Monotonicity of the power functions of some tests of hypotheses concerning multivariate complex normal distributions*

by

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1. Summary. Consider the test procedures invariant under certain groups of transformations[6]; (i) for testing the hypothesis $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ against one-sided alternatives, [2], [6], which is invariant under the transformation $\underset{\sim}{X}_j \rightarrow \underset{\sim}{B}\underset{\sim}{X}_j + \underset{\sim}{b}_j$, $j=1,2$, where $\underset{\sim}{X}_j$ are distributed as multivariate normal, and $\underset{\sim}{B}$ is any nonsingular matrix and $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$ are any vectors; (ii) for testing the general multivariate linear hypothesis, [3], [6], which is invariant under the transformation $(\underset{\sim}{X}_1(p \times s), \underset{\sim}{X}_2(p \times (n-r)), \underset{\sim}{X}_3(p \times (r-s)) \rightarrow (\underset{\sim}{B}\underset{\sim}{X}_1\underset{\sim}{F}_1, \underset{\sim}{B}\underset{\sim}{X}_2\underset{\sim}{F}_2, \underset{\sim}{B}\underset{\sim}{X}_3\underset{\sim}{F}_3 + \underset{\sim}{G})$ where $\underset{\sim}{B}$ is nonsingular and $\underset{\sim}{F}_1, \underset{\sim}{F}_2$ and $\underset{\sim}{F}_3$ are orthogonal matrices; and (iii) for testing independence between two sets of normally distributed variates, [1], [6], which is invariant under the transformation $\begin{pmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \end{pmatrix} \rightarrow \begin{pmatrix} \underset{\sim}{B}_1 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}_2 \end{pmatrix} \begin{pmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \end{pmatrix} \underset{\sim}{F}$, where $\underset{\sim}{B}_1, \underset{\sim}{B}_2$ are nonsingular matrices of order p and q respectively, and $\underset{\sim}{F}$ is orthogonal. In the real case, sufficient conditions on the procedure for the power function to be a monotonically increasing function of each of the parameters, for (i) are obtained by Anderson and Das Gupta [2]; for (ii), by Das Gupta, Anderson and Mudholkar [3]; and for (iii) by Anderson and Das Gupta [1]. Furthermore,

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for (ii) and (iii) Mudholkar [8] has shown that the power functions of the members of a class of invariant tests based on statistics, which are symmetric gauge functions of increasing convex functions of the maximal invariants, are monotone increasing functions of the relevant noncentrality parameters. The monotonicity of the power function of Roy's test has been shown by Roy and Mikhail [7,12]. Further, Pillai and Jayachandran, [9],[10], have carried out exact power function comparisons for these tests based on four criteria for the two-roots case.

In this paper, in addition to extending the above results to the complex case, the monotonicity of the power of Pillai's $V^{(p)}$ criterion with respect to each population root has been shown for the first time for (ii) and (iii). In fact, for (ii) and (iii) the monotonicity property of power of elementary symmetrical function of the ch. roots in the range zero to unity with respect to each population parameter has been shown. In Section 2, we derive some distributions in the complex case and in Section 3, prove a lemma, which helps to extend to the complex case, some results on convex sets in the real case. In Sections 4,5, and 6 are briefly stated the theorems which can be proved from the real case with necessary changes, while in Section 7 for tests (i), (ii) and (iii), Pillai's $V^{(p)}$ criterion is shown to have monotonicity property with respect to each population root, and finally, in Section 8 follows a discussion of special cases of tests: the likelihood-ratio test; Roy's maximum root test; and Hotelling's trace test for (i), (ii) and (iii).

2. Introduction and notations. Matrices will be denoted by bold face capital letters, and their dimensions will be indicated parenthetically. The pxp identity matrix will be denoted by $\underset{\sim}{I}_p$ and zero matrix by $\underset{\sim}{0}$. The complex conjugate of a matrix $\underset{\sim}{A}$ will be denoted by $\underset{\sim}{\bar{A}}$ and the conjugate transpose by $\underset{\sim}{\bar{A}}'$. The notation $d\underset{\sim}{A}$ denotes the volume element associated with $\underset{\sim}{A}$.

$\underset{\sim}{U}(p\times n)$ will denote a semi-unitary matrix, where $\underset{\sim}{U}\overline{\underset{\sim}{U}}' = I_p$ for $p<n$ or $\overline{\underset{\sim}{U}}'\underset{\sim}{U} = \underset{\sim}{I}_n$ for $n<p$, and $\underset{\sim}{U}(n\times n)$ is unitary matrix if $\underset{\sim}{U}\overline{\underset{\sim}{U}}' = \overline{\underset{\sim}{U}}'\underset{\sim}{U} = \underset{\sim}{I}_n$. The characteristic (ch.) roots of $\underset{\sim}{A}$ will be denoted by $ch[\underset{\sim}{A}]$ and $ch_j[\underset{\sim}{A}]$ denotes the jth ordered characteristic root of $\underset{\sim}{A}$ if $\underset{\sim}{A}$ has real roots.

Let $\underset{\sim}{\xi}' = (Z_1,\ldots,Z_p)$ be a p-variate complex normal random variable such that the vector of real and imaginary parts $\underset{\sim}{\eta}' = (X_1,Y_1,\ldots,X_p,Y_p)$ is 2p-variate normal distributed, where $Z_j = X_j + i\,Y_j$ $j = 1,\ldots,p$. Then the distribution of $\underset{\sim}{\xi}$ was found by Wooding [13] and Goodman [4] and is given by

$$(2.1) \qquad p(\underset{\sim}{\eta}) = p(\underset{\sim}{\xi}) = \pi^{-p}|\underset{\sim}{\Sigma}|^{-1}\, e^{-(\overline{\underset{\sim}{\xi}-\underset{\sim}{\nu}})'\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{\xi}-\underset{\sim}{\nu})}$$

where $\underset{\sim}{\nu} = E[\underset{\sim}{\xi}]$ and $\underset{\sim}{\Sigma} = \underset{\sim}{\Sigma}_{\xi}$ (pxp) is a positive definite hermitian matrix.

Now let $\underset{\sim}{Z}(p\times n)$ be a complex random matrix whose columns are independently distributed, each distributed as (2.1). Then the distribution of $\underset{\sim}{Z}$, is given by, [4], [5],

$$(2.2) \qquad p(\underset{\sim}{Z};\underset{\sim}{\Sigma},n) = \pi^{-pn}\,|\underset{\sim}{\Sigma}|^{-n}\, e^{-tr\,\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{Z}-\underset{\sim}{\mu})(\overline{\underset{\sim}{Z}-\underset{\sim}{\mu}})'}$$

where $\underset{\sim}{\mu} = E[\underset{\sim}{Z}]$ is a matrix of pn complex parameters. In the more general case, $\underset{\sim}{Z}(p\times n)$ can be assumed to be distributed as

$$(2.3) \qquad p(\underset{\sim}{Z};\underset{\sim}{\Sigma},n) = \pi^{-pn}\,|\underset{\sim}{\Sigma}|^{-n}\, e^{-tr\,\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{Z}-\underset{\sim}{\mu}\underset{\sim}{A})(\overline{\underset{\sim}{Z}-\underset{\sim}{\mu}\underset{\sim}{A}})'}$$

where $\underset{\sim}{A}$ is a known mxn matrix of rank r [assume $r \le \min(m,n-p)$] and $\underset{\sim}{\mu}$

is a pxm matrix of unknown parameters. If $\underset{\sim}{\mu} = \underset{\sim}{0}$, (2.2) and (2.3) reduce to

$$(2.4) \qquad p(\underset{\sim}{Z};\underset{\sim}{\Sigma},n) = \pi^{-pn} |\underset{\sim}{\Sigma}|^{-n} e^{-\mathrm{tr}\, \underset{\sim}{\Sigma}^{-1} \underset{\sim}{Z}\underset{\sim}{\bar{Z}}'} .$$

For later use, we use the same techniques as those in Roy's [11] to derive some distributions. Transform

$$\underset{\sim}{A}(mxn) = \binom{\underset{\sim}{\tilde{T}_1}}{\underset{\sim}{T_2}} \underset{\sim}{U_1}(rxn)$$

where $\underset{\sim}{\tilde{T}_1}(rxr)$ is nonsingular, $\underset{\sim}{T_2}$ is $(m-r)Xr$ matrix, and $\underset{\sim}{U_1}$ is a semi-unitary, i.e. $\underset{\sim}{U_1}\underset{\sim}{\bar{U}_1'} = \underset{\sim}{I_r}$. Let $\underset{\sim}{U_2}((n-r)Xn)$ be the completion of $\underset{\sim}{U_1}$. Then make the unitary transformation $\underset{\sim}{\Delta} = \underset{\sim}{Z}(\underset{\sim}{\bar{U}_1'}\underset{\sim}{\bar{U}_2'}) = (\underset{\sim}{\zeta}\ \underset{\sim}{Z_2})$ say i.e. $\underset{\sim}{Z} = \underset{\sim}{\zeta}\ \underset{\sim}{U_1} + \underset{\sim}{Z_2}\underset{\sim}{U_2}$ where $\underset{\sim}{\zeta}$ is pxr and $\underset{\sim}{Z_2}$ is px(n-r) matrix.

Making unitary transformation again $\underset{\sim}{\Delta_1} = \underset{\sim}{\zeta}(\underset{\sim}{\nu_1'}\underset{\sim}{\nu_2'}) = (\underset{\sim}{Z_1}\underset{\sim}{Z_3})$ say where $\underset{\sim}{\nu_1}$ is sxr, $\underset{\sim}{\nu_3}$ is (r-s)xr, $\underset{\sim}{Z_1}$ is pxs and $\underset{\sim}{Z_3}$ is px(r-s) matrix respectively, and $\binom{\underset{\sim}{\nu_1}}{\underset{\sim}{\nu_3}}$ is unitary, then

$$\underset{\sim}{\zeta} = \underset{\sim}{Z_1}\ \underset{\sim}{\nu_1} + \underset{\sim}{Z_3}\ \underset{\sim}{\nu_3}$$

Similarly put $\underset{\sim}{\mu_1}$ (pxs) $= \underset{\sim}{\mu}\binom{\underset{\sim}{\tilde{T}_1}}{\underset{\sim}{T_2}}\underset{\sim}{\bar{\nu}_1'}$, $\underset{\sim}{\mu_3}(px(r-s)) = \underset{\sim}{\mu}\binom{\underset{\sim}{\tilde{T}_1}}{\underset{\sim}{T_2}}\underset{\sim}{\bar{\nu}_3'}$, then we have

$$(2.5)\ p(\underset{\sim}{Z_1},\underset{\sim}{Z_2},\underset{\sim}{Z_3}) = \pi^{-pn}|\underset{\sim}{\Sigma}|^{-n} \exp[-\mathrm{tr}\ \Sigma^{-1}\{(\underset{\sim}{Z_1}-\underset{\sim}{\mu_1})(\underset{\sim}{\bar{Z}_1}-\underset{\sim}{\bar{\mu}_1})' + \underset{\sim}{Z_2}\underset{\sim}{\bar{Z}_2'} + (\underset{\sim}{Z_3}-\underset{\sim}{\mu_3})(\underset{\sim}{\bar{Z}_3}-\underset{\sim}{\bar{\mu}_3})'\}].$$

Put $\mu_1\bar{\mu}_1' = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} D_\theta(t\times t)\ (\bar{\xi}_1'\bar{\xi}_2')$, and $\Sigma(p\times p) = \begin{pmatrix}\xi_1 & \tilde{\xi}_3\\ \xi_2 & \xi_4\end{pmatrix}\begin{pmatrix}\bar{\xi}_1' & \bar{\xi}_2'\\ \bar{\tilde{\xi}}_3' & \bar{\xi}_4'\end{pmatrix} = \xi\,\bar{\xi}'$

where $\xi_1((p-t)\times t)$, $\xi_2(t\times t)$, $\tilde{\xi}_3((p-t)\times(p-t))$, and $\xi_4(t\times(p-t))$; and ξ and $\tilde{\xi}_3$ are nonsingular; and D_θ denotes the diagonal matrix with ch. roots $\theta_1 \geq \ldots \geq \theta_t$ of $\mu_1\bar{\mu}_1'\Sigma^{-1}$ as its diagonal elements, and $t=\min(p,s)$.

Put $\mu_1 = \begin{pmatrix}\mu_1^{(1)}\\ \mu_1^{(2)}\end{pmatrix} = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} D_{\sqrt{\theta}}\,\varphi\ (t\times s)$ where φ is determined by

$\varphi = D_{\sqrt{\theta}}^{-1}\ \xi_2^{-1}\ \mu_1^{(2)}$ and $\varphi\,\bar{\varphi}' = I_t$ and complete $\bar{\varphi}'(s\times t)$ into a unitary matrix $\bar{\psi}'(s\times s)$. Finally, let

$$\xi^{-1} Z_1 \bar{\psi}' = V\ , \qquad \xi^{-1} Z_2 = W$$

From (2.5) we obtain

$$(2.6)\quad p(V,W) = \pi^{-p(n-r+s)} \exp\{-\mathrm{tr}(W\bar{W}' + V\bar{V}' - 2\mathrm{Re}\ VD_{\sqrt{\theta}}^{**} + D_\theta^{*})\}$$

where $D_\theta^{*}(p\times p) = \begin{pmatrix}D_\theta & 0\\ 0 & 0\end{pmatrix}$ and $D_{\sqrt{\theta}}^{**} = \begin{pmatrix}D_{\sqrt{\theta}} & 0\\ 0 & 0_1\end{pmatrix}$ and 0_1 is $(s-t)\times(p-t)$ zero matrix.

If $V = (v_{jk})$ $j=1,\ldots,p$; $k=1,\ldots,s$, then (2.6) can be rewritten

$$(2.7)\qquad p(V,W) = \pi^{-p(n-r+s)}\exp\{-\mathrm{tr}W\bar{W}' - \sum_{j=1}^{t}(v_{jj} - \theta_j^{\frac{1}{2}})(\bar{v}_{jj} - \theta_j^{\frac{1}{2}}) - \sum_{j=t+1}^{p} v_{jj}\bar{v}_{jj}$$

$$- \sum_{\substack{j=1\\ j\neq k}}^{p}\sum_{k=1}^{s} v_{jk}\,\bar{v}_{jk}\ \}.$$

3. Tests of multivariate linear hypothesis. Let random complex matrix $\underset{\sim}{Z}(p \times n)$ have density (2.3) and we wish to test the hypothesis H_0: $\underset{\sim}{\mu}\underset{\sim}{C} = \underset{\sim}{0}(p \times s)$ where $\underset{\sim}{C}$ is a known $m \times s$ matrix of rank $s(\leq r)$ such that $\underset{\sim}{\mu}\underset{\sim}{C}$ is estimable, against all alternatives. By Section 2, this problem can be transformed into the canonical form

$$\underset{\sim}{Z} \to (\underset{\sim}{Z}_1(p \times s),\ \underset{\sim}{Z}_2(p \times (n-r)),\ \underset{\sim}{Z}_3(p \times (r-s))), \quad s \leq r \leq n-p$$

with expectations.

$$E[\underset{\sim}{Z}_1] = \underset{\sim}{\mu}_1(p \times s),\ E[\underset{\sim}{Z}_2] = \underset{\sim}{0}(p \times (n-r)),\ E[\underset{\sim}{Z}_3] = \underset{\sim}{\mu}_3(p \times (r-s)).$$

The hypothesis H_0 is equivalent to the hypothesis $\underset{\sim}{\mu}_1 = \underset{\sim}{0}(p \times s)$. The matrices of sums of products due to hypothesis and due to error are given by $\underset{\sim}{S}_h = \underset{\sim}{Z}_1\overline{\underset{\sim}{Z}}_1'$ and $\underset{\sim}{S}_e = \underset{\sim}{Z}_2\overline{\underset{\sim}{Z}}_2'$ respectively. The problem is invariant under the transformation

$$(\underset{\sim}{Z}_1, \underset{\sim}{Z}_2, \underset{\sim}{Z}_3) \to (\underset{\sim}{B}\underset{\sim}{Z}_1\underset{\sim}{F}_1, \underset{\sim}{B}\underset{\sim}{Z}_2\underset{\sim}{F}_2, \underset{\sim}{B}\underset{\sim}{Z}_3\underset{\sim}{F}_3 + \underset{\sim}{G})$$

where $\underset{\sim}{B}$ is nonsingular and $\underset{\sim}{F}_1, \underset{\sim}{F}_2$ and $\underset{\sim}{F}_3$ are unitary matrices. These invariant test procedures depend on $C_1 \geq \cdots \geq C_p$, the ch. roots of $\underset{\sim}{S}_h\underset{\sim}{S}_e^{-1}$, and it is known [5] that the power function of any such test depends on the parameters $\theta_1, \ldots, \theta_t$ where $\theta_1 \geq \cdots \geq \theta_t$ are the possible nonzero ch. roots of $\underset{\sim}{\mu}_1\overline{\underset{\sim}{\mu}}_1'\underset{\sim}{\Sigma}^{-1}$ and $t = \min(p, s)$.

Lemma 3.1. Let $\underset{\sim}{\xi}' = (Z_1, \ldots, Z_p)$ and $\underset{\sim}{\eta}' = (X_1, Y_1, \ldots, X_p, Y_p)$ where $Z_j = X_j + iY_j$ $j = 1, \ldots, p$, and let T be a one-one transformation between

$\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ such that $T[\underset{\sim}{\xi}] = \underset{\sim}{\eta}$ with the following properties:

(1) $T[\underset{\sim}{\xi}_1 + \underset{\sim}{\xi}_2] = T[\underset{\sim}{\xi}_1] + T[\underset{\sim}{\xi}_2]$ and

(2) $T[a\underset{\sim}{\xi}] = aT[\underset{\sim}{\xi}]$ where a is a real number.

Let ω be a subset of $\underset{\sim}{\xi}$'s in p-dimensional complex sample space C^p; and ω^* be its corresponding subset of $\underset{\sim}{\eta}$'s in the 2p-dimensional real sample space R^{2p}. If ω is convex in C^p and symmetric in $\underset{\sim}{\xi}$. Then ω^* is convex in R^{2p} and symmetric in $\underset{\sim}{\eta}$ and conversely.

<u>Proof:</u> Let $\underset{\sim}{\eta}_1, \underset{\sim}{\eta}_2 \epsilon\ \omega^*$ then $T^{-1}[\underset{\sim}{\eta}_k] = \underset{\sim}{\xi}_k$ for some $\underset{\sim}{\xi}_k \epsilon\ \omega$, k=1,2. Since ω is convex in C^p, hence $\alpha\underset{\sim}{\xi}_1 + (1-\alpha)\underset{\sim}{\xi}_2 \epsilon\, \omega$, $0 \leq \alpha \leq 1$, and $T[\alpha\underset{\sim}{\xi}_1 + (1-\alpha)\underset{\sim}{\xi}_2]\ \epsilon\,\omega^*$ i.e. $\alpha\underset{\sim}{\eta}_1 + (1-\alpha)\underset{\sim}{\eta}_2\ \epsilon\,\omega^*$. This shows ω^* is convex in R^{2p}.

Let ω^*_- be the set of all $-\underset{\sim}{\eta}$ such that $\underset{\sim}{\eta}\epsilon\omega^*$. If any $-\underset{\sim}{\eta}\epsilon\omega^*_-$ then $\underset{\sim}{\eta}\epsilon\omega^*$ and $T^{-1}[\underset{\sim}{\eta}] = \underset{\sim}{\xi}$ for some $\underset{\sim}{\xi}\epsilon\omega$. Since ω is symmetric in $\underset{\sim}{\xi}$, hence $\omega = \omega_-$, where ω_- is a set of all $-\underset{\sim}{\xi}$ for which $\underset{\sim}{\xi}\epsilon\omega$, implies $-\underset{\sim}{\xi}\epsilon\omega$, and then $T[-\underset{\sim}{\xi}]\ \epsilon\omega^*$, i.e. $-\underset{\sim}{\eta}\epsilon\omega^*$. Therefore $\omega^*_- \subset \omega^*$. Using the same argument, we can show $\omega^*_- \supset \omega$ and hence $\omega^* = \omega^*_-$.

Similarly for the converse.

<u>Theorem 3.1.</u> Let the random complex vectors $\underset{\sim}{\xi}_j$ (j=1,...,s) and the complex matrix $\underset{\sim}{\varphi}$ be mutually independent, the distribution of $\underset{\sim}{\xi}_j$ being $N(\ell_j \underset{\sim}{\nu}_j, \underset{\sim}{\Sigma}_j)$ j=1,...,s. If a set ω in the sample space is convex and symmetric in each $\underset{\sim}{\xi}_j$ given the other $\underset{\sim}{\xi}_h$'s and $\underset{\sim}{\varphi}$. Then $Pr(\omega)$ decreases with respect to each ℓ_j (≥ 0).

<u>Proof</u>: Let $\underset{\sim}{\xi}'_j = (Z_{1j}, \ldots, Z_{pj})$ and $\underset{\sim}{\eta}'_j = (X_{1j}, Y_{1j}, \ldots, X_{pj}, Y_{pj})$ where $Z_{kj} = X_{kj} + i\, Y_{kj}$ k=1,...,p; j=1,...,s and let ω^* be the corresponding

set of ω in the sample space R^{2p}. Then by Lemma 3.1 we know that ω^* is convex and symmetric in each $\underset{\sim}{\eta}_j$. Denote

$$\mathcal{D} = \omega\ \{\underset{\sim}{\zeta}_j \mid \underset{\sim}{\xi}_h,\ h\neq j,\ h=1,\dots,s;\ \underset{\sim}{\varphi}\} \text{ and}$$
$$\mathcal{D}^* = \omega^*\ \{\underset{\sim}{\eta}_j \mid \underset{\sim}{\eta}_h\ \ h\neq j,\ h=1,\dots,s;\ \underset{\sim}{X},\underset{\sim}{Y}\}$$

where $\underset{\sim}{\varphi} = \underset{\sim}{X} + i\,\underset{\sim}{Y}$. Since the $\underset{\sim}{\xi}_j$'s and $\underset{\sim}{\varphi}$ are mutually independent, hence the $\underset{\sim}{\eta}_j$'s and $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are mutually independent (but $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are not independent). Define $p_j(\underset{\sim}{\eta}_j)$ to be the density of $N(\underset{\sim}{0},\underset{\sim}{\Sigma}_j)$ at $\underset{\sim}{\eta}_j$. Then by Theorem 1 of [3], we have $\int_{\mathcal{D}^*} p_j(\underset{\sim}{\eta}_j + \ell_j\underset{\sim}{\zeta}_j)d\underset{\sim}{\eta}_j \geq \int_{\mathcal{D}^*} p_j(\underset{\sim}{\eta}_j + \ell_j^*\underset{\sim}{\zeta}_j)d\underset{\sim}{\eta}_j$ where $0 \leq \ell_j \leq \ell_j^*$, $\underset{\sim}{\nu}_j' = (\nu_{1j},\dots,\nu_{pj})$, $\nu_{kj} = \alpha_{kj} + i\,\beta_{kj}$ and $\underset{\sim}{\zeta}_j' = (\alpha_{1j},\beta_{1j},\dots,\alpha_{pj},\beta_{pj})$ $k=1,\dots,p;\ j=1,\dots,s$. But $\int_{\mathcal{D}} p_j(\underset{\sim}{\xi}_j + \ell_j\underset{\sim}{\nu}_j)d\underset{\sim}{\xi}_j = \int_{\mathcal{D}^*} p_j(\underset{\sim}{\eta}_j + \ell_j\underset{\sim}{\zeta}_j)d\underset{\sim}{\eta}_j$ and $\int_{\mathcal{D}} p_j(\underset{\sim}{\xi}_j + \ell_j^*\underset{\sim}{\nu}_j)d\underset{\sim}{\xi}_j = \int_{\mathcal{D}^*} p_j(\underset{\sim}{\eta}_j + \ell_j^*\underset{\sim}{\zeta}_j)d\underset{\sim}{\eta}_j$, hence

$$\int_{\mathcal{D}} p_j(\underset{\sim}{\xi}_j + \ell_j\underset{\sim}{\nu}_j)d\underset{\sim}{\xi}_j \geq \int_{\mathcal{D}} p_j(\underset{\sim}{\xi}_j + \ell_j^*\,\underset{\sim}{\nu}_j)d\underset{\sim}{\xi}_j. \tag{3.1}$$

Multiplying both sides of inequality (3.1) by the joint density of the temporarily fixed variables and integrating with respect to them we obtain $\Pr\{\omega \mid \ell_1,\dots,\ell_j,\dots,\ell_s\} \geq \Pr\{\omega \mid \ell_1,\dots,\ell_j^*,\dots,\ell_s\}$ for $0 \leq \ell_j \leq \ell_j^*$ and any ℓ_h's $(h\neq j)$.

<u>Theorem 3.2.</u> If the acceptance region of an invariant test is convex in the space of each column vector of $\underset{\sim}{V}$ for each set of fixed values of $\underset{\sim}{W}$ (see equation (2.6)) and of the other column vectors of $\underset{\sim}{V}$, then the power of the test increases monotonically in each θ_j.

The proof of the above theorem is as straight forward as [3].

<u>Corollary 3.1.</u> If the acceptance region of an invariant test is convex in $\underset{\sim}{V}$ for each fixed $\underset{\sim}{W}$, then the power of the test increases monotonically in each θ_j.

Lemma 3.2. For any hermitian matrix $\underset{\sim}{H}(n\times n)$ the region

$$\mathcal{D} = \{\underset{\sim}{A}(n\times s) \mid ch_1[\underset{\sim}{A}\underset{\sim}{\bar{A}}'\underset{\sim}{H}] \le \lambda\}$$

is convex in $\underset{\sim}{A}$.

Proof: Since the Cauchy-Schwarz inequality is also valid for complex vectors, hence the proof is as straight forward as Lemma 1 of [3].

Corollary 3.2. The maximum root test of Roy, the acceptance region of which is given by

$$ch_1[(\underset{\sim}{V}\underset{\sim}{\bar{V}}')(\underset{\sim}{W}\underset{\sim}{\bar{W}}')^{-1}] \le \lambda,$$

has a power function which is monotonically increasing in each θ_j.

The proof of the above corollary follows from Corollary 3.1 and Lemma 3.2.

Let $c_1 \ge \dots \ge c_p$ be the ch. roots of $(\underset{\sim}{V}\underset{\sim}{\bar{V}}')(\underset{\sim}{W}\underset{\sim}{\bar{W}}')^{-1}$, and $d_j = 1+c_j$ $(j=1,\dots,p)$. Let Q_k be the sum of all different products of $d_1,\dots,d_p$ taken k $(k=1,\dots,p)$ at a time. Consider a complex matrix $\underset{\sim}{M}(p\times n)=(\underset{\sim}{M}_1,\dots,\underset{\sim}{M}_n)$ where $\underset{\sim}{M}_\ell$'s are the column vectors of $\underset{\sim}{M}$. Define $Q_k(\underset{\sim}{M})$ as the sum of all k-rowed principal minors of $\underset{\sim}{M}\underset{\sim}{\bar{M}}' + \underset{\sim}{I}_p$, or equivalently as the sum of all different products of ch. roots of $\underset{\sim}{M}\underset{\sim}{\bar{M}}' + \underset{\sim}{I}_p$ taken k at a time.

Theorem 3.3. An invariant test having acceptance region $\sum_{k=1}^{p} a_k Q_k \le \lambda$ (a_k's ≥ 0) has a power function which is monotonically increasing in each θ_j.

The proof of Theorem 3.3 is analogous to that of Theorem 4 in [3].

In the real case, Das Gupta, Anderson and Mudholkar [3] have given another sufficient condition on the acceptance region. The same is true for the complex case, we only state the corresponding theorem, because the proof is quite similar in [3] with minor changes.

Theorem 3.4. For each j $(j=1,\ldots,s)$ and for each set of fixed values of $\underset{\sim}{V}_k$'s $(k \neq j)$ and $\underset{\sim}{W}$, suppose there exists a unitary transformation: $\underset{\sim}{V}_j \rightarrow \underset{\sim}{U}\underset{\sim}{V}_j = \underset{\sim}{V}_j^* = (V_{1j}^*, \ldots, V_{pj}^*)'$ such that the region $\omega_j(\underset{\sim}{V}_j)$ is transformed into the region $\omega_j^*(\underset{\sim}{V}_j^*)$ which has the following property: Any section of $\omega_j^*(\underset{\sim}{V}_j^*)$ for fixed values of $V_{\ell j}^*$ $(\ell \neq k)$ is a region symmetric about $V_{kj}^* = 0$. Then the power function of the test, having the acceptance region ω, monotonically increases in each θ_j.

4. Tests of independence between two sets of variates. Consider a $(p+q)\times(n+1)$ complex random matrix $\underset{\sim}{Z}$, $(p \leq q,\ p+q \leq n+1)$ whose column vectors $\underset{\sim}{Z}_j$'s $(j=1,\ldots,n+1)$ are independently distributed as a $(p+q)$-variate complex normal distribution $N(\underset{\sim}{\nu},\underset{\sim}{\Sigma})$ where $\underset{\sim}{\Sigma}((p+q)\times(p+q))$ is positive definite hermitian and be partitioned as follows:

$$\underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \bar{\underset{\sim}{\Sigma}}'_{12} & \underset{\sim}{\Sigma}_{22} \end{pmatrix},$$

where $\underset{\sim}{\Sigma}_{11}$ is $(p\times p)$, $\underset{\sim}{\Sigma}_{12}$ is $(p\times q)$ and $\underset{\sim}{\Sigma}_{22}$ is $(q\times q)$ matrices.

Consider the problem of testing the hypothesis

$$H_o:\ \underset{\sim}{\Sigma}_{12} = \underset{\sim}{0}\ (p\times q)$$

against all alternatives. Let the sample covariance matrix be $\underset{\sim}{S}$ which is similarly partitioned as

$$\underset{\sim}{S} = \begin{pmatrix} \underset{\sim}{S}_{11} & \underset{\sim}{S}_{12} \\ \bar{\underset{\sim}{S}}'_{12} & \underset{\sim}{S}_{22} \end{pmatrix}$$

where $n\,\underset{\sim}{S} = \underset{\sim}{Z}\bar{\underset{\sim}{Z}}' - (n+1)\,\underset{\sim}{Z}^*\bar{\underset{\sim}{Z}}^{*\prime}$ and $\underset{\sim}{Z}^* = \sum_{j=1}^{n+1} \underset{\sim}{Z}_j \,/\, (n+1)$. This problem is invariant under transformations

$$\underset{\sim}{Z} \to \begin{pmatrix} \underset{\sim}{B}_1 & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{B}_2 \end{pmatrix} \underset{\sim}{Z}\,\underset{\sim}{U} \qquad \underset{\sim}{Z}_j \to \underset{\sim}{Z}_j + \underset{\sim}{b} \qquad j=1,\ldots,n+1,$$

where $\underset{\sim}{B}_1$ and $\underset{\sim}{B}_2$ are nonsingular matrices of order p and q respectively, and $\underset{\sim}{U}$ is unitary. A test procedure which is invariant under these transformations depends only on the ch. roots $r_1^2 \geq \ldots \geq r_p^2$ of $\underset{\sim}{S}_{11}^{-1}\,\underset{\sim}{S}_{12}\,\underset{\sim}{S}_{22}^{-1}\,\bar{\underset{\sim}{S}}'_{12}$. For convenience let us denote $e_j = r_j^2$ $(j=1,\ldots,p)$. The power function of such a test depends only on the ch. roots $\rho_1^2 \geq \ldots \geq \rho_p^2$ of $\underset{\sim}{\Sigma}_{11}^{-1}\,\underset{\sim}{\Sigma}_{12}\,\underset{\sim}{\Sigma}_{22}^{-1}\,\bar{\underset{\sim}{\Sigma}}'_{12}$ which are the squares of the possible nonzero population canonical correlation coefficients [5]. The distribution of the ch $[\underset{\sim}{S}_{11}^{-1}\,\underset{\sim}{S}_{12}\,\underset{\sim}{S}_{22}^{-1}\,\bar{\underset{\sim}{S}}'_{12}]$ is the same as the distribution of the ch $[(\underset{\sim}{\xi}\bar{\underset{\sim}{\xi}}')^{-1}(\underset{\sim}{\xi}\bar{\underset{\sim}{\zeta}}')(\underset{\sim}{\zeta}\bar{\underset{\sim}{\zeta}}')^{-1}(\underset{\sim}{\zeta}\bar{\underset{\sim}{\xi}}')]$ where the density of the matrices $\underset{\sim}{\xi}(p\times n) = (\xi_{jk})$ and $\underset{\sim}{\zeta}(q\times n) = (\zeta_{jk})$ can be given in the form

$$\pi^{-(p+q)n} \prod_{j=1}^{p} (1-\rho_j^2)^{-n}$$

$$\cdot\exp\left\{-\sum_{j=1}^{p} (1-\rho_j^2)^{-1} \sum_{k=1}^{n} [\xi_{jk}\bar{\xi}_{jk} + \zeta_{jk}\bar{\zeta}_{jk} - 2\rho_j \mathrm{Re}(\xi_{jk}\bar{\zeta}_{jk})] - \sum_{j=k+1}^{q} \sum_{k=1}^{n} \zeta_{jk}\bar{\zeta}_{jk}\right\},$$

or

$$(4.1) \qquad \pi^{-(p+q)n} \prod_{j=1}^{p} (1-\rho_j^2)^{-n}$$

$$\cdot \exp\left\{-\sum_{j=1}^{p} (1-\rho_j^2)^{-1} \sum_{k=1}^{n} (\xi_{jk} - \rho_j \zeta_{jk})(\overline{\xi}_{jk} - \rho_j \overline{\zeta}_{jk}) - \sum_{j=1}^{q} \sum_{k=1}^{n} \zeta_{jk}\overline{\zeta}_{jk}\right\} ,$$

and H_o holds if and only if $\rho_1 = \dots = \rho_p = 0$.

From (4.1) we find that given ζ, the column vectors ξ_j's of ξ are independently distributed each according to a p-variate complex normal distribution with covariance matrix D which is a diagonal matrix with diagonal elements $1-\rho_1^2,\dots,1-\rho_p^2$. The marginal distribution of ζ does not depend on ρ_j's. Moreover, the conditional expectation of ξ given ζ is $E[\xi|\zeta] = A_1\zeta$ where $A_1(p\times q) = (\Delta\ 0)$ and Δ is the diagonal matrix with diagonal elements $\rho_1,\dots,\rho_p$.

Define $\qquad S_h = (\xi\overline{\zeta}')(\zeta\overline{\zeta}')^{-1}(\zeta\overline{\xi}')$

$$S_e = (\xi\overline{\xi}') - (\xi\overline{\zeta}')(\zeta\overline{\zeta}')^{-1}(\zeta\overline{\xi}').$$

If e_j is the jth largest root of $(\xi\overline{\xi}')^{-1}(\xi\overline{\zeta}')(\zeta\overline{\zeta}')^{-1}(\zeta\overline{\xi}')$, then $e_j(1-e_j)^{-1}$ is the jth largest root of $S_hS_e^{-1}$. Thus the class of test procedures based on the $ch[(\xi\overline{\xi}')^{-1}(\xi\overline{\zeta}')(\zeta\overline{\zeta}')^{-1}(\zeta\overline{\xi}')]$ is the same as the class of test procedures based on the $ch\,[S_hS_e^{-1}]$. Let

$$V(p\times q) = B\xi F, \qquad W(p\times(n-q)) = B\xi G$$

where $B(p\times p)$ is nonsingular, and $F(n\times q)$ and $G(n\times(n-q))$ are such that

$$\underset{\sim}{F}\bar{\underset{\sim}{F}}' = \bar{\underset{\sim}{\zeta}}'(\underset{\sim}{\zeta}\bar{\underset{\sim}{\zeta}}')^{-1}\underset{\sim}{\zeta}, \qquad \underset{\sim}{G}\bar{\underset{\sim}{G}}' = \underset{\sim}{I}_n - \bar{\underset{\sim}{\zeta}}'(\underset{\sim}{\zeta}\bar{\underset{\sim}{\zeta}}')^{-1}\underset{\sim}{\zeta}.$$

Then the roots of $\underset{\sim}{S}_h\underset{\sim}{S}_e^{-1}$ are the same as the roots of $(\underset{\sim}{V}\bar{\underset{\sim}{V}}')(\underset{\sim}{W}\bar{\underset{\sim}{W}}')^{-1}$. The matrices $\underset{\sim}{B}, \underset{\sim}{F}$ and $\underset{\sim}{G}$ can be found to use the methods in Section 2, such that the conditional density of $\underset{\sim}{V}=(v_{jk})$ and $\underset{\sim}{W} = (w_{jk})$ given $\underset{\sim}{\zeta}$ is

$$(4.2) \quad \pi^{-pn} \exp\left\{-\mathrm{tr}(\underset{\sim}{W}\bar{\underset{\sim}{W}}') - \sum_{j=1}^{p}(v_{jj}-\tau_j)(\bar{v}_{jj}-\tau_j) - \sum_{\substack{j=1 \\ j\neq k}}^{p}\sum_{k=1}^{q} v_{jk}\bar{v}_{jk}\right\}$$

where $\tau^2 \geq \dots \geq \tau_p^2$ are the ch. roots of $\underset{\sim}{A}\underset{\sim}{\zeta}\,\overline{\underset{\sim}{A}\underset{\sim}{\zeta}}'\underset{\sim}{D}^{-1}$.

Theorem 4.1. An invariant test for which the acceptance region is convex in each column vector of $\underset{\sim}{V}$ for each fixed $\underset{\sim}{W}$ and fixed values of the other column vectors of $\underset{\sim}{V}$ has a power function which is monotonically increasing in each ρ_j.

The proof of the above theorem is similar to that of Anderson and Das Gupta [1] with necessary changes.

Let $c_1 \geq \dots \geq c_p$ be the roots of $(\underset{\sim}{V}\bar{\underset{\sim}{V}}')(\underset{\sim}{W}\bar{\underset{\sim}{W}}')^{-1}$. Then $c_j = e_j(1-e_j)^{-1}$. Thus the relation $e_1 \leq \lambda$ is equivalent to the relation $c_1 \leq \lambda(1-\lambda)^{-1} = \lambda^*$ (say). Let $d_j = 1+c_j$ $(j=1,\dots,p)$ and let Q_k be the sum of all different products of $d_1,\dots,d_p$ taken k at a time $(k=1,\dots,p)$. In particular,

$$Q_p = \prod_{j=1}^{p} d_j = \prod_{j=1}^{p}(1-e_j)^{-1}.$$

The following theorem is obtained from Section 3 and Theorem 4.1.

Theorem 4.2. A test having the acceptance region $\sum_{j=1}^{p} a_j Q_j \leq \lambda$ (a_j's ≥ 0) has a power function which is monotonically increasing in each ρ_j.

5. Symmetric gauge functions and convex functions of matrices. A real valued function

$$\psi(\underset{\sim}{G}) = \psi(a_1,\ldots,a_p)$$

on the p-dimensional space of p-tuples of real numbers is said to be a gauge function if

(1) $\psi(a_1,\ldots,a_p) \geq 0$ with equality if and only if $a_1 = \ldots = a_p = 0$.

(2) $\psi(ca_1,\ldots,ca_p) = |c|\psi(a_1,\ldots,a_p)$ for any real number c.

(3) $\psi(a_1 + b_1,\ldots,a_p + b_p) \leq \psi(a_1,\ldots,a_p) + \psi(b_1,\ldots,b_p)$.

$\psi(G)$ is said to be a symmetric gauge function if, in addition to (1), (2) and (3), it also satisfies

(4) $\psi(\epsilon_1 a_{j1},\ldots,\epsilon_p a_{jp}) = \psi(a_1,\ldots,a_p)$ where $\epsilon_j = \pm 1$ for all j and $j_1,\ldots,j_p$ is a permutation of $1,\ldots,p$.

Let $\underset{\sim}{A}(p\times n), p \leq n$ be a complex matrix, then $\underset{\sim}{A}\bar{\underset{\sim}{A}}'$ is hermitian and all its ch. roots are non-negative. Let $\alpha_1 \geq \ldots \geq \alpha_p$ be its ordered roots. For any increasing convex function f on the positive half of the real line and any symmetric gauge function ψ of p variables, define

$$||\underset{\sim}{A}||_{\psi,f} = \psi(f(\alpha_1^{\frac{1}{2}}),\ldots,f(\alpha_p^{\frac{1}{2}})).$$

Theorem 5.1. $||\underset{\sim}{A}||_{\psi,f}$ is a convex function of $\underset{\sim}{A}$.

The proof is analogous to Theorem 4 of [8] with minor changes.

Let $c_1 \geq \ldots \geq c_p$ be the ch. roots of $\underset{\sim}{S}_h\underset{\sim}{S}_e^{-1}$ in Section 3 and let $\mathcal{D}=\mathcal{D}(c_1,\ldots,c_p)$ be a region in the space of $c_1,\ldots,c_p$.

Theorem 5.2. The power function of an invariant test, which accepts the general multivariate linear hypothesis over $\mathcal{D}$: $\psi(f(\alpha_1^{\frac{1}{2}}),\ldots,f(\alpha_p^{\frac{1}{2}})) \leq \lambda$, where ψ, f and λ are, respectively a symmetric gauge function of p variables, an increasing convex function on the positive half of the real line and a constant determined by the significance level of the test, is a monotonically increasing function in each θ_j.

The proof follows that of Theorem 5 of [8] with necessary changes.

Now let $e_1 \geq \ldots \geq e_p$ be the ch. roots of $(\underset{\sim}{\xi}\underset{\sim}{\bar{\xi}}')^{-1}(\underset{\sim}{\xi}\underset{\sim}{\bar{\zeta}}')(\underset{\sim}{\zeta}\underset{\sim}{\bar{\zeta}}')^{-1}(\underset{\sim}{\zeta}\underset{\sim}{\bar{\xi}}')$ in Section 5, and let $c_j = e_j(1-e_j)^{-1}$ $j=1,\ldots,p$. Then we have, in view of Theorem 4.1, the following:

Theorem 5.3. The power of an invariant test which accepts the independence hypothesis over $\mathcal{D}$, increases monotonically in each population canonical correlation coefficient ρ_j $(j=1,\ldots,p)$.

6. Tests of the equality of two covariance matrices. Samples of size N_1 and N_2 are drawn from $N(\underset{\sim}{\nu}_1,\underset{\sim}{\Sigma}_1)$ and $N(\underset{\sim}{\nu}_2,\underset{\sim}{\Sigma}_2)$ respectively, where $N(\underset{\sim}{\nu}_j,\underset{\sim}{\Sigma}_j)$ $j=1,2$ are (2.1). On the basis of these data we wish to test the null hypothesis:

$$H_o: \underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$$

Since the null hypothesis is invariant under the transformations

$$\underset{\sim}{\xi}_j \rightarrow \underset{\sim}{B}\underset{\sim}{\xi}_j + \underset{\sim}{b}_j \qquad j=1,2$$

where ξ_j are distributed as (2.1) and B is any non-singular matrix and b_1 and b_2 are any vectors. As in the real case, it is known [5] that the power of any invariant test depends on the parameters only through the ch. roots $\gamma_1 \geq \ldots \geq \gamma_p$ of $\Sigma_1\Sigma_2^{-1}$. The null hypothesis can then be restated as

$$H_o: \gamma_1 = \ldots = \gamma_p = 1$$

In this paper we consider the following alternatives

$$H_1: \gamma_j \geq 1 \quad j=1,\ldots,p \quad \sum_{j=1}^{p} \gamma_j > p$$

or

$$H_1^*: \gamma_j \leq 1 \quad j=1,\ldots,p \quad \sum_{j=1}^{p} \gamma_j < p$$

Consider only the problem of testing H_o against H_1 (for H_0 against H_1^*, we consider the test procedures having the above acceptance regions as rejection regions, then the power of such a test will decrease as each ordered root of $\Sigma_1\Sigma_2^{-1}$ increase.)

Theorem 6.1. Let $Z(p\times n)$, $p \leq n$, be a complex random matrix having density (2.4) and let $c_1 \geq \ldots \geq c_p$ be the ch. roots of $Z\bar{Z}'$ and ω be a set in the space of $c_1,\ldots,c_p$ such that when a point $(c_1,\ldots,c_p)$ is in ω so is every point $(c_1^*,\ldots,c_p^*)$ for $c_j^* \leq c_j$ $(j=1,\ldots,p)$. Then the probability of the set ω depends on Σ only through ch $[\Sigma]$ and is a monotonically decreasing function of each of the ch. roots of Σ.

Theorem 6.2. Let Z_1 and Z_2 are independently distributed as (2.4) i.e. $p(Z_1,\Sigma_1,n_1)$ and $p(Z_2;\Sigma_2,n_2)$ respectively, and let ω be a set in the space of ch. roots of $(Z_1\bar{Z}_1')(Z_2\bar{Z}_2')^{-1}$ [here also called the c_j's] satisfying the condition stated in Theorem 6.1. Then the probability of ω

depends on Σ_1 and Σ_2 only through ch $[\Sigma_1\Sigma_2^{-1}]$, and is a monotonically decreasing function of each of the ch. roots of $\Sigma_1\Sigma_2^{-1}$.

The proof of the above two theorems are analogous to those of Theorem 1 and 2 in [2] with necessary changes.

Corollary 6.1. If an invariant test has an acceptance region such that if $(c_1,\ldots,c_p)$ is in the region, so is $(c_1^*,\ldots,c_p^*)$ for $c_j^* \leq c_j$ $(j=1,\ldots,p)$, then the power of the test is a monotonically increasing function of each γ_j.

Corollary 6.2. If $g(c_1,\ldots,c_p)$ is monotonically increasing in each of the arguments, a test with acceptance region $g(c_1,\ldots,c_p) \leq \lambda$ has a monotonically increasing power function in each γ_j.

7. Pillai's $V^{(p)}$ test. Pillai and Jayachandran, [9], [10], have carried out exact power function comparisons for tests (i) to (iii) based on four criteria for the two-roots case. Now we show that the power function of $V^{(p)}$ test is monotonically increasing in each of the parameters. First we prove the following lemma:

Lemma 7.1. For any hermitian matrix $H(n\times n)$, the region $\mathcal{S}=\{K(n\times m)\,|\,\mathrm{tr}[(K\bar{K}')H]\leq \lambda\}$ is convex in K.

Proof. Let $H=\bar{T}'T$ where T is an nxn matrix, and let $k_1 \geq \ldots \geq k_n$ be the ch. roots of $(K\bar{K}')H$ then $\mathrm{tr}[(K\bar{K}')H] = \sum_{j=1}^{n} k_j$. Further let $K_1, K_2 \in \mathcal{S}$ and $K = \alpha K_1+(1-\alpha)K_2$ for $0 \leq \alpha \leq 1$. Then

$$\sum_{j=1}^{n} k_j^{(1)} \leq \lambda \quad \text{and} \quad \sum_{j=1}^{n} k_j^{(2)} \leq \lambda,$$

where $k_1^{(\ell)},\ldots,k_n^{(\ell)}$ are the ch. roots of $(\underset{\sim}{K}_\ell \bar{\underset{\sim}{K}}_\ell')\underset{\sim}{H}$, $\ell=1,2$. Now

$$\begin{aligned}
\mathrm{tr}[(\underset{\sim}{K}\bar{\underset{\sim}{K}}')\underset{\sim}{H}] &= \mathrm{tr}[\underset{\sim}{T}\underset{\sim}{K}\bar{\underset{\sim}{K}}'\bar{\underset{\sim}{T}}'] \\
&= \mathrm{tr}[\underset{\sim}{T}\{\alpha\underset{\sim}{K}_1 + (1-\alpha)\,\underset{\sim}{K}_2\}\{\alpha\bar{\underset{\sim}{K}}_1' + (1-\alpha)\,\bar{\underset{\sim}{K}}_2'\}\bar{\underset{\sim}{T}}'] \\
&= \mathrm{tr}[\alpha^2\underset{\sim}{T}\underset{\sim}{K}_1\bar{\underset{\sim}{K}}_1'\bar{\underset{\sim}{T}}' + (1-\alpha)^2\,\underset{\sim}{T}\underset{\sim}{K}_2\bar{\underset{\sim}{K}}_2'\bar{\underset{\sim}{T}}' + 2\alpha(1-\alpha)\,\mathrm{Re}\underset{\sim}{T}\underset{\sim}{K}_1\bar{\underset{\sim}{K}}_2'\bar{\underset{\sim}{T}}'] \\
&\le \alpha^2\sum_{j=1}^{n} k_j^{(1)} + (1-\alpha)^2\sum_{j=1}^{n} k_j^{(2)} + 2\alpha(1-\alpha)\sqrt{\sum_{j=1}^{n} k_j^{(1)}\sum_{j=1}^{n} k_j^{(2)}} \\
&= \left\{\alpha\Big(\sum_{j=1}^{n} k_j^{(1)}\Big)^{\frac{1}{2}} + (1-\alpha)\,\Big(\sum_{j=1}^{n} k_j^{(2)}\Big)^{\frac{1}{2}}\right\}^2 \\
&\le \lambda
\end{aligned}$$

Thus $\underset{\sim}{K}\in\mathcal{D}$.

For $V^{(p)}$ criterion, the acceptance regions for (i) to (iii) are $\sum_{j=1}^{p} e_j \le \lambda$ where $e_j = c_j(1+c_j)^{-1}$. For test (i), since $\sum_{j=1}^{p} e_j = \sum_{j=1}^{p} \frac{c_j}{1+c_j}$ and the latter is a monotonically increasing function in each c_j, hence the power function of such test is monotonically increasing in each of parameters, guaranteed by Corollary 6.2. As for (ii) and (iii), monotonicity property guaranteed by Lemma 7.1 and Corollary 3.1. It is also guaranteed by the following theorem:

Let Q_k^* be the sum of all different products of $e_1,\ldots,e_p$ taken $k(k=1,\ldots,p)$ at a time. Consider a matrix $\underset{\sim}{M}(p\times n)=(\underset{\sim}{M}_1,\ldots,\underset{\sim}{M}_n)$ where $\underset{\sim}{M}_j$'s are the column vectors of $\underset{\sim}{M}$. Define $Q_k^*(\underset{\sim}{M})$ as the sum of all k-rowed principal minors of $\underset{\sim}{M}\underset{\sim}{\bar{M}}'$, or equivalently as the sum of all different products of the roots of $\underset{\sim}{M}\underset{\sim}{\bar{M}}'$ taken k at a time.

Lemma 7.2. For any j and k ($j=1,\ldots,n$; $k=1,\ldots,p$) and for $\underset{\sim}{M}_\ell$'s fixed, $\ell \neq j$, $Q_k^*(\underset{\sim}{M})$ is a positive definite hermitian form in $\underset{\sim}{M}_j$ plus a constant.

The proof is similar to that of Lemma 2 of [3], except that in the present case the matrix $\underset{\sim}{B}$ of [3] does not have a second term (which is an identity matrix in their proof.)

Theorem 7.1. An invariant test having acceptance region $\sum_{k=1}^{p} a_k Q_k^* \leq \lambda$ (a_k's ≥ 0) has a power function which is monotonically increasing in each of the parameters.

The proof is analogous to Theorem 4 of [3].

8. Remarks. The following discussion of special cases of tests generalizes to the complex case, the results of previous authors in the real case.

(I) The likelihood-ratio test for (ii) and (iii) has the acceptance regions of the form

$$\prod_{j=1}^{p} (1+c_j) \leq \lambda_1.$$

The power function of such test is monotonically increasing in each of the parameters, for (ii) guaranteed by Theorem 3.3, and for (iii) by Theorem 4.2. However, for test (i), it is very difficult to investigate tests with reasonable power against all alternatives, because the acceptance region of such a test is

$$g(c_1,\ldots,c_p) = \prod_{j=1}^{p} \frac{(1+c_j)^{n_1+n_2}}{c_j^{n_1}} \leq \lambda_2$$

and $g(c_1,\ldots,c_p)$ is an increasing function of $c_1,\ldots,c_p$ or not, depending on the values of sample sizes n_1 and n_2.

(II). For Roy's maximum root test, the acceptance regions for (i) to (iii) are of the form

$$c_1 \leq \lambda_3 .$$

The power function of such test is monotonically increasing in each of the parameters, for (i) guaranteed by Corollary 6.2; for (ii) and (iii) by Corollary 3.2.

(III). For Hotelling's trace test, the acceptance regions for (i) to (iii) are of the form

$$\sum_{j=1}^{p} c_j \leq \lambda_4 .$$

The power function of such test is monotonically increasing in each of the parameters, for (i) guaranteed by Corollary 6.2; for (ii) by Theorem 3.3 and for (iii) by Theorem 4.2.

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