On Asymptotic Expansions of the Distributions of The Characteristic Roots of Two Matrices *

by

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Asymptotic representation of the distribution of the characteristic roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ when roots are not all distinct

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ERRATA

Page	Line	Incorrect	Correction
2	1	$\prod_{i=1}^{p}$	$\prod_{i<j}^{p}$
3	16	$O[\phi_n(x)]$	$o[\phi_n(x)]$
	20	$O(\phi_N(x))$	$o[\phi_N(x)]$
4	3	$O(\phi(x))$	$o[\phi(x)]$
	13-14	$\text{Cos}2\theta)^{-\frac{n}{2}}$	$\cos 2\theta)]^{-\frac{n}{2}}$
14	1	ai's	a_i's
15	4	ℓ's	ℓ_i's
16	10	1 2 ... p	1,2,...,p
	12	1 2 ... p	1,2,...,p
17	6	$\underset{\sim}{H}^{**}$	$\underset{\sim}{H}^{*}$
18	6	$\overset{A}{\Delta}$	$\underset{\sim}{A}$
19	18	$x=$	$\underset{\sim}{X} =$
20	1	: =	:
	13	$\lvert \underset{\sim}{A}_0'\underset{\sim}{A}_0 \rvert^{-\frac{1}{2}}$	$\lvert \underset{\sim}{A}_0'\underset{\sim}{A}_0 \rvert^{\frac{1}{2}}$
21	last line	ϕ	(
25	last line	: =	:
27	5	$a\vert_p$	$a\ell_p$
	10-11	(1+0)	1+0

33	1-2	$\frac{\pi^{\frac{1}{2}kp}}{\Gamma_k(\frac{p}{2})}$	$\frac{\Gamma_k(\frac{p}{2})}{\pi^{\frac{1}{2}kp}}$
	12-13	$\frac{\pi^{\frac{1}{2}kp}}{\Gamma_k(\frac{p}{2})}$	$\frac{\Gamma_k(\frac{p}{2})}{\pi^{\frac{1}{2}kp}}$
34	9-10	$\pi^{\frac{1}{4}p(p-1)+\frac{1}{2}kp}$	$\pi^{\frac{1}{4}p(p-1)-\frac{1}{2}kp}$
	10-11	$\Gamma_k(\frac{p}{2})$	$[\Gamma_k(\frac{p}{2})]^{-1}$
47	19	$\mu_i' \frac{w_i}{w_p}$	$\mu_i' = \frac{w_i}{w_p}$
52	7	ℓ's	ℓ_i's
53	1	Thomson	Thompson
	2	.2.5	2.5
54	18	Namda	Nanda
56	4	$(x;2,k+1)$	$(x;q,k+1)$

CHAPTER I

ASYMPTOTIC REPRESENTATION OF THE DISTRIBUTION OF THE CHARACTERISTIC ROOTS OF $S_1 S_2^{-1}$ IN THE DISTINCT ROOTS CASE

1. Introduction and Summary

Let S_i: $p \times p (i=1,2)$ be independently distributed as Wishart (n_i, p, Σ_i). Let the characteristic roots of $S_1 S_2^{-1}$ and $\Sigma_1 \Sigma_2^{-1}$ be denoted by $\ell_i (i=1,2,\ldots,p)$ and $\lambda_i (i=1,2,\ldots,p)$ respectively such that $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_p \geq 0$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0$. Then the distribution of $\ell_1, \ldots, \ell_p$ can be expressed in the form (Khatri [15])

$$(1.1) \qquad C|\Lambda|^{-\frac{1}{2}n_1} |L|^{\frac{1}{2}(n_1-p-1)} \prod_{i<j} (\ell_j - \ell_i) \int_{O(p)} |I + \Lambda^{-1} HLH'|^{-\frac{1}{2}(n_1+n_2)} (H' dH)$$

where

$$(1.2) \qquad C = 2^{-p} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma_p(\tfrac{1}{2}i) \Gamma_p(\tfrac{n_1+n_2}{2}) \{\Gamma_p(\tfrac{1}{2}p) \Gamma_p(\tfrac{1}{2}n_1) \Gamma_p(\tfrac{1}{2}n_2)\}^{-1},$$

$$(1.3) \qquad \Gamma_p(t) = \pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma(t - \tfrac{1}{2}j + \tfrac{1}{2}),$$

$L = \operatorname{diag}(\ell_1, \ell_2, \ldots, \ell_p)$ and $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$. The invariant Haar measure

$$(1.4) \qquad (\underset{\sim}{H}'d\underset{\sim}{H}) = \prod_{i=1}^{p} h_i' dh_j$$

is defined on the group $O(p)$ of $p \times p$ orthogonal matrices with h_i and dh_j the ith and jth column of $\underset{\sim}{H}$ and $d\underset{\sim}{H}$ respectively (James [11]). The group $O(p)$ has a volume

$$(1.5) \qquad V(p) = \int_{O(p)} (\underset{\sim}{H}'d\underset{\sim}{H}) = 2^p \, \Pi^{\frac{1}{4}p(p+1)} / \prod_{i=1}^{p} \Gamma(\tfrac{1}{2}i).$$

One of the approaches to the evaluation of the integral of (1.1) (from here on we denote it as E) is to expand it as a power series:

$$(1.6) \quad E = \int_{O(p)} |\underset{\sim}{I}+\underset{\sim}{\Lambda}^{-1}\underset{\sim}{H}\underset{\sim}{L}\underset{\sim}{H}'|^{-\frac{1}{2}(n_1+n_2)} (\underset{\sim}{H}'d\underset{\sim}{H})$$

$$= V(p) \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} \frac{C_\kappa(-\underset{\sim}{\Lambda}^{-1}) C_\kappa(\underset{\sim}{L}) (n_1+n_2)_\kappa}{C_\kappa(\underset{\sim}{I})}$$

with zonal polynomial $C_\kappa(\underset{\sim}{T})$ of any $p \times p$ symmetric matrix $\underset{\sim}{T}$ defined in James [14]. But its convergence is very slow unless the characteristic roots of the argument matrices are small. In the one sample case G.A. Anderson [1] has obtained a gamma type asymptotic expansion for the distribution of the characteristic roots of the estimated convariance matrix. In this chapter we obtain a beta type asymptotic representation of the roots distribution of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ involving linkage factors between sample roots and corresponding population roots. A study is also made of the approximation to the distribution of $w_1,\ldots,w_p$ where $w_i = \ell_i/(1+\ell_i)$ $(i=1,2,\ldots,p)$. If the roots are distinct the limiting distribution as n_2 tends to infinity has the same form as that of G.A. Anderson [1]. If, moreover, n_1 is assumed also large, then it agrees with Girshick's result [7].

In the following we will assume $\ell_1 > \ell_2 > \ldots > \ell_p > 0$ and $\lambda_1 > \lambda_2 > \ldots > \lambda_p > 0$, and for the simplification of notations we let $\underset{\sim}{A} = \underset{\sim}{\Lambda}^{-1}$, i.e., $a_i = 1/\lambda_i (i=1,\ldots,p), 0<a_1<a_2<\ldots<a_p<\infty$, and $n = n_1+n_2$.

The definitions below are taken from Erdélyi [6].

DEFINITION. The sequence of functions $\{\phi_n(x)\}$ is an asymptotic sequence as $x \to \infty$, if for each n

$$\phi_{n+1}(x) = o[\phi_n(x)] \quad \text{as} \quad x \to \infty,$$

Let $\{\phi_n\}$ be an asymptotic sequence.

DEFINITION. The (formal) series $\Sigma\, a_n\phi_n(x)$ is an asymptotic expansion to N terms of $f(x)$ as $x \to \infty$ if

$$f(x) = \sum^{N} a_n\phi_n(x) + o(\phi_N(x)) \qquad \text{as } x \to \infty.$$

This is written $f(x) \sim \Sigma^N a_n\phi_n(x)$.

DEFINITION. The function $\phi(x)$ is an asymptotic representation for $f(x)$ as $x \to \infty$ if

$$f(x) = \phi(x) + o(\phi(x)) \qquad \text{as} \quad x \to \infty.$$

This is written $f(x) \sim \phi(x)$.

2. The Asymptotic Representation of E when p = 2

An elementary method is used in this section to obtain the asymptotic expansion of E when $p = 2$.

Let $O^{\pm}(2) = \{\underset{\sim}{H} \epsilon O(2),\ |\underset{\sim}{H}| = \pm 1\}$ then

$$(2.1) \qquad E = 2\int_{O^+(2)} |\underset{\sim}{I} + \underset{\sim}{AHLH'}|^{-\frac{n}{2}} (\underset{\sim}{H}' d\underset{\sim}{H}).$$

Now let $\underset{\sim}{H} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad -\pi < \theta \leq \pi,$

So that $(\underset{\sim}{H}' d\underset{\sim}{H}) = d\theta$ and

$$(2.2) \quad E = 4[(1+a_1\ell_1)(1+a_2\ell_2)]^{-\frac{n}{2}} \int_{-\pi/2}^{\pi/2} [1+\tfrac{1}{2}C_{12}(1-\mathrm{Cos}\,2\theta)]^{-\frac{n}{2}} d\theta$$

where

$$(2.3) \qquad C_{12} = \frac{(a_2-a_1)(\ell_1-\ell_2)}{(1+a_1\ell_1)(1+a_2\ell_2)} .$$

The integrand of (2.2) has a maximum of unity at $\theta = 0$ and decreases to $(1+\frac{1}{2}C_{12})^{-\frac{1}{2}n}$ at $\theta = \pm\frac{\pi}{2}$. Write (2.2) as

$$(2.4)\quad 4[\prod_{i=1}^{2}(1+a_i\ell_i]^{-\frac{n}{2}}\int_{-\pi/2}^{\pi/2}\exp\{-\tfrac{n}{2}\log[1+\tfrac{1}{2}C_{12}(1-\cos 2\theta)]\}\,d\theta.$$

Since the integral is mostly concentrated in a small neighborhood of the origin, for large n, we can expand the argument of the exponential function and $\cos 2\theta$ in the usual power series and set the limit to be $\pm\infty$ (see Erdélyi [6]). Thus for large degrees of freedom E is approximately

$$(2.5)\quad 4[\prod_{i=1}^{2}(1+a_i\ell_i)]^{-\frac{n}{2}}\int_{-\infty}^{\infty}\exp\{-\tfrac{n}{2}C_{12}\theta^2\}\,d\theta\ \{1+O(\tfrac{1}{n})\}$$

or

$$(2.6)\qquad E \simeq 4[\prod_{i=1}^{2}(1+a_i\ell_i]^{-\frac{n}{2}}\left(\frac{2\pi}{nC_{12}}\right)^{\frac{1}{2}}\{1+O(\tfrac{1}{n})\}.$$

3. The Asymptotic Representation of E in General Case

Lemma (1.1) Let $\underset{\sim}{A}$ and $\underset{\sim}{L}$ are defined as before then $f(\underset{\sim}{H}) = |\underset{\sim}{I}+\underset{\sim}{AHLH}'|$ $\underset{\sim}{H}\epsilon O(p)$ attains its identical minimum value $|\underset{\sim}{I}+\underset{\sim}{AL}|$ when $\underset{\sim}{H}$ is of the form

$$(3.1) \qquad \underset{\sim}{H} = \begin{pmatrix} \pm 1 & & & \\ & \pm 1 & & 0 \\ & & \ddots & \\ 0 & & & \pm 1 \end{pmatrix}$$

Proof: $df = d\,|\underset{\sim}{I} + \underset{\sim}{AHLH'}|$

$$= d\,|\underset{\sim}{I} + \underset{\sim}{A}^{\frac{1}{2}}\underset{\sim}{HLH'}\underset{\sim}{A}^{\frac{1}{2}}|$$

$$= \mathrm{tr}\,(\underset{\sim}{I} + \underset{\sim}{A}^{\frac{1}{2}}\underset{\sim}{HLH'}\underset{\sim}{A}^{\frac{1}{2}})^{-1}(\underset{\sim}{A}^{\frac{1}{2}}d\underset{\sim}{HLH'}\underset{\sim}{A}^{\frac{1}{2}} + \underset{\sim}{A}^{\frac{1}{2}}\underset{\sim}{HL}d\underset{\sim}{H'}\underset{\sim}{A}^{\frac{1}{2}})$$

$$= 2\,\mathrm{tr}\,\underset{\sim}{LH'}\underset{\sim}{A}^{\frac{1}{2}}(\underset{\sim}{I} + \underset{\sim}{A}^{\frac{1}{2}}\underset{\sim}{HLH'}\underset{\sim}{A}^{\frac{1}{2}})^{-1}\,\underset{\sim}{A}^{\frac{1}{2}}\underset{\sim}{HH'}d\underset{\sim}{H}.$$

Note that $\underset{\sim}{H}'d\underset{\sim}{H}$ is a skew symmetric matrix, therefore, $df = 0$ implies that $\underset{\sim}{LH'}\underset{\sim}{A}^{\frac{1}{2}}\,(\underset{\sim}{I} + \underset{\sim}{A}^{\frac{1}{2}}\underset{\sim}{HLH'}\underset{\sim}{A}^{\frac{1}{2}})^{-1}\underset{\sim}{A}^{\frac{1}{2}}\underset{\sim}{H}$ is a symmetric matrix. But $\underset{\sim}{H}'\underset{\sim}{A}^{\frac{1}{2}}(\underset{\sim}{I} + \underset{\sim}{A}^{\frac{1}{2}}\underset{\sim}{HLH'}\underset{\sim}{A}^{\frac{1}{2}})^{-1}\,\underset{\sim}{A}^{\frac{1}{2}}\underset{\sim}{H}$ is itself a symmetric matrix and $\underset{\sim}{L}$ is a diagonal matrix with distinct positive roots, so $\underset{\sim}{H}'\underset{\sim}{A}^{\frac{1}{2}}(\underset{\sim}{I}+\underset{\sim}{A}^{\frac{1}{2}}\underset{\sim}{H}\,\underset{\sim}{LH'}\underset{\sim}{A}^{\frac{1}{2}})^{-1}\underset{\sim}{A}^{\frac{1}{2}}\underset{\sim}{H}$ has to be a diagonal matrix, say $\underset{\sim}{D}$. Thus $\underset{\sim}{I} = \underset{\sim}{A}^{\frac{1}{2}}\underset{\sim}{H}(\underset{\sim}{L}-\underset{\sim}{D}^{-1})\underset{\sim}{H}'\underset{\sim}{A}^{\frac{1}{2}}$. This can happen only if $\underset{\sim}{H}$ is of the form with ± 1 in one position in a column or a row and zero in other positions. After substituting those stationary values into $f(\underset{\sim}{H})$ we obtain a general form

$$(3.2) \qquad \prod_{i=1}^{p} (1 + a_i \ell_{\sigma_i}),$$

where ℓ_{σ_i} is any permutation of $\ell_i (i = 1,\ldots,p)$. It is easy to see that (3.2) attains its minimum value when $\ell_{\sigma_i} = \ell_i (i = 1,2,\ldots,p)$. Or $f(\underset{\sim}{H})$ attains its identical minimum value $|\underset{\sim}{I} + \underset{\sim}{AL}|$ when H is of the form of (3.1).

The above lemma enables us to claim that, for large n_1, the integrand of E is negligible except for small neighborhoods about each of these matrices of (3.1) and E consists of identical contributions from each of these neighborhoods so that

$$(3.3) \qquad E \simeq 2^p \int_{N(\underset{\sim}{I})} |\underset{\sim}{I} + \underset{\sim\sim\sim}{AHLH'}|^{-\frac{n}{2}} (\underset{\sim}{H}' d\underset{\sim}{H}),$$

where $N(\underset{\sim}{I})$ is a neighborhood of the identity matrix on the orthogonal manifold. Since any proper orthogonal matrix can be written as the exponential of a skew symmetric matrix we transform E under

$$(3.4) \qquad \underset{\sim}{H} = \exp \underset{\sim}{S}, \ \underset{\sim}{S} \text{ a pxp skew symmetric matrix,}$$

So that $N(\underset{\sim}{I}) \to N(\underset{\sim}{S} \to \underset{\sim}{0})$. The Jacobian of this transformation has been computed by G.A. Anderson [1],

$$(3.5) \qquad J = 1 + \frac{p-2}{24} \operatorname{tr} \underset{\sim}{S}^2 + \frac{8-p}{4 \cdot 6!} \operatorname{tr} \underset{\sim}{S}^4 + \cdots\cdots$$

Direct substitution of (3.4) into (3.3) yields

$$(3.6) \qquad |\underset{\sim}{I} + \underset{\sim\sim\sim}{AHLH'}|^{-\frac{n}{2}}$$

$$= |\underset{\sim}{I} + \underset{\sim\sim}{AL} + \underset{\sim\sim\sim}{ASL} - \underset{\sim\sim\sim}{ALS} + \underset{\sim\sim\sim}{ALS}^2 - \underset{\sim\sim\sim\sim}{ASLS} + \cdots|^{-\frac{n}{2}}$$

$$= |\underset{\sim}{I} + \underset{\sim\sim}{AL}|^{-\frac{n}{2}} |\underset{\sim}{I} + (\underset{\sim}{I} + \underset{\sim\sim}{AL})^{-1} (\underset{\sim\sim\sim}{ASL} - \underset{\sim\sim\sim}{ALS} + \underset{\sim\sim\sim}{ALS}^2 - \underset{\sim\sim\sim\sim}{ASLS} + \cdots)|^{-\frac{n}{2}}.$$

Lemma (1.2) For any pxp matrix $\underset{\sim}{B}$ and its characteristic roots $b_i (i=1,\ldots,p)$, if $\max_{1\le i\le p} |b_i| < 1$ then

$$|\underset{\sim}{I}+\underset{\sim}{B}|^{-\frac{1}{2}n} = \exp\{-\tfrac{1}{2}n\ \mathrm{tr}(\underset{\sim}{B} - \frac{\underset{\sim}{B}^2}{2} + \frac{\underset{\sim}{B}^3}{3} \ldots)\}.$$

Proof: $|\underset{\sim}{I}+\underset{\sim}{B}|^{-\frac{1}{2}n} = \exp\{-\tfrac{1}{2}n \log \prod_{i=1}^{p} (1+b_i)\}$.

If $\max_{1\le i\le p} |b_i| < 1$ then

$$|\underset{\sim}{I}+\underset{\sim}{B}|^{-\frac{1}{2}n} = \exp\{-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{B} - \frac{\underset{\sim}{B}^2}{2} + \frac{\underset{\sim}{B}^3}{3} - \ldots)\}.$$

Apply lemma (1.2) to (3.6) and the maximum characteristic roots of $(\underset{\sim}{I}+\underset{\sim}{AL})^{-1}(\underset{\sim}{ASL}-\underset{\sim}{ALS} + \ldots)$ can be shown to be less than unity. Since we are only interested in the first term we need to investigate the group of terms up to order of $\underset{\sim}{S}^2$ which is denoted by $\{\underset{\sim}{S}^2\}$. Let $\underset{\sim}{R} = (\underset{\sim}{I}+\underset{\sim}{AL})^{-1}$, then

$$(3.7)\qquad \mathrm{tr}\{\underset{\sim}{S}^2\} = \mathrm{tr}[\underset{\sim}{R}(\underset{\sim}{ALS}^2 - \underset{\sim}{ASLA}) - \tfrac{1}{2}(\underset{\sim}{RALSRALS} + \underset{\sim}{RASLRASL} - \underset{\sim}{RASLRALS} - \underset{\sim}{RALSRASL})].$$

after simplification (3.7) reduces to

$$(3.8)\qquad \mathrm{tr}[\underset{\sim}{R}(\underset{\sim}{ALS}^2 - \underset{\sim}{ASLS}) - (\underset{\sim}{LS} - \underset{\sim}{SL})\ \underset{\sim}{RALSRA}]$$

or

$$(3.9)\qquad \mathrm{tr}\{\underset{\sim}{S}^2\} = \sum_{i<j}^{p} c_{ij}\, s_{ij}^2$$

where

$$(3.10) \quad c_{ij} = (a_j - a_i)(\ell_i - \ell_j)/[(1 + a_i\ell_i)(1 + a_j\ell_j)].$$

Direct substitution into E yields

$$(3.11) \quad E \simeq 2^p \prod_{i=1}^{p} (1+a_i\ell_i)^{-\frac{n}{2}} \int_{N(\underset{\sim}{S}=\underset{\sim}{0})} \exp\{-\frac{n}{2}\sum_{i<j}^{p} c_{ij}s_{ij}^2\} \prod_{i<j} ds_{ij}\{1+O(\frac{1}{n})\}.$$

For large n the limits for each s_{ij} can be put to $\pm\infty$. We finally have the following theorem.

<u>Theorem:</u> The asymptotic distribution of the roots, $\ell_1 > \ell_2 > \ldots > \ell_p > 0$, of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ for large degrees of freedom $n = n_1+n_2$ when the roots of $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$ are $\lambda_1 > \lambda_2 > \ldots > \lambda_p > 0$ and $a_i = 1/\lambda_i (i=1,\ldots,p)$, is given by

$$(3.12) \quad C2^p \prod_{i<j} (\ell_j - \ell_i) \prod_{i=1}^{p} [\ell_i^{\frac{n_1-p-1}{2}} a_i^{\frac{1}{2}n_1} (1+a_i\ell_i)^{\frac{-(n_1+n_2)}{2}}] \prod_{i<j}^{p} [\frac{2\pi}{c_{ij}(n_1+n_2)}]^{\frac{1}{2}}.$$

The asymptotic formula shows that the distribution function of a group of adjacent roots is sensitive only to those other roots which are close to them.

4. A Dual Expansion of E and Some Remarks

If we let $\underset{\sim}{W} = \underset{\sim}{L}(\underset{\sim}{I} + \underset{\sim}{L})^{-1}$ in (1.1), i.e., $w_i = \ell_i/(1+\ell_i)$ $(i=1,2,\ldots,p)$ where $\underset{\sim}{W} = \text{diag}(w_1,\ldots,w_p)$, then the joint distribution of w_i's is given by

$$(4.1) \quad C|\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1}|\underset{\sim}{W}|^{\frac{1}{2}(n_1-p-1)}|\underset{\sim}{I}-\underset{\sim}{W}|^{-\frac{1}{2}(n_1+p+1)}\prod_{i<j}(w_j-w_i)$$

$$\int_{O(p)}|\underset{\sim}{I}+\underset{\sim}{AHLH'}|^{-\frac{1}{2}(n_1+n_2)}(\underset{\sim}{H}'d\underset{\sim}{H})$$

$$1 > w_1 > w_2 > \ldots > w_p > 0.$$

Application of lemmas (1.1) and (1.2) to (4.1) yields its asymptotic representation

$$(4.2) \quad 2^p C|\underset{\sim}{A}|^{\frac{1}{2}n_1}|\underset{\sim}{W}|^{\frac{1}{2}(n_1-p-1)}|\underset{\sim}{I}-\underset{\sim}{W}|^{\frac{1}{2}(n_2-p-1)} \quad \cdots$$

$$\prod_{i<j}(w_j-w_i)\prod_{i=1}^{p}[1+(a_i-1)w_i]^{-\frac{1}{2}(n_1+n_2)}\prod_{i<j}^{p}\left[\frac{2\pi}{C^*_{ij}(n_1+n_2)}\right]^{\frac{1}{2}}$$

when $C^*_{ij} = \dfrac{(a_j-a_i)(w_i-w_j)}{[1+(a_i-1)w_i][1+(a_j-1)w_j]}$.

Now let us proceed to look at (3.12) once again. The asymptotic distribution of characteristic roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ given there can be rewritten as

$$(4.3) \quad F_1(A)\prod_{i<j}(\ell_i-\ell_j)^{\frac{1}{2}}\prod_{i=1}^{p}[\ell_i^{\frac{n_1-p-1}{2}}(1+a_i\ell_i)^{\frac{-(n_1+n_2)}{2}+p-1}]\prod_{i=1}^{p}d\ell_i$$

where $F_i(A)$ $(i=1,2,3)$ depend on a_i but not on $\ell_i(i=1,2,\ldots,p)$. If we make $g_i = \ell_i/n_2$ $(i=1,2,\ldots,p)$ and let n_2 tends to infinity