On an Asymptotic Representation of the Distribution of the Characteristic Roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$

by

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1. Introduction and Summary. Let $\underset{\sim}{S}_i$: pxp (i = 1, 2) be independently distributed as Wishart $(n_i, p, \underset{\sim}{\Sigma}_i)$. Let the characteristic roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ and $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$ be denoted by ℓ_i $(i = 1, 2, \ldots, p)$ and λ_i $(i = 1, 2, \ldots, p)$ respectively such that $\ell_1 > \ell_2 > \ldots > \ell_p > 0$ and $\lambda_1 > \lambda_2 > \ldots > \lambda_p > 0$. Then the distribution of $\ell_1, \ldots, \ell_p$ can be expressed in the form (Khatri [8])

$$(1.1) \qquad C|\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1} |\underset{\sim}{L}|^{\frac{1}{2}(n_1-p-1)} \alpha_p(\underset{\sim}{L}) \int_{O(p)} |\underset{\sim}{I}_p + \underset{\sim}{\Lambda}^{-1}\underset{\sim}{H}\underset{\sim}{L}\underset{\sim}{H}'|^{-\frac{1}{2}(n_1+n_2)} (\underset{\sim}{H}'d\underset{\sim}{H})$$

where $C = 2^{-p}\pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma(\frac{i}{2})\Gamma_p(\frac{1}{2}n_1 + \frac{1}{2}n_2) \left\{\Gamma_p(\frac{1}{2}p)\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2)\right\}^{-1}$,

$\Gamma_p(t) = \pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma(t-\frac{1}{2}j + \frac{1}{2})$, $\alpha_p(\underset{\sim}{L}) = \prod_{i<j} (\ell_j - \ell_i)$,

$\underset{\sim}{L} = \mathrm{diag}\,(\ell_i, \ldots \ell_p)$, $\underset{\sim}{\Lambda} = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_p)$ and $(\underset{\sim}{H}'d\underset{\sim}{H})$ is the invariant measure on the group $O(p)$. However, this form is not convenient for further development. Also, since

$$(1.2) \qquad I = \int_{O(p)} |\underset{\sim}{I}_p + \underset{\sim}{\Lambda}^{-1}\underset{\sim}{H}\underset{\sim}{L}\underset{\sim}{H}'|^{\frac{1}{2}(n_1+n_2)} (\underset{\sim}{H}'d\underset{\sim}{H}) = c'\sum_{k=0} \frac{1}{k!} \sum_{\kappa} \frac{C_\kappa(-\underset{\sim}{\Lambda}^{-1})C_\kappa(\underset{\sim}{L})(n_1+n_2)_\kappa}{C_\kappa(\underset{\sim}{I}_p)}$$

where

$$c' = 2^p\pi^{p(p+1)/4} \prod_{i=1}^{p} \Gamma(\frac{i}{2}) .$$

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and the zonal polynomial $C_\kappa(\underset{\sim}{T})$ of any pxp symmetric matrix $\underset{\sim}{T}$ is defined in James [7], the use of (1.2) in (1.1) gives a power series expansion, but the convergence of this series is very slow. In the one sample case G. A. Anderson [1] has obtained a gamma type asymptotic expansion for the distribution of the characteristic roots of the estimated covariance matrix. In this paper we obtain a beta type asymptotic representation of the roots distribution of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ involving linkage factors between sample roots and corresponding population roots. A study is also made of the approximation to the distribution of $w_1, \dots w_p$ where $w_i = \ell_i/(1 + \ell_i)$, $(i = 1, 2 \dots, p)$. If the roots are distinct the limiting distribution as n_2 tends to infinity has the same form as that of Anderson [1]. If, moreover, n_1 is assumed also large, then it agrees with Girshick's result [4].

2. <u>The asymptotic representation of I</u>. The procedure used to find the expansion of (1.2) is an extension of the method sketched below for the case $p = 2$. In the asymptotic theory it is necessary to assume $\ell_1 > \ell_2 > \dots \ell_p > 0$ and $\lambda_1 > \lambda_2 > \dots > \lambda_p > 0$. For the simplification of notations we let $\underset{\sim}{A} = \Lambda^{-1}$, i.e. $a_i = 1/\lambda_i$ $(i = 1, \dots p)$, $0 < a_1 < a_2 < \dots < a_p < \infty$, and $n = n_1 + n_2$. Thus for $p = 2$, let $0 \pm (2) = \left\{\underset{\sim}{H} \epsilon O(2), |\underset{\sim}{H}| = \pm 1\right\}$ then

$$(2.1) \qquad I = 2 \int_{O^+(2)} |\underset{\sim}{I}_p + \underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{L}\,\underset{\sim}{H}'|^{-\frac{n}{2}} (\underset{\sim}{H}'d\underset{\sim}{H}).$$

Now let $\underset{\sim}{H} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad -\pi < \theta \leq \pi$,

so that $(\underset{\sim}{H}'d\underset{\sim}{H}) = d\theta$ and

$$\text{(2.2)} \qquad I = 4\left[(1 + a_1\ell_1)(1 + a_2\ell_2)\right]^{-\frac{n}{2}}\int_{-\pi/2}^{\pi/2}\left[1 + \tfrac{1}{2}C_{12}(1 - \text{Cos}2\theta)\right]^{-\frac{n}{2}}d\theta$$

where

$$C_{12} = \frac{(a_2 - a_1)(\ell_1 - \ell_2)}{(1+a_1\ell_1)(1+a_2\ell_2)} \quad .$$

The integrand has a maximum of unity at $\theta = 0$ and then decreases to $(1 + \frac{1}{2}C_{12})$ at $\theta = \pm\frac{\pi}{2}$. Write (2.2) as

$$\text{(2.3)} \qquad 4\left(\prod_{i=1}^{2} (1 + a_i\ell_i)\right)^{-\frac{n}{2}}\int_{-\pi/2}^{\pi/2} \exp\left[-\frac{n}{2}\log(1 + C_{12}(1 - \text{Cos}2\theta)\right]d\theta$$

Since the integral is mostly concentrated in a small neighborhood of the origin, for large n we can expand the argument of the exponential function and $\text{Cos}2\theta$ in the usual power series and set the limit to be $\pm\infty$ (see Erdélyi [3]). Thus for large degrees of freedom I is approximately

$$4\left[\prod_{i=1}^{2}(1 + a_i\ell_i)\right]^{-\frac{n}{2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{n}{2}C_{12}\theta^2\right\}d\theta,\left\{1 + O(\tfrac{1}{n})\right\}$$

or

$$I \sim 4\left[\prod_{i=1}^{2} (1 + a_i\ell_i)\right]^{-\frac{n}{2}}\left(\frac{2\pi}{nC_{12}}\right)^{\frac{1}{2}}\left\{1 + O(\tfrac{1}{n})\right\} \quad .$$

Lemma 1. Let $\underset{\sim}{A}$ and $\underset{\sim}{L}$ are defined as before then $f(\underset{\sim}{H}) = |\underset{\sim}{I}_p + \underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{L}\,\underset{\sim}{H}'|$ $\underset{\sim}{H} \in O(p)$ attains its identical minimum value $|\underset{\sim}{I}_p + \underset{\sim}{A}\underset{\sim}{L})|$ when $\underset{\sim}{H}$ is of the form

$$(2.4)\qquad H = \begin{pmatrix} \pm 1 & & & 0 \\ & \pm 1 & & \\ & & \ddots & \\ 0 & & & \pm 1 \end{pmatrix} .$$

Proof: $$\begin{aligned} df &= d|\underset{\sim}{I}_p + \underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{L}\,\underset{\sim}{H}'| \\ &= d|\underset{\sim}{I}_p + \underset{\sim}{A}^{\frac{1}{2}}\,\underset{\sim}{H}\,\underset{\sim}{L}\,\underset{\sim}{H}'\underset{\sim}{A}^{\frac{1}{2}}| \\ &= \text{tr}\,(\underset{\sim}{I}_p + \underset{\sim}{A}^{\frac{1}{2}}\,\underset{\sim}{H}\,\underset{\sim}{L}\,\underset{\sim}{H}'\underset{\sim}{A}^{\frac{1}{2}})^{-1}(\underset{\sim}{A}^{\frac{1}{2}}\,d\,\underset{\sim}{H}\,\underset{\sim}{L}\,\underset{\sim}{H}'\,\underset{\sim}{A}^{\frac{1}{2}} + \underset{\sim}{A}^{\frac{1}{2}}\,\underset{\sim}{H}\,\underset{\sim}{L}\,d\,\underset{\sim}{H}'\,\underset{\sim}{A}^{\frac{1}{2}}) \\ &= 2\text{tr}\,\underset{\sim}{L}\,\underset{\sim}{H}'\,\underset{\sim}{A}^{\frac{1}{2}}(\underset{\sim}{I}_p + \underset{\sim}{A}^{\frac{1}{2}}\,\underset{\sim}{H}\,\underset{\sim}{L}\,\underset{\sim}{H}'\,\underset{\sim}{A}^{\frac{1}{2}})^{-1}\,\underset{\sim}{A}^{\frac{1}{2}}\,\underset{\sim}{H}\,\underset{\sim}{H}'\,d\,\underset{\sim}{H} . \end{aligned}$$

Note that $\underset{\sim}{H}'\,d\,\underset{\sim}{H}$ is a skew symmetric matrix therefore, $df = 0$ implies that $\underset{\sim}{L}\,\underset{\sim}{H}'\,\underset{\sim}{A}^{\frac{1}{2}}(\underset{\sim}{I}_p + \underset{\sim}{A}^{\frac{1}{2}}\,\underset{\sim}{H}\,\underset{\sim}{L}\,\underset{\sim}{H}'\,\underset{\sim}{A}^{\frac{1}{2}})^{-1}\,\underset{\sim}{A}^{\frac{1}{2}}\,\underset{\sim}{H}$ is a symmetric matrix. But $\underset{\sim}{H}'\,\underset{\sim}{A}^{\frac{1}{2}}(\underset{\sim}{I}_p + \underset{\sim}{A}^{\frac{1}{2}}\,\underset{\sim}{H}\,\underset{\sim}{L}\,\underset{\sim}{H}'\,\underset{\sim}{A}^{\frac{1}{2}})^{-1}\,\underset{\sim}{A}^{\frac{1}{2}}\,\underset{\sim}{H}$ is itself a symmetric matrix and $\underset{\sim}{L}$ is a diagonal matrix with distinct positive roots,
so $\underset{\sim}{H}'\,\underset{\sim}{A}^{\frac{1}{2}}(\underset{\sim}{I}_p + \underset{\sim}{A}^{\frac{1}{2}}\,\underset{\sim}{H}\,\underset{\sim}{L}\,\underset{\sim}{H}'\,\underset{\sim}{A}^{\frac{1}{2}})^{-1}\,\underset{\sim}{A}^{\frac{1}{2}}\,\underset{\sim}{H}$ has to be a diagonal matrix, say $\underset{\sim}{D}$. Thus $\underset{\sim}{I}_p = \underset{\sim}{A}^{\frac{1}{2}}\,\underset{\sim}{H}(\underset{\sim}{L} - \underset{\sim}{D}^{-1})\,\underset{\sim}{H}'\,\underset{\sim}{A}^{\frac{1}{2}}$. This can happen only if $\underset{\sim}{H}$ is of the form with ± 1 in one position in a column or a row and zero in other positions. After substituting those stationary values into $f(\underset{\sim}{H})$ we obtain a general form

$$(2.5)\qquad \prod_{i=1}^{p} (1 + a_i \ell_{\sigma_i}),$$

where ℓ_{σ_i} is any permutation of $\ell_i (i = 1, \ldots, p)$. It is easy to see that (2.5) attains its minimum value when $\ell_{\sigma_i} = \ell_i (i = 1, 2, \ldots, p)$. Or $f(\underset{\sim}{H})$ attains its identical minimum value $|\underset{\sim}{I}_p + \underset{\sim}{A}\,\underset{\sim}{L}|$ when $\underset{\sim}{H}$ is of the form of (2.4).

The above lemma enables us to claim that, for large n, the integrand of I is negligible except for small neighborhoods about each of these matrices of (2.4) and I consists of identical contributions from each of these neighborhoods so that

$$(2.6) \qquad I \cong 2^p \int_{N(\underset{\sim}{I})} |\underset{\sim}{I}_p + \underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{L}\,\underset{\sim}{H}'|^{-\frac{n}{2}} (\underset{\sim}{H}'\, d\,\underset{\sim}{H}),$$

where $N(I)$ is a neighborhood of the identity matrix on the orthogonal manifold. Since any proper orthogonal matrix can be written as the exponential of a skew symmetric matrix we transform I under

$$(2.7) \qquad \underset{\sim}{H} = \exp \underset{\sim}{S}, \quad \underset{\sim}{S} \text{ a } p\times p \text{ skew symmetric matrix,}$$

so that $N(\underset{\sim}{I}) \to N(\underset{\sim}{S} = 0)$. The Jacobian of this transformation has been computed by G. A. Anderson [1],

$$(2.8) \qquad J = 1 + \frac{p-2}{24} \operatorname{tr} \underset{\sim}{S}^2 + \frac{8-p}{4\times 6!} \operatorname{tr} \underset{\sim}{S}^4 + \ldots$$

Direct substitution of (2.7) into (2.6) yields

$$\begin{aligned} (2.9) \qquad & |\underset{\sim}{I}_p + \underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{L}\,\underset{\sim}{H}'|^{-\frac{n}{2}} \\ & = |\underset{\sim}{I}_p + \underset{\sim}{A}\,\underset{\sim}{L} + \underset{\sim}{A}\,\underset{\sim}{S}\,\underset{\sim}{L} - \underset{\sim}{A}\,\underset{\sim}{L}\,\underset{\sim}{S} + \underset{\sim}{A}\,\underset{\sim}{L}\,\underset{\sim}{S}^2 - \underset{\sim}{A}\,\underset{\sim}{S}\,\underset{\sim}{L}\,\underset{\sim}{S} + \ldots.|^{-\frac{n}{2}} \\ & = |\underset{\sim}{I}_p + \underset{\sim}{A}\,\underset{\sim}{L}|^{-\frac{n}{2}} |\underset{\sim}{I}_p + (\underset{\sim}{I}_p + \underset{\sim}{A}\,\underset{\sim}{L})^{-1} (\underset{\sim}{A}\,\underset{\sim}{S}\,\underset{\sim}{L} - \underset{\sim}{A}\,\underset{\sim}{L}\,\underset{\sim}{S} + \underset{\sim}{A}\,\underset{\sim}{L}\,\underset{\sim}{S}^2 \\ & \qquad - \underset{\sim}{A}\,\underset{\sim}{S}\,\underset{\sim}{L}\,\underset{\sim}{S} + \ldots)|^{-\frac{n}{2}}. \end{aligned}$$

Lemma 2. For any pxp matrix $\underset{\sim}{B}$ and its characteristic roots $b_i (i = 1 \ldots p)$, if $\max_{1 \le i \le p} |b_i| < 1$ then

$$(2.10) \qquad |\underset{\sim}{I}_p + \underset{\sim}{B}|^{-\frac{n}{2}} = \exp\left\{-\frac{n}{2}\operatorname{tr}\left(\underset{\sim}{B} - \frac{\underset{\sim}{B}^2}{2} + \frac{\underset{\sim}{B}^3}{3} \cdots\right)\right\}.$$

Proof:

$$(2.11) \qquad |\underset{\sim}{I}_p + \underset{\sim}{B}|^{-\frac{n}{2}} = \exp\left\{-\frac{n}{2}\log \prod_{i=1}^{p}(1 + b_i)\right\} .$$

If $\max_{1 \le i \le p} |b_i| < 1$ then

$$|\underset{\sim}{I}_p + \underset{\sim}{B}|^{-\frac{n}{2}} = \exp\left\{-\frac{n}{2}\operatorname{tr}\left(\underset{\sim}{B} - \frac{\underset{\sim}{B}^2}{2} + \frac{\underset{\sim}{B}^3}{3} \cdots\right)\right\}.$$

Apply lemma 2 to (2.9) and the maximum characteristic roots of $(\underset{\sim}{I}_p + \underset{\sim}{A}\,\underset{\sim}{L})^{-1}(\underset{\sim}{A}\,\underset{\sim}{S}\,\underset{\sim}{L} - \underset{\sim}{A}\,\underset{\sim}{L}\,\underset{\sim}{S} + \ldots)$ can be assumed to be less than unity. Since we are only interested in the first term we need to investigate the group of terms up to order of $\underset{\sim}{S}^2$ which is denoted by $\{\underset{\sim}{S}^2\}$. Let $\underset{\sim}{R} = (\underset{\sim}{I} + \underset{\sim}{A}\,\underset{\sim}{L})^{-1}$, then

$$(2.12) \qquad \operatorname{tr}\{\underset{\sim}{S}^2\} = \operatorname{tr}\Big[\underset{\sim}{R}(\underset{\sim}{A}\,\underset{\sim}{L}\,\underset{\sim}{S}^2 - \underset{\sim}{A}\,\underset{\sim}{S}\,\underset{\sim}{L}\,\underset{\sim}{S})$$

$$- \tfrac{1}{2}(\underset{\sim}{R}\,\underset{\sim}{A}\,\underset{\sim}{L}\,\underset{\sim}{S}\,\underset{\sim}{R}\,\underset{\sim}{A}\,\underset{\sim}{L}\,\underset{\sim}{S} + \underset{\sim}{R}\,\underset{\sim}{A}\,\underset{\sim}{S}\,\underset{\sim}{L}\,\underset{\sim}{R}\,\underset{\sim}{A}\,\underset{\sim}{L}\,\underset{\sim}{S} - \underset{\sim}{R}\,\underset{\sim}{A}\,\underset{\sim}{S}\,\underset{\sim}{L}\,\underset{\sim}{R}\,\underset{\sim}{A}\,\underset{\sim}{L}\,\underset{\sim}{S}$$

$$- \underset{\sim}{R}\,\underset{\sim}{A}\,\underset{\sim}{L}\,\underset{\sim}{S}\,\underset{\sim}{R}\,\underset{\sim}{A}\,\underset{\sim}{S}\,\underset{\sim}{L})\Big] .$$

After simplification (2.12) reduces to

$$(2.13) \qquad \operatorname{tr}\Big[\underset{\sim}{R}(\underset{\sim}{A}\,\underset{\sim}{L}\,\underset{\sim}{S}^2 - \underset{\sim}{A}\,\underset{\sim}{S}\,\underset{\sim}{L}\,\underset{\sim}{S}) - (\underset{\sim}{L}\,\underset{\sim}{S} - \underset{\sim}{S}\,\underset{\sim}{L})\underset{\sim}{R}\,\underset{\sim}{A}\,\underset{\sim}{L}\,\underset{\sim}{S}\,\underset{\sim}{R}\,\underset{\sim}{A}\Big]$$

$$\text{or} \quad \operatorname{tr}\{\underset{\sim}{S}^2\} = \sum_{i<j}^{p} C_{ij} s_{ij}^{\,2}$$

(2.14) where $C_{ij} = (a_j - a_i)(\ell_i - \ell_j) / ((1 + a_i\ell_i)(1 + a_j\ell_j))$.

Direct substitution into (2.1) yields

(2.15) $$I = 2^p \prod_{i=1}^{p} (1 + a_i\ell_i)^{-\frac{n}{2}} \int_{N(s=0)} \exp\left\{-\frac{n}{2}\sum_{i<j}^{p} C_{ij}s_{ij}^2\right\} \prod_{i<j} ds_{ij}\left\{1 + O(\tfrac{1}{n})\right\} .$$

For large n the limits for each s_{ij} can be put to $\pm\infty$. We finally have the following theorem.

<u>Theorem</u>: The asymptotic distribution of the roots, $\ell_1 > \ell_2 > \ldots > \ell_p > 0$, of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ for large degrees of freedom $n = n_1 + n_2$ when the roots of $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$ are $\lambda_1 > \lambda_2 > \ldots \lambda_p > 0$ and $a_i = 1/\lambda_i (i = 1, \ldots p)$, is given by

(2.16) $$C2^p \alpha_p(\underset{\sim}{L}) \prod_{i=1}^{p}\left[(\ell_i)^{\frac{n_1-p-1}{2}} (a_i)^{+\frac{1}{2}n_1} (1 + a_i\ell_i)^{\frac{-(n_1+n_2)}{2}}\right] \prod_{i<j}^{p} \left(\frac{2\pi}{C_{ij}(n_1 + n_2)}\right)^{\frac{1}{2}} .$$

The asymptotic formula shows that the distribution function of a group of adjacent roots is sensitive only to those other roots which are close to them.

3. <u>A Dual Expansion of $\underset{\sim}{I}$ and Some Remarks.</u> If we let $\underset{\sim}{W} = \underset{\sim}{L}(\underset{\sim}{I}_p + \underset{\sim}{L})^{-1}$ in (1.1) i.e. $w_i = \ell_i/(1 + \ell_i)(i = 1, 2, \ldots p)$ where $\underset{\sim}{W} = \text{diag}\,(w_1, \ldots w_p)$, then the joint distribution of w_i's is given by

(3.1) $$C|\underset{\sim}{\Lambda}|^{-\frac{n_1}{2}} |\underset{\sim}{W}|^{\frac{1}{2}(n_1-p-1)} |\underset{\sim}{I}_p - \underset{\sim}{W}|^{-\frac{1}{2}(n_1+p+1)} \alpha_p(\underset{\sim}{W}) \int_{O(p)} |\underset{\sim}{I}_p + \underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{L}\,\underset{\sim}{H}'|^{-\frac{n}{2}} (\underset{\sim}{H}d\underset{\sim}{H})$$

$$1 > w_1 > w_2 > \ldots > w_p \geq 0.$$

Application of lemmas 1 and 2 to (3.1) yields its asymptotic representation

$$(3.2)\qquad C|\underset{\sim}{A}|^{+\frac{n_1}{2}}|\underset{\sim}{W}|^{\frac{1}{2}(n_1-p-1)}|\underset{\sim}{I}_p-\underset{\sim}{W}|^{\frac{1}{2}(n_2-p-1)}\alpha_p(\underset{\sim}{W})\prod_{i=1}^{p}(1+(a_i-1)w_i)^{-\frac{1}{2}(n_1+n_2)}$$

$$\prod_{i<j}^{p}\left(\frac{2\pi}{C_{ij}^{*}n}\right)^{\frac{1}{2}}$$

where $C_{ij}^{*} = \dfrac{(a_j - a_i)(w_i - w_j)}{[1+(a_i-1)w_i][1+(a_j-1)w_j]}$.

Now let us proceed to look at (2.16) once again. The asymptotic distribution of characteristic roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ given there can be rewritten as

$$(3.3)\qquad F_1(\underset{\sim}{A})\prod_{i<j}(\ell_i-\ell_j)^{\frac{1}{2}}\prod_{i=1}^{p}\left[\ell_i^{\frac{n_1-p-1}{2}}(1+a_i\ell_i)^{-\frac{(n_1+n_2)}{2}+p-1}\right]\prod_{i=1}^{p}d\ell_i$$

where $F_i(A)$ $(i = 1, 2, 3)$ depends on a_i but not on ℓ_i . If we make $g_i = \ell_i/n_2$ $(i = 1, 2, \ldots p)$ and let n_2 tends to infinity then (3.3) reduces to the limiting form

$$(3.4)\qquad F_2(A)\prod_{i=1}^{p}g_i^{\frac{1}{2}(n_1-p-1)}\,e^{-\frac{1}{2}\sum_{i=1}^{p}a_ig_i}\prod_{i<j}^{p}(g_i-g_j)^{\frac{1}{2}}\ .$$

Moreover, let $\ell_i^{*} = n_1g_i$ $(i = 1, 2, \ldots p)$, then (3.4) becomes

$$(3.5)\qquad F_3(A)\prod_{i=1}^{p}\ell_i^{*\,\frac{1}{2}(n_1-p-1)}\,e^{-\frac{n_1}{2}\sum_{i=1}^{p}a_i\ell_i^{*}}\prod_{i<j}^{p}(\ell_i^{*}-\ell_j^{*})^{\frac{1}{2}}$$

Note that ℓ_i^{*}'s here are, in limiting sense, the characteristic roots of $\underset{\sim}{S}_1^{*}\underset{\sim}{\Sigma}_2^{-1}$ where $\underset{\sim}{S}_1^{*}$ is the covariance matrix of the first sample.

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