A Class of Selection Procedures Including Procedures for Restricted Families of Distributions*

by

S. Panchapakesan

Department of Statistics

Division of Mathematical Sciences

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S. Panchapakesan

Purdue University

1. Introduction and Summary

Gupta (1966) defined a class of selection procedures and considered some of its properties. Some additional results concerning the properties of this class of procedures were obtained by Gupta and Panchapakesan (1968b). In the present paper we define a class of selection procedures, which is a natural generalization of the class considered by Gupta (1966). Let $\pi_1,\ldots,\pi_k$ be k continuous populations. Let Λ be an interval on the real line. Associated with π_i is the random variable X_i with distribution function F_{λ_i}, $\lambda_i \in \Lambda$, $i=1,\ldots,k$. It is assumed that the functional forms of F_{λ_i} are known, but the values of λ_i are unknown. Let $\lambda_{[1]} \leq \lambda_{[2]} \leq \cdots \leq \lambda_{[k]}$ be the ordered λ_i's. The correct pairing of the ordered and the unordered λ's is not known. Based on the observations $x_1,\ldots,x_k$ from $\pi_1,\ldots,\pi_k$, we want to define a class of procedures for selecting a non-empty subset of the k populations such that the probability is at least $P^*(\frac{1}{k} < P^* < 1)$ that the population associated with $\lambda_{[k]}(\lambda_{[1]})$ is included in the selected subset. If there are more than one population with $\lambda_i= \lambda_{[k]}(\lambda_i= \lambda_{[1]})$, then it is assumed that one of them is tagged as the 'best'. If we let CS stand for a correct selection, i.e., the selection of a subset which includes the best population and $P(CS|R)$ denote the probability of a correct selection using the procedure R, then the probability requirement stated above can be written

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$$\inf_{\Omega} P(CS|R) \geq P^*, \tag{1.1}$$

where Ω is the space of all k-tupels $(F_{\lambda_1},\ldots,F_{\lambda_k})$. The requirement (1.1) will be referred to as the basic probability requirement or the P*-condition.

Section 2 defines a class of procedures R_h for selecting the population associated with $\lambda_{[k]}$ and deals with the expression for the probability of a correct selection. Section 3 contains a theorem which is a generalization of a result of Lehmann and uses it to obtain a sufficient condition for the probability of a correct selection when $\lambda_1 = \ldots = \lambda_k = \lambda$ to be nondecreasing (increasing) in λ. This result provides the infimum of the probability of a correct selection over Ω. Also some special cases of interest are discussed. Some properties of the procedure R_h are investigated in Section 4. A sufficient condition is obtained for the supremum of the expected size of the subset selected to take place when $\lambda_1 = \ldots = \lambda_k$ and it is shown that this condition implies the condition obtained in Section 3 for the monotonicity of the probability of a correct selection when the λ's are equal. Specific results are obtained in some special cases. Also shown is the fact that the same sufficient condition assures that the supremum of the expected number of non-best populations included in the selected subset takes place when the parameters are equal. Section 5 defines a class of procedures R_H for the selection of the population associated with $\lambda_{[1]}$ and briefly discusses the infimum of the probability of a correct selection and the suprema of the expected size of the selected subset and the expected number of non-best populations in the selected subset. Section 6 is concerned with selection procedures for restricted families of distributions. On the space of probability distributions a partial ordering (h-ordering) is defined. The star-shaped and tail orderings are shown to be particular cases of this ordering.

The problem of selecting a subset containing the stochastically largest (assumed to exist) among k populations F_i, $i=1,\dots,k$, is discussed when the forms of the F_i are not known and each F_i is h-ordered with respect to a known distribution G. It is shown that some of the procedures discussed by Barlow and Gupta (1969) fall under this case for particular choices of h.

2. The Class of Procedures R_h and the Probability of a Correct Selection

Let $h \equiv h_{c,d}$; $c \in [1,\infty)$, $d \in [0,\infty)$ be a function defined on the real line satisfying the following properties: For every real x,

(2.1)
(i) $h_{c,d}(x) \geq x$
(ii) $h_{1,0}(x) = x$
(iii) $h_{c,d}(x)$ is continuous in c and d
(iv) $h_{c,d}(x) \uparrow \infty$ as $d \to \infty$ and/or $xh_{c,d}(x) \uparrow \infty$ as $c \to \infty$, $x \neq 0$.

Some of the functions satisfying these properties that will be of interest to us are cx, $x+d$ and $cx+d$.

Now we define a class of procedures R_h as follows.

R_h: Include π_i in the selected subset iff

$$h(x_i) \geq \max_{1 \leq r \leq k} x_r. \tag{2.2}$$

Because of (2.1)-(i), the procedure R_h will select a non-empty subset.

Denoting the random variable associated with $\lambda_{[j]}$ by $X_{(j)}$ and its cdf by $F_{[j]} \equiv F_{\lambda_{[j]}}(x)$, we have

$$P(CS|R_h) = P(h(X_{(k)}) \geq X_{(r)},\ r=1,\dots,k-1) \tag{2.3}$$

$$= \int_{-\infty}^{\infty} \prod_{r=1}^{k-1} F_{[r]}(h(x))\, dF_{[k]}(x).$$

We now assume that the distributions are stochastically increasing in λ. To put it more specifically, we assume that, for $\lambda < \lambda'$, F_λ and $F_{\lambda'}$ are distinct and

(2.4) $$F_\lambda(x) \geq F_{\lambda'}(x) \quad \text{for all } x.$$

Then

(2.5) $$P(CS|R_h) \geq \int_{-\infty}^{\infty} F_{[k]}^{k-1}(h(x))\, dF_{[k]}(x).$$

Hence

(2.6) $$\inf_{\Omega} P(CS|R_h) = \inf_{\lambda\epsilon\Lambda} \Psi(\lambda;c,d,k),$$

where

(2.7) $$\Psi(\lambda;c,d,k) = \int_{-\infty}^{\infty} F_\lambda^{k-1}(h(x))\, dF_\lambda(x)$$

and $\Omega = \{\underline{\lambda} | \underline{\lambda}' = (\lambda_1,\ldots,\lambda_k),\ \lambda_i \epsilon \Lambda,\ i=1,\ldots,k\}$. Because of (2.1)-(i) and (ii), we get

(2.8) $$\Psi(\lambda;c,d,k) \geq \frac{1}{k}$$

and

(2.9) $$\Psi(\lambda;1,0,k) = \frac{1}{k}\ .$$

Further (2.1)-(iii) and (iv) yield

(2.10) $$\lim_{d\to\infty} \Psi(\lambda;\ c,d,k) = 1$$

and/or

(2.11) $$\lim_{c\to\infty} \Psi(\lambda;c,d,k) = 1-F_\lambda(0).$$

If (2.10) holds, then for given λ, c and k, we can choose d such that the P*-condition is satisfied, since $\frac{1}{k} < P^* < 1$. If (2.11) holds, then for given λ, d and k we can find c to satisfy the P*-condition provided that $1-F_\lambda(0) \geq P^*$. Since we can choose P* as close to 1 as we want, this would mean that we should have $F_\lambda(0) = 0$. Hence if (2.11) holds but not (2.10), then the constants for the procedure can be evaluated whatever $P^* \epsilon(\frac{1}{k}, 1)$ be only if the random variables are non-negative.

3. <u>A Sufficient Condition for the Monotonicity of $\Psi(\lambda;c,d,k)$</u>

We start with a result in Lehmann (1959, p. 112), which is stated below without proof as

<u>Lemma 3.1</u>. Let F_0, F_1 be two cumulative distribution functions on the real line such that $F_1 \underset{st}{\geq} F_0$($F_1$ is stochastically larger than F_0), i.e., $F_1(x) \leq F_0(x)$ for all x. Then $E_0\Psi(X) \leq E_1\Psi(X)$ for any non-decreasing function Ψ. An immediate consequence of Lemma 3.1 is

<u>Lemma 3.2</u>. Let $\{F_\lambda\}$ be a family of distribution functions on the real line which are stochastically increasing in λ. Then $E_\lambda\Psi(X)$ is non-decreasing in λ for any non-decreasing function Ψ.

In the following theorem, we want to obtain a more general result, which gives a sufficient condition for $E_\lambda\Psi(x,\lambda)$ to be non-decreasing in λ.

<u>Theorem 3.1</u>. Let $\{F_\lambda\}$, $\lambda\epsilon\Lambda$, be a family of continuous distributions on the real line and $\Psi(x,\lambda)$ be a differentiable function in x and λ. Then

$E_\lambda \Psi(X,\lambda)$ is non-decreasing in λ provided that

(3.1) $$\left|\frac{\partial(F,\Psi)}{\partial(x,\lambda)}\right| \geq 0,$$

where

(3.2) $$\left|\frac{\partial(F,\Psi)}{\partial(x,\lambda)}\right| = \begin{vmatrix} \frac{\partial}{\partial x} F_\lambda(x) & \frac{\partial}{\partial x} \Psi(x,\lambda) \\ \frac{\partial}{\partial \lambda} F_\lambda(x) & \frac{\partial}{\partial \lambda} \Psi(x,\lambda) \end{vmatrix}$$

and $E_\lambda(\Psi(X,\lambda))$ is strictly increasing in λ if strict inequality holds in (3.1) on a set of positive measure.

Before we proceed with the proof of this theorem, we introduce some notations and obtain some useful lemmas. We assume that F_λ for $\lambda \epsilon \Lambda$ has the support I. Let

(3.3) $$A(\lambda) = \int_I \Psi(x,\lambda)\, dF_\lambda(x) \equiv E_\lambda(\Psi(X,\lambda)).$$

Consider $\lambda_1, \lambda_2 \epsilon \Lambda$ such that $\lambda_1 \leq \lambda_2$ and define

(3.4) $$A_i(\lambda_1,\lambda_2) = \int_I \prod_{\substack{r=1 \\ r \neq i}}^{2} \Psi(x,\lambda_r)\, dF_{\lambda_i}(x), \quad i=1,2$$

and

(3.5) $$B(\lambda_1,\lambda_2) = A_1(\lambda_1,\lambda_2) + A_2(\lambda_1,\lambda_2).$$

Note that when $\lambda_1 = \lambda_2 = \lambda$,

(3.6) $$B(\lambda,\lambda) = 2A(\lambda).$$

Lemma 3.3. $B(\lambda_1,\lambda_2)$ is non-decreasing in λ_1, when λ_2 is kept fixed, provided that, for $\lambda_1 \leq \lambda_2$,

$$(3.7) \qquad \frac{\partial}{\partial\lambda_1} \Psi(x,\lambda_1)\, f_{\lambda_2}(x) - \frac{\partial}{\partial\lambda_1} F_{\lambda_1}(x)\, \frac{\partial}{\partial x} \Psi(x,\lambda_2) \geq 0 \quad \text{for all } x$$

where $f_\lambda(x) \equiv \frac{\partial}{\partial x} F_\lambda(x)$.

Proof. Integrating by parts, we obtain

$$(3.8) \qquad A_1(\lambda_1,\lambda_2) = F_{\lambda_1}(x)\, \Psi(x,\lambda_2)\big|^{*} - \int_I F_{\lambda_1}(x)\, \Psi'(x,\lambda_2)dx$$

where $\Psi'(x,\lambda) = \frac{\partial}{\partial x} \Psi(x,\lambda)$ and the asterisk in the first term indicates that it is evaluated between the proper limits. However, we note that this term will be independent of λ_1. Using (3.8) in (3.5), we obtain

$$(3.9) \qquad B(\lambda_1,\lambda_2) = \text{a term independent of } \lambda_1 + \int_I \{\Psi(x,\lambda_1) f_{\lambda_2}(x) - F_{\lambda_1}(x)\Psi'(x,\lambda_2)\}dx.$$

Hence

$$(3.10) \qquad \frac{\partial}{\partial\lambda_1} B(\lambda_1,\lambda_2) = \int_I \{\frac{\partial}{\partial\lambda_1} \Psi(x,\lambda_1)\, f_{\lambda_2}(x) - \frac{\partial}{\partial\lambda_1} F_{\lambda_1}(x)\, \Psi'(x,\lambda_2)\}\, dx$$

and the partial derivative on the left side of (3.10) is non-negative if (3.7) holds for any pair of $\lambda_1,\lambda_2 \epsilon \Lambda$ such that $\lambda_1 \leq \lambda_2$. This completes the proof of Lemma 3.3.

Lemma 3.4. If $\lambda_1 = \lambda_2 = \lambda$, then $B(\lambda,\lambda)$ is non-decreasing in λ if (3.7) holds.

Proof. We note the following properties of $B(\lambda_1,\lambda_2)$ which can be verified easily.

$$(3.11) \qquad \frac{\partial}{\partial\lambda} B(\lambda,\lambda) = \sum_{i=1}^{2} \frac{\partial}{\partial\lambda_i} B(\lambda_1,\lambda_2)\Bigg|_{\lambda_1=\lambda_2=\lambda}$$

and $B(\lambda_1,\lambda_2)$ as a function of λ_1 and λ_2 (ignoring the fact that $\lambda_1 \leq \lambda_2$) remains unchanged when λ_1 and λ_2 are interchanged (denoted by $\lambda_1 \leftrightarrow \lambda_2$). Hence

$$(3.12) \qquad \frac{\partial}{\partial\lambda_2} B(\lambda_1,\lambda_2) = \frac{\partial}{\partial\lambda_2} B(\lambda_2,\lambda_1) = \frac{\partial}{\partial\lambda_1} B(\lambda_1,\lambda_2)\Bigg|_{\lambda_1 \leftrightarrow \lambda_2}$$

Now, from (3.11) and (3.12), we get

$$(3.13) \qquad \frac{\partial}{\partial\lambda} B(\lambda,\lambda) = 2\,\frac{\partial}{\partial\lambda_1} B(\lambda_1,\lambda_2)\Bigg|_{\lambda_1=\lambda_2=\lambda}$$

$$\geq 0, \quad \text{if} \quad \frac{\partial}{\partial\lambda_1} B(\lambda_1,\lambda_2) \geq 0,$$

which is ensured by Lemma 3.3 if (3.7) holds. This completes the proof of Lemma 3.4.

Now we are ready for the

Proof of Theorem 3.1. By Lemma 3.4, $B(\lambda,\lambda) = 2A(\lambda)$ is non-decreasing in λ if (3.7) is satisfied. In the hypothesis of the theorem we have only F_λ, considering $\lambda_1 \leq \lambda_2$ is only an artificial device to obtain the desirable result. We are interested only in the pairs of λ_1,λ_2 such that $\lambda_1=\lambda_2=\lambda$. Thus for $A(\lambda)$ to be non-decreasing in λ, it is sufficient if (3.7) holds for $\lambda_1=\lambda_2=\lambda$, i.e., if

$$(3.14) \qquad \frac{\partial}{\partial\lambda} \Psi(x,\lambda)\, f_\lambda(x) - \frac{\partial}{\partial\lambda} F_\lambda(x)\, \frac{\partial}{\partial x} \Psi(x,\lambda) \geq 0,$$

which is the same as (3.1). The strict inequality part is now obvious.

Remark 3.1. In the proof of Lemma 3.3 we have made use of the assumption that F_λ, $\lambda\epsilon\Lambda$ have all the same support I. But the result is true even if the support changes with λ. If (a_1,b_1) and (a_2,b_2) are the supports of F_{λ_1} and F_{λ_2}, (3.8) will be

$$A_1(\lambda_1,\lambda_2) = \Psi(b_1,\lambda_2) - \int_{a_1}^{b_1} F_{\lambda_1}(x)\ \Psi'(x,\lambda_2)dx \tag{3.15}$$

and this yields

$$\begin{aligned}\frac{\partial}{\partial\lambda_1} A_1(\lambda_1,\lambda_2) &= \frac{\partial}{\partial b_1}\Psi(b_1,\lambda_2)\frac{db_1}{d\lambda_1} - \int_{a_1}^{b_1}\frac{\partial}{\partial\lambda_1}F_{\lambda_1}(x)\Psi'(x,\lambda_2)dx \\ &\quad - \frac{db_1}{d\lambda_1}\frac{\partial}{\partial b_1}\Psi(b_1,\lambda_2) \\ &= -\int_{a_1}^{b_1}\frac{\partial}{\partial\lambda_1}F_{\lambda_1}(x)\ \Psi'(x,\lambda_2)dx\ .\end{aligned} \tag{3.16}$$

Hence,

$$\frac{\partial}{\partial\lambda_1} B(\lambda_1,\lambda_2) = \int_{a_2}^{b_2}\frac{\partial}{\partial\lambda_1}\Psi(x,\lambda_1)\ f_{\lambda_2}(x)dx - \int_{a_1}^{b_1}\frac{\partial}{\partial\lambda_1}F_{\lambda_1}(x)\ \Psi'(x,\lambda_2)dx \tag{3.17}$$

and it can be seen that (3.7) is sufficient to make $\frac{\partial}{\partial\lambda_1}B(\lambda_1,\lambda_2) \geq 0$. Lemma 3.4 also holds. Hence Theorem 3.1 is true even when the supports are not the same.

Corollary 3.1. If $\Psi(x,\lambda) = \Psi(x)$ for $\lambda\epsilon\Lambda$, i.e., $\Psi(x,\lambda)$ is independent of λ, then $E_\lambda\Psi(X)$ is non-decreasing in λ if

$$(3.18) \qquad \frac{\partial}{\partial\lambda} F_\lambda(x) \frac{d}{dx} \Psi(x) \leq 0.$$

If we assume in the above corollary that $\{F_\lambda\}$ is a family of distributions stochastically increasing in λ, then (3.18) is equivalent to $\frac{d}{dx} \Psi(x) \geq 0$, since $\frac{\partial}{\partial\lambda} F_\lambda(x) \leq 0$. Hence $E_\lambda(\Psi(X))$ is non-decreasing in λ if $\Psi(x)$ is non-decreasing in x. So we get Lemma 3.2 as a special case of our Theorem.

Corollary 3.2. Let $\{F_\lambda\}$ and $\Psi(x,\lambda)$ be as in the hypothesis of Theorem 3.1. In addition let $\Psi(x,\lambda) \geq 0$. Then, for any positive integer t, $E_\lambda(\Psi^t(X,\lambda))$ is non-decreasing in λ if (3.7) holds.

Proof. Let $\varphi(x,\lambda) = \Psi^t(x,\lambda)$ play the role of $\Psi(x,\lambda)$ in Theorem 3.1. Then $E_\lambda(\varphi(X,\lambda))$ is nondecreasing in λ if $\left|\frac{\partial(F,\varphi)}{\partial(x,\lambda)}\right| \geq 0$ which can be written as $t\Psi^{t-1}(x,\lambda) \left|\frac{\partial(F,\Psi)}{\partial(x,\lambda)}\right| \geq 0$, which is equivalent to (3.7) since $\Psi(x,\lambda) \geq 0$.

Now we prove a theorem giving a sufficient condition for the monotonicity of $\Psi(\lambda;c,d,k)$ defined with reference to the procedure R_h.

Theorem 3.2. For the procedure R_h, $\Psi(\lambda;c,d,k)$ is non-decreasing in λ provided that

$$(3.19) \qquad \frac{\partial}{\partial\lambda} F_\lambda(h(x)) f_\lambda(x) - h'(x) f_\lambda(h(x)) \frac{\partial}{\partial\lambda} F_\lambda(x) \geq 0,$$

where $h'(x) = \frac{d}{dx} h(x)$ and, $\Psi(\lambda;c,d,k)$ is strictly increasing in λ if strict inequality holds in (3.19) on a set of positive measure.

Proof. The proof is immediate by letting $\Psi(x,\lambda) = F_\lambda(h(x))$ in Corollary 3.2.

Before proceeding to some special cases, we note that $\Psi(\lambda;c,d,k)$ is independent of λ if

$$(3.20) \qquad \frac{\partial}{\partial\lambda} F_\lambda(h(x)) f_\lambda(x) - h'(x) f_\lambda(h(x)) \frac{\partial}{\partial\lambda} F_\lambda(x) = 0 \qquad \text{for all } x.$$

Special Cases

(a) λ is a location parameter and $h(x) = x+d,\ d > 0$.

In this case, $F_\lambda(x) = F(x-\lambda)$ and $\frac{\partial}{\partial\lambda} F_\lambda(x) = -f(x-\lambda) = -f_\lambda(x)$. This ensures (3.20) and $\Psi(\lambda;c,d,k)$ is independent of λ.

(b) $\lambda > 0$ is a scale parameter and $h(x) = cx,\ c > 1$.

Here $f_\lambda(x) = \frac{1}{\lambda} f(\frac{x}{\lambda}),\ x \geq 0$ and $F_\lambda(x) = F(\frac{x}{\lambda})$. Thus $\frac{\partial}{\partial\lambda} F_\lambda(x) = -\frac{x}{\lambda^2} f(\frac{x}{\lambda}) = -\frac{x}{\lambda} f_\lambda(x)$. Noting that $h(x) = xh'(x)$, we see that (3.20) holds and hence $\Psi(\lambda;c,d,k)$ is independent of λ.

(c) $f_\lambda(x)$ is a convex mixture of a sequence of density functions .

In this case, we assume that

$$f_\lambda(x) = \sum_{j=0}^{\infty} W(\lambda,j)\ g_j(x), \tag{3.21}$$

where $g_j(x)$, $j=0,1,\ldots$ is a sequence of density functions and $W(\lambda,j)$ are non-negative weights such that $\sum_{j=0}^{\infty} W(\lambda,j) = 1$. We restrict ourselves to weight functions given by

$$W(\lambda,j) = \frac{a_j\ \lambda^j}{A(\lambda)j!}\,, \quad A(\lambda) > 0. \tag{3.22}$$

Because of the non-negativity of $W(\lambda,j)$, we have either $\lambda \geq 0$, $a_j \geq 0$ or $\lambda \leq 0$, $(-1)^j\ a_j \geq 0$. Also, since the weights add up to unity,

$$A(\lambda) = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}\ a_j\ . \tag{3.23}$$

We will assume that $\lambda \geq 0$ in what follows. Define

$$r_\lambda(x) = A(\lambda)\ f_\lambda(x) \tag{3.24}$$

and

$$R_\lambda(x) = A(\lambda)\ F_\lambda(x). \tag{3.25}$$

Then (3.19) can be written as

$$r_\lambda(x)[\tfrac{\partial}{\partial\lambda} R_\lambda(h(x)) - A'(\lambda)\ F_\lambda(h(x))] - h'(x)\ r_\lambda(h(x))\ [\tfrac{\partial}{\partial\lambda} R_\lambda(x) - A'(\lambda)\ F_\lambda(x)] \geq 0, \tag{3.26}$$

where $A'(\lambda) \equiv \frac{\partial}{\partial\lambda} A(\lambda)$. Using (3.25), we can rewrite (3.26) equivalently as

$$r_\lambda(x)\ [A(\lambda)\ \tfrac{\partial}{\partial\lambda} R_\lambda(h(x)) - A'(\lambda)\ R_\lambda(h(x))] - h'(x)\ r_\lambda(h(x))\ [A(\lambda)\ \tfrac{\partial}{\partial\lambda} R_\lambda(x) - A'(\lambda) R_\lambda(x)] \geq 0. \tag{3.27}$$

Now $A(\lambda)\ \frac{\partial}{\partial\lambda} R_\lambda(x) - A'(\lambda)\ R_\lambda(x)$

$$= \Big(\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_j\Big)\Big(\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_{j+1} G_{j+1}(x)\Big) - \Big(\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_{j+1}\Big)\Big(\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_j G_j(x)\Big)$$

$$= \sum_{\alpha=0}^{\infty} \frac{\lambda^j}{\alpha!} B_\alpha(x), \quad \text{where}$$

$G_j(x)$ is the cdf corresponding to $g_j(x)$ and

$$B_\alpha(x) = \sum_{j=0}^{\alpha} \binom{\alpha}{j} a_j\ a_{\alpha-j+1}\ (G_{\alpha-j+1}(x) - G_j(x)). \tag{3.28}$$

Using the above result in (3.27) and expanding the products and rewriting we obtain

$$\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} C_i \geq 0, \tag{3.29}$$

where

$$C_i = \sum_{\alpha=0}^{i} \binom{i}{\alpha} [a_{i-\alpha} g_{i-\alpha}(x) B_\alpha(h(x)) - h'(x) a_\alpha g_\alpha(h(x)) B_{i-\alpha}(x)]. \tag{3.30}$$

We see that (3.29) holds and consequently (3.19) holds if $C_i \geq 0$ for every non-negative integer i, that is,

$$\sum_{\alpha=0}^{i} \binom{i}{\alpha} [a_{i-\alpha} g_{i-\alpha}(x) B_\alpha(h(x)) - h'(x) a_\alpha g_\alpha(h(x)) B_{i-\alpha}(x)] \geq 0 \tag{3.31}$$

for every integer $i \geq 0$.

At this stage we consider a more special case where

$$a_{j+1} = (q+pj)a_j \quad , \qquad j = 0,1,\ldots; \; p,q \geq 0. \tag{3.32}$$

Successive applications of (3.32) yield

$$a_{j+1} = a_0 q(q+p)(q+2p) \ldots (q+jp), \quad j = 0,1,\ldots. \tag{3.33}$$

Hence

$$A(\lambda) = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_0 q(q+p) \ldots (q+jp) \tag{3.34}$$

$$= a_0(1-\lambda p)^{-q/p}, \text{ provided that } \lambda < \frac{1}{p} .$$

Then $A(\lambda)\frac{\partial}{\partial\lambda}R_\lambda(x)-A'(\lambda)R_\lambda(x)=a_0(1-\lambda p)^{-q/p}\,Q_\lambda(x)$, where

$$(3.35)\qquad Q_\lambda(x) = (1-\lambda p)\sum_{j=1}^{\infty}\frac{\lambda^{j-1}}{(j-1)!}\,a_jG_j(x)-q\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\,a_jG_j(x)$$

$$=\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\,a_{j+1}G_{j+1}(x)-\sum_{j=1}^{\infty}\frac{\lambda^j}{j!}(pj+q)a_jG_j(x)-qa_0G_0(x)$$

$$=\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\,a_{j+1}(G_{j+1}(x)-G_j(x)).$$

Using this result, (3.27) holds and consequently (3.19) holds if

$$(3.36)\qquad Q_\lambda(h(x))\,r_\lambda(x)-h'(x)Q_\lambda(x)\,r_\lambda(h(x))\geq 0.$$

Letting $\Delta G_j(x) = G_{j+1}(x)-G_j(x)$, (3.36) can be written as

$$(3.37)\qquad \sum_{i=0}^{\infty}\frac{\lambda^i}{i!}\,E_i\geq 0,$$

where

$$(3.38)\qquad E_i=\sum_{\alpha=0}^{i}\binom{i}{\alpha}[a_{i-\alpha}a_{\alpha+1}g_{i-\alpha}(x)\Delta G_\alpha(h(x))-h'(x)a_{i-\alpha+1}a_\alpha g_\alpha(h(x))\Delta G_{i-\alpha}(x)].$$

Hence a sufficient condition for (3.19) to hold is that, for every integer $i\geq 0$, $E_i\geq 0$, which in view of (3.32) can be written as

$$(3.39)\qquad \sum_{\alpha=0}^{i}\binom{i}{\alpha}\,a_\alpha a_{i-\alpha}[(q+\alpha p)g_{i-\alpha}(x)\Delta G_\alpha(h(x))-h'(x)(q+(i-\alpha)p)g_\alpha(h(x))\Delta G_{i-\alpha}(x)]\geq 0.$$

Summarizing the above discussion, we state

Theorem 3.3. For the procedure R_h, when $f_\lambda(x)$ is given by (3.21) with weight functions defined in (3.22) where the a_j are governed by the relation (3.32), $\Psi(\sigma;c,d,k)$ is non-decreasing in λ provided that, for every integer $i \geq 0$, (3.39) holds and $\Psi(\lambda;c,d,k)$ is strictly increasing in λ if strict inequality holds in (3.39) for some i.

If $q=1$, $a_0=1$ and $p=0$, $A(\lambda) = \lim_{p\to 0}(1-\lambda p)^{-1/p} = e^{\lambda}$ and $\lambda \geq 0$. Also $a_j = 1$ for all j and $W(\lambda,j) = \frac{e^{-\lambda}\lambda^j}{j!}$. Thus, $g_j(x)$ are weighted by Poisson weights. In this case (3.39) becomes

$$(3.40) \qquad \sum_{\alpha=0}^{i} \binom{i}{\alpha} [g_{i-\alpha}(x)\Delta G_\alpha(h(x)) - h'(x)g_\alpha(h(x))\Delta G_{i-\alpha}(x)] \geq 0.$$

This special case has been considered by Gupta and Studden (1965) who have obtained the condition (3.40) with $h(x) = cx$.

If $p=1$ and $a_0=1$, $a_j=q(q+1)\ldots(q+j-1)$ and $A(\lambda)=(1-\lambda)^{-q}$, $0 \leq \lambda \leq 1$. Then $W(\lambda,j) = \frac{\Gamma(q+j)}{\Gamma(q)}\frac{\lambda^j}{j!}(1-\lambda)^q$. The weights in this case are negative binomial weights. In this case (3.39) becomes

$$(3.41) \qquad \sum_{\alpha=0}^{i} \binom{i}{\alpha}(q)_\alpha(q)_{i-\alpha}[(q+\alpha)g_{i-\alpha}(x)\Delta G_\alpha(h(x)) - h'(x)(q+i-\alpha) g_\alpha(h(x))\Delta G_{i-\alpha}(x)] \geq 0,$$

where

$$(3.42) \qquad (q)_\alpha = q(q+1)\ldots(q+\alpha-1).$$

This special case has been considered by Gupta and Panchapakesan (1968a) who have obtained (3.41) with $h(x)=cx$.

Both (3.40) and (3.41) were obtained by Gupta and Studden (1965) and

Gupta and Panchapakesan (1968a) respectively by a different approach.

Before we proceed to the next section wherein we discuss the properties of the procedure R_h we want to make a few remarks concerning the Lemmas 3.3 and 3.4. Suppose we consider $\lambda_1, \lambda_2, \ldots, \lambda_{t+1} \epsilon \Lambda$ such that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{t+1}$ and assume that the F_λ have the same support. Define

$$(3.43) \qquad A_i(\lambda_1, \ldots, \lambda_{t+1}) = \int_I \prod_{\substack{r=1 \\ r \neq i}}^{t+1} \Psi(x, \lambda_r) dF_{\lambda_i}(x), \quad i=1,\ldots,t+1$$

and

$$(3.44) \qquad B(\lambda_1, \ldots, \lambda_{t+1}) = \sum_{i=1}^{t+1} A_i(\lambda_1, \ldots, \lambda_{t+1}).$$

Then, as in the proof of Lemma 3.3, if we integrate $A_1(\lambda_1, \ldots, \lambda_{t+1})$ by parts and use the result in (3.44), corresponding to (3.9) we will obtain

$$(3.45) \qquad B(\lambda_1, \ldots, \lambda_{t+1}) = \text{a term independent of } \lambda_1 + \sum_{i=2}^{t+1} \int \{ \prod_{\substack{r=2 \\ r \neq i}}^{t+1} \Psi(x, \lambda_r) \} [\Psi(x, \lambda_1) f_{\lambda_i}(x) - F_{\lambda_1}(x) \Psi'(x, \lambda_i)] dx.$$

If we further assume that $\Psi(x, \lambda) \geq 0$ for all $\lambda \epsilon \Delta$, then $\frac{\partial}{\partial \lambda_1} B(\lambda_1, \ldots, \lambda_{t+1}) \geq 0$ if

$$(3.46) \qquad \frac{\partial}{\partial \lambda_1} \Psi(x, \lambda_1) \; f_{\lambda_i}(x) - \frac{\partial}{\partial \lambda_1} F_{\lambda_1}(x) \; \Psi'(x, \lambda_i) \geq 0 \quad \text{for } i=2,\ldots,t+1.$$

Since $\lambda_1, \ldots, \lambda_{t+1}$ are chosen arbitrarily in Λ subject only to the condition that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{t+1}$, (3.46) is satisfied if (3.7) is satisfied. If we now set $\lambda_1 = \cdots = \lambda_m = \lambda \leq \lambda_{m+1} \leq \cdots \leq \lambda_{t+1}$, $1 \leq m \leq t+1$, we note that

$$(3.47)\qquad \frac{\partial}{\partial\lambda} B(\lambda,\ldots,\lambda,\lambda_{m+1},\ldots,\lambda_{t+1}) = \sum_{i=1}^{m} \frac{\partial}{\partial\lambda_i} B(\lambda_1,\ldots,\lambda_{t+1})\Big|_{\lambda_1=\cdots=\lambda_m=\lambda}$$

and $B(\lambda_1,\ldots,\lambda_{t+1})$ as a function of $\lambda_1,\ldots,\lambda_{t+1}$ (ignoring the fact that $\lambda_1 \le \cdots \le \lambda_{t+1}$) is unaltered by interchanging any two of the λ's. Hence

$$(3.48)\qquad \frac{\partial}{\partial\lambda} B(\lambda,\ldots,\lambda,\lambda_{m+1},\ldots,\lambda_{t+1})$$

$$= m \frac{\partial}{\partial\lambda_1} B(\lambda_1,\ldots,\lambda_{t+1})\Big|_{\lambda_1=\cdots=\lambda_{t+1}=\lambda}$$

$$\ge 0, \text{ if (3.7) is satisfied.}$$

We summarize the above results in

Lemma 3.5. $B(\lambda_1,\ldots,\lambda_{t+1})$ as defined in (3.44) with $\Psi(x,\lambda) \ge 0$ for $\lambda\epsilon\Lambda$, is nondecreasing in λ when $\lambda_1=\cdots=\lambda_m=\lambda \le \lambda_{m+1}\le\cdots\le \lambda_{t+1}$, $1 \le m \le t+1$ provided that (3.7) holds.

As a consequence of Lemma 3.5, we can state

Lemma 3.6. The supremum of $B(\lambda_1,\ldots,\lambda_{t+1})$ over the space of $(\lambda_1,\ldots,\lambda_{t+1})$ where $\lambda_1\le\cdots\le \lambda_{t+1}$, takes place when $\lambda_1=\cdots= \lambda_{t+1}$ if (3.7) holds.

Proof. The proof follows by successive applications of Lemma 3.6 with $m=1,\ldots,t$.

4. Properties of the Procedure R_h

Unbiasedness. A procedure defined for selecting a subset containing the population associated with $\lambda_{[k]}$ is said to be unbiased if, for $1\le i< j\le k$, the population associated with $\lambda_{[j]}$ has at least as much probability of being included in the selected subset as the population associated with $\lambda_{[i]}$. If

p_r, r=1,...,k, is the probability of including the population associated with $\lambda_{[r]}$, then the procedure is unbiased if $p_i \le p_j$ for $1 \le i < j \le k$.

Theorem 4.1. The procedure R_h is unbiased, if h(x) is non-decreasing in x. We omit the proof, since it is the same as in the case of the procedure R_{h_b} of Gupta (1966).

Expected Subset Size. Let S denote the size of the subset selected by the procedure R_h. We are interested in $E_{\underline{\lambda}}(S|R_h)$, the expected size of the selected subset using the procedure R_h over $\Omega = \{\underline{\lambda}|\underline{\lambda}'=(\lambda_1,\dots,\lambda_k)\}$. It is easy to see that

$$(4.1) \qquad E_{\underline{\lambda}}(S|R_h) = \sum_{i=1}^{k} p_i,$$

where

$$(4.2) \qquad p_i = \int \prod_{\substack{r=1 \\ r\neq i}}^{k} F_{[r]}(h(x))\, dF_{[i]}(x), \quad i=1,\dots,k.$$

Let

$$(4.3) \qquad \Psi(x,\lambda_{[r]}) = F_{[r]}(h(x)), \quad r = 1,\dots,k.$$

Then, in the notations of Section 3,

$$(4.4) \qquad p_i = A_i(\lambda_{[1]},\dots,\lambda_{[k]}), \quad i = 1,\dots,k$$

and

$$(4.5) \qquad E_{\underline{\lambda}}(S|R_h) = B(\lambda_{[1]},\dots,\lambda_{[k]}).$$

Hence Lemmas 3.5 and 3.6 apply and we get

Theorem 4.2. $E_{\underline{\lambda}}(S|R_h)$ is non-decreasing in λ, where $\lambda_{[1]}=\cdots=\lambda_{[m]}=\lambda\leq\lambda_{[m+1]}\leq\cdots\leq\lambda_{[k]}$, $1\leq m\leq k$ and consequently $\sup_{\Omega} E_{\underline{\lambda}}(S|R_h)$ takes place at a point where $\lambda_{[1]}=\cdots=\lambda_{[k]}$ provided that, for $\lambda_1\leq\lambda_2$,

$$\frac{\partial}{\partial\lambda_1} F_{\lambda_1}(h(x))\, f_{\lambda_2}(x)-h'(x)\, \frac{\partial}{\partial\lambda_1} F_{\lambda_1}(x)\, f_{\lambda_2}(h(x)) \geq 0. \tag{4.6}$$

Remark 4.1. When λ is a location parameter and $h(x) = x+d$ or when λ is a scale parameter and $h(x) = cx$, it is easy to see that (4.6) is equivalent to the condition that, for $\lambda_1 \leq \lambda_2$ and $x_1 \leq x_2$,

$$f_{\lambda_1}(x_1)\, f_{\lambda_2}(x_2) - f_{\lambda_1}(x_2)\, f_{\lambda_2}(x_1) \geq 0, \tag{4.7}$$

which is the condition for $f_\lambda(x)$ to have a monotone likelihood ratio.

Remark 4.2. It is to be noted that, if (4.6) is satisfied, then (3.19) is satisfied. If we denote the probability of a correct selection when $\underline{\lambda} = (\lambda,\ldots,\lambda)$ by $P_\lambda(CS|R_h)$, then (4.6) implies that

$$\sup_{\Omega} E_{\underline{\lambda}}(S|R_h) = k \sup_{\lambda} P_\lambda(CS|R_h) \tag{4.8}$$

and that $P_\lambda(CS|R_h)$ is nondecreasing in λ.

Remark 4.3. In the cases of location and scale parameters we saw that $P_\lambda(CS|R_h)$ is independent of λ. Hence its infimum and supremum over λ are equal. Thus we get

$$\sup_{\Omega} E_{\underline{\lambda}}(S|R_h) = kP^*. \tag{4.9}$$

Remark 4.4. In the cases of location and scale parameters discussed above, if there is any other procedure R for which the basic P*-condition is satisfied and

(4.10) $$E_\lambda(S|R) = kP_\lambda(CS|R),$$

then

(4.11) $$\sup_\Omega E_{\underline{\lambda}}(S|R) - \sup_\Omega E_{\underline{\lambda}}(S|R_h)$$

$$\geq E_{\lambda_0}(S|R)-kP^*, \text{ where } \lambda_0 \text{ is any point } \Lambda$$

$$= k(P_{\lambda_0}(CS|R)-P^*)$$

$$\geq 0, \text{ since } R \text{ satisfies the } P^*\text{-condition.}$$

Hence, in the cases of location and scale parameters, R_h with $h(x)=x+d$ and $h(x)=cx$ respectively is minimax in the sense of (4.11) among the procedures satisfying the condition (4.10) and the P*-condition. Gupta and Studden (1966) have defined an invariance property of a selection procedure and showed that for an invariant procedure the condition (4.10) is satisfied and that (4.11) follows as a consequence.

Remark 4.5. For any procedure R_h satisfying (4.6), if $\sup_\lambda P_\lambda(CS|R_h) = 1$, then $\sup_\Omega E_{\underline{\lambda}}(S|R_h) = k$.

The expected size of the subset selected is a reasonable performance characteristic of a procedure and serves a criterion to compare two procedures both of which satisfy the P*-condition. Associated with the subset size is S', the number of non-best populations in the subset selected. Obviously S-S' takes values 0 and 1 with probabilities $1-p_k$ and p_k respectively. Hence for R_h,

(4.12) $$E_{\underline{\lambda}}(S'|R_h) = E_{\underline{\lambda}}(S|R_h) - p_k$$

where

$$(4.13) \qquad p_k = \int \prod_{r=1}^{k-1} F_{[r]}(h(x))\, dF_{[k]}(x).$$

From (4.13) we see that $\frac{\partial}{\partial\lambda_{[1]}} p_k \le 0$ and consequently p_k is non-increasing in $\lambda_{[1]}$, when other λ's are kept fixed. It is easily seen that the above fact together with Theorem 4.2 give

Lemma 4.1. $E(S'|R_h)$ is non-decreasing in $\lambda_{[1]}$ provided that (4.6) is satisfied.

On the lines similar to the proof of Theorem 4.2 we can show that $E_{\underline{\lambda}}(S'|R_h)$ is non-decreasing in λ, where $\lambda_{[1]}=\dots=\lambda_{[m]}=\lambda\le \lambda_{[m+1]}\le \dots\le \lambda_{[k]}$, $1\le m\le k-1$ provided that (4.6) is satisfied. Consequently we get

Theorem 4.3. $\sup_{\Omega} E_{\underline{\lambda}}(S'|R_h)$ takes place at a point where $\underline{\lambda}$ has all its components equal provided that (4.6) is satisfied.

Another property of $E_{\underline{\lambda}}(S'|R_h)$ which is true with no further assumption beyond the stochastic ordering of F_λ is stated below.

Theorem 4.4. $E_{\underline{\lambda}}(S'|R_h)$ is non-increasing in $\lambda_{[k]}$, when other λ's are kept fixed.

Proof. It is easy to see that $\frac{\partial}{\partial\lambda_{[k]}} p_i \le 0$ for $i=1,\dots,k-1$. Hence $E_{\underline{\lambda}}(S'|R_h) = p_1+\dots+p_{k-1}$ is non-increasing in $\lambda_{[k]}$.

The Case of $f_\lambda(x)$ Being a Convex Mixture. We are interested here in the case where $f_\lambda(x)$ is given by (3.21) and (3.22) where the a_j are governed by (3.32). Following our earlier notations used in Section 3, (4.6) is equivalent to

$$(4.14) \qquad r_{\lambda_2}(x)\left[A(\lambda_1)\frac{\partial}{\partial\lambda_1}R_{\lambda_1}(h(x))-A'(\lambda_1)R_{\lambda_1}(h(x))\right]$$

$$-h'(x)r_{\lambda_2}(h(x))\left[A(\lambda_1)\frac{\partial}{\partial\lambda_1}R_{\lambda_1}(x)-A'(\lambda_1)R_{\lambda_1}(x)\right] \ge 0.$$

We know that $A(\lambda_1) = a_0(1-\lambda_1 p)^{-q/p}$ and

$$(4.15) \qquad A(\lambda_1)\frac{\partial}{\partial\lambda_1}R_{\lambda_1}(x)-A'(\lambda_1)R_{\lambda_1}(x) = a_0(1-\lambda_1 p)^{-q/p}\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}a_{j+1}\Delta G_j(x).$$

Using this we see that (4.14) holds if, for $\lambda_1 \leq \lambda_2$,

$$(4.16) \qquad Q_{\lambda_1}(h(x))\, r_{\lambda_2}(x)-h'(x)\, Q_{\lambda_1}(x)\, r_{\lambda_2}(h(x)) \geq 0.$$

Setting $\lambda_2 = b\lambda_1$, $b \geq 1$, we can rewrite (4.16) in the equivalent form

$$(4.17) \qquad Q_\lambda(h(x))\, r_{b\lambda}(x)-h'(x)\, Q_\lambda(x)\, r_\lambda(h(x)) \geq 0.$$

We note that (4.17) is same as (3.36) except that in the place of $g_j(\cdot)$ we have $b^j g_j(\cdot)$. Hence, following the same line of argument as before, we can say that (4.17) holds if, for $b \geq 1$ and every integer $i \geq 0$,

$$(4.18) \qquad \sum_{\alpha=0}^{i}\binom{i}{\alpha}\left[a_{\alpha+1}a_{i-\alpha}b^{i-\alpha}g_{i-\alpha}(x)\Delta G_\alpha(h(x))-h'(x)a_{i-\alpha+1}a_\alpha b^\alpha g_\alpha(h(x))\Delta G_{i-\alpha}(x)\right] \geq 0.$$

Because of (3.32), (4.18) can be written as

$$(4.19) \qquad \sum_{\alpha=0}^{i}\binom{i}{\alpha}a_\alpha a_{i-\alpha}\left[b^{i-\alpha}(q+p\alpha)g_{i-\alpha}(x)\Delta G_\alpha(h(x))-h'(x)b^\alpha(q+(i-\alpha)p)g_\alpha(h(x))\Delta G_{i-\alpha}(x)\right] \geq 0.$$

Since (4.19) implies (4.6), we can state the following

<u>Theorem 4.5</u>. For the procedure R_h, when $f_\lambda(x)$ is given by (3.21) and (3.22) where the a_j are governed by (3.32), $\sup_\Omega E_{\underline{\lambda}}(S|R_h)$ takes place for $\lambda_1=\dots=\lambda_k$ provided that, for $b \geq 1$ and every integer $i \geq 0$, (4.19) is satisfied.

Now we let

(4.20) $T_\alpha = b^{i-\alpha}(q+p\alpha)g_{i-\alpha}(x)\Delta G_\alpha(h(x)) - h'(x)b^\alpha(q+(i-\alpha)p)g_\alpha(h(x))\Delta G_{i-\alpha}(x)$.

Then the left side of (4.19)

$$= \sum_{\alpha=0}^{i} \binom{i}{\alpha} a_\alpha a_{i-\alpha} T_\alpha$$

$$= \begin{cases} \displaystyle\sum_{\alpha=0}^{\frac{i-1}{2}} \binom{i}{\alpha} a_\alpha a_{i-\alpha}(T_\alpha + T_{i-\alpha}), & \text{if } i \text{ is odd} \\ \displaystyle\sum_{\alpha=0}^{\frac{i}{2}-1} \binom{i}{\alpha} a_\alpha a_{i-\alpha}(T_\alpha + T_{i-\alpha}) + \binom{i}{i/2} a_{i/2}^2 T_{i/2}, & \text{if } i \text{ is even.} \end{cases}$$

Hence (4.19) holds if $T_\alpha + T_{i-\alpha} \geq 0$ for $\alpha = 0,1,\ldots,[\frac{i}{2}]$, where $[\frac{i}{2}]$ stands for the largest integer $\leq \frac{i}{2}$. To put it explicitly, (4.19) holds if, for $b \geq 1$ and every integer $i \geq 0$,

(4.21) $$b^{i-\alpha}(q+p\alpha)[g_{i-\alpha}(x)\Delta G_\alpha(h(x)) - h'(x)g_{i-\alpha}(h(x))\Delta G_\alpha(h(x))]$$

$$+ b^\alpha(q+p(i-\alpha))[g_\alpha(x)\Delta G_{i-\alpha}(h(x)) - h'(x)g_\alpha(h(x))\Delta G_{i-\alpha}(x)] \geq 0,$$

$$\alpha = 0,1,\ldots,[\tfrac{i}{2}].$$

Remark 4.6. As we have seen (4.21) is a stronger condition than (4.19), both of them implying (4.6). But we have several cases where (4.21) is satisfied. Gupta and Studden (1965) and Gupta and Panchapakesan (1968a) have discussed selection procedures involving $f_\lambda(x)$ as in Theorem 4.5. In all these cases, the condition (4.21) is verified (not shown here) to be satisfied. Hence in all those cases Theorem 4.5 applies.

5. Selection of the Population Associated with $\lambda_{[1]}$.

The case where the best population is defined to be the one associated with $\lambda_{[1]}$ is analogous to the case of $\lambda_{[k]}$. We need of course make certain modifications. We will briefly mention them and state the results without proofs unless there be a need to the contrary.

Let $H \equiv H_{c,d}$; $c \in [1,\infty)$, $d \in [0,\infty)$ be a function defined on the real line satisfying the following conditions. For every real x

$$(5.1)\qquad \begin{cases} \text{(i)} & H_{c,d}(x) \leq x \\ \text{(ii)} & H_{1,0}(x) = x \\ \text{(iii)} & H_{c,d}(x) \downarrow \text{ is continuous in } c \text{ and } d \\ \text{(iv)} & H_{c,d}(x) \downarrow -\infty \text{ as } d \to \infty \text{ and/or} \\ & xH_{c,d}(x) \downarrow 0 \text{ as } c \to \infty. \end{cases}$$

Of particular interest are the functions $\frac{x}{c}$, $x-d$ and $\frac{x}{c}-d$.

A class of procedures R_H for selecting a subset containing the best is defined as follows.

R_H: Include π_i in the selected subset iff

$$(5.2)\qquad H(x_i) \leq \min_{1 \leq r \leq k} x_r.$$

The procedure R_H obviously selects a non-empty subset because of (5.1)-(i). The probability of a correct selection is given by

$$(5.3)\qquad P(CS|R_H) = \int_{-\infty}^{\infty} \prod_{r=2}^{k} \bar{F}_{[r]}(h(x))\, dF_{[1]}(x),$$

where $\bar{F}_\lambda(x) = 1-F_\lambda(x)$. Because of the assumption (2.4) about the stochastic ordering of the distributions,

$$(5.4)\qquad P(CS|R_H) \geq \int_{-\infty}^{\infty} \bar{F}_{[1]}^{k-1}(H(x))\, dF_{[1]}(x).$$

Hence

$$(5.5)\qquad \inf_{\Omega} P(CS|R_H) = \inf_{\lambda} \int_{-\infty}^{\infty} \bar{F}_{\lambda}^{k-1}(H(x))dF_{\lambda}(x) = \inf_{\lambda} \varphi(\lambda;c,d,k), \quad \text{say.}$$

Because of (5.1)-(i) and (ii),

$$(5.6)\qquad \varphi(\lambda;c,d,k) \geq \frac{1}{k}$$

and

$$(5.7)\qquad \varphi(\lambda;1,0,k) = \frac{1}{k}\,.$$

Properties (5.1)-(iii) and (iv) yield

$$(5.8)\qquad \lim_{d\to\infty} \varphi(\lambda;c,d,k) = 1$$

and/or

$$(5.9)\qquad \lim_{c\to\infty} \varphi(\lambda;c,d,k) = (1-F_{\lambda}(0))^{k-1}.$$

If (5.8) holds, then for every λ, c and k, we choose d such that the P*-condition is satisfied. If (5.9) holds but not (5.8) then for every λ, d and k, we can choose c subject to the P*-condition whatever P* is chosen between $\frac{1}{k}$ and 1 provided that $F_{\lambda}(0) = 0$.

Corresponding to Theorem 3.2, we get

Theorem 5.1. For the procedure R_H, $\varphi(\lambda;c,d,k)$ is non-decreasing in λ provided that

$$(5.10)\qquad H'(x)f_{\lambda}(h(x)) \frac{\partial}{\partial\lambda} F_{\lambda}(x) - f_{\lambda}(x) \frac{\partial}{\partial\lambda} F_{\lambda}(H(x)) \geq 0$$

where $H'(x) \equiv \frac{d}{dx} H(x)$, and $\varphi(\lambda;c,d,k)$ is strictly increasing in λ if strict inequality holds in (5.10) on a set of positive measure.

Proof. The proof is immediate by using Corollary 3.2 with $\Psi(x,\lambda) = \bar{F}_\lambda(H(x))$.

Now we can state the following results analogous to the case of $\lambda_{[k]}$.

Theorem 5.2. For the procedure R_H, when $f_\lambda(x)$ is given by (3.21) and (3.22) where the a_j are governed by (3.32), $\varphi(\lambda;c,d,k)$ is non-decreasing in λ provided that, for every integer $i \geq 0$,

$$(5.11) \quad \sum_{\alpha=0}^{i} \binom{i}{\alpha} a_\alpha a_{i-\alpha} [H'(x)(q+(i-\alpha)p)g_\alpha(H(x))\Delta G_{i-\alpha}(x)-(q+\alpha p)g_{i-\alpha}(x)\Delta G_\alpha(H(x))] \geq 0$$

and strict inequality in (5.11) for some i implies that $\varphi(\lambda;c,d,k)$ is strictly increasing in λ.

Remark 5.1. Suppose we use the procedure R_H with $H(x) = \frac{x}{c}$ (in the case of nonnegative random variables) or $H(x)=x-d$. Then (5.10) reduces respectively to

$$(5.12) \quad \frac{1}{c} f_\lambda(\tfrac{x}{c}) \frac{\partial}{\partial\lambda} F_\lambda(x)-f_\lambda(x) \frac{\partial}{\partial\lambda} F_\lambda(\tfrac{x}{c}) \geq 0$$

or

$$(5.13) \quad f_\lambda(x-d) \frac{\partial}{\partial\lambda} F_\lambda(x)-f_\lambda(x) \frac{\partial}{\partial\lambda} F_\lambda(x-d) \geq 0.$$

Setting $\frac{x}{c} = y$ or $x-d=y$ as the case may be, we get

$$(5.14) \quad f_\lambda(y) \frac{\partial}{\partial\lambda} F_\lambda(cy) - cf_\lambda(cy) \frac{\partial}{\partial\lambda} F_\lambda(y) \geq 0$$

or

$$(5.15) \quad f_\lambda(y) \frac{\partial}{\partial\lambda} F_\lambda(y+d)-f_\lambda(y+d) \frac{\partial}{\partial\lambda} F_\lambda(y) \geq 0.$$

We note that (5.14) and (5.15) are sufficient conditions for $\Psi(\lambda;c,d,k)$ to be non-decreasing in λ in the case of the procedure R_h with $h(x)=cx$ and $h(x)=x+d$ respectively.

Remark 5.2. We also note that $\varphi(\lambda;c,d,k)$ is independent of λ if the left side of (5.10) vanishes and this happens as we know in the cases of location and scale parameters with $H(x)=x-d$ and $H(x)=\frac{x}{c}$ respectively.

Also, using the same method of proof as in the case of R_h, we obtain the following results concerning $E_{\underline{\lambda}}(S|R_H)$ and $E_{\underline{\lambda}}(S'|R_H)$.

Theorem 5.3. $E_{\underline{\lambda}}(S|R_H)$ is non-decreasing in $\lambda_{[1]}$ when other λ's are kept fixed provided that, for $\lambda_1 \leq \lambda_2$,

$$(5.16) \qquad H'(x)\, f_{\lambda_2}(H(x)) \frac{\partial}{\partial\lambda_1} F_{\lambda_1}(x) - f_{\lambda_2}(x) \frac{\partial}{\partial\lambda_1} F_{\lambda_1}(H(x)) \geq 0.$$

Theorem 5.4. $E_{\underline{\lambda}}(S|R_H)$ attains its supremum when $\lambda_1=\dots=\lambda_k$ if (5.16) holds.

Remark 5.3. In the cases of location and scale parameters with $H(x)=x-d$ and $H(x)=\frac{x}{c}$ respectively, (5.16) is the condition that $f_\lambda(x)$ has a monotone likelihood ratio .

Also, a remark similar to Remark 5.1 can be made about (5.16). For the procedure R_H,

$$(5.17) \qquad E_{\underline{\lambda}}(S'|R_H) = p_2+\dots+p_k = E_{\underline{\lambda}}(S|R_H)-p_1$$

From the proof of Theorems 5.3 and 5.4, it is easy to see that $E_{\underline{\lambda}}(S'|R_H)$ is non-decreasing in $\lambda_{[2]}$ and non-decreasing in λ where $\lambda_{[1]}\leq \lambda_{[2]}=\dots\lambda_{[m]}=\lambda\leq \lambda_{[m+1]}\leq \dots \leq \lambda_{[k]}$ for $2\leq m\leq k$ provided that (5.16) holds. Hence, if (5.16) holds, we have

$$(5.18) \qquad \sup_{\Omega} E_{\underline{\lambda}}(S') = \sup_{\Omega'} E_{\underline{\lambda}}(S')$$

where $\Omega'=\{\underline{\lambda}|\lambda_{[1]}\leq \lambda_{[2]}=\dots=\lambda_{[k]}\}$. Because of the stochastic ordering of F_λ, p_i, $i=2,\dots,k$ and hence $E_{\underline{\lambda}}(S'|R_H)$ is non-decreasing in $\lambda_{[1]}$ when other λ's are kept fixed. Hence we have

Theorem 5.5. $\sup_{\Omega} E_{\underline{\lambda}}(S'|R_H)$ takes place for $\lambda_1=\dots=\lambda_k$ if (5.16) holds.

6. Selection Procedures for Restricted Families of Distributions

Selection procedures for restricted families of distributions were first considered by Barlow and Gupta (1969). These selection procedures are distribution-free in the sense that we do not assume any knowledge about the form of the distribution functions F_{λ_i}, $i = 1,\ldots,k$. However, we do assume that these distributions are partially ordered in some sense with respect to a known distribution G. Though we assumed that the distributions F_{λ_i}, $i = 1,\ldots,k$ are all stochastically ordered in our earlier discussions, it is sufficient to assume that there exists one population which is stochastically larger than any other and is the best. Then for selecting a subset containing that population the infimum of $P(CS|R_h)$ takes place when $\lambda_1 = \cdots = \lambda_k$. When we do not know the form of the distribution F_λ, we need further assumptions in order to evaluate the infimum of the probability of a correct selection. The existence of a partial ordering of F_λ with respect to a known G makes it possible to obtain the infimum which can be evaluated with the knowledge of G.

For the sake of a self-contained discussion, we start with certain known definitions.

Definition 6.1. A relation $\precsim$ on the space of probability distributions is said to be a partial ordering if (i) $F \precsim F$ for all distributions F and (ii) $F \precsim G$ and $G \precsim H$ together imply that $F \precsim H$. We note that $F \precsim G$ and $G \precsim F$ do not necessarily imply $F \equiv G$.

In what follows F and G denote continuous distributions and I denote the support of F. Let $\varphi \equiv G^{-1}F$.

Definition 6.2. F is star-shaped with respect to G (written $F \underset{*}{<} G$) iff $F(0) = G(0) = 0$ and $\varphi(\alpha x) \leq \alpha\varphi(x)$ whenever $x \in I$, $\alpha x \in I$ and $0 \leq \alpha \leq 1$.

Definition 6.3. F is said to be r-ordered with respect to G (written $F \underset{r}{<} G$) iff $F(0) = G(0) = \beta (0 < \beta < 1)$, and for $0 \leq \alpha \leq 1$, $\varphi(\alpha x) \leq \alpha\varphi(x)$ for $x \in I \cap (0,\infty)$ and $\varphi(\alpha x) \geq \alpha\varphi(x) \geq \alpha\varphi(x)$ for $x \in I \cap (-\infty,0)$.

In the above definition we do not mean any specific β but 'some' $\beta \in (0,1)$. Lawrence (1966) defines r-ordering with $\beta = \frac{1}{2}$, but it is not crucial for our discussions. Barlow and Gupta (1969) consider selection procedures with respect to the medians of distributions which when centered at their medians are r-ordered with respect to a known distribution G assuming that $\varphi(x)$ has a slope not less than unity at the origin. But from their proof it can be seen that their result can still be obtained if we assume only that $G^{-1}F(x + \Delta) - x$ is non-decreasing in $x \in I$ where Δ is the median of F. In view of this we give

Definition 6.4. F is said to be tail-ordered with respect to G (written $F \underset{t}{<} G$) iff $\varphi(x) - x$ is non-decreasing in $x \in I$. This definition has been mentioned by Doksum (1969) in a different context.

The following lemma shows that the selection procedure of Barlow and Gupta (1969) mentioned above applies to a wider family of distributions.

Lemma 6.1. If $F \underset{r}{<} G$ and $\varphi'(0) \geq 1$, then $F \underset{t}{<} G$.

Proof. It is easy to see from the definition of r-ordering that $\frac{\varphi(x)}{x}$ is non-decreasing (non-increasing) in $x > 0$ $(x < 0)$, $x \in I$. Hence $x\,\varphi'(x) \geq (\leq)\ \varphi(x)$ for $x > 0$ $(x < 0)$, $x \in I$. Now suppose for any $x_0 \in I$ $x_0 \in I \cap (0,\infty)$ we have $\varphi'(x_0) < \varphi'(0)$. Then $\varphi(x_0) \leq x_0\varphi'(x_0) < x_0\varphi'(0) = x_0 \lim_{x \downarrow 0} \frac{\varphi(x)}{x}$. This implies that there exists an $x_1 < x_0$ such that

(6.1) $$\varphi(x_0) < \frac{x_0}{x_1} \varphi(x_1) \quad .$$

Letting $\alpha = \frac{x_1}{x_0} < 1$, (6.1) becomes $\alpha\varphi(x_0) < \varphi(\alpha x_0)$, which is a contradiction. Hence $\varphi'(x) \geq \varphi'(0)$ for $x \in I \cap (0,\infty)$. A similar argument gives the result for $x \in I \cap (-\infty,0)$. Since $\varphi'(0) \geq 1$, we have $\varphi'(x) \geq 1$ for all $x \in I$, which means $F \underset{t}{<} G$.

Remark 6.1. That the converse of Lemma 6.1 is not always true can be seen by letting

(6.2) $$\varphi(x) = \begin{cases} \frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} + x \ , & x \geq 0 \\ -\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^2}{2} + x \ , & x < 0 \end{cases} \quad .$$

It can be verified that $\varphi'(x) \geq 1$ for all x, which shows that $F \underset{t}{<} G$. Setting $x = 1$ and $\alpha = \frac{1}{2}$, we can show that $\alpha\varphi(x) < \varphi(\alpha x)$, violating the condition in the definition of r-ordering.

Now we define a more general ordering.

Definition 6.5. Let $h \equiv h_{a,b}$; $a \geq 1$, $b \geq 0$, be a real-valued function defined on the real line. F is said to be h-ordered with respect to G (written $F \underset{h}{<} G$) iff $\varphi(h(x)) \geq h(\varphi(x))$ whenever $x \in I$, $h(x) \in I$, $a \geq 1$ and $b \geq 0$.

Corollary 6.1. Let $h(x) = ax$, $a \geq 1$ and $F(0) = G(0) = 0$. Then h-ordering becomes star ordering.

Corollary 6.2. Let $h(x) = x + b$, $b \geq 0$. Then h-ordering becomes tail ordering.

The proofs of the above corollaries are omitted.

Remark 6.2. The h-ordering defined above is a partial ordering. All we need verify is that, if $G^{-1}F(h(x)) \geq h(G^{-1}F(x))$ and $H^{-1}G(h(x)) \geq h(H^{-1}G(x))$, then $H^{-1}F(h(x)) \geq h(H^{-1}F(x))$. Now,

$$\begin{aligned} H^{-1}F(h(x)) &= H^{-1}G\, G^{-1}F(h(x)) \\ &\geq H^{-1}G\, h(G^{-1}F(x)), \quad \text{since } H^{-1}G \text{ is an } \uparrow \text{ function} \\ &\geq h(H^{-1}G\, G^{-1}F(x)) \\ &= h(H^{-1}F(x)) \quad . \end{aligned}$$

Remark 6.3. If $G^{-1}F(h(x)) \leq h(G^{-1}F(x))$, then $G \underset{h}{<} F$.

Lemma 6.2. If $F \underset{h}{<} G$, then for any positive integer t

$$(6.3) \qquad \int F^t(h(x))\, dF(x) \geq \int G^t(h(x))\, dG(x)$$

where the integrals are over the range of x.

Proof. Let $X_1, \ldots, X_{t+1}$ be independent and identically distributed random variables with cdf $F(x)$. Let

$$(6.4) \qquad Y_i = \varphi(X_i), \qquad i = 1,\ldots,t+1 \quad ,$$

where $\varphi \equiv G^{-1}F$. Then Y_i, $i = 1,\ldots,t+1$ are independent and identically distributed with cdf $G(x)$ and (6.3) is same as

$$(6.5) \qquad P(h(X_{t+1}) > \max_{1\leq r\leq t+1} X_r) \geq P(h(Y_{t+1}) \geq \max_{1\leq r\leq t+1} Y_r) \quad .$$

To prove this, we first let $\Psi = F^{-1}G$ and note that $\Psi(Y_i) = X_i$, $i = 1,\ldots, t+1$ and $h(\Psi(x)) \geq \Psi(h(x))$. Now suppose

$$(6.6) \qquad h(Y_{t+1}) \geq \max_{1 \leq r \leq t+1} Y_r \quad .$$

Since Ψ is an increasing function,

$$(6.7) \qquad \Psi(h(Y_{t+1})) \geq \Psi(\max_{1 \leq r \leq t+1} Y_r) = \max_{1 \leq r \leq t+1} \Psi(Y_r) \quad .$$

Hence

$$(6.8) \qquad h(\Psi(Y_{t+1})) \geq \max_{1 \leq r \leq t+1} \Psi(Y_r)$$

which is same as

$$(6.9) \qquad h(X_{t+1}) \geq \max_{1 \leq r \leq t+1} X_r \quad .$$

Hence (6.5) follows.

Remark 6.4. Gupta (1966) has a lemma concerning his procedure R_{h_b}, where it can be seen that the conditions under which he obtains the inequality (6.3) amount to having h-ordering with $h = h_b$.

Now we discuss a general selection problem. Let $\pi_1, \ldots, \pi_k$ be k populations. The random variable X_i associated with π_i has a continuous distribution F_i, i = 1,..., k+1. We assume that there exists one among the k populations which is stochastically larger than any other. Let us denote the distribution of that population by $F_{[k]}$. Then the assumption made above can be expressed as

$$(6.10) \qquad F_i(x) \geq F_{[k]}(x) \quad \text{for} \quad i = 1,\ldots,k \quad \text{and all} \quad x.$$

We also assume that there exists a continuous distribution G such that

(6.11) $$F_i \underset{h}{<} G \quad \text{for } i = 1,\ldots,k \;,$$

where $h = h_{c,d}$ defined in (2.1). Let $\underline{X}_i = (X_{i1},\ldots, X_{in})$ be the observed sample from π_i and $T_i = T(\underline{X}_i)$ be a statistic that preserves both the ordering relations (6.10) and (6.11), i.e.,

(6.12) $$P_{F_i}(T(\underline{X}) \leq x) \geq P_{F_{[k]}}(T(\underline{X} \leq x)) \quad \text{for } i = 1,\ldots,k \text{ and all } x$$

and

(6.13) $$F_{T(\underline{X}_i)} \underset{h}{<} G_{T(\underline{Y})} \;, \qquad i = 1,\ldots,k$$

where $F_{T(\underline{X}_i)}$ represents the cdf of $T(\underline{X}_i)$ under F_i and $G_{T(\underline{Y})}$ is the distribution of $T(\underline{Y})$ under G, $\underline{Y} = (Y_1,\ldots,Y_n)$ being a random sample from G.

Now, for selecting a subset containing the population associated with $F_{[k]}$, we propose the rule

R: Include π_i in the selected subset iff

(6.14) $$h(T_i) \geq \max_{1\leq r\leq k} T_r \;.$$

Denoting by $T_{(k)}$ the T_i associated with $F_{[k]}$ and by $T_{(r)}$, $r = 1,\ldots, k-1$, the other T_i's, we have

(6.15) $$P(CS|R) = P(h(T_{(k)}) \geq \max_{1\leq r\leq k-1} T_{(r)})$$

$$= \int_{-\infty}^{\infty} \prod_{r=1}^{k-1} F_{T_{(r)}}(h(x))\, dF_{T_{(k)}}(x)$$

$$\geq \int_{-\infty}^{\infty} F^{k-1}_{T_{(k)}}(h(x))\, dF_{T_{(k)}}(x), \quad \text{by (6.12).}$$

Since $F_{T_{[k]}} \underset{h}{<} G_{T(\underline{Y})}$, using Lemma 6.2 we obtain

$$(6.16) \qquad P(CS|R) \geq \int_{-\infty}^{\infty} G_T^{k-1}(h(x))\, dG_T(x) \quad ,$$

where $G_T \equiv G_{T(\underline{Y})}$. The constants of the procedure are determined to satisfy

$$(6.17) \qquad \int_{-\infty}^{\infty} G_T^{k-1}(h(x))\, dG_T(x) = P^* \quad .$$

We now state a few facts by way of remarks.

Remark 6.5. If $F \underset{h}{<} G$, then $F_j \underset{h}{<} G_j$, where F_j and G_j are the distributions of the jth order statistic in a sample of size n from F and G respectively. To see this, we note that $F_j(x) = B_{j,n}(F(x)) \equiv B_{j,n}F(x)$ where

$$(6.18) \qquad B_{j,n}(x) = j\binom{n}{j} \int_0^x u^{j-1}(1-u)^{n-j}\, du \quad .$$

Hence $G_j^{-1}F_j(x) = [B_{j,n}G]^{-1}B_{j,n}F(x) = G^{-1}F(x)$, which gives the desired result.

Remark 6.6. If we take $h(x) = cx$, $c \geq 1$, then $F_i \underset{*}{<} G$ and the constant c of the procedure is obtained from

$$(6.19) \qquad \int_0^{\infty} G_T^{k-1}(cx)\, dG_T(x) = P^* \quad .$$

Remark 6.7. If $h(x) = x+d$, $d \geq 0$, then $F_i \underset{t}{<} G$ and the constant d is chosen to satisfy

$$(6.20) \qquad \int_{-\infty}^{\infty} G_T^{k-1}(x+d)\, dG_T(x) = P^* \quad .$$

The procedures of Barlow and Gupta (1969) in terms of the quantiles of distributions star-shaped with respect to G and in terms of the medians of distributions which are contained in the family of distributions tail-ordered with respect to G are special cases of our general problem in view of Remarks 6.5 through 6.7.

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$\pi_1, \pi_2, \ldots, \pi_k$ are k continuous populations. Associated with π_i is the r.v. X_i with cdf $F_{\lambda_i}(x)$, $\lambda_i \epsilon \Lambda$, an interval on the real line, $i=1,2,\ldots,k$. A class of procedures R_h is defined in Section 2 for selecting a non-empty subset containing the population associated with $\lambda_{[k]}$, the largest λ, subject to the usual probability requirement. Section 3 contains a theorem generalizing a result of Lehmann. This theorem provides a sufficient condition for the probability of a correct selection when $\lambda_1=\ldots=\lambda_k=\lambda$ to be increasing (non-decreasing) in λ. Section 4 investigates the properties R_h and a sufficient condition is obtained for the supremum of the expected size of the selected subset to take place when the λ's are equal. Section 5 briefly discusses the procedure R_H for selecting the population associated with $\lambda_{[1]}$, the smallest λ. A general partial ordering (h-ordering) is considered on the space of probability distributions. This and a selection problem for distributions h-ordered w.r.t. a specified distribution G form the contents of Section 6.

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