_i and b_i are defined in (3.6).

It is clear $G^{3,0}_{3,3}\left(W^{(3)}\middle|{a_1,a_2,a_3 \atop b_1,b_2,b_3}\right)$ could be written out in terms of the hypergeometric function using Theorem 1, for computation purposes.

Also letting $p = 4$ in (3.7) yields

$$(3.12) \quad f(W^{(4)}) = \frac{\Gamma_4(\frac{1}{2}n)}{\Gamma_4(\frac{1}{2}n_2)} |\lambda\Lambda|^{-\frac{1}{2}n_1} \{W^{(4)}\}^{\frac{1}{2}(n_2-5)}$$

$$\cdot \sum_{k=o}^{\infty} \sum_{K} \frac{(\frac{1}{2}n)_K(\frac{1}{2}n_1)_K}{k!} C_k(I_4-(\lambda\Lambda)^{-1})\, G^{4,0}_{4,4}\left(W^{(4)}\middle|{a_1,a_2,a_3,a_4 \atop b_1,b_2,b_3,b_4}\right)$$

where a_i's and b_i's are defined in (3.6).

4. <u>The Non-Central Distribution of $W^{(p)}$ in Case 2.</u> Let $\Lambda = W^{(p)} = \prod_{i=1}^{p} (1-\ell_i)$ where $\ell_1,\ell_2,\ldots,\ell_p$ are the ch. roots of the determinantal equation

$$(4.1) \qquad |\underset{\sim}{S}_1 - \ell(\underset{\sim}{S}_1 + \underset{\sim}{S}_2)| = 0$$

where $\underset{\sim}{S}_1$ is a (pxp) matrix distributed as non-central Wishart with s degrees of freedom, $\underset{\sim}{\Omega}$ is a matrix of non-centrality parameters and $\underset{\sim}{S}_2$ has the Wishart distribution with t degrees of freedom, the covariance matrix in each case being $\underset{\sim}{\Sigma}$. Pillai and Al-Ani [8] obtained the density of $W^{(p)}$ for $p=2,3,4$. Here we obtain the density of $W^{(p)}$ in general in terms of Meijer's G-functions. As in section 3, applying Lemma 1 to the expression for $E[W^{(p)}]^h$ given by Constantine [1] and using (2.4) we find

$$(4.2) \quad f(W^{(p)}) = C_p\{W^{(p)}\}^{\frac{1}{2}(t-p-1)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\nu)_\kappa C_\kappa(\underset{\sim}{\Omega})}{k!}\, G^{p,0}_{p,p}\left(W^{(p)}\middle|{a_1,a_2,\ldots,a_p \atop b_1,b_2,\ldots,b_p}\right)$$

where

$$\nu=\tfrac{1}{2}(s+t), \quad C_p=\frac{\Gamma_p(\nu)}{\Gamma_p(\frac{1}{2}t)}\, e^{-\mathrm{tr}\underset{\sim}{\Omega}}, \quad b_i=\tfrac{1}{2}(i-1), \quad a_i=\tfrac{1}{2}s+k_{p-i+1}+b_i.$$

The probability that $W^{(2)} \le w(\le 1)$ can be obtained by using (2.5) in (4.2), integrating by parts a_1 times when s is even, then using (3.9) we get the c.d.f. of $W^{(2)}$ as

$$(4.3)\quad \Pr\{W^{(2)} \le w\} = e^{-\mathrm{tr}\underset{\sim}{\Omega}} \sum_{k=o}^{\infty} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{\Omega})}{k!} w^{\frac{1}{2}(t-1)} \Big\{ \frac{\Gamma_2(\nu)(\nu)_\kappa}{\Gamma_2(\frac{1}{2}t)\Gamma(a_1+a_2-b_1-b_2)}$$

$$\cdot \sum_{r=o}^{a} \frac{(a_1+a_2-b_1-b_2-r)_r}{\{\frac{1}{2}(t-1)\}_{r+1}} w^r (1-w)^{a_1+a_2-b_1-b_2-r-1} {}_2F_1(a_2-b_2, a_1-b_2;\ a_1+a_2-b_1-b_2-r; 1-W^{(2)})$$

$$+ I_w(\tfrac{1}{2}t, b)\Big\}$$

where

$a = a_1-1$, $b = a_2-b_2$ and the a_i's and b_i's are defined in (4.2). When s is odd, we integrate (4.2) by parts a_2 times and find the c.d.f. is (4.3) with $a=a_2-1$, $b=a_1-b_2$.

The densities of $W^{(3)}$ and $W^{(4)}$ obtained by Pillai and Al-Ani [8] are special cases of (4.2) as can be verified by letting $p=3,4$ in (4.2), applying Theorem 1 and making the substitution

$$(a_1, a_3, b_1, b_3) \rightarrow (a_3, a_1, b_3, b_1).$$

5. <u>The Non-Central Distribution of $W^{(p)}$ in Case 3.</u> Let the columns of $\begin{pmatrix} \underset{\sim}{X}_1 \\ \underset{\sim}{X}_2 \end{pmatrix}$ be independent normal $(p+q)$ - variates ($p \le q$, $p+q \le n$, n is the sample size) with zero means and covariance matrix

(5.1) $$\underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}'_{12} & \underset{\sim}{\Sigma}_{22} \end{pmatrix}.$$

Let $\underset{\sim}{R}^2 = \text{diag}(r_1^2, r_2^2, \ldots, r_p^2)$ where r_i^2 are the ch. roots of

(5.2) $$|\underset{\sim}{X}_1 \underset{\sim}{X}'_2 (\underset{\sim}{X}_2 \underset{\sim}{X}'_2)^{-1} \underset{\sim}{X}_2 \underset{\sim}{X}'_1 - r^2 \underset{\sim}{X}_1 \underset{\sim}{X}'_1| = 0$$

and $\underset{\sim}{P}^2 = \text{diag}\ (\rho_1^2, \rho_2^2, \ldots, \rho_p^2)$ where ρ_i^2 are the ch. roots of

(5.3) $$|\underset{\sim}{\Sigma}_{12} \underset{\sim}{\Sigma}_{22}^{-1} \underset{\sim}{\Sigma}_{12} - \rho^2 \underset{\sim}{\Sigma}_{11}| = 0\ .$$

Constantine [1] obtained the density of $r_1^2, r_2^2, \ldots, r_p^2$ as

(5.4) $$f(r_1^2, r_2^2, \ldots, r_p^2) = C\ |\underset{\sim}{I}_p - \underset{\sim}{P}^2|^{\frac{1}{2}n}\ |\underset{\sim}{R}^2|^{\frac{1}{2}(q-p-1)}\ |\underset{\sim}{I}_p - \underset{\sim}{R}^2|^{\frac{1}{2}(n-q-p-1)}$$

$$\cdot \prod_{i<j} (r_i^2 - r_j^2) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}n)_\kappa (\frac{1}{2}n)_\kappa\ C_\kappa(\underset{\sim}{R}^2) C_\kappa(\underset{\sim}{P}^2)}{(\frac{1}{2}q)_\kappa\ C_\kappa(\underset{\sim}{I}_p)\ k!}$$

where

$$C = \pi^{\frac{1}{2}p^2}\ \Gamma_p(\tfrac{1}{2}n)\ [\Gamma_p(\tfrac{1}{2}q)\Gamma_p(\tfrac{1}{2}(n-q))\Gamma_p(\tfrac{1}{2}p)]^{-1}.$$

To find $E[W^{(p)}]^h$, $W^{(p)} = \prod_{i=1}^{p}(1-r_i^2)$, we multiply (5.4) by $|\underset{\sim}{I}_p-\underset{\sim}{R}^2|^h$, proceed as in section 3 for case 1 and we find

$$(5.5)\quad E[W^{(p)}]^h = \frac{\Gamma_p(\frac{1}{2}n)\Gamma_p(\frac{1}{2}(n-q)+h)}{\Gamma_p(\frac{1}{2}(n-q))\Gamma_p(\frac{1}{2}n+h)}|\underset{\sim}{I}_p-\underset{\sim}{P}^2|^{\frac{1}{2}n}\,{}_2F_1(\tfrac{1}{2}n,\ \tfrac{1}{2}n;\ \tfrac{1}{2}n+h;\ \underset{\sim}{P}^2).$$

Noting that (5.5) can be obtained from (3.5) by substituting

$$(5.6)\qquad (n_2,n_1,\ (\lambda\underset{\sim}{\Lambda})^{-1}) \rightarrow (n-q,n,\underset{\sim}{I}_p-\underset{\sim}{P}^2)$$

it can be verified that the density of $W^{(p)}$ in this case is

$$(5.7)\quad f(W^{(p)}) = C_p\{W^{(p)}\}^{\frac{1}{2}(n-q-p-1)}\sum_{k=o}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}n)_\kappa(\frac{1}{2}n)_\kappa C_\kappa(\underset{\sim}{P}^2)}{k!}\,G_{p,p}^{p,o}\left(W^{(p)}\middle|{}^{a_1,a_2,\ldots,a_p}_{b_1,b_2,\ldots,b_p}\right)$$

where

$$C_p = \frac{\Gamma_p(\frac{1}{2}n)}{\Gamma_p(\frac{1}{2}(n-q))}\ |\underset{\sim}{I}_p-\underset{\sim}{P}^2|^{\frac{1}{2}n},\quad a_i = \tfrac{1}{2}q+k_{p-i+1}+b_i,\quad b_i = \tfrac{1}{2}(i-1).$$

The c.d.f. of $W^{(2)}$ is obtained from (3.10) when q is even by substituting as in (5.6) and using the a_i's as just defined.

For q odd the c.d.f. of $W^{(2)}$ follows from that of case 1 for n_1 odd by making the substitution (5.6) and using the a_i's just defined. The densities of $W^{(p)}$ for $p=2,3,4$ follow from (3.8), (3.11), (3.12) respectively making substitution (5.6).

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