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Some Contributions to Multiple Decision (Selection and Ranking) Procedures*

by

S. Panchapakesan

Department of Statistics

Division of Mathematical Sciences

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INTRODUCTION

A common situation in practice that confronts an experimenter is the necessity of making decisions regarding k given populations (categories, varieties, processes, candidates etc.). Suppose $\theta_1,\theta_2,\ldots,\theta_k$ are the characteristics of the populations in which the experimenter is interested. These may be, for example, the means of the populations. The classical tests of homogeneity, i.e. the test for the hypothesis of equality of the parameters never answered the question of what next if the null hypothesis was rejected. Attempts were made to overcome the inadequacy of the tests of homogeneity by formulating the problem in a more meaningful and realistic way. A partial answer was provided by Mosteller [60] who tested homogeneity against slippage alternatives. Contributions to the theory of slippage tests have been made by many authors, notably Doornbos and Prius [21,22,23], Kartin and Truax [49], Kudô [54], Paulson [64], and Pfanzagl [68], to mention a few. A fuller answer came in the form of selection and ranking procedures, otherwise known as multiple decision procedures. Bahadur [3] and Paulson [49] are among the earliest authors to make contribution in this area. Since then many authors have contributed to various aspects and modifications of the basic problem. References could be made to Bechhofer [9], Bechhofer, Kiefer and Sobel [14], Gupta [32], Gupta and Panchapakesan [37] and Lehmann [56].

Generally, problems of selection and ranking have been formulated in two ways. Suppose $\pi_1,\ldots,\pi_k$ are k populations with distributions

Patterson [61], and Rizvi and Sobel [71]. Trawinski and David [79] discuss the problem of selecting the best treatment by paired-comparisons. Selection procedures for restricted families of probability distributions where the distributions are partially ordered in some sense with respect to a known distribution G have been studied by Barlow and Gupta [4], and Barlow, Gupta, and Panchapakesan [5]. A decision theoretic approach to subset selection has been made by Deely and Gupta [19], and Studden [78].

A usual modification in the selection problems is to select the populations better than a control or standard population. Contributions to this aspect have been made by Dunnett [24], Gupta and Sobel [40], Krishnaiah [51], Krishnaiah and Rizvi [53], and Paulson [63,65].

Multiple decision procedures have also been examined from a Bayesian point of view. Deely and Gupta [19], Dunnett [25], and Guttman and Tias [46] have studied the problem by assuming a prior distribution on the parameter space. A more meaningful situation in practice is the one where only the existence of the a priori distribution is known, but not the specific form of that distribution. Deely [18] studies the selection procedures in this case using the empirical Bayes technique of Robbins [72].

Recently, Sobel [76] has made an attempt in combining the fixed subset size approach with indifference zone and the (random) subset selection approach. His goal is to select from k populations of which t are considered best a subset of size $s(s \leq k-t)$ so as to include any one of the t best populations with a minimum probability P^* under an indifference zone set-up.

The present thesis relates to the subset selection approach of Gupta. Suppose $\pi_1,\ldots,\pi_k$ are k populations with absolutely continuous distributions F_{λ_i}, $\lambda_i \epsilon \Lambda$, an interval on the real line. The λ_i are unknown and $\lambda_{[1]} \leq \cdots \leq \lambda_{[k]}$ are the ordered λ's. Chapter I defines a class of selection procedures R_h for selecting the population associated with $\lambda_{[k]}$. This class of procedures R_h is in a way a natural generalization of the class of procedures considered by Gupta [34]. Under the assumption of stochastic ordering of the populations, the infimum of the probability of a correct selection (PCS) over the parameter space $\Omega = \{\underline{\lambda}:\underline{\lambda}' = (\lambda_1,\ldots,\lambda_k), \lambda_i \epsilon \Lambda\}$ is attained when $\lambda_1 = \cdots = \lambda_k = \lambda$, say. A result of Lehmann has been generalized (Theorem 1.4.2) and this is used to obtain a sufficient condition for the probability of a correct selection when the parameters are equal to λ to be non-decreasing (non-increasing) in λ. This result provides the infimum of the PCS over Ω. The properties of procedure R_h are also studied. A sufficient condition is obtained in order to guarantee that the supremum of the expected size of the subset selected and the supremum of the expected number of non-best populations selected are attained when the parameters are equal. It turns out that this sufficient condition includes the condition which guarantees the monotonicity of the PCS in λ. For the problem of selecting the population associated with $\lambda_{[1]}$, a class of procedures R_H is defined and the properties of R_H are briefly discussed. More specific results are obtained in the case where λ is a location or scale parameter and in the case where $f_\lambda(x) = \sum_{j=0}^{\infty} w(\lambda,j)\, g_j(x)$, where $w(\lambda,j)$ are non-negative weights adding up to unity and $g_j(x)$, $j=0,1,\ldots,$ is a sequence of density functions.

Chapter II considers the selection of multivariate normal populations in terms of multiple correlation coefficient. The so-called 'conditional' and 'unconditional' cases are both considered using procedures based on sample multiple correlation coefficients. Some asymptotic results are obtained. The investigations in these cases illustrate the applications of the general results in Chapter I. Tables of constants are given for many of the procedures for selected values of known constants and probability levels. This chapter also includes selection of p-variate normal populations in terms of $|\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}|$ where $\Sigma_{11}, \Sigma_{12}, \Sigma_{21}$ and Σ_{22} are partitions of the covariance of matrix corresponding to a partition of the p variables into two sets of q_1 and q_2 variables, $q_1 + q_2 = p$. The criterion of ranking then represents the conditional generalized variance of the q_2 set when q_1 set is held fixed.

Chapter III deals with selection procedures for restricted families of distributions. In these problems we assume that there exists a population among the k given populations which is stochastically larger than any other. But we do not know the form of the distributions. We assume that these distributions are partially ordered in some sense with respect to a specified distribtion G. A general partial ordering called h-ordering is defined on the space of distributions. It is shown that star-ordering and tail-ordering are particular cases of h-ordering. A general selection problem is considered with h-ordering. Some implications of tail-ordering for certain chocies of G are studied. The selection problem in terms of the medians is considered for families of distributions tail-ordered with respect to G. Formulae are obtained for computing the constant defining the procedure when G is logistic.

Chapter IV embodies a brief discussion on the problem of selecting a subset containing at least one of the $t (t < k)$ best populations. A change in the usual probability requirement is also considered. Some other possible procedures are indicated.

CHAPTER I

A CLASS OF SELECTION RULES AND ITS PROPERTIES

1.1. Introduction and Summary

In this chapter we define a class of selection procedures which is a natural generalization of a class of procedures defined by Gupta [34]. Let $\pi_1, \pi_2, \ldots, \pi_k$ be k continuous populations. Let Λ be an interval on the real line. Associated with π_i is the real valued random variable X_i with an absolutely continuous distributions $F_i \equiv F_{\lambda_i}$, $\lambda_i \epsilon \Lambda$ and density function $f_i \equiv f_{\lambda_i}$. It is assumed that the functional forms of F_{λ_i} are known, but not the values of λ_i. Let $\lambda_{[1]} \leq \lambda_{[2]} \leq \cdots \leq \lambda_{[k]}$ be the ordered λ's. The correct pairing of the ordered and the unordered λ's is unknown. It is also assumed that F_λ is differentiable in λ and that $\{F_\lambda\}$ $\lambda \epsilon \Lambda$ is a stochastically increasing family of distributions which means that for $\lambda < \lambda'$, F_λ and $F_{\lambda'}$ are distinct and

$$(1.1.1) \qquad F_\lambda(x) \geq F_{\lambda'}(x) \quad \text{for all } x.$$

Let $x_1, x_2, \ldots, x_k$ be observations on $X_1, X_2, \ldots, X_k$, respectively. Based on these observations, we are interested in selecting a non-empty subset of the k populations such that the probability is at least P^*

that the best population, i.e., the population associated with $\lambda_{[k]}(\lambda_{[1]})$ is included in the selected subset. For the problem to be meaningful we should have $\frac{1}{k} < P^* < 1$. If there are more than one populations with $\lambda_i = \lambda_{[k]}(\lambda_i = \lambda_{[1]})$, then one of them will be assumed to have been tagged as the best population. If we let CS stand for a correct selection, i.e. the selection of a subset which includes the best population and $P(CS|R)$ denote the probability of a correct selection using the procedure R, then the probability requirement stated above can be written as

$$\inf_{\Omega} P(CS|R) \geq \underline{P}^* \quad , \tag{1.1.2}$$

where Ω is the space of all k-tuples $(F_1, F_2, \ldots, F_k)$. This requirement (1.1.2) will be referred to hereafter as the basic probability requirement or $\underline{P}^*$ - condition.

In the next section a procedure R_h is defined using a function $h \equiv h_{c,d}$ defined on the real line for the selection of the population associated with $\lambda_{[k]}$ and the probability of a correct selection is obtained. The procedure R_h is a natural generalization of a class of procedures considered by Gupta [34]. Section 1.3 discusses the infimum of $P(CS|R_h)$ and a relevant lemma. A theorem which is more general than a result of Lehmann [55] and its application for obtaining a sufficient condition for the monotonicity of a probability integral leading to the evaluation of $\inf_{\Omega} P(CS|R_h)$ form the contents of Section 1.4. The succeeding section spells out more specific results concerning the infimum of $P(CS|R_h)$ when $f_\lambda(x)$ is a convex mixture of a sequence of density functions. It is shown that certain results of Gupta and Panchapakesan [38], and Gupta and Studden [45] follow as particular cases. The properties of the procedure R_h are investigated in Section 1.6. A procedure R_H using a function $H \equiv H_{c,d}$ defined on the real line is defined in the following section for the selection of the population associated with $\lambda_{[1]}$. This section briefly discusses the infimum of $P(CS|R_H)$ and the properties of R_H. The chapter ends with a short section which reviews the essential results concerning the procedures R_h and R_H.

1.2. Definition of the Procedure R_h and the Expression for the Probability of a Correct Selection

Let $h \equiv h_{c,d}$, $c\epsilon[1,\infty)$, $d\epsilon[0,\infty)$ be a class of functions defined on the real line such that for every x belonging to the support of F_λ,

$$(1.2.1)\quad \begin{cases} \text{(i)} & h_{c,d}(x) \geq x \\ \text{(ii)} & h_{1,0}(x) = x \\ \text{(iii)} & h_{c,d}(x) \text{ is continuous in } c \text{ and } d \\ \text{(iv)} & h_{c,d}(x) \uparrow \infty \text{ as } d \to \infty \text{ and/or} \\ & h_{c,d}(x) \uparrow \infty \text{ as } c \to \infty,\ x \neq 0 . \end{cases}$$

The procedure R_h is defined as follows.

R_h: Include π_i in the selected subset iff

$$(1.2.2) \qquad h(x_i) \geq \max_{1 \leq r \leq k} x_r .$$

This procedure will obviously select a non-empty subset in view of (1.2.1). (i). Letting $X_{(r)}$ denote that random variable among $X_1, X_2, \ldots, X_k$ which is associated with $\lambda_{[r]}$ and $F_{[r]}(x) \equiv F_{\lambda_{[r]}}(x)$ denote the cdf, we have

$$(1.2.3) \qquad P(CS|R_h) = P(h(X_{(k)}) \geq X_{(r)}, \quad r=1,\ldots,k-1)$$

$$= \int \left\{ \prod_{r=1}^{k-1} F_{[r]}(h(x)) \right\} f_{[k]}(x)\, dx,$$

where $f_{[r]}(r=1,\ldots,k)$ denotes the density corresponding to $F_{[r]}(x)$ and the integral is taken over the (common) support of the distributions. Because of the assumption (1.1.1) regarding the stochastic ordering of the distributions,

$$(1.2.4) \qquad P(CS|R_h) \geq \int F_{[k]}^{k-1}(h(x))\, f_{[k]}(x)\, dx.$$

Hence

$$(1.2.5) \qquad \inf_{\Omega} P(CS|R_h) = \inf_{\lambda \in \Lambda} \int F_{\lambda}^{k-1}(h(x))\, f_{\lambda}(x)\, dx,$$

where $\Omega = (\underline{\lambda} | \underline{\lambda}' = (\lambda_1, \lambda_2, \ldots, \lambda_k),\ \lambda_i \in \Lambda,\ i=1,2,\ldots,k)$. Let

$$(1.2.6) \qquad \psi(\lambda; c, d, t+1) = \int F_{\lambda}^{t}(h(x))\, f_{\lambda}(x)\, dx.$$

Because of (1.2.1) - (i) and (ii), we get

(1.2.7)
$$\psi(\lambda;c,d,k) \geq \frac{1}{k}$$

and

(1.2.8)
$$\psi(\lambda;1,0,k) = \frac{1}{k} .$$

The properties (1.1.3) - (iii) and (iv) yield

(1.2.9)
$$\lim_{d\to\infty} \psi(\lambda;c,d,k) = 1$$

and/or

(1.2.10)
$$\lim_{c\to\infty} \psi(\lambda;c,d,k) = 1.$$

If (1.2.9) holds, then for a given set of λ,c,k and P^* we can choose d such that the P^* - condition is satisifed. If (1.2.10) holds but not (1.2.9), then for a given set of λ,d,k and P^* we can find c subject to the P^*-condition.

1.3 Infimum of the Probability of a Correct Selection

We see from the last section that the constants of the procedure must be evaluated satisfying the consition

(1.3.1)
$$\inf_{\lambda} \psi(\lambda;c,d,k) \geq P^* .$$

Hence our attention in this and several subsequent sections is focussed on the question of evaluating the infimum of $\psi(\lambda;c,d,k)$ for $\lambda\epsilon\Lambda$. Presently we will consider a lemma concerning the infimum of $\psi(\lambda;c,d,k)$ which is analogous to a lemma of Gupta [34].

<u>Lemma 1.3.1</u>. Let $\psi(\lambda;c,d,t+1)$ be defined as in (1.2.6). Suppose there exists a density $f(x)$ with cdf $F(x)$ such that for a and b real,

$$h(ag_\lambda(x) + b) \geq ag_\lambda(h(x)) + b, \tag{1.3.2}$$

where $g_\lambda(x)$ is defined by

$$F_\lambda(ag_\lambda(x) + b) = F(x) \quad \text{for all } x. \tag{1.3.3}$$

Then, for any $t > 0$,

$$\psi(\lambda;c,d,t+1) \geq \int F^t(h(x))\, f(x)\, dx, \tag{1.3.4}$$

where the integral extends over the range of x.

<u>Proof</u>.
$$\int F_\lambda^t\, (h(x))\, f_\lambda(x)\, dx$$
$$= \int F_\lambda^t\, (h(ag_\lambda(z) + b))\, dF_\lambda(ag_\lambda(z) + b), \quad \text{setting } x=ag_\lambda(z)+b$$
$$= \int F_\lambda^t\, (h(ag_\lambda(z) + b))\, dF(z), \quad \text{because of (1.3.3)}$$
$$\geq \int F_\lambda^t\, (ag_\lambda(h(z)) + b)\, dF(z), \quad \text{because of (1.3.2)}$$
$$= \int F^t(z)\, dF(z), \quad \text{using (1.3.3) again.}$$

Though we obtained Lemma 1.3.1 in a form analogous to that of Gupta [34], we can prove a more general result, which throws more light on the nature of the result we are after. The importance of this general lemma stated below will become more clear when we discuss the selection problems for restricted families of probability distributions in Chapter III.

Lemma 1.3.2. Let X and Y be random variables having densities $f_\lambda(x)$ and $f(y)$ and cdf's $F_\lambda(x)$ and $F(y)$ respectively. Let $h(x)$ be a function such that

$$h(\varphi(x)) \geq \varphi(h(x)), \tag{1.3.5}$$

where $\varphi \equiv F_\lambda^{-1} F$. Then, for any $t > 0$,

$$\int F_\lambda^t (h(x))\, f_\lambda(x)\, dx \geq \int F^t(h(x))\, f(x)\, dx. \tag{1.3.6}$$

Proof. X is stochastically equal ($\underset{st}{=}$) to $\varphi(Y)$, since $P(\varphi(Y) \leq x) = P(F(Y) \leq F_\lambda(x)) = F_\lambda(x)$ because $F(Y)$ is uniformly distributed in $(0,1)$. Hence

$$\begin{aligned} &\int F_\lambda^t (h(x))\, dF_\lambda(x) \\ &= \int F_\lambda^t (h(x))\, dF(\varphi^{-1}(x)) \\ &= \int F_\lambda^t (h(\varphi(y)))\, dF(y), \qquad \text{setting } x = \varphi(y) \\ &\geq \int F_\lambda^t (\varphi(h(y)))\, dF(y), \qquad \text{because of (1.3.5)} \\ &= \int F^t (h(y))\, dF(y) . \end{aligned}$$

If we assume that t is a positive integer as is the case in our selection problems, Lemma 1.3.2 can be proved in a more elegant way using probability arguments.

Alternative Proof of Lemma 1.3.2.

Let $Y_1, Y_2, \ldots, Y_{t+1}$ be independent and identically distributed (i.i.d.) random variables with cdf $F(x)$. Let

$$X_i = \varphi(Y_i), \qquad i=1,2,\ldots,t+1. \tag{1.3.7}$$

Then X_i, $i=1,\ldots,t+1$ are i.i.d. with cdf $F_\lambda(x)$ and (1.3.6) is same as

$$P(h(X_{t+1}) \geq \max_{1\leq r\leq t} X_r) \geq P(h(Y_{t+1}) \geq \max_{1\leq r\leq t} Y_r). \tag{1.3.8}$$

To prove this, suppose $h(Y_{t+1}) \geq \max_{1\leq r\leq t} Y_r$. Since $\varphi(x)$ is an increasing function,

$$\varphi(h(Y_{t+1})) \geq \varphi(\max_{1\leq r\leq t} Y_r) = \max_{1\leq r\leq t} \varphi(Y_r) . \tag{1.3.9}$$

Then (1.3.5) and (1.3.9) imply that

$$h(\varphi(Y_{t+1})) \geq \max_{1\leq r\leq t} \varphi(Y_r), \tag{1.3.10}$$

which is same as

$$h(X_{t+1}) \geq \max_{1\leq r\leq t} X_r \tag{1.3.11}$$

Thus $h(Y_{t+1}) \geq \max_{1\leq r\leq t} Y_r \Rightarrow h(X_{t+1}) \geq \max_{1\leq r\leq t} X_r$, which yields (1.3.8).

Remark 1.3.1. It is readily seen that Lemma 1.3.1 is a particular case of Lemma 1.3.2 by setting $\varphi(x) = a\, g_\lambda(x) + b$ which gives $F(x) = F_\lambda(a\, g_\lambda(x) + b)$, the assumption (1.3.3).

Examples of Application of Lemma 1.3.2 to Selection Problems.

(1) Let $F_{\lambda_i}(x) = F(x - \lambda_i)$ and $\Lambda = (-\infty,\infty)$, i.e. λ_i are location parameters. Suppose we use the procedure R_h with $h(x) = x+d$. For any fixed i, let $\varphi \equiv F_{\lambda_i}^{-1} F_0$, where $F_0(x) = F(x)$. Then, $\varphi(x) = F_{\lambda_i}^{-1} F_0(x) = F^{-1}F(x) + \lambda_i = x+\lambda_i$. Hence $h(\varphi(x) = x+\lambda_i + d = \varphi(h(x))$ and the lemma applies. Thus, for $i=1,2,\ldots,k$, $\psi(\lambda_i;c,d,k) \geq \psi(0;c,d,k) = \int_{-\infty}^{\infty} F^{k-1}(x+d)\, dF(x)$. Hence

$$\inf_\lambda \psi(\lambda;c,d,k) = \psi(0;c,d,k), \tag{1.3.12}$$

i.e. the infimum is attained for $\lambda=0$.

(2) Let $F_{\lambda_i}(x) = F(\frac{x}{\lambda_i})$, $x \geq 0$ and $\Lambda = (0,\infty)$, i.e. λ_i are scale parameters. If we use the procedure R_h with $h(x) = cx$. For a given i, let $\varphi \equiv F_{\lambda_i}^{-1}F_1$, where $F_1(x) = F(x)$. Then, $\varphi(x) = F_{\lambda_i}^{-1}F(x) = \lambda_i F^{-1}F(x) = \lambda_i x$. Hence $h(\varphi(x)) = c\lambda_i x = \varphi(h(x))$ and the lemma applies. Thus, for $i=1,2,\ldots,k$, $\psi(\lambda_i;c,d,k) \geq \psi(1;c,d,k) = \int_0^\infty F^{k-1}(cx)f(x)dx$.

Hence

$$\inf_\lambda \psi(\lambda; c,d,k) = \psi(1;c,d,k), \tag{1.3.13}$$

i.e. the infimum is attained for $\lambda=1$.

1.4. Sufficient Condition for the Monotonicity of $\psi(\lambda;c,d,k)$

A. Some Preliminary Results

We start with a result in Lehmann [55, p.112], which we state below as a lemma without proof.

Lemma 1.4.1. Let F_0 and F_1 be two cdf's on the real line such that F_1 is stochastically larger than F_0, i.e. $F_1(x) \leq F_0(x)$ for all x. Then $E_0\psi(X) \leq E_1\psi(X)$ for any non-decreasing function ψ. As an immediate consequence of the above lemma, we obtain the following theorem.

Theorem 1.4.1. Let $\{F_\lambda\}$ be a family of distribution functions on the real line which are stochastically increasing in λ, i.e. for $\lambda_2 > \lambda_1$, F_{λ_1} and F_{λ_2} are distinct and $F_{\lambda_1}(x) \geq F_{\lambda_2}(x)$ for all x. Then $E_\lambda\psi(X)$ is non-decreasing in λ for any non-decreasing function ψ.

A lemma in Lehmann [55, p.74] establishes the same result under a stronger hypothesis that $\{F_\lambda\}$ is a family of distribution functions having the property of monotone likelihood ratio (MLR), which implies the stochastic ordering.

As our next step, we obtain a more general result, which gives a sufficient condition for $E_\lambda\psi(X,\lambda)$ to be non-decreasing in λ.

Theorem 1.4.2 Let $\{F_\lambda\}$, $\lambda\epsilon\Lambda$, be a family of absolutely continuous distributions on the real line and $\psi(x,\lambda)$ be a differentiable function of x and λ. Then $E_\lambda\psi(X,\lambda)$ is non-decreasing in λ provided that

$$(1.4.1) \qquad \left|\frac{\partial(F,\psi)}{\partial(x,\lambda)}\right| \geq 0,$$

where

$$(1.4.2)\qquad \left|\frac{\partial(F,\psi)}{\partial(x,\lambda)}\right| = \begin{vmatrix} \frac{\partial}{\partial x} F_\lambda(x) & \frac{\partial}{\partial x}\psi(x,\lambda) \\ \frac{\partial}{\partial\lambda} F_\lambda(x) & \frac{\partial}{\partial\lambda}\psi(x,\lambda) \end{vmatrix}$$

and the derivative of F_λ w.r.t. λ is assumed to exist. Further $E_\lambda\psi(X,\lambda)$ is strictly increasing in λ if (1.4.1) holds with strict inequality on a set of positive Lebesgue measure.

To facilitate the proof of this theorem, we need some more notation and lemmas. Let

$$(1.4.3)\qquad A(\lambda) = \int_I \psi(x,\lambda)\, dF_\lambda(x) \equiv E_\lambda\psi(X,\lambda),$$

where I is the support of F_λ. Let us consider $\lambda_1,\lambda_2 \epsilon\ \Lambda$ such that $\lambda_1 \leq \lambda_2$ and define

$$(1.4.4)\qquad A_i(\lambda_1,\lambda_2) = \int_I \prod_{\substack{r=1 \\ r\neq i}}^{2} \psi(x,\lambda_r)\, dF_i(x), \quad i=1,2$$

and

$$(1.4.5)\qquad B(\lambda_1,\lambda_2) = \sum_{i=1}^{2} A_i(\lambda_1,\lambda_2),$$

where $F_i(x) \equiv F_{\lambda_i}(x)$, $i=1,2$. We note that when $\lambda_1 = \lambda_2 = \lambda$,

$$(1.4.6)\qquad B(\lambda,\lambda) = 2A(\lambda)\ .$$

Lemma 1.4.2. $B(\lambda_1,\lambda_2)$ is non-decreasing in λ_1, when λ_2 is kept fixed, provided that, for $\lambda_1 \leq \lambda_2$,

$$(1.4.7) \qquad \frac{\partial}{\partial\lambda_1} \psi(x,\lambda_1)\ f_{\lambda_2}(x) - \frac{\partial}{\partial\lambda_1} F_{\lambda_1}(x) \frac{\partial}{\partial x} \psi(x,\lambda_2) \geq 0 .$$

Proof. Integrating by parts, we obtain

$$(1.4.8) \qquad A_1(\lambda_1,\lambda_2) = F_1(x)\ \psi(x,\lambda_2) \Big|^{*} - \int F_1(x)\ d\psi(x,\lambda_2),$$

where the asterisk in the first term indicates that it has to be evaluated between proper limits. However, we note that this term will be independent of λ_1. Using (1.4.8) in (1.4.5), we obtain

$$(1.4.9) \qquad B(\lambda_1,\lambda_2) = \text{ a term independent of } \lambda_1 + \int\{\psi(x,\lambda_1)\ f_2(x) - F_1(x)\ \psi'(x,\lambda_2)\}\ dx,$$

where $\psi'(x,\lambda_2) = \frac{\partial}{\partial x} \psi(x,\lambda_2)$. Hence,

$$(1.4.10) \quad \frac{\partial}{\partial\lambda_1} B(\lambda_1,\lambda_2) = \int\{\frac{\partial}{\partial\lambda_1} \psi(x,\lambda_1)\ f_2(x) - \frac{\partial}{\partial\lambda_1} F_1(x)\ \psi'(x,\lambda_2)\}\ dx$$

and this is non-negative if

$$(1.4.11) \quad \frac{\partial}{\partial\lambda_1} \psi(x,\lambda_1)\ f_2(x) - \frac{\partial}{\partial\lambda_1} F_1(x)\ \psi'(x,\lambda_2) \geq 0 \quad \text{for all } x \in I.$$

Since our selection of λ_1 and λ_2 in Λ is subject only to the condition that $\lambda_1 \leq \lambda_2$ and is otherwise arbitrary, (1.4.11) is satisfied if (1.4.7) holds. This completes the proof of Lemma 1.4.2.

Lemma 1.4.3. If $\lambda_1 = \lambda_2 = \lambda$, then $B(\lambda,\lambda)$ is non-decreasing in λ provided that (1.4.7) is satisfied.

Proof. We note the following properties of $B(\lambda_1,\lambda_2)$ which can be easily verified.

$$\frac{d}{d\lambda} B(\lambda,\lambda) = \sum_{i=1}^{2} \frac{\partial}{\partial\lambda_i} B(\lambda_1,\lambda_2)\Big|_{\lambda_1=\lambda_2=\lambda} . \tag{1.4.12}$$

$$\frac{\partial}{\partial\lambda_2} B(\lambda_1,\lambda_2) = \frac{\partial}{\partial\lambda_2} B(\lambda_2,\lambda_1) = \frac{\partial}{\partial\lambda_1} B(\lambda_1,\lambda_2)\Big|_{\lambda_1 \leftrightarrow \lambda_2} , \tag{1.4.13}$$

where $\lambda_1 \leftrightarrow \lambda_2$ indicates that after differentiation λ_1 and λ_2 are interchanged in the final expression. Hence

$$\sum_{i=1}^{2} \frac{\partial}{\partial\lambda_i} B(\lambda_1,\lambda_2)\Big|_{\lambda_1=\lambda_2=\lambda} = 2\frac{\partial}{\partial\lambda_1} B(\lambda_1,\lambda_2)\Big|_{\lambda_1=\lambda_2=\lambda} . \tag{1.4.14}$$

Thus $\frac{d}{d\lambda} B(\lambda,\lambda) \geq 0$ if $\frac{\partial}{\partial\lambda_1} B(\lambda_1,\lambda_2) \geq 0$ for $\lambda_1 \leq \lambda_2$. This completes the proof of Lemma 1.4.3.

Now we give the proof of Theorem 1.4.2.

Proof. By Lemma 1.4.3, $B(\lambda,\lambda) = 2A(\lambda)$ is non-decreasing in λ if (1.4.7) is satisfied. In the theorem we are concerned only with the monotonicity of $A(\lambda)$. For the purpose of this theorem choosing

$\lambda_1, \lambda_2 \epsilon \Lambda$ is only an artifical device. Obviously, for $A(\lambda)$ to be non-decreasing in λ, we only need the condition (1.4.7) with $\lambda_1 = \lambda_2 = \lambda$. Hence the sufficient condition needed is

$$\frac{\partial}{\partial\lambda} \psi(x,\lambda)\, f_\lambda(x) - \frac{\partial}{\partial\lambda} F_\lambda(x) \frac{\partial}{\partial x} \psi(x,\lambda) \geq 0,$$

which is same as (1.4.1). The strict inequality part is now obvious.

Remark 1.4.1. In the proof of Lemma 1.4.2 we have made use of the assumption that $F_\lambda, \lambda\epsilon\Lambda$, have all the same support I. But the result is true even if the support changes with λ. If (a_1,b_1) and (a_2,b_2) are the supports of F_{λ_1} and F_{λ_2}, (1.4.8) will be

$$(1.4.15) \qquad A_1(\lambda_1,\lambda_2) = \psi(b_1,\lambda_2) - \int_{a_1}^{b_1} F_{\lambda_1}(x)\, \psi'(x,\lambda_2)\, dx$$

and this yields

$$(1.4.16) \quad \frac{\partial}{\partial\lambda_1} A_1(\lambda_1,\lambda_2) = \frac{\partial}{\partial b_1} \psi(b_1,\lambda_2) \frac{db_1}{d\lambda_1} - \int_{a_1}^{b_1} \frac{\partial}{\partial\lambda_1} F_{\lambda_1}(x)\psi'(x,\lambda_2)\, dx$$

$$- \frac{db_1}{d\lambda_1} \frac{\partial}{\partial b_1} \psi(b_1,\lambda_2)$$

$$= - \int_{a_1}^{b_1} \frac{\partial}{\partial\lambda_1} F_{\lambda_1}(x)\, \psi'(x,\lambda_2)\, dx.$$

Hence,

$$(1.4.17) \quad \frac{\partial}{\partial\lambda_1} B(\lambda_1,\lambda_2) = \int_{a_2}^{b_2} \frac{\partial}{\partial\lambda_1} \psi(x,\lambda_1) f_{\lambda_2}(x)\, dx$$

$$- \int_{a_1}^{b_1} \frac{\partial}{\partial\lambda_1} F_{\lambda_1}(x)\psi'(x,\lambda_2)\, dx$$

and it can be seen that (1.4.7) is sufficient to make $\frac{\partial}{\partial\lambda_1}B(\lambda_1,\lambda_2) \geq 0$. Lemma 1.4.3 also holds and hence Theorem 1.4.2 is true when the supports are not same.

Corollary 1.4.1. If $\psi(x,\lambda) = \psi(x)$ for all $\lambda \in \Lambda$, i.e. $\psi(x,\lambda)$ is independent of λ, then $E_\lambda \psi(X)$ is non-decreasing in λ if

$$(1.4.18) \qquad \frac{\partial}{\partial\lambda} F_\lambda(x) \frac{\partial}{\partial x} \psi(x) \leq 0.$$

The proof is omitted, since it is immediate from Theorem 1.4.2.

Remark 1.4.2. We see that, if we assume in Corollary 1.4.1 that $\{F_\lambda\}$, $\lambda \in \Lambda$ is a stochastically increasing family in λ, then (1.4.18) is equivalent to $\frac{\partial}{\partial x} \psi(x) \geq 0$, which means that $\psi(x)$ is non-decreasing in x. Thus we obtain Theorem 1.4.1 as a particular case.

Corollary 1.4.2. Let $\{F_\lambda\}$ and $\psi(x,\lambda)$ be as in the hypothesis of Theorem 1.4.2 with the additional condition that $\psi(x,\lambda) \geq 0$. Then, for any positive integer t, $E_\lambda \psi^t(X,\lambda)$ is non-decreasing in λ provided that (1.4.1) holds and is strictly increasing in λ if strict inequality in (1.4.1) holds on a set of positive measure.

Define

$$(1.4.21)\qquad A_i(\lambda_1,\ldots,\lambda_k) = \int \prod_{\substack{r=1\\ r\neq i}}^{k} \psi(x,\lambda_i)\, dF_i(x), \quad i=1,\ldots,k$$

where $F_i(x) \equiv F_{\lambda_i}(x)$,

$$(1.4.22)\qquad B(\lambda_1,\ldots,\lambda_k) = \sum_{i=1}^{k} A_i(\lambda_1,\ldots,\lambda_k).$$

and

$$(1.4.23)\qquad A(\lambda,k) = \int \psi^{k-1}(x,\lambda)\, dF_\lambda(x).$$

Then

$$(1.4.24)\qquad B(\lambda,\ldots,\lambda) = k\, A(\lambda,k).$$

Integrating $A_1(\lambda_1,\ldots,\lambda_k)$ by parts and using it in (1.4.22) and then differentiating w.r.t. λ_1 we get

$$(1.4.25)\ \frac{\partial}{\partial\lambda_1} B(\lambda_1,\ldots,\lambda_k) =$$

$$\sum_{\alpha=2}^{k} \int \prod_{\substack{r=2\\ r\neq\alpha}}^{k} \psi(x,\lambda_r)\Big\{\frac{\partial}{\partial\lambda_1}\psi(x,\lambda_1)\, f_\alpha(x) - \frac{\partial}{\partial\lambda_1} F_1(x)\, \psi'(x,\lambda_\alpha)\Big\}dx.$$

Hence, similar to Lemma 1.4.2 we will get the following result.

Lemma 1.4.4. $B(\lambda_1,\ldots,\lambda_k)$ is non-decreasing in λ_1, when $\lambda_2,\ldots,\lambda_k$ are kept fixed, provided that $\psi(x,\lambda) \geq 0$ for $\lambda \in \Lambda$ and (1.4.7) holds.

If $\lambda_1 = \ldots = \lambda_m = \lambda \leq \lambda_{m+1} \leq \cdots \leq \lambda_k$, $1 \leq m \leq k$, then by a reasoning similar to the one employed in the proof of Lemma 1.4.3, we can see that

$$\frac{\partial}{\partial\lambda} B(\lambda,\ldots,\lambda,\lambda_{m+1},\ldots,\lambda_k) \equiv m \frac{\partial}{\partial\lambda_1} B(\lambda_1,\ldots,\lambda_k)\Bigg|_{\lambda_1=\ldots=\lambda_m=\lambda} . \tag{1.4.26}$$

Hence we can state the following lemma.

Lemma 1.4.5. If $\lambda_1 = \ldots = \lambda_m = \lambda \leq \lambda_{m+1} \leq \cdots \leq \lambda_k$, $1 \leq m \leq k$, then $B(\lambda,\ldots,\lambda,\lambda_{m+1},\ldots,\lambda_k)$ is non-decreasing in λ when $\lambda_{m+1},\ldots,\lambda_k$ are kept fixed, provided that $\psi(x,\lambda)$ is nonnegative and (1.4.7) holds.

As a consequence of Lemma 1.4.5, we can state the following theorem.

Theorem 1.4.3 Let $B(\lambda_1,\ldots,\lambda_k)$ be defined by (1.4.21) and (1.4.22). Then the supremum of $B(\lambda_1,\ldots,\lambda_k)$ over $\lambda_1,\ldots,\lambda_k \in \Lambda$ subject to the condition $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$ takes place for $\lambda_1 = \lambda_2 = \ldots = \lambda_k$ provided $\psi(x,\lambda)$ is non-negative and (1.4.7) holds.

B. Monotonicity of $\psi(\lambda;c,d,k)$

Theorem 1.4.4. For the procuedure R_h defined in Section 1.2, $\psi(\lambda;c,d,k)$ is non-decreasing in λ provided that

$$\frac{\partial}{\partial\lambda}F_\lambda(h(x))\, f_\lambda(x) - h'(x)\, f_\lambda(h(x)) \frac{\partial}{\partial\lambda} F_\lambda(x) \geq 0, \tag{1.4.27}$$

where $h'(x) = \frac{d}{dx} h(x)$ and $\psi(\lambda;c,d,k)$ is strictly increasing in λ if strict inequality holds in (1.4.27) on a set of positive Lebesgue measure.

Proof. The proof is immediate from Corollary 1.4.2 with $\psi(x,\lambda)=F_\lambda(h(x))$.

Remark 1.4.4. From Remark 1.4.3, we see that $\psi(\lambda;c,d,k)$ is non-increasing if the inequality is reversed in (1.4.27) and consequently equality in (1.4.27) implies that $\psi(\lambda;c,d,k)$ is independent of λ.

Now we discuss some special cases of interest.

Case (1): If λ is a location parameter and $h(x) = x+d$, $d > 0$, $F_\lambda(x) = F(x-\lambda)$ and $\frac{\partial}{\partial\lambda} F_\lambda(x) = -f(x-\lambda) = -f_\lambda(x)$. Also $h'(x) = 1$ and hence the left hand side of (1.4.27) vanishes. Thus $\psi(\lambda;c,d,k)$ is independent of λ. Thus $\psi(\lambda;c,d,k) = \psi(0;c,d,k)$. Hence $\inf_\lambda \psi(\lambda;c,d,k)$ $= \psi(0;c,d,k)$, a fact established in Section 1.3.

Case (2): If λ is a scale parameter and $h(x) = cx$, $c > 1$, $F_\lambda(x) = F(\frac{x}{\lambda})$ and $\frac{\partial}{\partial\lambda} F_\lambda(x) = -\frac{x}{\lambda^2} f(\frac{x}{\lambda}) = -\frac{x}{\lambda} f_\lambda(x)$. Also $h'(x) = c$ and hence the left hand side of (1.4.27) vanishes. Thus $\psi(\lambda;c,d,k)$ is independent of λ. Thus $\psi(\lambda;c,d,k) = \lambda(1;c,d,k)$, yielding $\inf_\lambda \psi(\lambda;c,d,k) = \psi(1;c,d,k)$, which is again a fact established in Section 1.3.

We are also interested in another case where $f_\lambda(x)$ is a convex mixture of a sequence of known density functions and λ is involved in the weights. Some selection procedures for distributions of this form are discussed in Chapter II. So we discuss this case in some detail in the next section.

1.5. Selection Procedures for Distributions which are Convex Mixtures

A. Preliminary Discussions and Main Theorems

We are presently concerned with the procedure R_h for selecting the population with the largest λ_i, where

$$f_\lambda(x) = \sum_{j=0}^{\infty} w(\lambda,j)g_j(x), \tag{1.5.1}$$

where $g_j(x)$, $j=0,1,\ldots$ is a sequence of density functions and $w(\lambda,j)$ are non-negative weight functions such that $\sum_{j=0}^{\infty} w(\lambda,j) = 1$. We recall that the first stage in obtaining the infimum of $P(CS|R_h)$, namely (1.2.5), is based on the assumption of the stochastic ordering of F_{λ_i}, $i=1,\ldots,k$. We state below a lemma which gives a set of sufficient conditions on $w(\lambda,j)$ and $g_j(x)$ which will guarantee the stochastic ordering of F_λ with respect to λ. In fact it guarantees more.

<u>Lemma 1.5.1.</u> Let $f_\lambda(x)$ be a density function given by (1.5.1). Then $f_\lambda(x)$ is totally positive of order 2 (TP_2), i.e., for $\lambda_1 < \lambda_2$ and $x_1 < x_2$, $\begin{vmatrix} f_{\lambda_1}(x_1) & f_{\lambda_1}(x_2) \\ f_{\lambda_2}(x_1) & f_{\lambda_2}(x_2) \end{vmatrix} \geq 0$ provided that $g_j(x)$ and $w(\lambda,j)$ are TP_2.

<u>Proof.</u> The proof is a straightforward consequence of the basic composition formula of Polyá and Szegö (see Karlin [48], p. 17), which in the present case is

$$(1.5.2)\quad \begin{vmatrix} f_{\lambda_1}(x_1) & f_{\lambda_1}(x_2) \\ f_{\lambda_2}(x_1) & f_{\lambda_2}(x_2) \end{vmatrix} = \sum_{j_1 < j_2} \begin{vmatrix} g_{j_1}(x_1) & g_{j_1}(x_2) \\ g_{j_2}(x_1) & g_{j_2}(x_2) \end{vmatrix} \begin{vmatrix} w(\lambda_1,j_1) & w(\lambda_1,j_2) \\ w(\lambda_2,j_1) & w(\lambda_2,j_2) \end{vmatrix}$$

for $x_1 < x_2$ and $\lambda_1 < \lambda_2$.

Remark 1.5.1. If the density function $f_\lambda(x)$ of a random variable X is TP_2, then equivalently it has MLR in x and consequently the distribution of X is stochastically increasing in λ.

From now on we assume that

$$(1.5.3)\qquad w(\lambda,j) = \frac{a_j\lambda^j}{A(\lambda)j!}, \quad A(\lambda) \geq 0, \quad \lambda \geq 0.$$

Because of the non-negativity of $w(\lambda,j)$, $a_j \geq 0$. Since the weights add up to unity,

$$(1.5.4)\qquad A(\lambda) = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_j.$$

The weight distribution considered here is the general power series distribution. Let us define

$$(1.5.5)\qquad r_\lambda(x) = A(\lambda)\, f_\lambda(x)$$

and

$$(1.5.6)\qquad R_\lambda(x) = A(\lambda)\, F_\lambda(x)\,.$$

Using (1.5.5) and (1.5.6), (1.4.27) can be rewritten as

$$Q_\lambda(h(x))\, r_\lambda(x) - h'(x)\, Q_\lambda(x)\, r_\lambda(h(x)) \geq 0, \tag{1.5.7}$$

where

$$Q_\lambda(x) = A(\lambda) \frac{\partial}{\partial\lambda} R_\lambda(x) - A'(\lambda)\, R_\lambda(x)\ , \tag{1.5.8}$$

the prime over $A(\lambda)$ denoting the derivative with respect to λ. Now by series multiplication (they are all absolutely convergent series) we obtain

$$Q_\lambda(x) = \sum_{\alpha=0}^{\infty} \frac{\lambda^\alpha}{\alpha!} B_\alpha(x), \tag{1.5.9}$$

where

$$B_\alpha(x) = \sum_{j=0}^{\alpha} \binom{\alpha}{j} a_j a_{\alpha-j+1}(G_{\alpha-j+1}(x) - G_j(x))\ . \tag{1.5.10}$$

Using (1.5.9) in (1.5.7) and simplifying we get

$$\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}\ C_i \geq 0, \tag{1.5.11}$$

where

$$C_i = \sum_{\alpha=0}^{i} \binom{i}{\alpha}[a_{i-\alpha}B_\alpha(h(x))g_{i-\alpha}(x) - h'(x)a_\alpha g_\alpha(h(x))\ B_{i-\alpha}(x)]\ . \tag{1.5.12}$$

Thus (1.5.11) holds and consequently (1.4.27) holds if $C_i \geq 0$, i.e. for every integer $i \geq 0$,

$$(1.5.13) \quad \sum_{\alpha=0}^{i} \binom{i}{\alpha}[a_{i-\alpha}B_{\alpha}(h(x))g_{i-\alpha}(x)-h'(x)a_{\alpha}g_{\alpha}(h(x))B_{i-\alpha}(x)] \geq 0.$$

The above discussions could be summarized as follows.

Theorem 1.5.1. For the procedure R_h, when $f_\lambda(x)$ is given by (1.5.1) with weight functions given by (1.5.3), $\psi(\lambda;c,d,k)$ is non-decreasing in λ provided that for every integer $i \geq 0$, (1.5.13) holds and $\psi(\lambda;c,d,k)$ is strictly increasing in λ if strict inequality holds in (1.5.13) for some i.

What is of more interest to us is the case where the a_j are governed by the relation

$$(1.5.14) \qquad a_{j+1} = (q+jp)a_j, \qquad j=0,1,\ldots; \quad p,q \geq 0.$$

This on successive applications yield

$$(1.5.15) \qquad a_{j+1} = a_0 q(q+2p) \ldots (q+jp), \quad j=0,1,\ldots \quad .$$

Hence,

$$A(\lambda) = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_0 q(q+2p) \ldots (q+jp)$$

$$= a_0(1-\lambda p)^{-q/p}, \qquad \text{provided that } \lambda < \frac{1}{p} .$$

Further,

$$Q_\lambda(x) = a_0(1-\lambda p)^{-1-q/p}[(1-\lambda p)\sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{(j-1)!} a_j G_j(x) - q \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_j G_j(x)].$$

But the expression inside the last brackets

$$= \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_{j+1} G_{j+1}(x) - \sum_{j=1}^{\infty} (q+jp) \frac{\lambda^j}{j!} a_j G_j(x) - q\, a_0 G_0(x)$$

$$= \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_{j+1} G_{j+1}(x) - \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_{j+1}\, G_j(x)$$

$$= \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_{j+1}\, \Delta G_j(x),$$

where

(1.5.25) $$\Delta G_j(x) = G_{j+1}(x) - G_j(x).$$

Hence, (1.5.7) holds if

(1.5.26) $$\left(\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_{j+1}\, \Delta G_j(h(x))\right)\left(\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_j g_j(x)\right) -$$

$$h'(x)\left(\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_{j+1}\, \Delta G_j(x)\right)\left(\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_j g_j(h(x))\right) \geq 0.$$

This can be simplified and rewritten using (1.5.14) as

(1.5.27)
$$\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \sum_{\alpha=0}^{i} \binom{i}{\alpha} a_\alpha a_{i-\alpha} T_\alpha(x) \geq 0$$

where

(1.5.28) $T_\alpha(x) =$

$$(q+\alpha p)g_{i-\alpha}(x)\ \Delta G_\alpha(h(x))-h'(x)(q+(i-\alpha)p)g_\alpha(h(x))\ \Delta G_{i-\alpha}(x).$$

Obviously, for (1.5.7) to hold, it is sufficient that, for every integer $i \geq 0$,

(1.5.29)
$$\sum_{\alpha=0}^{i} \binom{i}{\alpha} a_\alpha a_{i-\alpha} T_\alpha(x) \geq 0.$$

Before we summarize our results, we will obtain a condition which is more stringent than (1.5.29) but which is verified to hold in several cases. Grouping the terms corresponding to α and $i-\alpha$, (1.5.29) becomes

(1.5.30)
$$\sum_{\alpha=0}^{[\frac{i}{2}]} \binom{i}{\alpha} a_\alpha a_{i-\alpha}(T_\alpha(x) + T_{i-\alpha}(x)) \geq 0,$$

where $[s]$ denotes the largest integer $\leq s$. Thus (1.5.29) holds, if $T_\alpha(x) + T_{i-\alpha}(x) \geq 0$ for $\alpha=0,1,\ldots,[\frac{i}{2}]$, i.e., if

$$(1.5.31)\quad (q+\alpha p)[g_{i-\alpha}(x)\ \Delta G_{\alpha}(h(x))-h'(x)g_{i-\alpha}(h(x))\ \Delta G_{\alpha}(x)]\ +$$

$$(q+(i-\alpha)p)[g_{\alpha}(x)\ \Delta G_{i-\alpha}(h(x))-h'(x)g_{\alpha}(h(x))\ \Delta G_{i-\alpha}(x)] \geq 0.$$

Summarizing all our results, we have the following theorem.

Theorem 1.5.2. For the procedure R_h, when $f_\lambda(x)$ is given by (1.5.1) with weight functions given by (1.5.3) and (1.5.14), $\psi(\lambda;c,d,k)$ is non-decreasing in λ provided that for every integer $i \geq 0$, (1.5.29) is satisfied or more strongly (1.5.31) is satisified for $\alpha=0,\ldots,[\frac{i}{2}]$ and $\psi(\lambda;c,d,k)$ is strictly increasing in λ if strict inequality holds in (1.5.29) or (1.5.31) for some i.

B. Some Special Cases

We are interested in some special choices of p and q in (1.5.14). These special cases arise in the next chapter when we consider selection procedures for multivariate normal populations in terms of multiple correlation coefficient.

Case (1): $q=1$, $a_0= 1$, $p=0$.

In this case we have $A(\lambda) = \lim_{p\to 0} (1-\lambda p)^{-1/p} = e^{\lambda}$, $\lambda \geq 0$, $w(\lambda,j) = \frac{e^{-\lambda}\lambda^j}{j!}$ and $a_j= 1$ for all j. Thus the densities $g_j(x)$ are weighted by Poisson weights. Familiar examples of $f_\lambda(x)$ in this case are the densities of a non-central chi-square of F variable with non-centrality parameter λ. The sufficient condition (1.5.29) becomes

$$(1.5.32)\ \sum_{\alpha=0}^{i} \binom{i}{\alpha}[g_{i-\alpha}(x)\ \Delta G_{\alpha}(h(x))-h'(x)g_{\alpha}(h(x))\ \Delta G_{i-\alpha}(x)] \geq 0,$$

which is same as the condition obtained by Gupta and Studden [45] in whose procedure $h(x) = cx$, $c \geq 1$. Also falling in this category is the distribution of R^2, the square of the sample multiple correlation coefficient in the so-called conditional case.

Case (2): $p=1$ and $a_0=1$.

This gives $a_j = q(q+1),..(q+j-1)$ and $A(\lambda) = (1-\lambda)^{-q}$, $0 \leq \lambda \leq 1$.

Then $w(\lambda,j) = \frac{\Gamma(q+j)\lambda^j}{\Gamma(q)\ j!}(1-\lambda)^q$, i.e. the weights in this case are negative binomial weights. The density of R^2 in the 'unconditional' case is of this form and comes up in the selection problems discussed in the next chapter. The sufficient condition (1.5.29) becomes

$$(1.5.33)\quad \sum_{\alpha=0}^{i} \binom{i}{\alpha}(q)_\alpha (q)_{i-\alpha}[(q+\alpha)g_{i-\alpha}(x)\ \Delta G_\alpha(h(x)) - h'(x)(q+i-\alpha)g_\alpha(h(x))\ \Delta G_{i-\alpha}(x)] \geq 0,$$

where

$$(1.5.34)\qquad (q)_\alpha = q(q+1)\ \ldots\ (q+\alpha-1).$$

Case (3): Binomial Weights

Suppose we have

$$(1.5.35)\qquad f_\lambda(x) = \sum_{j=0}^{N} w(\lambda,j)g_j(x),$$

where

$$(1.5.36)\qquad w(\lambda,j) = \binom{N}{j} \lambda^j (1-\lambda)^{N-j}, \quad j=0,\ldots,N;\ 0 \le \lambda \le 1.$$

Let $\mu = \lambda/(1-\lambda)$. Then

$$(1.5.37)\qquad w(\lambda,j) = \frac{N(N-1)\ldots(N-j+1)}{j!} \frac{\mu^j}{(1+\mu)^N} = \frac{\mu^j}{j!} \frac{a_j}{A(\mu)},$$

where

$$(1.5.38)\qquad a_j = \begin{cases} 1 & , \ j=0 \\ N(N-1)\ldots(N-j+1) & , \ j=1,\ldots,N \\ 0 & , \ j=N+1,\ldots \end{cases}$$

and $$A(\mu) = (1+\mu)^N .$$

Thus, the density in this case becomes

$$(1.5.39)\qquad f_\mu(x) = \sum_{j=0}^{N} \frac{\mu^j}{j!} \frac{a_j}{A(\mu)} g_j(x).$$

Letting $\varphi(\mu;c,d,k) = \psi(\mu/(1+\mu);\ c,d,k)$ and noting that μ is an increasing function of λ, it can be seen that $\psi(\lambda;c,d,k)$ is non-decreasing (increasing) in λ iff $\varphi(\mu;c,d,k)$ is non-decreasing (increasing) in μ. In our present case

$$\frac{a_{j+1}}{a_j} = \begin{cases} N-j & , \ j=0,1,\ldots,N \\ 0 & , \ j=N+1,\ldots \end{cases} .$$

We can write it in the form

(1.5.40) $$\frac{a_{j+1}}{a_j} = \begin{cases} q+jp & , j=0,1,\dots,N \\ 0 & , j=N+1,\dots \end{cases},$$

where $q=N$ and $p=-1$. In Theorem 1.5.2, we assumed that p and q are non-negative. This was in view of the infinite mixture. When we have a finite mixture as is the case now, all we need is that $q+jp$ be non-negative for all j. Then corresponding to (1.5.26) we will have

(1.5.41) $$\left(\sum_{j=0}^{N-1} \frac{\mu^j}{j!} a_{j+1} \Delta G_j(h(x))\right)\left(\sum_{j=0}^{N} \frac{\mu^j}{j!} a_j g_j(x)\right) - h'(x)\left(\sum_{j=0}^{N-1} \frac{\mu^j}{j!} a_{j+1} \Delta G_j(x)\right)\left(\sum_{j=0}^{N} \frac{\mu^j}{j!} a_j g_j(h(x))\right) \geq 0.$$

We can rewrite (1.5.41) as follows:

(1.5.42) $$\sum_{i=0}^{2N-1} \frac{\mu^i}{i!} E_i \geq 0,$$

where

(1.5.43) $$E_i = \begin{cases} \sum_{\alpha=0}^{i} \binom{i}{\alpha} M_\alpha(x) & , i=0,1,\dots,N-1 \\ \sum_{\alpha=i+1-N}^{N} \binom{i}{\alpha} M_\alpha(x) & , i=N, N+1,\dots,2N-1 \end{cases}$$

and

$$(1.5.44) \quad M_\alpha(x) = a_\alpha a_{i-\alpha+1}[g_\alpha(x)\ \Delta G_{i-\alpha}(h(x)) - h'(x) g_\alpha(h(x))\ \Delta G_{i-\alpha}(x)].$$

We can rewrite (1.5.43) as

$$(1.5.45) \qquad E_i = \sum_{\alpha=\max(0,i+1-N)}^{\min(i,N)} \binom{i}{\alpha} M_\alpha(x).$$

Thus the sufficient condition for $\psi(\lambda;c,d,k)$ to be non-decreasing in λ becomes

$$(1.5.46) \qquad \sum_{\alpha=\max(0,i+1-N)}^{\min(i,N)} \binom{i}{\alpha} M_\alpha(x) \geq 0 \quad \text{for } i=0,.,\ldots,2N-1.$$

A very special case which may be of interest is where $N=1$, i.e., $f_\lambda(x) = (1-\lambda)\ g_0(x) + \lambda g_1(x)$. We have this type of situation when we have systems whose life distributions are mixtures of two distributions and the preference for a system is in terms of the proportion in which the two distributions are mixed. The condition (1.5.46) reduces in this case to

$$(1.5.47) \qquad g_\alpha(x)\ \Delta G_0(x) - h'(x) g_\alpha(h(x))\ \Delta G_0(x) \geq 0, \quad \alpha=0,1.$$

C. An Alternative Proof of Theorem 1.5.2.

We give below another way of obtaining the sufficient condition (1.5.29) which assures that $\psi(\lambda;c,d,k)$ is non-decreasing in λ under the hypothesis of Theorem 1.5.2. This method is the one which was employed by Gupta and Studden [45] for the case of Poisson weights and later used by Gupta and Panchapakesan [38] for the case of negative binomial weights. This proof is direct but does not bring out the general result. We avoid unnecessary details and give only the main trend of the results.

Following our earlier notations

(1.5.48) $\psi(\lambda;c,d,k)$

$$= \frac{1}{[A(\lambda)]^k} \int [\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_j G_j(h(x))]^{k-1} [\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_j g_j(x)]\, dx$$

$$= \frac{1}{[A(\lambda)]^k} \sum_{\alpha=0}^{\infty} c_\alpha \frac{\lambda^\alpha}{\alpha} ,$$

where

(1.5.49) $$c_\alpha = \sum_{P(\alpha)} \binom{\alpha}{\alpha_1 \alpha_2 \ldots \alpha_k} \int [\prod_{j=1}^{k-1} a_{\alpha_j} G_{\alpha_j}(h(x))]\, a_{\alpha_k} g_{\alpha_k}(x)\, dx,$$

where $P(\alpha)$ denotes the set of all partitions $(\alpha_1,\alpha_2,\ldots,\alpha_k)$ of α such that $\alpha_i, i=1,\ldots,k$ are non-negative integers and $\sum_{i=1}^{k} \alpha_i = \alpha$, and $\binom{\alpha}{\alpha_1\alpha_2 \ldots \alpha_k} = \frac{\alpha}{\alpha_1! \ldots \alpha_k!}$. Because of (1.5.14), we know that $A(\lambda) = (1-\lambda p)^{-q/p}$. Hence

(1.5.50) $$\psi(\lambda;c,d,k) = \frac{(1-\lambda p)^{qk/p}}{a_0^k} \sum_{\alpha=0}^{\infty} c_\alpha \frac{\lambda^\alpha}{\alpha!} .$$

Lemma 1.5.2. $\psi(\lambda;c,d,k)$ is non-decreasing in λ provided that

(1.5.51) $c_{\alpha+1} - (qk+p\alpha)\, c_\alpha \geq 0$ for $\alpha=0,1,\ldots$ and

$\psi(\lambda;c,d,k)$ is strictly increasing in λ if strict inequality holds in (1.5.51) for some α.

Proof: $\frac{\partial}{\partial\lambda}\psi(\lambda;c,d,k) \geq 0$ if

$$(1.5.52) \qquad (1-\lambda p)\sum_{\alpha=1}^{\infty} c_\alpha \frac{\lambda^{\alpha-1}}{(\alpha-1)!} - qk\sum_{\alpha=0}^{\infty} c_\alpha \frac{\lambda^\alpha}{\alpha!} \geq 0.$$

The left hand side of the above inequality

$$= \sum_{\alpha=0}^{\infty} c_{\alpha+1}\frac{\lambda^\alpha}{\alpha!} - \sum_{\alpha=0}^{\infty}(qk+p\alpha)\, c_\alpha \frac{\lambda^\alpha}{\alpha!}$$

$$= \sum_{\alpha=0}^{\infty}[c_{\alpha+1} - (qk+p\alpha)\, c_\alpha]\frac{\lambda^\alpha}{\alpha!}$$

Hence (1.5.51) implies (1.5.52). This completes the proof. The strict inequality part is obvious.

Lemma 1.5.3. For each set of integers $\alpha_1,\ldots,\alpha_k$ we have

$$\int \prod_{i=1}^{k-1} G_{\alpha_i}(h(x))\, g_{\alpha_k+1}(x)dx = \int \prod_{i=1}^{k-1} G_{\alpha_i}(h(x))\, g_{\alpha_k}(x)dx$$

$$- \int \frac{d}{dx}\Big[\prod_{i=1}^{k-1} G_{\alpha_i}(h(x))\Big](G_{\alpha_k+1}(x) - G_{\alpha_k}(x))dx.$$

This lemma is substially Lemma 3.3 of Gupta and Studden [45] and hence the proof is omitted.

Lemma 1.5.4. For $\alpha=0,1,2,\ldots,$

$$c_{\alpha+1}-(qk+p\alpha)c_{\alpha} = \sum_{P(\alpha)} \binom{\alpha}{\alpha_1\alpha_2\ldots\alpha_k} \sum_{j=1}^{k-1} \int \{\prod_{\substack{i=1\\ i\neq j}}^{k-1} a_{\alpha_i} G_{\alpha_i}(h(x))\} M_{\alpha_j,\alpha_k}(x)\, dx,$$

where

$$M_{\alpha_j;\alpha_k}(x) = a_{\alpha_j+1} a_{\alpha_k} g_{\alpha_k}(x)\, \Delta G_{\alpha_j}(h(x)) - h'(x)\, a_{\alpha_j} a_{\alpha_k+1} g_{\alpha_j}(h(x)) \Delta G_{\alpha_k}(x).$$

Proof. Using the fact that

$$\binom{\alpha+1}{\alpha_1\alpha_2\ldots\alpha_k} = \binom{\alpha}{\alpha_1-1\ \alpha_2\ldots\alpha_k} + \binom{\alpha}{\alpha_1\alpha_2-1\ldots\alpha_k} + \ldots + \binom{\alpha}{\alpha_1\alpha_2\ldots\alpha_k-1},$$

we obtain

$$c_{\alpha+1} = \sum_{P(\alpha)} \binom{\alpha}{\alpha_1\alpha_2\ldots\alpha_k} [\sum_{j=1}^{k-1} \int \{\prod_{\substack{i=1\\ i\neq j}}^{k-1} a_{\alpha_i} G_{\alpha_i}(h(x))\} a_{\alpha_j+1} a_{\alpha_k} g_{\alpha_k}(x) G_{\alpha_j+1}(h(x))dx$$

$$+ \int \{\prod_{i=1}^{k-1} a_{\alpha_i} G_{\alpha_i}(h(x))\} a_{\alpha_k+1} g_{\alpha_k+1}(x)dx].$$

Hence

$$c_{\alpha+1}-(qk+p\alpha)c_\alpha = \sum_{P(\alpha)}\binom{\alpha}{\alpha_1\alpha_2\ldots\alpha_k}\Big[\sum_{j=1}^{k-1}\int\{\prod_{\substack{i=1\\i\neq j}}^{k-1} a_{\alpha_i}G_{\alpha_i}(h(x))\}\{a_{\alpha_j+1}G_{\alpha_j+1}(h(x)) -$$

$$(q+p\alpha_j)a_{\alpha_j}G_{\alpha_j}(h(x))\}a_{\alpha_k}g_{\alpha_k}(x)dx + \int\{\prod_{\substack{i=1\\i\neq j}}^{k-1} a_{\alpha_i}G_{\alpha_i}(h(x))\}$$

$$\{a_{\alpha_k+1}g_{\alpha_k+1}(x) - (q+p\alpha_k)a_{\alpha_k}g_{\alpha_k}(x)\}dx\Big]$$

$$= \sum_{P(\alpha)}\binom{\alpha}{\alpha_1\alpha_2\ldots\alpha_k}\Big[\sum_{j=1}^{k-1}\int\{\prod_{\substack{i=1\\i\neq j}}^{k-1} a_{\alpha_i}G_{\alpha_i}(h(x))\}a_{\alpha_j+1}a_{\alpha_k}g_{\alpha_k}(x)\Delta G_{\alpha_j}(h(x))dx$$

$$+ \int\{\prod_{i=1}^{k-1} a_{\alpha_i}G_{\alpha_i}(h(x))\}a_{\alpha_k+1}(g_{\alpha_k+1}(x)-g_{\alpha_k}(x))dx\Big] .$$

Using Lemma 1.5.3, we get

$$c_{\alpha+1}- (qk+p\alpha)\, c_\alpha$$

$$= \sum_{P(\alpha)}\binom{\alpha}{\alpha_1\alpha_2\ldots\alpha_k}\Big[\sum_{j=1}^{k-1}\int\{\prod_{\substack{i=1\\i\neq j}}^{k-1} a_{\alpha_i}G_{\alpha_i}(h(x))\}\, a_{\alpha_j+1}a_{\alpha_k}g_{\alpha_k}(x)\ \Delta G_{\alpha_j}(h(x))dx$$

$$- \int \frac{d}{dx}\Big[\prod_{i=1}^{k-1} G_{\alpha_i}(h(x))\Big]\Big(\prod_{i=1}^{k-1} a_{\alpha_i}\Big)a_{\alpha_k+1}\ \Delta G_{\alpha_k}(x)dx$$

$$= \sum_{P(\alpha)} \binom{\alpha}{\alpha_1 \alpha_2 \ldots \alpha_k} [\sum_{j=1}^{k-1} \int \{ \prod_{\substack{i=1 \\ i \neq j}}^{k-1} a_{\alpha_i} G_{\alpha_i}(h(x)) \} a_{\alpha_j+1} a_{\alpha_k} g_{\alpha_k}(x) \ \Delta G_{\alpha_j}(h(x)) dx$$

$$- \sum_{j=1}^{k-1} \int \{ \prod_{\substack{i=1 \\ i \neq j}}^{k-1} a_{\alpha_i} G_{\alpha_i}(h(x)) \} \ h'(x) \ a_{\alpha_j} a_{\alpha_k+1} g_{\alpha_j}(h(x)) \Delta G_{\alpha_k}(x) dx]$$

$$= \sum_{P(\alpha)} \binom{\alpha}{\alpha_1 \alpha_2 \ldots \alpha_k} [\sum_{j=1}^{k-1} \int \{ \prod_{\substack{i=1 \\ i \neq j}}^{k-1} a_{\alpha_i} G_{\alpha_i}(h(x)) \} \ M_{\alpha_j, \alpha_k}(x) dx] \ .$$

Now we proceed to the alternative proof of Theorem 1.5.2. By Lemmas 1.5.2 and 1.5.4, $\psi(\lambda;c,d,k)$ is non-decreasing in λ if for $\alpha=0,1,2,\ldots,$

$$(1.5.53) \quad \sum_{P(\alpha)} \binom{\alpha}{\alpha_1 \alpha_2 \ldots \alpha_k} [\sum_{j=1}^{k-1} \int \{ \prod_{\substack{i=1 \\ i \neq j}}^{k-1} a_{\alpha_i} G_{\alpha_i}(h(x)) \} M_{\alpha_j, \alpha_k}(x)] \geq 0.$$

In (1.5.53) interchanging the summation, fixing $\alpha_i, i=1,2,\ldots,k-1;$ $i \neq j$, and summing over α_j and α_k with $\alpha_j + \alpha_k = \ell$, we find that (1.5.53) holds if, for every interger $\ell \geq 0$,

$$(1.5.54) \sum_{\alpha=0}^{\ell} \binom{\ell}{\alpha} [a_{\ell+1} a_{\ell-\alpha} g_{\ell-\alpha}(x) \Delta G_{\alpha}(h(x)) - h'(x) a_{\alpha} a_{\ell-\alpha+1} g_{\alpha}(h(x) \Delta G_{\ell-\alpha}(x)] \geq 0.$$

which is same as (1.5.29). This completes the proof of Theorem 1.5.2.

1.6. Properties of the Procedure R_h

In this section we will examine some of the important properties of the procedure R_h.

A. Monotonicity of R_h

The procedure R_h defined in Section 1.2 was meant for selecting a subset including the one associated with $\lambda_{[k]}$. Hence a desirable property is that the larger the λ value of a population the higher the probability of including it in the selected subset. To put it mathematically, for $1 \leq i < j \leq k$,

$$P\{\pi_{(i)} \text{ is included } \} \leq P\{\pi_{(j)} \text{ is included}\}, \text{ where}$$

$\pi_{(r)}(r=1,2,\ldots,k)$ is the population associated with $\lambda_{[r]}$. This is known as the monotonicity property. It implies unbiasedness which means that $P\{\pi_{(k)}$ is included$\} \geq P\{\pi_{(j)}$ is included$)$ for $j=1,2,\ldots,k-1$.

Theorem 1.6.1. The procedure R_h has monotonicity, if $h(x)$ is non-decreasing in x.

Proof. Let $p_r = P\{\pi_{(r)}$ is included$\}$, $r=1,2,\ldots,k$. Then for $1 \leq i < j \leq k$,

$$p_j = \int \{\prod_{\substack{r=1 \\ r\neq i,j}}^{k} F_{[r]}(h(x))\} F_{[i]}(h(x)) f_{[j]}(x)dx$$

$$\geq \int \{\prod_{\substack{r=1 \\ r\neq i,j}}^{k} F_{[r]}(h(x))\} F_{\lambda}(h(x)) f_{[j]}(x)dx, \text{ for } \lambda > \lambda_{[i]}$$

since F_λ is stochastically increasing in λ. Now by Theorem 1.4.1, the last integral is non-decreasing in $\lambda_{[j]}$. Hence

$$p_j \geq \int \{ \prod_{\substack{r=1 \\ r \neq i,j}}^{k} F_{[r]}(h(x)) \} F_\lambda(h(x)) f_{[i]}(x)dx. \tag{1.6.1}$$

Since (1.6.1) is true for any $\lambda > \lambda_{[i]}$, we get

$$p_j \geq \int \{ \prod_{\substack{r=1 \\ r \neq i,j}}^{k} F_{[r]}(h(x)) \} F_{[j]}(h(x)) f_{[i]}(x)dx = p_i. \tag{1.6.2}$$

This completes the proof of Theorem 1.6.1.

B. Expected Subset Size and Related Concepts

In the subset selection formulation any procedure passes as a candidate if it satisfies the P* condition. So we do need a criterion to compare prospective procedures. The very spirit of the subset selection approach makes it desirable that the expected size of the subset selected be as small as possible. Thus we are led to the following considerations. Let S be the size of the subset selected. S is a random variable with possible values $1,2,\ldots,k$. As before we let p_r stand for the probability that $\pi_{(r)}$ is included. Then it is easy to see that $E(S) \equiv E(S|R_h)$, the expected subset size using the procedure R_h is given by

$$E(S) = p_1 + p_2 + \ldots + p_k. \tag{1.6.3}$$

Of course, $E(S)$ depends upon the configuration of the λ's and whenever it is necessary to emphasize it we will write $E_{\underline{\lambda}}(S)$. To compare two procedures R_1 and R_2 we would like to compare the suprema of $E(S|R_1)$ and $E(S|R_2)$ over all possible configurations of the parameters. Thus our first concern is to evaluate the $\sup E(S|R_h)$.

We know that

$$p_i = \int \prod_{\substack{r=1 \\ r \neq i}}^{k} F_{[r]}(h(x))\, dF_{[i]}(x), \quad i=1,2,\ldots,k. \tag{1.6.4}$$

We can see that in the notations of section 1.4

$$\begin{cases} F_{[i]}(h(x)) = \psi(x,\lambda_{[i]}), \\ p_i = A_i(\lambda_{[1]},\ldots,\lambda_{[k]}), \quad i=1,\ldots,k, \\ E(S|R_h) = B(\lambda_{[1]},\ldots,\lambda_{[k]}). \end{cases} \tag{1.6.5}$$

The condition (1.4.7) becomes in this case

$$\frac{\partial}{\partial\lambda_1} F_{\lambda_1}(h(x)) f_{\lambda_2}(x) - h'(x) \frac{\partial}{\partial\lambda_1} F_{\lambda_1}(x) f_{\lambda_2}(h(x)) \geq 0, \; \lambda_1 \leq \lambda_2. \tag{1.6.6}$$

Hence we can state the following theorem which is a consequence of Lemma 1.4.5 and Theorem 1.4.3 in the present case.

<u>Theorem 1.6.2.</u> If $\lambda_{[1]} = \cdots = \lambda_{[m]} = \lambda \leq \lambda_{[m+1]} \leq \cdots \leq \lambda_{[k]}$, then $E(S|R_h)$ is non-decreasing in λ, when $\lambda_{[m+1]},\ldots,\lambda_{[k]}$ are kept fixed, provided that (1.6.6) holds. Also, consequently, the supremum

of $E(S|R_h)$ is attained for $\lambda_1 = \lambda_2 = \dots = \lambda_k$ if (1.6.6) holds.

Remark 1.6.1. We note that the condition (1.6.6) implies (1.4.27) which assures that $\psi(\lambda,c,d,k)$ is non-decreasing in λ. Thus (1.6.6) simplifies the evaluation of the supremum of $E(S)$.

Special Cases.

Case (1): $f_\lambda(x) = f(x-\lambda)$, $\Lambda=(-\infty,\infty)$ and $h(x) = x+d$, $d > 0$. In this case, (1.6.6) becomes

$$f_{\lambda_2}(x+d)\, f_{\lambda_1}(x) - f_{\lambda_2}(x)\, f_{\lambda_1}(x+d) \geq 0, \quad \lambda_1 \leq \lambda_2 \tag{1.6.7}$$

and this is equivalent to saying that, for $\lambda_1 \leq \lambda_2$ and $x_1 \leq x_2$,

$$f_{\lambda_2}(x_2)\, f_{\lambda_1}(x_1) - f_{\lambda_2}(x_1)\, f_{\lambda_1}(x_2) \geq 0, \tag{1.6.8}$$

which is the condition for f_λ to have the MLR property.

Case (2): $f_\lambda(x) = \frac{1}{\lambda} f(\frac{x}{\lambda})$, $x \geq 0$, $\Lambda = (0,\infty)$ and $h(x) = cx$, $c > 1$.

Then (1.6.6) becomes

$$f_{\lambda_2}(cx)\, f_{\lambda_1}(x) - f_{\lambda_2}(x)\, f_{\lambda_1}(cx) \geq 0, \tag{1.6.9}$$

which is equivalent to (1.6.8), the condition for f_λ to have the MLR property.

Case (3): $f_\lambda(x)$ given by (1.5.1) with weights defined by (1.5.3) and (1.5.14).

Following the notations of Section 1.5, the condition (1.6.6) is equivalent to

$$(1.6.10)\quad r_{\lambda_2}(x)[A(\lambda_1)R'_{\lambda_1}(h(x))-A'(\lambda_1)R_{\lambda_1}(h(x))] - h'(x)r_{\lambda_2}(h(x))[A(\lambda_1)R'_{\lambda_1}(x)-A'(\lambda_1)R_{\lambda_1}(x)] \geq 0$$

where the primes over R and A denote derivatives w.r.t. λ_1. We know that in our special case $A(\lambda_1) = a_0(1-\lambda_1 p)^{-q/p}$ and

$$(1.6.11)\quad A(\lambda_1)\, R'_{\lambda_1}(x)-A'(\lambda_1)R_{\lambda_1}(x) = a_0(1-\lambda_1 p)^{-q/p}\sum_{j=0}^{\infty}\frac{\lambda_1^j}{j!}\, a_{j+1}\, \Delta G_j(x).$$

Using this, (1.6.10) holds if

$$(1.6.12)\quad \Big(\sum_{j=0}^{\infty}\frac{\lambda_1^j}{j!}\, a_{j+1}\, \Delta G_j(h(x))\Big)\Big(\sum_{j=0}^{\infty}\frac{\lambda_2^j}{j!}\, a_j g_j(x)\Big) - h'(x)\Big(\sum_{j=0}^{\infty}\frac{\lambda_1^j}{j!}\, a_{j+1}\Delta G_j(x)\Big)\Big(\sum_{j=0}^{\infty}\frac{\lambda_2^j}{j!}\, a_j g_j(h(x))\Big) \geq 0.$$

Since (1.6.12) should hold for $\lambda_1 \leq \lambda_2$, we can set $\lambda_2 = b\lambda_1$, $b \geq 1$ and write in the equivalent form

$$(1.6.13)\quad \Big(\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\, a_{j+1}\, \Delta G_j(h(x))\Big)\Big(\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\, a_j b^j g_j(x)\Big) - h'(x)\Big(\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\, a_{j+1}\, \Delta G_j(x)\Big)\Big(\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\, a_j b^j g_j(h(x))\Big) \geq 0.$$

We note that (1.6.13) is similar to (1.5.26), the only difference being $b^j g_j(\cdot)$ in the place of $g_j(\cdot)$. Hence, following the same line of reasoning as before, we can say that (1.6.13) holds if, for $b \geq 1$ and every integer $i \geq 0$,

$$(1.6.14)\quad \sum_{\alpha=0}^{i} \binom{i}{\alpha}[a_{\alpha+1}a_{i-\alpha}b^{i-\alpha}g_{i-\alpha}(x)\Delta G_\alpha(h(x)) - h'(x)\, a_{i-\alpha+1}a_\alpha b^\alpha g_\alpha(h(x))\, \Delta G_{i-\alpha}(x)] \geq 0.$$

Because of the relation among a_j's, (1.6.14) can be thrown in the form

$$(1.6.15)\quad \sum_{\alpha=0}^{i} \binom{i}{\alpha}a_\alpha a_{i-\alpha}[b^{i-\alpha}(q+p\alpha)g_{i-\alpha}(x)\ \Delta G_\alpha(h(x)) - h'(x)b^\alpha(q+p(i-\alpha))g_\alpha(h(x))\ \Delta G_{i-\alpha}(x)] \geq 0.$$

A stronger condition which will imply (1.6.15) will be that for $\alpha=0,1,2,\ldots,[\frac{i}{2}]$,

$$(1.6.16)\quad b^{i-\alpha}(q+p\alpha)\ [g_{i-\alpha}(x)\ \Delta G_\alpha(h(x))-h'(x)g_{i-\alpha}(h(x))\ \Delta G_\alpha(x)] + b^\alpha(q+p(i-\alpha))\ [g_\alpha(x)\ \Delta G_{i-\alpha}(h(x))-h'(x)g_\alpha(h(x))\ \Delta G_{i-\alpha}(x)] \geq 0.$$

Remark 1.6.2. A sufficient condition for $\psi(\lambda;c,d,k)$ to be non-increasing in λ is either (1.6.15) or (1.6.16) with $b=1$ which is implied by (1.6.15) or (1.6.16), respectively.

For the case when $h(x) = cx$, (1.6.16) becomes

$$(1.6.17) \quad b^{i-\alpha}(q+p\alpha)\,[g_{i-\alpha}(x)\ \Delta G_{\alpha}(cx) - cg_{i-\alpha}(cx)\ \Delta G_{\alpha}(x)] +$$
$$b^{\alpha}(q+p(i-\alpha))\,[g_{\alpha}(x)\ \Delta G_{i-\alpha}(cx) - cg_{\alpha}(cx)\ \Delta G_{i-\alpha}(x)] \geq 0.$$

Though we will be dealing with distributions and procedures falling under this special case in the next chapter, we will consider at this time an example where (1.6.17) is satisfied. Gupta and Studden [45] consider selection of non-central chi-square populations in terms of their non-centrality parameters. In this case $w(\lambda,j) = \frac{e^{-\lambda}\lambda^j}{j!}$, $\lambda \geq 0$ and $g_j(x) = \frac{e^{-x}x^{\mu+j-1}}{\Gamma(\mu+j)}$, $\mu > 0$. We know that $q=1$ and $p=0$ for the Poisson weights. It is known in this case that $\Delta G_j(x) = -g_{j+1}(x)$. Thus (1.6.17) reduces to

$$(1.6.18) \quad b^{i-\alpha}\,[c\ g_{i-\alpha}(cx)\ g_{\alpha+1}(x) - g_{i-\alpha}(x)\ g_{\alpha+1}(cx)] +$$
$$b^{\alpha}\quad [c\ g_{\alpha}(cx)\ g_{i-\alpha+1}(x) - g_{\alpha}(x)\ g_{i-\alpha+1}(cx)] \geq 0.$$

Let $Q(\alpha) = c\ g_{i-\alpha}\,(cx)\ g_{\alpha+1}(x) - g_{i-\alpha}(x)\ g_{\alpha+1}(cx)$. Then

$$Q(\alpha) = \frac{e^{-(c+1)x}\ x^{2\mu+i-1}c^{\mu}}{\Gamma(\mu+i-\alpha)\ \Gamma(\mu+\alpha+1)}\quad (c^{i-\alpha} - c^{\alpha}).$$

≥ 0, since $c \geq 1$ and $i-\alpha \geq \alpha$. Hence (1.6.18) holds.

The above example shows that for the procedure of Gupta and Studden [45] for non-central chi-square populations the $\sup_{\Omega} E(S) = \sup_{\lambda} k\ \psi(\lambda;c,d,k)$.

Associated with $E(S)$, we can define another criterion for the efficiency of a procedure. Let S' be the number of non-best populations included in the selected subset. Then S' is an integer-valued random variable taking on values 0 through $k-1$. Obviously $S-S'$ is a random variable taking values 0 and 1 with probabilities $1-p_k$ and p_k respectively. Hence

$$E(S) = E(S') + p_k. \tag{1.6.19}$$

since $p_k = \int \prod_{r=1}^{k-1} F_{[r]}(h(x))\, dF_{[k]}(x)$, we see that $\frac{\partial p_k}{\partial \lambda_{[1]}} \leq 0$. Thus p_k is non-increasing in $\lambda_{[1]}$, when other λ's are kept fixed. Consequently we have the following result.

Lemma 1.6.1. $E(S')$ is non-decreasing in $\lambda_{[1]}$, the other λ's kept fixed, provided that (1.6.6) is satisfied.

Proof. Since $E(S') = E(S) - p_k$, the result follows immediately because $E(S)$ is non-decreasing in $\lambda_{[1]}$ by Theorem 1.6.2 and p_k is non-increasing in $\lambda_{[1]}$.

Lemma 1.6.2. When $\lambda_{[1]} = \cdots = \lambda_{[m]} = \lambda \leq \lambda_{[m+1]} \leq \cdots \leq \lambda_{[k]}$, $1 \leq m \leq k-1$, $E(S'|R_h)$ is non-decreasing in λ, provided that (1.6.6) is satisfied.

Proof. From the proofs of Lemmas 1.4.4 and 1.4.5, it can be easily seen that $E(S') \equiv \sum_{i=1}^{k-1} A_i(\lambda_{[1]},\ldots,\lambda_{[k]})$ can be shown to be non-decreasing in λ if (1.6.6) holds. The details of the proof are omitted.

Theorem 1.6.3. For the procedure R_h, $E_{\underline{\lambda}}(S')$ attains its supremum for $\lambda_1 = \lambda_2 = \ldots = \lambda_k$, provided the density $f_\lambda(x)$, $\lambda \epsilon \Lambda$ satisfies, the condition (1.6.6), which reduces to the condition for $f_\lambda(x)$ to have the MLR property in the cases of location and scale parameters with $h(x) = x+d$ and $h(x) = cx$ respectively.

Proof. The conclusions of the theorem follow from successive applications of Lemma 1.6.2.

Another property of $E_{\underline{\lambda}}(S')$ which is true without any further condition beyond the stochastic ordering of F_λ, $\lambda \epsilon \Lambda$ is as follows.

Theorem 1.6.4. For the procedure R_h, $E_{\underline{\lambda}}(S')$ is non-increasing in $\lambda_{[k]}$, when other λ's are kept fixed.

Proof. From (1.6.4) we see that $\frac{\partial p_i}{\partial \lambda_{[k]}} \leq 0$ for $i=1,\ldots,k-1$, since $\frac{\partial}{\partial \lambda_{[k]}} F_{[k]}(h(x)) \leq 0$. Hence the conclusion of the theorem follows since, $E(S) = \sum_{i=1}^{k-1} p_i$.

C. Invariance and Minimax Properties

We define below an invariance or symmetry property used by Gupta and Studden [44].

Definition 1.6.1. Let $X_1,\ldots,X_k$ be a set of observations from k populations $\pi_1,\ldots,\pi_k$ and R be a procedure which selects π_i with probability $\varphi_i(X_1,\ldots,X_k)$. Then the procedure R is said to be invariant or symmetric if

$$\varphi_i(X_1,\ldots,X_i,\ldots,X_j,\ldots,X_k) = \varphi_j(X_1,\ldots,X_j,\ldots,X_i,\ldots,X_k) \tag{1.6.18}$$

for all i and j.

If $\underline{\lambda}_0$ is the point Ω where $\lambda_1 = \lambda_2 = \cdots = \lambda_k = \lambda_0$, i.e. $\underline{\lambda}_0 = (\lambda_0, \lambda_0, \ldots, \lambda_0)$, then Gupta and Studden [44] have shown that, for any invariant procedure R,

$$E_{\underline{\lambda}_0}(S|R) = k\, P_{\underline{\lambda}_0}(CS|R). \tag{1.6.19}$$

The invariance property defined by (1.6.18) says that if the ith and jth observations are interchanged, then we select the jth population with probability equal to that of selecting π_i before the interchanging. Suppose we consider the class of invariant rules satisfying the basic P* condition and R is a rule of the class for which

$$\inf_{\Omega} P_{\underline{\lambda}}(CS|R) = P_{\underline{\lambda}_0}(CS|R) = P^* \tag{1.6.20}$$

and

$$\sup_{\Omega} E_{\underline{\lambda}}(S|R) = E_{\underline{\lambda}_0}(S|R). \tag{1.6.21}$$

Then it has been shown by Gupta and Studden that R is minimax in the class of invariant rules in the sense that for any other procedure R' in the class

$$\sup_{\Omega} E_{\underline{\lambda}}(S|R') \geq \sup_{\Omega} E_{\underline{\lambda}}(S|R). \tag{1.6.22}$$

As one can see from their proof, this minimax property of R which satisfies (1.6.20) and (1.6.21) can be shown for any class of rules for which, under the equal parameter configuration, the expected subset is k times the probability of a correct selection. The procedure R_h is invariant and if $P_{\underline{\lambda}_0}(CS|R_h)$ is independent of $\underline{\lambda}_0$ then (1.6.20) and (1.6.21) are satisfied in the case of R_h. Hence under these conditions R_h is minimax in the class of invariant procedures.

1.7. Selection of the Population Associated with $\lambda_{[1]}$

A. The Procedure R_H and Probability of a Correct Selection

The case where the best population is defined to be the one associated with $\lambda_{[1]}$ is analogous to the case of $\lambda_{[k]}$. We need of course to make certain modifications. We will briefly mention them and state the results without proofs wherever they are exactly similar to those in the case of $\lambda_{[k]}$.

Let $H \equiv H_{c,d}$; $c \in [1,\infty)$, $d \in [0,\infty)$ be a function defined on the real line such that for every x belonging to the support of F_λ,

(1.7.1)

(i) $H_{c,d}(x) \leq x$

(ii) $H_{1,0}(x) = x$

(iii) $H_{c,d}(x)$ is continuous in c and d

(iv) $H_{c,d}(x) \downarrow -\infty$ as $d \to \infty$ and/or

$H_{c,d}(x) \downarrow 0$ as $c \to \infty$.

A class of procedures R_H for selecting a subset containing the best is defined as follows.

R_H: Include π_i in the selected subset iff

(1.7.2) $$H(x_i) \leq x_{min},$$

where $x_{min} = \min_{1 \leq r \leq k} x_r$.

This procedure obviously selects a non-empty subset in view of (1.7.1)-(i). The probability of a correct selection is given by

(1.7.3) $$P(CS|R_{H_{c,d}}) = \int \prod_{r=1}^{k-1} \bar{F}_{[r]}(H(x))\, dF_{[1]}(x),$$

where $\bar{F}_\lambda(x) = 1-F_\lambda(x)$. Because of the assumption (1.1.1) regarding the stochastic ordering of the distributions,

(1.7.4) $$P(CS|R_{H_{c,d}}) \geq \int \bar{F}_{[1]}^{k-1}(H(x))\, dF_{[1]}(x).$$

Hence we have

(1.7.5) $$\inf_{\Omega} P(CS|R_{H_{c,d}}) = \inf_{\lambda \in \Lambda} \int \bar{F}_{\lambda}^{k-1}(H(x))\, dF_{\lambda}(x).$$

Let

(1.7.6) $$\varphi(\lambda;c,d,k) = \int \bar{F}_{\lambda}^{k-1}(H(x))\, dF_{\lambda}(x).$$

Because of (1.7.1)-(i) and (ii),

(1.7.7) $$\varphi(\lambda;c,d,k) \geq \frac{1}{k} \quad \text{and}$$

(1.7.8) $$\varphi(\lambda;1,0,k) = \frac{1}{k}.$$

Properties (1.7.1)-(iii) and (iv)

(1.7.9) $$\lim_{d\to\infty} \varphi(\lambda;\ c,d,k) = 1 \quad \text{and/or}$$

(1.7.10) $$\lim_{c\to\infty} \varphi(\lambda;\ c,d,k) = [1-F_\lambda(0)]^{k-1}.$$

If (1.7.9) holds, then for every λ, c and k, we can choose d such that the basic probability requirement is satisfied. If (1.7.10) alone holds, then for every λ, d and k, we can choose c in order to satisfy the basic probability requirement provided that $[1-F_\lambda(0)]^{k-1} \geq P^*$ for all admissible λ and P^*. Since P^* can be as close to 1 as we desire, this means that we should have $F_\lambda(0) = 0$. Hence, if (1.7.10) holds but not (1.7.9), then to evaluate the constants of the procedure whatever P^*, we must have non-negative random variables. Corresponding to Lemma 1.3.2 we have the following lemma.

<u>Lemma 1.7.1</u>. Let X and Y be random variables having densities f(x) and g(y) and cdf's F(x) and G(y) respectively. Let H(x) be a function such that

(1.7.11) $$H(\varphi(x)) \leq \varphi(H(x)),$$

where $\varphi \equiv G^{-1} F$. Then, for any $t > 0$,

(1.7.12) $$\int \bar{F}^t(H(x))\ f(x)dx \leq \int \bar{G}^t(H(x))\ g(x)dx.$$

The proof runs on similar lines and so is omitted. A sufficient condition for the monotonicity of $\varphi(\lambda;c,d,k)$ is given in the following theorem.

<u>Theorem 1.7.1</u>. For the procedure R_H, $\varphi(\lambda;c,d,k)$ is non-decreasing in λ provided that

$$(1.7.13) \qquad H'(x)\, f_\lambda(H(x))\, \frac{\partial}{\partial\lambda} F_\lambda(x) - f_\lambda(x)\, \frac{\partial}{\partial\lambda} F_\lambda(H(x)) \geq 0,$$

where $H'(x) = \frac{d}{dx} H(x)$ and $\varphi(\lambda;c,d,k)$ is strictly increasing in λ if strict inequality holds in (1.7.13) on a set of positive Lebesgue measure.

The proof is immediate by using Corollary 1.4.2 with $\Psi(x,\lambda)= \bar{F}_\lambda(H(x))$.

Now we state other results without proofs, since they are all analogous to the case of R_h.

<u>Theorem 1.7.2</u>. For the procedure R_H, when $f_\lambda(x)$ is given by (1.5.1) with weight function given by (1.5.3), $\varphi(\lambda;c,d,k)$ is non-decreasing in λ provided that for every non-negative integer i,

$$(1.7.14) \qquad \sum_{\alpha=0}^{i} \binom{i}{\alpha} [H'(x) a_\alpha g_\alpha(H(x)) B_{i-\alpha}(x) - a_{i-\alpha} g_{i-\alpha}(x) B_\alpha(H(x))] \geq 0$$

where $B_\alpha(x)$ is given by (1.5.10).

<u>Theorem 1.7.3</u>. For the procedure R_H, when $f_\lambda(x)$ is given by (1.5.1) with weights defined by (1.5.3) and (1.5.14), $\varphi(\lambda;c,d,k)$ is non-decreasing in λ provided that for every integer $i \geq 0$,

$$(1.7.15) \qquad \sum_{\alpha=0}^{i} \binom{i}{\alpha} a_\alpha a_{i-\alpha} [H'(x)(q+\overline{i-\alpha}\,p) g_\alpha(H(x)) \Delta G_{i-\alpha}(x) - (q+\alpha p) g_{i-\alpha}(x)\, \Delta G_\alpha(H(x))] \geq 0.$$

Strict inequality in (1.7.15) for some i implies that $\varphi(\lambda;c,d,k)$ is strictly increasing in λ.

Remark 1.7.1. Suppose we use the procedure R_H with $H(x) = \frac{x}{c}$ (in the case of non-negative r.v.'s) or $H(x) = x-d$. Then (1.7.13) reduces to

(1.7.16) $$\frac{1}{c} f_\lambda(\tfrac{x}{c}) \frac{\partial}{\partial\lambda} F_\lambda(x) - f_\lambda(x) \frac{\partial}{\partial\lambda} F_\lambda(\tfrac{x}{c}) \geq 0$$

or

(1.7.17) $$f_\lambda(x-d) \frac{\partial}{\partial\lambda} F_\lambda(x) - f_\lambda(x) \frac{\partial}{\partial\lambda} F_\lambda(x-d) \geq 0.$$

Setting $\frac{x}{c} = y$ or $x-d = y$, we get

(1.7.18) $$f_\lambda(y) \frac{\partial}{\partial\lambda} F_\lambda(cy) - c f_\lambda(cy) \frac{\partial}{\partial\lambda} F_\lambda(y) \geq 0$$

or

(1.7.19) $$f_\lambda(y) \frac{\partial}{\partial\lambda} F_\lambda(y+d) - f_\lambda(y+d) \frac{\partial}{\partial\lambda} F_\lambda(y) \geq 0.$$

(1.7.18) and (1.7.19) are sufficient conditions for $\Psi(\lambda;c,d,k)$ to be non-decreasing in λ in the case of the procedure R_h with $h(x) = cx$ and $h(x) = x+d$ respectively. Hence, we see that the sufficient condition has to be verified only once.

B. Properties of the Procedure R_H

In this case

(1.7.20) $$p_i = \int \{ \prod_{\substack{r=1 \\ r \neq i}}^{k} \bar{F}_{[r]} (H(x)) \} f_{[i]}(x)dx, \quad i=1,\ldots,k.$$

We first state a modification of Theorem 1.4.1.

Lemma 1.7.2. Let $\{F_\lambda\}$ be a family of distribution functions on the real line which are stochastically increasing in λ. Then $E_\lambda \Psi(x)$ is non-increasing in λ for any non-increasing function $\Psi(x)$. Using the above lemma, we obtain the following result.

Theorem 1.7.4. The procedure R_H has monotonicity, if $H(x)$ is non-decreasing in x.

Further, using the same method of proof as in the case of R_h, we obtain the following results concerning the expected size of the selected subset.

Theorem 1.7.5. For the procedure R_H, $E_{\underline{\lambda}}(s)$ is non-decreasing in $\lambda_{[1]}$ when other λ's are kept fixed provided that, for $\lambda_1 \leq \lambda_2$,

$$H'(x)f_{\lambda_2}(H(x)) \frac{\partial}{\partial\lambda_1} F_{\lambda_1}(x) - f_{\lambda_2}(x) \frac{\partial}{\partial\lambda_1} F_{\lambda_1}(H(x)) \geq 0. \tag{1.7.21}$$

Theorem 1.7.6. $E_{\underline{\lambda}}(s)$ attains its supremum at a point in Ω where $\underline{\lambda}$ has equal components provided that (1.7.21) holds.

Remark 1.7.2. In the cases of location and scale parameters with $H(x) = x-d$ and $H(x) = \frac{x}{c}$ respectively, (1.7.21) is the condition that $f_\lambda(x)$ has the MLR property.Also a remark similar to Remark 1.7.1 applies to the condition (1.7.21).

For the procedure R_H,

$$E(S') = p_2 + \cdots + p_k = E(S) - p_1. \tag{1.7.22}$$

From the proofs of Theorems 1.7.5 and 1.7.6, it will be easy to see that $E(S')$ is non-decreasing in $\lambda_{[2]}$ and increases in λ where $\lambda_{[1]} \leq \lambda_{[2]} = \cdots = \lambda_{[m]} = \lambda \leq \lambda_{[m+1]} \leq \cdots \leq \lambda_{[k]}$ for $2 \leq m \leq k$ provided that (1.7.21) holds. Hence, if (1.7.21) holds, we have

$$\text{(1.7.22)} \qquad \sup_{\Omega} E_{\underline{\lambda}}(S') = \sup_{\Omega'} E_{\underline{\lambda}}(S'),$$

$\Omega' = \{\underline{\lambda} | \lambda_{[1]} \leq \lambda_{[2]} = \cdots = \lambda_{[k]}\}$. Because of the stochastic ordering p_i, $i=2,\ldots,k$ and hence $E_{\underline{\lambda}}(S')$ is non-decreasing in $\lambda_{[1]}$ when other λ's are kept fixed. Hence we obtain the following result.

Theorem 1.7.7. For the procedure R_H, $E(S')$ attains its supremum at a point in Ω where $\underline{\lambda}$ has all its components equal if (1.7.21) holds.

1.8. A Review

In this section, we want to collect some essential results and present them together in a summary form. Let $\pi_1, \pi_2, \ldots, \pi_k$ be k continuous populations with distributions F_{λ_i}, $i=1,2,\ldots,k$; $\lambda_i \in \Lambda$, an interval on the real line. We assume that $\{F_\lambda\}$ is a stochastically increasing family. Let $h(x) \equiv h_{c,d}(x)$ be a non-decreasing function in x satisfying the properties in (1.2.1). Assume that, for $\lambda_1 \leq \lambda_2$, (1.6.6) holds. That is,

$$f_{\lambda_2}(x) \frac{\partial}{\partial \lambda_1} F_{\lambda_1}(h(x)) - h'(x) f_{\lambda_2}(h(x)) \frac{\partial}{\partial \lambda_1} F_{\lambda_1}(x) \geq 0.$$

Then, for selecting a subset of the k populations including the population associated with $\lambda_{[k]}$ subject to the basic P* condition (1.1.2), we propose the rule R_h which includes π_i in the selected subset iff $h(x_i) \geq \max_{1 \leq r \leq k} x_r$, where $x_1, \ldots, x_k$ are a set of observations from $\pi_1, \ldots, \pi_k$ respectively. For this rule,

(1) $\inf_{\Omega} P_{\underline{\lambda}}(CS|R_h) = \inf_{\lambda} P_{\lambda}(CS|R_h)$, where $P_{\lambda}(CS|R_h)$ denotes the probability of a correct selection when $\underline{\lambda} = (\lambda,\lambda,\ldots,\lambda)$;

(2) $P_{\lambda}(CS|R_h)$ is non-decreasing in λ;

(3) Prob$\{\pi_{(i)}$ is included$\} \leq P\{\pi_{(j)}$ is included$\}$ for $1 \leq i < j \leq k$, where $\pi_{(r)}$ is the population associated with $\lambda_{[r]}$, $r=1,\ldots,k$;

(4) $E_{\underline{\lambda}}(S)$ is non-decreasing in λ, where $\lambda_{[1]}=\cdots=\lambda_{[m]}=\lambda\leq\lambda_{[m+1]}\leq\cdots\leq\lambda_{[k]}$, for $1 \leq m \leq k$;

(5) $E_{\underline{\lambda}}(S)$ attains its supremum at a point of Ω where $\underline{\lambda}$ has all its components equal and hence $\sup_{\Omega} E_{\underline{\lambda}}(S) = k \sup_{\lambda} P_{\lambda}(CS|R_h)$;

(6) $E_{\lambda}(S')$ attains its supremum at a point of Ω where $\underline{\lambda}$ has all its components equal and hence $\sup_{\Omega} E_{\underline{\lambda}}(S') = (k-1) \sup_{\lambda} P_{\lambda}(CS|R_h)$.

(7) In the class of invariant rules (see (1.6.32)) satisfying the basic P* condition, the rule R_h is minimax in the sense that, for any other rule R' in the class, $\sup_{\Omega} E_{\underline{\lambda}}(S|R') \geq \sup_{\Omega} E_{\underline{\lambda}}(S|R_h)$, provided that $P(CS|R_h)$ is independent of $\lambda\epsilon\Lambda$.

CHAPTER II

SOME SELECTION PROCEDURES FOR MULTIVARIATE NORMAL POPULATIONS

2.1 Introduction

Most of the earlier work in the area of selection and ranking problems pertained to univariate populations. During the last few years some work has been done on selection and ranking problems for multivariate normal populations. Selection in terms of Mahalanobis distance function has been considered by Alam and Rizvi [1], Gupta [34] and Gupta and Studden [45]. Krishnaiah [51] and Krishnaiah and Rizvi [53] have investigated procedures for selection in terms of linear combinations of components of the mean vector and elements of the covariance matrix. Gnanadesikan [26] and Gnanadesikan and Gupta [27] have studied selection in terms of the generalized variance. Some other problems such as selection with respect to the means of correlated normal populations have also been considered by Gnanadesikan [26]. Selection in terms of the cell probabilities in a multinomial distribution has been discussed by Bechhofer, Elmaghraby and Morse [13], Cacoullos and Sobel [16], and Gupta and Nagel [36]. In many of the multivariate problems of selection and ranking, certain probability integrals arise; these and other related topics are discussed in Gupta [30, 31].

Although the procedures are for multivariate populations, the ranking of the populations is in terms of a scalar function of the parameters and the statistic used in any procedure is one which has a univariate distribution involving the parameters through that scalar function. So these procedures are also useful in the situations where the observations come from the respective univariate distributions.

Most of the present chapter is devoted to the selection problem for multivariate normal populations in terms of multiple correlation coefficient. The multiple correlation coefficient is a measure of the dependence of one variable on the remaining and is used in studies connected with behavioral sciences. Formulae have been obtained for computing the constants used in the procedures. Tables have been constructed in certain cases.

The last section deals with k p-variate normal populations in which the interest is in selection in terms of $|\Sigma|/|\Sigma_{11}|=|\Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}|$, where Σ_{11}, Σ_{12}, Σ_{21} and Σ_{22} are the covariance matrices corresponding to a partition of the p variables into two sets of q_1 and q_2 variables, $q_1+q_2=p$. Then the criterion represents the conditional generalized variance of the q_2 set when the q_1 set is held fixed. (See Anderson [2]).

2.2. Selection in Terms of Multiple Correlation Coefficient

A. The Set-up and the Notations

Let $\underline{X}_i' = (X_{i1},X_{i2},\ldots,X_{ip})$, $i=1,2,\ldots,k$ be random vectors with p-variate normal distributions with unknown mean vectors $\underline{\mu}_i$ and unknown positive definite covariance matrices Σ_i. The multiple correlation coefficient between, say, X_{i1} and $X_{i2},\ldots,X_{ip}$ denoted by $\rho_{1.2\ldots p}^{(i)} \equiv \rho_i$, is defined by

$$(2.2.1) \qquad 1-\rho_i^2 = \frac{|\Sigma_i|}{\sigma_{i11}|\Sigma_{i(11)}|},$$

where σ_{i11} is the leading element of Σ_i and $\Sigma_{i(11)}$ is the matrix obtained from Σ_i by deleting the first row and the first column. This ρ_i (taken to be the positive square root of ρ_i^2) is the maximum of the correlation between X_{i1} and a linear combination of $X_{i2},\dots,X_{ip}$ over all possible linear combinations. We assume that $\rho_i < 1$ for $i=1,2,\dots,k$. Let $0 \le \rho_{[1]} \le \rho_{[2]} \le \dots \le \rho_{[k]} < 1$ be the ordered ρ's. $R^{(i)}_{1.2\dots p} \equiv R_i$, the sample multiple correlation coefficient between X_{i1} and $X_{i2},\dots,X_{ip}$ is defined analogous to ρ_i by replacing Σ_i by the sample covariance matrix S_i. Two cases arise: (1) the case when $X_{i2},\dots,X_{ip}$ are fixed, called the 'conditional case' (2) the case when $X_{i2},\dots,X_{ip}$ are random, called the 'unconditional case'. (See Kendall and Stuart [50]). Under each case we will discuss subset selection for the population associated with $\rho_{[k]}$ as well as $\rho_{[1]}$. Other general notations are carried over from Chapter I.

B. Selection for the Population Associated with $\rho_{[k]}$

a. <u>A Procedure Based on R^2 (Unconditional Case).</u> Based on R_i^2 obtained from a sample of size $n > p$ from π_i, $i=1,2,\dots,k$, we propose the following procedure.

R_1: Select π_i iff $cR_i^2 \ge \max_{1\le r\le k} R_r^2$,

where $c = c(k,P^*,p,n) > 1$ is chosen so that the P^*-condition is satisfied. Then, letting $\lambda_i = \rho_i^2$, $i=1,\dots,k$,

$$(2.2.2) \qquad P(CS|\Omega_1) = \int_0^1 \prod_{r=1}^{k-1} F_{\lambda_{[r]}}(cx)\, dF_{\lambda_{[k]}}(x),$$

where $F_{\lambda_{[i]}}$ is the distribution function of $R^2_{(i)}$, $i=1,2,\dots,k$. The density $f_\lambda(x)$ of R^2 in the unconditional case is given by

$$(2.2.3) \qquad f_\lambda(x) = \sum_{j=0}^{\infty} w(q,m;\lambda,j)\, b(x;q+j,m), \quad 0 \le x \le 1,$$

where

$$(2.2.4) \quad \begin{cases} q = \frac{p-1}{2} \quad , \qquad m = \frac{n-p}{2} \\ b(x;q+j,m) = \frac{\Gamma(q+j+m)}{\Gamma(q+j)\,\Gamma(m)}\, x^{q+j-1}\,(1-x)^{m-1}, \quad 0 \le x \le 1 \\ w(q,m;\lambda,j) \equiv w(\lambda,j) = \frac{\Gamma(q+m+j)}{\Gamma(q+m)}\,(1-\lambda)^{q+m}\,\frac{\lambda^j}{j!}\,, \quad 0 \le \lambda < 1 \end{cases}$$

Lemma 2.2.1. The density $f_\lambda(x)$ given by (2.2.3) has MLR in x and consequently the distribution of R^2 is stochastically increasing in λ.

Proof. For $\lambda_1 < \lambda_2$ and $j_1 < j_2$, it can be easily verified that $w(\lambda_1,j_1)\, w(\lambda_2,j_2) - w(\lambda_2,j_1)\, w(\lambda_1,j_2) > 0$ and for $j_1 < j_2$ and $x_1 < x_2$, $b(x_1;q+j_1,m)\, b(x_2;q+j_2,m) - b(x_1;q+j_2,m)\, b(x_2;q+j_1,m) > 0$. Hence by Lemma 1.5.1, $f_\lambda(x)$ has the MLR property, which implies the stochastic increasing in λ.

Lemma 2.2.2. For every integer $i \ge 0$ and $\alpha = 0,1,\dots,[\frac{i}{2}]$,

$$b^{i-\alpha}(q+m+\alpha)[g_{i-\alpha}(x)\Delta G_{\alpha}(cx)-cg_{i-\alpha}(cx)\Delta G_{\alpha}(x)] + \tag{2.2.5}$$
$$b^{\alpha}(q+m+i-\alpha)[g_{\alpha}(x)\Delta G_{i-\alpha}(cx)-cg_{\alpha}(cx)\Delta G_{i-\alpha}(x)] \geq 0$$

where $b \geq 1$ and $g_j(x) = b(x;q+j,m)$.

Proof. Obviously, (2.2.5) holds if $cx \geq 1$. So we assume $cx < 1$. For $0 \leq y \leq 1$, let $I_y(u,v)$ denote the incomplete beta function defined by

$$I_y(u,v) = \int_0^y b(t;u,v)\, dt. \tag{2.2.6}$$

Then

$$\Delta G_j(y) = I_y(q+j+1,m)-I_y(q+j,m) \tag{2.2.7}$$
$$= -\frac{\Gamma(q+j+m)}{\Gamma(q+j+1)\,\Gamma(m)}\, y^{q+j}\,(1-y)^m.$$

Hence

$$g_{i-\alpha}(x)\Delta G_{\alpha}(cx)-cg_{i-\alpha}(cx)\Delta G_{\alpha}(x) \tag{2.2.8}$$

$$= \frac{\Gamma(q+\alpha+m)\,\Gamma(q+i-\alpha+m)c^q c^{2q+i-1}(1-x)^{m-1}}{\Gamma(q+\alpha+1)\,\Gamma(q+i-\alpha)\,\{\Gamma(m)\}^2}[c^{i-\alpha}(1-x)-c^{\alpha}(1-cx)].$$

Using (2.2.8) on the left hand side of (2.2.5) and taking the common factors out, we see that (2.2.5) holds if

$$b^{i-\alpha}(q+m+\alpha)(q+i-\alpha)[c^{i-\alpha}(1-x)-c^{\alpha}(1-cx)] + \tag{2.2.9}$$
$$b^{\alpha}(q+m+i-\alpha)(q+\alpha)[c^{\alpha}(1-x)-c^{i-\alpha}(1-cx)] \geq 0.$$

Regrouping the terms on the lefthand side of (2.2.9) we have $(1-x)[c^{\alpha}b^{\alpha}(q+\alpha)(q+m+i-\alpha)+c^{i-\alpha}b^{i-\alpha}(q+m+\alpha)(q+i-\alpha)]-(1-cx)[c^{\alpha}b^{i-\alpha}(q+m+\alpha)(q+i-\alpha)+c^{i-\alpha}b^{\alpha}(q+m+i-\alpha)(q+\alpha)]$. To show that this is ≥ 0, we note that $1-cx \leq 1-x$, since $c \geq 1$. So it is sufficient if we show that

$$c^{\alpha}b^{\alpha}(q+\alpha)(q+m+i-\alpha)+c^{i-\alpha}b^{i-\alpha}(q+m+\alpha)(q+i-\alpha) \geq$$
$$c^{\alpha}b^{i-\alpha}(q+m+\alpha)(q+i-\alpha)+c^{i-\alpha}b^{\alpha}(q+m+i-\alpha)\ (q+\alpha)$$

which is same as

$$b^{\alpha}(q+\alpha)(q+m+i-\alpha)c^{\alpha}-c^{i-\alpha}) \geq b^{i-\alpha}(q+m+\alpha)(q+i-\alpha)(c^{\alpha}-c^{i-\alpha}). \tag{2.2.10}$$

Since $\alpha \leq i-\alpha$ and $c \geq 1$, $c^{\alpha}-c^{i-\alpha} \leq 0$. Hence (2.2.10) holds if

$$b^{\alpha}(q+\alpha)(q+mi-\alpha) \leq b^{i-\alpha}(q+m+\alpha)(q+i-\alpha). \tag{2.2.11}$$

Now $(q+m+\alpha)(q+i-\alpha)-(q+\alpha)(q+m+i-\alpha)= m(i-2\alpha) \geq 0$. Thus $(q+\alpha)(q+m+i-\alpha) \leq (q+m+\alpha)(q+i-\alpha)$ and also $b^{\alpha} \leq b^{i-\alpha}$. These two imply (2.2.11). This completes the proof of Lemma 2.2.2.

Theorem 2.2.1. Let Ω represent the space of $\underline{\rho} = (\rho_1,\rho_2,\ldots,\rho_k)$. Then

$$\inf_{\Omega} P(CS|\Omega_1) = \int_0^1 F^{k-1}(cx)\ dF(x), \tag{2.2.12}$$

where $F(x) \equiv F_0(x)$ is the cdf of R^2 when $\rho^2= 0$ and the corresponding density is given by

$$f(x) = \frac{\Gamma(q+m)}{\Gamma(q)\ \Gamma(m)}\ x^{q-1}\ (1-x)^{m-1},\quad 0 \leq x \leq 1,\ q,m > 0. \tag{2.2.13}$$

<u>Proof</u>. $P(CS|\mathcal{R}_1) \geq \int_0^1 F_{\lambda_{[k]}}^{k-1}(cx)dF_{\lambda_{[k]}}(x)$ by Lemma 2.2.1. Hence

(2.2.14) $$\inf_{\Omega} P(CS|\mathcal{R}_1) = \inf_{\lambda} \int_0^1 F_{\lambda}^{k-1}(cx)\, dF_{\lambda}(x).$$

By Lemma 2.2.2 and Remark 1.6.2 and noting that p=1 and q is to be replaced by q+m in (1.6.17) for the given weights, the integral on the right hand side of (2.2.14) is non-decreasing in λ. Hence the infimum takes place for $\lambda = 0$. This completes the proof of Theorem 2.2.1.

<u>Corollary 2.2.1</u>. The constant c defining the procedure $\mathcal{R}_1$ is given by

(2.2.15) $$\int_0^1 F^{k-1}(cx)\, dF(x) = P^*.$$

b. <u>A Procedure Based on R^2 (Conditional Case)</u>. In this case the density of R^2 is given by

(2.2.16) $$h_{\lambda}(x) = \sum_{j=0}^{\infty} w(\lambda,j)\, b(x;q+j,m), \quad 0 \leq x \leq 1,$$

where $b(x;q+j,m)$ is as defined in (2.2.4), $\lambda = \rho^2$ and

(2.2.17) $$w(\lambda,j) = e^{-m\lambda} \frac{(m\lambda)^j}{j!}.$$

Let $H_{\lambda}(x)$ denote the corresponding cdf. Then we use the procedure

$\mathcal{R}_2$: Select π_i iff $c'R_i^2 \geq \max_{1\leq r\leq k} R_r^2$,

where $c' = c'(k,P^*,p,n) > 1$ is chosen so as to satisy the P*-condition. The procedure $\mathcal{R}_2$ is of the same type as $\mathcal{R}_1$, but called differently in order to differentiate the conditional and the unconditional cases. As we will see the constants used in the two cases turn out to be the same.

Lemma 2.2.3. The density $h_\lambda(x)$ given by (2.2.16) has m.l.r. in x.

Proof. It is easily verified that $g_j(x) \equiv b(x;q+j,m)$ and $w(\lambda,j)$ are TP_2. Hence, by Lemma 1.5.1, $h_\lambda(x)$ has m.l.r. in x.

Lemma. 2.2.4. For every integer $i \geq 0$ and $\alpha = 0,1,\ldots,[\frac{i}{2}]$,

$$b^{i-\alpha}[g_{i-\alpha}(x)\Delta G_\alpha(cx)-cg_{i-\alpha}(cx)\Delta G_\alpha(x)] + \tag{2.2.18}$$

$$b^\alpha[g_\alpha(x)\Delta G_{i-\alpha}(cx)-cg_\alpha(cx)\Delta G_{i-\alpha}(cx)] \geq 0$$

where $b \geq 1$.

Proof. Obviously (2.2.18) holds if $cx \geq 1$. So we consider $cx < 1$. Using (2.2.8), we can see that (2.2.18) holds if

$$b^{i-\alpha}(q+i-\alpha)[c^{i-\alpha}(1-x)-c^\alpha(1-cx)] + \tag{2.2.19}$$

$$b^\alpha(q+\alpha)[c^\alpha(1-x)-c^{i-\alpha}(1-cx)] \geq 0.$$

Regrouping the terms on the left hand side of (2.2.19) we have

$$(1-x)[c^\alpha b^\alpha(q+\alpha)+c^{i-\alpha}b^{i-\alpha}(q+i-\alpha)]-(1-cx)[c^\alpha b^{i-\alpha}(q+i-\alpha)+c^{i-\alpha}b^\alpha(q+\alpha)] \geq 0.$$

Since $1-cx < 1-x$, it is sufficient if we show that

$$c^\alpha b^\alpha(q+\alpha)+c^{i-\alpha}b^{i-\alpha}(q+i-\alpha) \geq c^\alpha b^{i-\alpha}(q+i-\alpha)+c^{i-\alpha}b^\alpha(q+\alpha), \tag{2.2.20}$$

which is same as

$$b^\alpha(q+\alpha)(c^\alpha-c^{i-\alpha}) \geq b^{i-\alpha}(q+i-\alpha)(c^\alpha-c^{i-\alpha}). \tag{2.2.21}$$

Since $c^{\alpha}-c^{i-\alpha} \leq 0$, (2.2.21) holds if

$$b^{\alpha}(q+\alpha) \leq b^{i-\alpha}(q+i-\alpha). \tag{2.2.22}$$

Now, since $\alpha \leq i-\alpha$, $(q+\alpha) \leq (q+i-\alpha)$. Further, $b^{\alpha} \leq b^{i-\alpha}$, since $b \geq 1$. These two together imply (2.2.22). This completes the proof of Lemma 2.2.4.

Theorem 2.2.2. For Procedure Ω_2,

$$\inf_{\Omega} P(CS|\Omega_2) = \int_0^1 H^{k-1}(c'x)\, dH(x), \tag{2.2.23}$$

where $H(x) \equiv H_0(x)$ is the cdf of R^2 in the conditional case when $\rho^2 = 0$ and the corresponding density is given by

$$h(x) = \frac{\Gamma(q+m)}{\Gamma(q)\,\Gamma(m)}\, x^{q-1}\,(1-x)^{m-1}, \quad 0 \leq x \leq 1. \tag{2.2.24}$$

Proof.

$$P(CS|\Omega_2) = \int_0^1 \prod_{r=1}^{k-1} H_{\lambda_{[r]}}(cx)\, dH_{\lambda_{[k]}}(x)$$

$$\geq \int_0^1 H^{k-1}_{\lambda_{[k]}}(cx)\, dH_{\lambda_{[k]}}(x) \quad \text{by Lemma 2.2.3.}$$

Hence

$$\inf_{\Omega} P(CS|\Omega_2) = \inf_{\lambda} \int_0^1 H_{\lambda}^{k-1}(cx)\, dH_{\lambda}(x). \tag{2.2.25}$$

By Lemma 2.2.4 and Remark 1.6.2 and noting that q=1 and p=0 in (1.6.17) for the case of Poisson weights, we see that the integral on the right hand side of (2.2.25) is non-decreasing in λ. Hence the infimum takes place for $\lambda = 0$. This completes the proof of the theorem.

Corollary 2.2.2. The constant c' defining the procedure Ω_2 satisfies the equation

$$\int_0^1 H^{k-1}(c'x)\, dH(x) = P^*. \tag{2.2.26}$$

Remark 2.2.1. The distribution of R^2 when $\rho^2 = 0$ is the same in the conditional as well as the unconditional case. Thus the equations (2.2.15) and (2.2.26) are the same and $c' = c$.

c. Formulae for Evaluating the Constant Defining R_1 and Ω_2. The equation (2.2.15) can be rewritten as

$$\int_0^{c^{-1}} I_{cx}^{k-1}(q,m)\, b(x;q,m)\, dx + 1 - I_{c^{-1}}(q,m) = P^*. \tag{2.2.27}$$

When q and m are integers, (2.2.27) can be written in the form

$$1-P^* = I_{c^{-1}}(q,m) - \int_0^{c^{-1}} \left\{ \sum_{t=q}^{q+m-1} \binom{q+m-1}{t} (cx)^t (1-cx)^{q+m-1-t} \right\}^{k-1} \frac{\Gamma(q+m)}{\Gamma(q)\Gamma(m)} x^{q-1}(1-x)^{m-1} dx. \tag{2.2.28}$$

Let $\Lambda(q,m,k,c)$ denote the integral on the right hand side of (2.2.28). Then, setting $\ell = k-1$ and $y = 1-xc_o$ where $c_o = c^{-1}$, (2.2.29) A(q,m,k,c)

$$= \frac{\Gamma(q+m)}{\Gamma(q)\,\Gamma(m)}\, c_0^q (1-c_0)^{m-1} \int_0^1 \Big[\sum_{t=q}^{q+m-1} \binom{q+m-1}{t} y^{q+m-1-t} (1-y)^t \Big]^{\ell} (1-y)^{q-1} \Big[1+\frac{c_0 y}{1-c_0}\Big]^{m-1} dy$$

$$= c_0\, b(c_0;q,m) \int_0^1 \Big[\sum_{r=0}^{\ell(q+m-1)} C(r,\ell;q+m-1,q) y^r \Big] (1-y)^{q-1} \Big[1+ \frac{c_0 y}{1-c_0}\Big]^{m-1} dy$$

where $C(r,\ell;n,j)$ is the coefficient of y^r in the expansion of $[\sum_{t=j}^{n} \binom{n}{t}(1-y)^t y^{n-t}]^{\ell}$. The coefficients $C(r,\ell;n,j)$ are given by the following recursive relations:

$$(2.2.30)\qquad C(r,1;n,j) = \begin{cases} 1 & r = 0 \\ 0 & 1 \le r \le n-j \\ \binom{n}{r} \sum_{t=0}^{n-j} (-1)^{r-t} \binom{r}{t} & n-j+1 \le r \le n \end{cases}$$

and for $\ell > 1$

$$(2.2.31)\quad C(r,\ell;n,j) = \begin{cases} 1 & r = 0 \\ 0 & 1 \le r \le n-j \\ C(r,\ell-1;n,j)[1-\ \epsilon(r-n\ell+n-1)] & n-j+1 \le r \le n\ell \\ \quad + \sum_{t=\max(n-j+1,\,r-n\ell+n)}^{\min(r,n)} C(t,1;n,j)\, C(r-t,\ell-1;n,j) & \end{cases}$$

where $\epsilon(x) = 1$ if $x \geq 0$ if $x < 0$. Now (2.2.29) can be written as

$$A(q,m,k,c)$$

$$= c_0\, b(c_0;q,m) \sum_{r=0}^{\ell(q+m-1)} \sum_{\alpha=0}^{m-1} \binom{m-1}{\alpha} \left(\frac{c_0}{1-c_0}\right)^{\alpha} C(r,\ell;q+m-1,q) \int_0^1 z^{\alpha+r}(1-z)^{q-1}\, dz.$$

Since $\int_0^1 z^{\alpha+r}(1-z)^{q-1}\, dz = \frac{\Gamma(\alpha+r+1)\,\Gamma(q)}{\Gamma(\alpha+r+1+q)}$, we get

$$(2.2.32)\quad A(q,m,k,c) = c_0^q(1-c_0)^{m-1} \sum_{r=0}^{\ell(q+m-1)} \sum_{\alpha=0}^{m-1} \left(\frac{c_0}{1-c_0}\right)^{\alpha}$$

$$\frac{\Gamma(q+m)\Gamma(\alpha+r+1)C(r;\ell;q+m-1,q)}{\Gamma(\alpha+1)\Gamma(m-\alpha)\Gamma(\alpha+r+1+q)}\,.$$

Thus, if q and m are integers, c is given by

$$(2.2.33)\qquad 1-P^* = I_c(q,m) - A(q,m,k,c)$$

where $A(q,m,k,c)$ is given by (2.2.32) and $c_0 = c^{-1}$.

d. Asymptotic Results. When p is fixed and $m \to \infty$ (i.e. $n \to \infty$), the asymptotic distribution of nR^2 is the non-central chi-square with $(p-1)$ degrees of freedom and non-centrality parameter $n\rho^2$ both in the conditional and unconditional cases. Then, for the procedure Ω (which stands for Ω_1 or Ω_2), namely,

Ω: Select π_i iff $cR_i^2 \geq \max_{1\leq r\leq k} R_r^2$, we get

$$(2.2.34) \qquad P(CS|R) \approx \int_0^\infty \prod_{r=1}^{k-1} G_{n\rho_{[j]}^2, p-1}(cx)\, dG_{n\rho_{[k]}^2, p-1}(x),$$

where $G_{\mu,\nu}(x)$ is the cdf of the non-central chi-square with ν d.f. and non-centrality parameter μ. In this case the infimum of $P(CS|R)$ takes place when $\rho_1^2 = \rho_2^2 = \ldots = \rho_k^2 = 0$. A detailed discussion of this is given in Gupta and Studden [45]. For selected values of k,p and P*, the c-values are tabulated by Gupta [29].

e. <u>A Procedure Based on a Transform of R^2 (Unconditional Case)</u>. The transform of R^2, which we are concerned with here, is $\frac{R^2}{1-R^2} = R^{*2}$ (say). The exact distribution of R^{*2} in the unconditional case has the density

$$(2.2.35) \qquad u_\lambda(x) = \sum_{j=0}^{\infty} w(\lambda, j)\, g_j(x),$$

where

$$(2.2.36) \quad \begin{cases} q = \frac{p-1}{2}, \qquad m = \frac{n-p}{2}, \qquad \lambda = \rho^2 \\ g_j(x) = \frac{\Gamma(q+j+m)}{\Gamma(q+j)\,\Gamma(m)} \frac{x^{q+j-1}}{(1+x)^{q+j+m}}, \qquad x \geq 0 \\ w(\lambda, j) = \frac{\Gamma(q+m+j)}{\Gamma(q+m)} (1-\lambda)^{q+m} \frac{\lambda^j}{j!}, \quad 0 \leq \lambda < 1. \end{cases}$$

Then, for selecting the population with $\rho_{[k]}$, we propose the procedure R_3: Select π_i iff $c_1 R_i^{*2} \geq \max_{1 \leq r \leq k} R_j^{*2}$,

where $c_1 = c_1(k, P^*, q, m) > 1$ is chosen to satisfy the P*-condition. Then, denoting $\frac{R^2_{(j)}}{1-R^2_{(j)}}$ by $R^{*2}_{(j)}$ ($R^{*2}_{(j)}$ corresponds to the population with $\rho_{[j]}$),

we have

$$(2.2.37) \qquad P(CS|\Omega_3) = P(c_1 R^{*2}_{(k)} \geq R^{*2}_{(j)}, \quad j = 1,2,\ldots,k-1)$$

$$= \int_0^\infty \prod_{r=1}^{k-1} U_{\lambda_{[j]}}(c_1 x)\, dU_{\lambda_{[k]}}(x),$$

where $U_\lambda(x)$ is the cdf corresponding to $u_\lambda(x)$.

<u>Lemma 2.2.5</u>. The density $u_\lambda(x)$ given by (2.2.36) has MLR in x.

<u>Proof</u>. The proof is by simple and direct verification of the conditions of Lemma 1.5.1. and hence is omitted.

<u>Lemma 2.2.6</u>. For every integer $i \geq 0$ and $\alpha = 0,1,\ldots,[\frac{i}{2}]$,

$$(2.2.38) \qquad b^{i-\alpha}(q+m+\alpha)[g_{i-\alpha}(x)\Delta G_\alpha(c_1x)-c_1g_{i-\alpha}(c_1x)\Delta G_\alpha(x)] +$$

$$b^\alpha(q+m+i-\alpha)[g_\alpha(x)\Delta G_{i-\alpha}(c_1x)-c_1g_\alpha(c_1x)\Delta G_{i-\alpha}(x)] \geq 0$$

where $b \geq 1$ and $g_j(x)$ is given by (2.2.36).

<u>Proof</u>. It can be easily seen by integration by parts that

$$(2.2.39) \qquad \Delta G_j(x) = -\frac{\Gamma(q+m+j)}{\Gamma(q+j+1)\,\Gamma(m)} \frac{x^{q+j}}{(1+x)^{q+m+j}} .$$

Using (2.2.39), we obtain

(2.2.40) $g_{i-\alpha}(x)\Delta G_{\alpha}(c_1x)-c_1g_{i-\alpha}(c_1x)\Delta g_{\alpha}(x)$

$$= \frac{\Gamma(q+m+\alpha)\Gamma(q+m+i-\alpha)c_1^q\, x^{2q+i-1}}{\Gamma(q+\alpha+1)\Gamma(q+i-\alpha)\{\Gamma(m)\}^2(1+x)^{q+m}(1+c_1x)^{q+m}} \times$$

$$\{\frac{c_1^{i-\alpha}}{(1+x)^{\alpha}(1+c_1x)^{i-\alpha}} - \frac{c_1^{\alpha}}{(1+c_1x)^{\alpha}(1+x)^{i-\alpha}}\} .$$

Using (2.2.40) on the left hand side of (2.2.38) and taking the common factors out, we see that (2.2.38) holds if

(2.2.41)
$$\frac{q+m+\alpha}{q+\alpha}\, b^{i-\alpha}\{\frac{c_1^{i-\alpha}}{(1+x)^{\alpha}(1+c_1x)^{i-\alpha}} - \frac{c_1^{\alpha}}{(1+c_1x)^{\alpha}(1+x)^{i-\alpha}}\}$$

$$+ \frac{q+m+i-\alpha}{q+i-\alpha}\, b^{\alpha}\{\frac{c_1^{i-\alpha}}{(1+x)^{i-\alpha}(1+c_1x)^{\alpha}} - \frac{c_1^{i-\alpha}}{(1+c_1x)^{i-\alpha}(1+x)^{\alpha}}\} \geq 0$$

The left hand side of (2.2.41)

$$= \frac{(q+m+\alpha)b^{i-\alpha}}{(q+\alpha)(1+x)^i}\,[\,\{\frac{c_1(1+x)}{1+c_1x}\}^{i-\alpha} - \{\frac{c_1(1+x)}{1+c_1x}\}^{\alpha}\,] +$$

$$\frac{(q+m+i-\alpha)b^{\alpha}}{(q+i-\alpha)(1+x)^i}\,[\,\{\frac{c_1(1+x)}{1+c_1x}\}^{\alpha} - \{\frac{c_1(1+x)}{1+c_1x}\}^{i-\alpha}\,]$$

$$= [\,\{\frac{c_1(1+x)}{1+c_1x}\}^{i-\alpha} - \{\frac{c_1(1+x)}{1+c_1x}\}^{\alpha}\,][\,\frac{q+m+\alpha}{q+\alpha}\, b^{i-\alpha} - \frac{q+m+i-\alpha}{q+i-\alpha}\, b^{\alpha}]\; \frac{1}{(1+x)^i} .$$

since $c_1 > 1$, $\frac{c_1(1+x)}{1+c_1x} > 1$ and $\alpha \leq i-\alpha$. Hence the first factor in the above expression is non-negative. In the second factor, $b^{i-\alpha} \geq b^{\alpha}$ and $\frac{q+m+\alpha}{q+\alpha} \geq \frac{q+m+i-\alpha}{q+i-\alpha}$ for $\alpha \leq i-\alpha$. Hence the second factor is non-negative. Thus (2.2.41) holds and so does (2.2.38).

Theorem 2.2.3. $\inf_{\Omega} P(CS|\Omega_3) = \int_0^{\infty} F_{2q,2m}^{k-1}(c_1x)dF_{2q,2m}(x)$, where $F_{2q,2m}(x)$ is the cdf of the central F random variable with 2q and 2m degrees of freedom.

Proof. $P(CS|\Omega_3) \geq \int_0^{\infty} U_{\lambda[k]}^{k-1}(c_1x)dU_{\lambda[k]}(x)$ by Lemma 2.2.5. Hence

$$\inf_{\Omega} P(CS|\Omega_3) = \inf_{\lambda} \int_0^{\infty} U_{\lambda}^{k-1}(c_1x)dU_{\lambda}(x). \tag{2.2.42}$$

By Lemma 2.2.6 and Remark 1.6.2 and noting that p=1 and q is to be replaced by q+m in (1.6.17) for the weights in the present case, the integral on the right hand side of (2.2.42) is non-decreasing in λ. Hence the infimum takes place when $\lambda = 0$. $U_{\lambda}(x)$ for $\lambda = 0$ is the cdf of a constant multiple of a F variable with 2q and 2m d.f. This completes the proof of Theorem 2.2.3.

Corollary 2.2.3. The constant c_1 defining Ω_3 is given by

$$\int_0^{\infty} F_{2q,2m}^{k-1}(c_1x)\, dF_{2q,2m}(x) = P^*. \tag{2.2.43}$$

f. A Procedure Based on the Transform of R^2 (Conditional Case).

In the conditional case the density of R^{*2} is given by

$$v_{\lambda}(x) = \sum_{j=0}^{\infty} w(\lambda,j)\, g_j(x), \tag{2.2.44}$$

where $w(\lambda,j) = \frac{e^{-\lambda}\lambda^j}{j!}$ and $g_j(x)$ is as defined in (2.2.36). Then, we propose the following procedure.

R_4: Select π_i iff $c_2 R_i^{*2} \geq \max_{1 \leq r \leq k} R_r^{*2}$,

where $c_2 = c_2(k,P^*,q,m) > 1$ is chosen so as to satisfy the P*-condition.

Lemma 2.2.7. The density $v_\lambda(x)$ given by (2.2.44) has MLR in x.

This is a known result. Further a simple proof can be given by verifying the conditions of Lemma 1.5.1.

Lemma 2.2.8. For every integer $i \geq 0$ and $\alpha = 0,1,\ldots,[\frac{i}{2}]$,

$$(2.2.45) \qquad b^{i-\alpha}[g_{i-\alpha}(x)\Delta G_\alpha(c_2x) - c_2 g_{i-\alpha}(c_2x)\Delta G_\alpha(x)] + b^\alpha[g_\alpha(x)\Delta G_{i-\alpha}(c_2x) - c_2 g_\alpha(c_2x)\Delta G_{i-\alpha}(x)] \geq 0$$

where $b \geq 1$ and $g_j(x)$ is given by (2.2.36).

Proof. Using (2.2.40), we see that (2.2.45) holds if

$$(2.2.46) \qquad \frac{b^{i-\alpha}}{q+\alpha}\left\{\frac{c_2^{i-\alpha}}{(1+x)^\alpha(1+c_2x)^{i-\alpha}} - \frac{c_2^\alpha}{(1+c_2x)^\alpha(1+x)^{i-\alpha}}\right\} + \frac{b^\alpha}{q+i-\alpha}\left\{\frac{c_2^\alpha}{(1+x)^{i-\alpha}(1+c_2x)^\alpha} - \frac{c_2^{i-\alpha}}{(1+c_2x)^{i-\alpha}(1+x)^\alpha}\right\} \geq 0.$$

From the earlier results, we see that the left hand side of (2.2.46)

$$= \left[\left\{\frac{c_2(1+x)}{1+c_2x}\right\}^{i-\alpha} - \left\{\frac{c_2(1+x)}{1+c_2x}\right\}^{\alpha}\right]\left[\frac{b^{i-\alpha}}{q+\alpha} - \frac{b^\alpha}{q+i-\alpha}\right]\frac{1}{(1+x)^i}.$$

It is easy to see that the above is non-negative. This completes the proof of the lemma.

Theorem 2.2.4. $\inf_{\Omega} P(CS|R_4) = \int_0^\infty F_{2q,2m}^{k-1}(x)\, dF_{2q,2m}(x)$, where $F_{2q,2m}(x)$ is the cdf of central F with 2q and 2m d.f.

Proof. $P(CS|R_4) = \int_0^\infty \prod_{r=1}^{k-1} V_{\lambda_{[r]}}(c_2 x)\, dV_{\lambda_{[k]}}(x)$, where $V_{\lambda_{[r]}}$ is the cdf of $R_{(r)}^{*2}$

$$\geq \int_0^\infty V_{\lambda_{[k]}}^{k-1}(c_2 x)\, dV_{\lambda_{[k]}}(x), \text{ by Lemma 2.2.7.}$$

Hence

$$(2.2.47) \qquad \inf_{\Omega} P(CS|R_4) = \inf_{\lambda} \int_0^\infty V_\lambda^{k-1}(c_2 x)\, dV_\lambda(x).$$

Noting that for the Poisson weights q=1 and p=0 in (1.6.17) and using Lemma 2.2.8 and Remark 1.6.3, we see that the integral on the right hand side of (2.2.47) is non-decreasing in λ. Hence the infimimum is attained at $\lambda=0$. Also $V_\lambda(x)$ for $\lambda=0$ is the same as $U_\lambda(x)$ for $\lambda=0$. This completes the proof of Theorem 2.2.4.

Corollary 2.2.4. The constant c_2 defining the procedure R_4 is given by

$$(2.2.48) \qquad \int_0^\infty F_{2q,2m}^{k-1}(c_2 x)\, dF_{2q,2m}(x) = P^*.$$

Remark 2.2.2. It can be seen from (2.2.43) and (2.2.48) that $c_1 = c_2$, i.e. the constants defining the procedures R_3 and R_4 are the same.

g. Formulae for Evaluating the Constants Defining R_3 and R_4. If we assume that q and m are integers, i.e., p and n are odd, we can write (2.2.43) as

$$(2.2.49)\quad P^* = \int_0^\infty [(1-t)^q\{1+ \frac{(q)_1}{1!}t+\ldots+\frac{(q)_{m'}}{m'!}t^{m'}]^{k-1} \frac{\Gamma(q+m)c_1'(2q)^q(2m)^m(c_1'z)^{q-1}}{\Gamma(q)\Gamma(m)(2m+2qc_1'z)^{m+q}}\,dz$$

where $t = \frac{m}{m+qz}$, $m' = m-1$, $c_1' = c_1^{-1}$ and $(q)_j = q(q+1)\ldots(q+j-1)$. By changing the variable of integration from z to t, we get after some simplifications

$$(2.2.50)\qquad P^* = \int_0^1 \frac{\Gamma(q+m)t^{m-1}(1-t)^{qk-1}c_1^m}{\Gamma(q)\Gamma(m)\{1+(c_1-1)t\}^{m+q}}\{1+\frac{(q)_1}{1!}t+\ldots+\frac{(q)_{m'}}{m'!}t^{m'}\}^{k-1}\,dt.$$

If we expand $[1+\frac{(q)_1}{1!}t+\ldots+\frac{(q)_{m'}}{m'!}]^r$ in powers of t as $\sum_{j=0}^{r(m-1)} a(r,j)t^j$, the coefficients a(r,j) are given by the following recursive relations:

$$(2.2.51)\qquad a(1,j) = \begin{cases} 1 & j = 0 \\ \frac{(q)_j}{j!} & 1 \le j \le m' \end{cases}$$

and for $r > 1$

$$(2.2.52)\qquad a(r,j) = \begin{cases} 1 & j = 0 \\ \sum_{S=\max(j-(r-1)m',0)}^{\min(m',j)} a(1,s)a(r-1),j-s) & 1 \le j \le rm' \end{cases}$$

Using the above expansion and also the binomial expansion of $(1-t)^{qk-1}$ in (2.2.50), setting $(c_1-1)t = \xi$ and integrating term by term, we obtain

(2.2.53) $$P^* = \frac{\Gamma(q+m)\, c_1^m}{\Gamma(q)\Gamma(m)(c_1-1)^m} \sum_{\alpha=0}^{qk-1} \sum_{j=0}^{(k-1)m'} (-1)^{\alpha}\binom{qk-1}{\alpha} a(k-1,j) \frac{K(c_1,m,q,\alpha,j)}{(c_1-1)^{\alpha+j}}$$

where

(2.2.54) $$K(c,m,q,\alpha,j)=\begin{cases} \dfrac{\Gamma(m+\alpha+j)\Gamma(q-\alpha-j)}{\Gamma(m+q)} I_{1-c^{-1}}(m+\alpha+j,q-\alpha-j), & q > \alpha+j \\[2ex] \displaystyle\sum_{\ell=1}^{m+q-1} \binom{m+q-1}{\ell}(-1)^{\ell} \frac{\{1-c^{-\ell}\}}{\ell} + \log c, & q=\alpha+j \\[2ex] \displaystyle\sum_{\substack{\ell=0 \\ \ell\neq\alpha+j-q}}^{m+\alpha+j-1} \binom{m+\alpha+j-1}{\ell}(-1)^{\ell} \frac{\{1-c^{\alpha+j-\ell-q}\}}{\ell-\alpha-j+q} + \binom{m+\alpha+j-1}{\alpha+j-q}(-1)^{\alpha+j-q} \log c, & q< \alpha+j. \end{cases}$$

h. <u>Properties of the Procedures R_1 through R_4.</u> These procedures come under the class of procedures R_h discussed in Chapter I. The function $h(x)$ chosen in each case is of the form $h(x) = cx$ and so increasing in x. So by Theorem 1.6.1 they are all unbiased.

The Lemmas 2.2.2, 2.2.4, 2.2.6 and 2.2.8 show that in each case the condition (1.6.6) is satisfied and hence by Theorem 1.6.2,

(2.2.55) $$\sup_{\Omega} E(S|R_i) = k \sup_{\lambda} P_{\lambda}(CS|R_i), \quad i = 1,\dots,4$$

when $\lambda \to 1$, $P_{\lambda}(CS|R_i) \to 1$. Thus in all the cases

(2.2.56) $$\sup_{\Omega} E(S|\mathcal{R}_i) = k \quad , \qquad i = 1,\ldots,4.$$

By Theorem 1.6.3, for the procedures $\mathcal{R}_i$, $i=1,\ldots,4$,

(2.2.57) $$\sup_{\Omega} E(S'|\mathcal{R}_i) = k-1.$$

C. Selection for the Population Associated with $\rho_{[1]}$

a. <u>The Procedures</u>. Here also we have four procedures to consider corresponding to $\mathcal{R}_1$ through $\mathcal{R}_4$. But in the light of our results in Chapter I and Section B of the present chapter we can state these results.

In the case of procedures based on R^2, we propose in the unconditional case the procedure

$\mathcal{R}_5$: Select π_i iff $\frac{1}{d} R_i^2 \leq \min\limits_{1\leq r\leq k} R_r^2$

and in the conditional case the procedure

$\mathcal{R}_6$: Select π_i iff $\frac{1}{d'} R_i^2 \leq \min\limits_{1\leq r\leq k} R_r^2$

where $d=d(k,P^*,n,p) > 1$ and $d' = d'(k,P^*,n,p) > 1$ are to be chosen so as to satisfy the P^* condition.

In the case of procedures based on the transform $R^{*2} = \frac{R^2}{1-R^2}$, we propose in the unconditional case the procedure

$\mathcal{R}_7$: Select π_i iff $\frac{1}{d_1} R_i^{*2} \leq \min\limits_{1\leq r\leq k} R_r^{*2}$

and in the conditional case the procedure

$\mathcal{R}_8$: Select π_i iff $\frac{1}{d_2} R_i^{*2} \leq \min\limits_{1\leq r\leq k} R_r^{*2}$, where $d_1 = d_1(k,P^*,n,p) > 1$ and $d_2 = d_2(k,P^*,n,p)$ are to be chosen so that the P^*-condition is satisfied.

In order to write explicitly the expression for the probability of a correct selection, we state it in the case of $\mathcal{R}_5$.

$$P(CS|\mathcal{R}_5) = \int_0^1 \prod_{r=1}^{k-1} \bar{F}_{\lambda_{[r]}}(x/d)\, dF_{\lambda_{[k]}}(x), \tag{2.2.58}$$

where $\bar{F}_\lambda(x) = 1-F_\lambda(x)$.

In view of Theorem 1.7.1, Remark 1.7.1 and Lemmas 2.2.1 through 2.2.8, we can see with no difficulty that the constants d, d', d_1 and d_2 defining the procedures $\mathcal{R}_5$ through $\mathcal{R}_8$ respectively are given by

$$\left\{\begin{array}{l} \int_0^1 I^{k-1}_{1-(x/d)}(m,q)\, b(x;q,m)\, dx = P^* \\ \int_0^1 I^{k-1}_{1-(x/d')}(m,q)\, b(x;q,m)\, dx = P^* \\ \int_0^\infty \bar{F}^{k-1}_{2q,2m}(x/d_1)\, f_{2q,2m}(x)\, dx = P^* \\ \int_0^\infty \bar{F}^{k-1}_{2q,2m}(x/d_2)\, f_{2q,2m}(x)\, dx = P^*. \end{array}\right. \tag{2.2.59}$$

It is clear from the above equations that $d = d'$ and $d_1 = d_2$.

b. <u>Formulae for Evaluating the Constants for the Procedures $\mathcal{R}_5$ and $\mathcal{R}_6$.</u> If q and m are integers, d is given by

$$P^* = \frac{\Gamma(q+m)}{\Gamma(q)\Gamma(m)} \int_0^1 \Big[\sum_{t=0}^{q-1} \binom{q+m-1}{t} \left(\frac{x}{d}\right)^t \left(1-\frac{x}{d}\right)^{q+m-1-t}\Big]^{k-1} x^{q-1}(1-x)^{m-1} dx. \tag{2.2.60}$$

Setting $y = 1-\frac{x}{d}$ and $\ell = k-1$, we obtain

$$(2.2.61)\quad P^* = \frac{\Gamma(q+m)}{\Gamma(q)\Gamma(m)}\, d^{q+m+1}\int_{1-d^{-1}}^{1} \Big[\sum_{r=0}^{\ell(q+m-1)} C'(r,\ell;q+m-1,q)y^r\Big](1-y)^{q-1} \{y-(1-d^{-1})\}^{m-1}\, dy$$

where $C'(r,\ell,n,j)$ is the coefficient of y^r in the expansion of $[\sum_{t=0}^{j-1} \binom{n}{t}(1-y)^t\, y^{n-t}]^p$ and is given by the following recursive relations:

$$(2.2.62)\quad C'(r,1;n,j) = \begin{cases} 0 & , \; 0 \le r \le n-j \\ \displaystyle\sum_{k=0}^{j-1-n+r} (-1)^k \binom{n}{n-r+k}\binom{n-r+k}{k} & , \; n-j+1 \le r \le n \end{cases}$$

and for $\ell > 1$

$$(2.2.63)\quad C'(r,\ell;n,j) = \begin{cases} 0 & , \; 0 \le r \le (n-j+1)\ell-1 \\ \displaystyle\sum_{s=\max(n-j+1,r-n(\ell-1))}^{\min(n,r-(\ell-1)(n-j+1)} C'(s,1;n,j)\, C'(r-s,\ell-1;n,j), & (n-j+1)\ell \le r \le n\ell. \end{cases}$$

Expanding $\{y-(1-d^{-1})\}^{m-1}$ in powers of y and integrating term by term, (2.2.61) yields

$$(2.2.64)\quad P^* = \frac{\Gamma(q+m)}{\Gamma(q)\Gamma(m)}\, d^{q+m+1} \sum_{r=0}^{\ell(q+m-1)} \sum_{\alpha=0}^{m-1} (-1)^{m-1-\alpha}\binom{m-1}{\alpha}(1-d^{-1})^{m-1-\alpha}\, C'(r,\ell;q+m-1,q)\, \frac{\Gamma(q)\Gamma(r+\alpha+1)}{\Gamma(q+r+\alpha+1)}\, I_{d^{-1}}(q,r+\alpha+1).$$

c. <u>Formulae for Evaluating the Constants Defining $\mathcal{R}_7$ and $\mathcal{R}_8$.</u> The constant d_1 is given by

(2.2.65) $$\int_0^\infty \bar{F}^{k-1}_{2q,2m}(x/d_1)\, f_{2q,2m}(x) = P^*.$$

It is easy to see that

(2.2.66) $$1-F_{2q,2m}(x/d_1) = F_{2m,2q}(d_1/x) \qquad \text{and}$$

(2.2.67) $$f_{2q,2m}(1/x) = x^2\, f_{2m,2q}(x).$$

Using (2.2.66) and (2.2.67), (2.2.65) becomes

(2.2.68) $$P^* = \int_0^\infty F^{k-1}_{2m,2q}(d_1/x)\, f_{2m,2q}(1/x)x^{-2}\, dx$$

$$= \int_0^\infty F^{k-1}_{2m,2q}(yd_1)\, f_{2m,2q}(y)\, dy.$$

Thus for a given set of q,m,k and P*, the constant d_1 of the procedures $\mathcal{R}_7$ and $\mathcal{R}_8$ is the same as the constant d of the procedures $\mathcal{R}_3$ and $\mathcal{R}_4$ with q and m interchanged.

d. <u>An Asymptotic Result.</u> When p is fixed and $m \to \infty$, then for the procedure $\mathcal{R}'$ (which stands for $\mathcal{R}_5$ or $\mathcal{R}_6$) which selects π_i iff $\frac{R_i^2}{d} \le \min_{1\le r\le k} R_r^2$, we have corresponding to (2.2.34),

(2.2.69) $$P(CS|\mathcal{R}') \approx \int_0^\infty \prod_{r=1}^{k-1} \bar{G}_{n\rho^2_{[j]},p-1}(x/d)\, dG_{n\rho^2_{[k]},p-1}(x)$$

and the infimum of the right hand side of (2.2.69) takes place when $\rho_1^2 = \dots = \rho_k^2 = 0$. Hence d is given by

(2.2.70) $$\int_0^\infty \bar{G}_{p-1}^{k-1}(x/d)\, dG_{p-1}(x) = P^*,$$

where $G_\nu(x)$ is the cdf of the central chi-square with ν d.f. For selected values of k,p and P*,the reciprocals of the d-values are tabulated in Gupta and Sobel [43].

e. Some Remarks on the Procedures R_1 through R_8. The constants of these procedures are the constants we need for similar procedures for selection for non-central beta or non-central F distributions in terms of their non-centrality parameters. They are also the percentage points or reciprocals of percentage points of the distributions of $Z_1 = \max(\frac{Y_1}{Y_k}, \frac{Y_2}{Y_k}, \dots, \frac{Y_{k-1}}{Y_k})$ and $Z_2 = \min(\frac{Y_1}{Y_k}, \frac{Y_2}{Y_k}, \dots, \frac{Y_{k-1}}{Y_k})$ where $Y_1, Y_2, \dots, Y_k$ are k independent random variables identically distributed as beta or F with 2q and 2m d.f.

D. Tables of the Constants for the Procedures

As we remarked elsewhere in this chapter the constants are the same for the unconditional and the conditional cases. Further the constants for the procedure R_7 are related to the constants for the procedure R_3.

Table 1 gives the reciprocals of c_1 and d_1. The table ranges over k=2(1)5 and P* = .75, .90, .95 and .99. The reciprocals of c_1 are directly readable for specified $q = \frac{p-1}{2}$ and $m = \frac{n-p}{2}$. The reciprocals of d_1 are read from the table after interchanging q and m. For the computational purposes, equation (2.2.53) has been used to solve for $c_1' = c_1^{-1}$.

Table 2 gives the values of $c_0 = c^{-1}$. This table ranges over $k = 2(1)5$ and $P^* = .75, .90, .95$ and $.99$. The constant c_0 is used in procedures R_1 and R_2. For computational purposes, equation (2.2.33) was used to solve for $c_0 = c^{-1}$.

As we pointed out earlier, $c_1' = c_1^{-1}$ tabulated in Table 1 is the 100 P* percentage point of the distribution of $Z_1 = \max(\frac{Y_1}{Y_k}, \ldots, \frac{Y_{k-1}}{Y_k})$ where $Y_1, \ldots, Y_k$ are i.i.d. central F variables with 2q and 2m d.f. The constant c_1' for a given pair of q and m, is also the 100(1-P*) percentage point of the distribution of $Z_2 = \min(\frac{Y_1}{Y_k}, \ldots, \frac{Y_{k-1}}{Y_k})$, where the Y_i are i.i.d. central F with 2m and 2q d.f. Also the constant c_0 of Table 2 is the 100 P* percentage point of Z_1 when Y_i's are i.i.d. beta variables with parameters q and m. These tables are also useful in testing of hypotheses because the percentage points of the statistics Z_1 and Z_2 are obtained under the assumption that the non-centrality parameter is zero.

2.3. Selection in Terms of the Conditional Generalized Variance

Let $\pi_1, \pi_2, \ldots, \pi_k$ be p-variate normal populations, where π_i is $N_p(\underline{\mu}_i, \Sigma_i)$, $i=1,2,\ldots,k$. We consider a partition of the p variables into two sets of q_1 and q_2 components respectively, where $q_1 + q_2 = p$. The corresponding partition of Σ_i is denoted by

$$(2.3.1) \qquad \Sigma_i = \begin{pmatrix} \Sigma_{11}^{(i)} & \Sigma_{12}^{(i)} \\ \Sigma_{21}^{(i)} & \Sigma_{22}^{(i)} \end{pmatrix}, \quad i = 1,2,\ldots,k.$$

Footnote for Table 1

1. The four entries in each cell refer to $P^* = .99, .95, .90, .75$ from top to bottom.

2. The number in the parentheses refers to the power of 10^{-1} by which each entry should be multiplied to get the reciprocal of appropriate constant.

q \ m	1	2	3	4	5	6	7	8
1	2.4867 (3) 3.0066 (2) 4.1222 (2) 1.9548 (1)	5.2940 (3) 3.0525 (2) 7.0383 (2) 2.5680 (1)	6.8360 (3) 3.7366 (2) 8.2871 (2) 2.8065 (1)	7.6400 (3) 4.1035 (2) 8.9587 (2) 2.9323 (1)	8.1290 (3) 4.3298 (2) 9.3746 (2) 3.0097 (1)	8.4560 (3) 4.4826 (2) 9.6567 (2) 3.0622 (1)	8.6900 (3) 4.5926 (2) 9.8604 (2) 3.1000 (1)	8.8660 (3) 4.6756 (2) 1.0014 (1) 3.1286 (1)
2	5.2940 (3) 3.0525 (2) 7.0383 (2) 2.5680 (1)	2.1570 (2) 7.1916 (2) 1.3183 (1) 3.4925 (1)	3.2095 (2) 9.4110 (2) 1.6190 (1) 3.8816 (1)	3.8543 (2) 1.0729 (1) 1.7935 (1) 4.0964 (1)	4.2792 (2) 1.1594 (1) 1.9069 (1) 4.2327 (1)	4.5781 (2) 1.2203 (1) 1.9864 (1) 4.3269 (1)	4.7992 (2) 1.2654 (1) 2.0451 (1) 4.3960 (1)	4.9690 (2) 1.3001 (1) 2.0903 (1) 4.4488 (1)
3	6.8353 (3) 3.7366 (2) 8.2871 (2) 2.8065 (1)	3.2096 (2) 9.4109 (2) 1.6190 (1) 3.8816 (1)	5.1008 (2) 1.2714 (1) 2.0306 (1) 4.3534 (1)	6.3604 (2) 1.4777 (1) 2.2789 (1) 4.6209 (1)	7.2352 (2) 1.6176 (1) 2.4446 (1) 4.7938 (1)	7.8725 (2) 1.7185 (1) 2.5630 (1) 4.9150 (1)	8.3559 (2) 1.7946 (1) 2.6519 (1) 5.0047 (1)	8.7344 (2) 1.8540 (1) 2.7210 (1) 5.0739 (1)
4	7.6401 (3) 4.1036 (2) 8.9587 (2) 2.9323 (1)	3.8543 (2) 1.0729 (1) 1.7935 (1) 4.0964 (1)	6.3604 (2) 1.4777 (1) 2.2789 (1) 4.6209 (1)	8.1157 (2) 1.7385 (1) 2.5791 (1) 4.9236 (1)	9.3760 (2) 1.9194 (1) 2.7831 (1) 5.1219 (1)	1.0317 (1) 2.0519 (1) 2.9309 (1) 5.2623 (1)	1.1043 (1) 2.1532 (1) 3.0429 (1) 5.3671 (1)	1.1620 (1) 2.2331 (1) 3.1308 (1) 5.4484 (1)
5	8.1286 (3) 4.3298 (2) 9.3747 (2) 3.0097 (1)	4.2792 (2) 1.1594 (1) 1.9069 (1) 4.2327 (1)	7.2352 (2) 1.6176 (1) 2.4446 (1) 4.7938 (1)	9.3760 (2) 1.9194 (1) 2.7831 (1) 5.1219 (1)				
6	8.4557 (3) 4.4827 (2) 9.6568 (2) 3.0622 (1)	4.5781 (2) 1.2203 (1) 1.9864 (1) 4.3269 (1)	7.8725 (2) 1.7185 (1) 2.5630 (1) 4.9150 (1)	1.0317 (1) 2.0520 (1) 2.9309 (1) 5.2623 (1)				
7	8.6900 (3) 4.5927 (2) 9.8604 (2) 3.1000 (1)	4.7992 (2) 1.2654 (1) 2.0451 (1) 4.3960 (1)	8.3559 (2) 1.7946 (1) 2.6519 (1) 5.0047 (1)	1.1043 (1) 2.1532 (1) 3.0429 (1) 5.3671 (1)				
8	8.8659 (3) 4.6756 (2) 1.0014 (1) 3.1286 (1)	4.9691 (2) 1.3001 (1) 2.0903 (1) 4.4487 (1)	8.7344 (2) 1.8540 (1) 2.7210 (1) 5.0739 (1)	1.1620 (1) 2.2331 (1) 3.1308 (1) 5.4484 (1)				

For fixed k, P*, q and m, the above table gives the values of $c_1 (= c_2)$ satisfying

$P\{F_k \geq c_1 \max_{1 \leq j \leq k-1} F_j\} = \int_0^\infty [F_{2q,2m}(\frac{x}{c_1})]^{k-1} dF_{2q,2m}(x) = P^*$ where F_i, $i=1,2,\ldots,k$ are i.i.d. central F random variables with 2q and 2m degrees of freedom. Values of $d_1 = d_2$ are given by $d_1(k,P^*,q,m) = c_1(k,P^*,m,q)$.

TABLE 1 (continued)

q \ m	1	2	3	4	5	6	7	8
1	9.0500 (4) 7.0579 (3) 1.8811 (2) 8.4314 (2)	3.1383 (3) 1.7622 (2) 3.9626 (2) 1.3550 (1)	4.2460 (3) 2.2752 (2) 4.9377 (2) 1.5733 (1)	4.8400 (3) 2.5568 (2) 5.4745 (2) 1.6915 (1)	5.2055 (3) 2.7322 (2) 5.8103 (2) 1.7652 (1)	5.4524 (3) 2.8513 (2) 6.0392 (2) 1.8155 (1)	5.6302 (3) 2.9375 (2) 6.2050 (2) 1.8519 (1)	5.7643 (3) 3.0026 (2) 6.3306 (2) 1.8794 (1)
2	2.6290 (3) 1.4934 (2) 3.3966 (2) 1.1979 (1)	1.4154 (2) 4.5506 (2) 8.1345 (2) 2.0451 (1)	2.2570 (2) 6.3957 (2) 1.0738 (1) 2.4437 (1)	2.7904 (2) 7.5320 (2) 1.2305 (1) 2.6723 (1)	3.1472 (2) 8.2895 (2) 1.3341 (1) 2.8202 (1)	3.4003 (2) 8.8276 (2) 1.4074 (1) 2.9235 (1)	3.5886 (2) 9.2285 (2) 1.4620 (1) 2.9998 (1)	3.7340 (2) 9.5384 (2) 1.5041 (1) 3.0584 (1)
3	3.4078 (3) 1.8462 (2) 4.0568 (2) 1.3415 (1)	2.1582 (2) 6.1120 (2) 1.0277 (1) 2.3553 (1)	3.7150 (2) 8.9373 (2) 1.3948 (1) 2.8564 (1)	4.7965 (2) 1.0779 (1) 1.6259 (1) 3.1529 (1)	5.5615 (2) 1.2053 (1) 1.7835 (1) 3.3489 (1)	6.1243 (2) 1.2983 (1) 1.8974 (1) 3.4881 (1)	6.5542 (2) 1.3689 (1) 1.9837 (1) 3.5921 (1)	6.8925 (2) 1.4244 (1) 2.0511 (1) 3.6728 (1)
4	3.8132 (3) 2.0353 (2) 4.4129 (2) 1.4182 (1)	2.6174 (2) 7.0528 (2) 1.1542 (1) 2.5312 (1)	4.7053 (2) 1.0560 (1) 1.5932 (1) 3.0982 (1)	6.2376 (2) 1.2926 (1) 1.8775 (1) 3.4407 (1)	7.3618 (2) 1.4605 (1) 2.0753 (1) 3.6706 (1)	8.2110 (2) 1.5852 (1) 2.2206 (1) 3.8357 (1)	8.8714 (2) 1.6813 (1) 2.3318 (1) 3.9603 (1)	9.3990 (2) 1.7576 (1) 2.4196 (1) 4.0576 (1)
5	4.0590 (3) 2.1518 (2) 4.6336 (2) 1.4658 (1)	2.9208 (2) 7.6740 (2) 1.2371 (1) 2.6444 (1)	5.3987 (2) 1.1672 (1) 1.7274 (1) 3.2575 (1)	7.2850 (2) 1.4436 (1) 2.0513 (1) 3.6334 (1)				
6	4.2235 (3) 2.2304 (2) 4.7833 (2) 1.4982 (1)	3.1344 (2) 8.1128 (2) 1.2956 (1) 2.7234 (1)	5.9064 (2) 1.2480 (1) 1.8241 (1) 3.3704 (1)	8.0728 (2) 1.5553 (1) 2.1784 (1) 3.7716 (1)				
7	4.3413 (3) 2.2870 (2) 4.8914 (2) 1.5216 (1)	3.2924 (2) 8.4386 (2) 1.3389 (1) 2.7817 (1)	6.2923 (2) 1.3092 (1) 1.8971 (1) 3.4548 (1)	9.6830 (2) 1.6411 (1) 2.2756 (1) 3.3759 (1)				
8	4.4297 (3) 2.3296 (2) 4.9731 (2) 1.5393 (1)	3.4139 (2) 8.6899 (2) 1.3724 (1) 2.8265 (1)	6.5949 (2) 1.3571 (1) 1.9541 (1) 3.5203 (1)	9.1698 (2) 1.7090 (1) 2.3521 (1) 3.9574 (1)				

Table 1 (continued)

k = 4

q \ m	1	2	3	4	5	6	7	8
1	5.9490 (4) 4.5910 (3) 1.2145 (2) 5.3430 (2)	2.3670 (3) 1.3156 (2) 2.9307 (2) 9.7883 (2)	3.3067 (3) 1.7590 (2) 3.7871 (2) 1.1806 (1)	3.8216 (3) 2.0069 (2) 4.2672 (2) 1.2919 (1)	4.1419 (3) 2.1626 (2) 4.5701 (2) 1.3619 (1)	4.3597 (3) 2.2691 (2) 4.7775 (2) 1.4099 (1)	4.5172 (3) 2.3463 (2) 4.9283 (2) 1.4448 (1)	4.6364 (3) 2.4048 (2) 5.0426 (2) 1.4712 (1)
2	1.7486 (3) 9.8820 (3) 2.2370 (2) 7.8003 (2)	1.1166 (2) 3.5345 (2) 6.2529 (2) 1.5397 (1)	1.8592 (2) 5.1926 (2) 8.6307 (2) 1.9241 (1)	2.3412 (2) 6.2388 (2) 1.0097 (1) 2.1500 (1)	2.6672 (2) 6.9441 (2) 1.1079 (1) 2.2979 (1)	2.8999 (2) 7.4484 (2) 1.1778 (1) 2.4020 (1)	3.0739 (2) 7.8257 (2) 1.2300 (1) 2.4792 (1)	3.2084 (2) 8.1183 (2) 1.2704 (1) 2.5387 (1)
3	2.2696 (3) 1.2258 (2) 2.6852 (2) 8.8086 (2)	1.7219 (2) 4.8022 (2) 7.9956 (2) 1.7988 (1)	3.1121 (2) 7.3698 (2) 1.1388 (1) 2.2885 (1)	4.1044 (2) 9.0894 (2) 1.3583 (1) 2.5862 (1)	4.8150 (2) 1.0295 (1) 1.5099 (1) 2.7857 (1)	5.3420 (2) 1.1182 (1) 1.6204 (1) 2.9286 (1)	5.7460 (2) 1.1859 (1) 1.7045 (1) 3.0360 (1)	6.0650 (2) 1.2392 (1) 1.7705 (1) 3.1196 (1)
4	2.5406 (3) 1.3532 (2) 2.9271 (2) 9.3498 (2)	2.0979 (2) 5.5712 (2) 9.0329 (2) 1.9473 (1)	3.9719 (2) 8.7742 (2) 1.3112 (1) 2.5051 (1)	5.3880 (2) 1.0999 (1) 1.5831 (1) 2.8518 (1)	6.4410 (2) 1.2601 (1) 1.7750 (1) 3.0879 (1)	7.2430 (2) 1.3800 (1) 1.9172 (1) 3.2591 (1)	7.8702 (2) 1.4730 (1) 2.0266 (1) 3.3890 (1)	8.3730 (2) 1.5471 (1) 2.1134 (1) 3.4909 (1)
5	2.7048 (3) 1.4316 (2) 3.0771 (2) 9.6862 (2)	2.3466 (2) 6.0805 (2) 9.7159 (2) 2.0435 (1)	4.5767 (2) 9.7429 (2) 1.4285 (1) 2.6489 (1)	6.3275 (2) 1.2352 (1) 1.7395 (1) 3.0312 (1)				
6	2.8147 (3) 1.4845 (2) 3.1789 (2) 9.9151 (2)	2.5219 (2) 6.4409 (2) 1.0199 (1) 2.1109 (1)	5.0205 (2) 1.0448 (1) 1.5134 (1) 2.7515 (1)	7.0360 (2) 1.3357 (1) 1.8546 (1) 3.1609 (1)				
7	2.8934 (3) 1.5226 (2) 3.2523 (2) 1.0081 (1)	2.6515 (2) 6.7089 (2) 1.0557 (1) 2.1608 (1)	5.3580 (2) 1.0984 (1) 1.5777 (1) 2.8285 (1)	7.5870 (2) 1.4131 (1) 1.9428 (1) 3.2591 (1)				
8	2.9524 (3) 1.5513 (2) 3.3078 (2) 1.0206 (1)	2.7512 (2) 6.9157 (2) 1.0834 (1) 2.1992 (1)	5.6232 (2) 1.1405 (1) 1.6279 (1) 2.8883 (1)	8.0260 (2) 1.4746 (1) 2.0125 (1) 3.3361 (1)				

Table [illegible] (continued)

q \ m	1	2	3	4	5	6	7	8
1	4.5000 (4) 3.4003 (3) 8.9608 (3) 3.9061 (2)	1.9550 (3) 1.0807 (2) 2.3950 (2) 7.8984 (2)	2.8000 (3) 1.4836 (2) 3.1806 (2) 9.8000 (2)	3.2711 (3) 1.7124 (2) 3.6276 (2) 1.0866 (1)	3.5667 (3) 1.8572 (2) 3.9116 (2) 1.1540 (1)	3.7687 (3) 1.9566 (2) 4.1069 (2) 1.2004 (1)	3.9153 (3) 2.0289 (2) 4.2491 (2) 1.2342 (1)	4.0261 (3) 2.0838 (2) 4.3573 (2) 2.8043 (1)
2	1.3099 (3) 7.3835 (3) 1.6675 (2) 5.7813 (2)	9.4700 (3) 2.9706 (2) 5.2240 (2) 1.2714 (1)	1.6280 (2) 4.5087 (2) 7.4508 (2) 1.6418 (1)	2.0783 (2) 5.4973 (2) 8.8501 (2) 1.8636 (1)	2.3855 (2) 6.1696 (2) 9.7945 (2) 2.0102 (1)	2.6060 (2) 6.6528 (2) 1.0471 (1) 2.1139 (1)	2.7713 (2) 7.0155 (2) 1.0978 (1) 2.1911 (1)	2.8995 (2) 7.2973 (2) 1.1371 (1) 2.2507 (1)
3	1.7014 (3) 9.1752 (3) 2.0066 (2) 6.5562 (2)	1.4704 (2) 4.0636 (2) 6.7270 (2) 1.4973 (1)	2.7537 (2) 6.4597 (2) 9.9237 (2) 1.9726 (1)	3.6885 (2) 8.0976 (2) 1.2034 (1) 2.2675 (1)	4.3645 (2) 9.2577 (2) 1.3507 (1) 2.4671 (1)	4.8681 (2) 1.0115 (1) 1.4587 (1) 2.6110 (1)	5.2560 (2) 1.0773 (1) 1.5412 (1) 2.7195 (1)	5.5630 (2) 1.1292 (1) 1.6061 (1)
4	1.9049 (3) 1.0135 (2) 2.1898 (2) 6.9731 (2)	1.7965 (2) 4.7294 (2) 7.6260 (2) 1.6277 (1)	3.5313 (2) 7.7268 (2) 1.1481 (1) 2.1709 (1)					
5	2.0282 (3) 1.0726 (2) 2.3033 (2) 7.2325 (2)	2.0125 (2) 5.1712 (2) 8.2196 (2) 1.7124 (1)	4.0800 (2) 8.6039 (2) 1.2545 (1) 2.3033 (1)					
6	2.1107 (3) 1.1125 (2) 2.3804 (2) 7.4092 (2)	2.1647 (2) 5.4843 (2) 8.6396 (2) 1.7720 (1)	4.4830 (2) 9.2442 (2) 1.3317 (1) 2.3980 (1)					
7	2.1697 (3) 1.1411 (2) 2.4360 (2) 7.5371 (2)	2.2773 (2) 5.7172 (2) 8.9520 (2) 1.8161 (1)	4.7905 (2) 9.7312 (2) 1.3902 (1) 2.4692 (1)					
8	2.2141 (3) 1.1628 (2) 2.4780 (2) 7.6340 (2)	2.3639 (2) 5.8970 (2) 9.1933 (2) 1.8500 (1)	5.0315 (2) 1.0114 (1) 1.4360 (1) 2.5247 (1)					

q \ m	k = 2: 1	k = 2: 2	k = 2: 3	k = 2: 4	k = 3: 1	k = 3: 2	k = 3: 3	k = 3: 4
1	2.0000 (2)	1.5056 (2)	1.3406 (2)	1.2580 (2)	1.5000 (2)	1.0748 (2)	9.3740 (3)	8.6980 (3)
	1.0000 (1)	7.6461 (2)	6.8524 (2)	6.4546 (2)	7.5000 (2)	5.4418 (2)	4.7705 (2)	4.4390 (2)
	2.0000 (1)	1.5609 (1)	1.4111 (1)	1.3359 (1)	1.5000 (1)	1.1064 (1)	9.7642 (2)	9.1204 (2)
	5.0000 (1)	4.1886 (1)	3.9030 (1)	3.7593 (1)	3.7500 (1)	2.9226 (1)	2.6385 (1)	2.4964 (1)
2	1.4142 (1)	1.0804 (1)	9.4690 (2)	8.7430 (2)	1.2247 (1)	9.0170 (2)	7.7620 (2)	7.0910 (2)
	3.1623 (1)	2.5000 (1)	2.2284 (1)	2.0794 (1)	2.7386 (1)	2.0788 (1)	1.8156 (1)	1.6732 (1)
	4.4722 (1)	3.6409 (1)	3.2930 (1)	3.1009 (1)	3.8730 (1)	3.0165 (1)	2.6673 (1)	2.4768 (1)
	7.0711 (1)	6.1904 (1)	5.8072 (1)	5.5934 (1)	6.1237 (1)	5.0723 (1)	4.6216 (1)	4.3715 (1)
3	2.7144 (1)	2.1511 (1)	1.8970 (1)	1.7496 (1)	2.4662 (1)	1.8982 (1)	1.6480 (1)	1.5051 (1)
	4.6416 (1)	3.8195 (1)	3.4375 (1)	3.2133 (1)	4.2171 (1)	3.3596 (1)	2.9700 (1)	2.7442 (1)
	5.8480 (1)	4.9497 (1)	4.5237 (1)	4.2717 (1)	5.3133 (1)	4.3412 (1)	3.8897 (1)	3.6255 (1)
	7.9370 (1)	7.1533 (1)	6.7690 (1)	6.5392 (1)	7.2112 (1)	6.2227 (1)	5.7420 (1)	5.4557 (1)
4	3.7606 (1)	3.0771 (1)	2.7413 (1)	2.5368 (1)	3.4997 (1)	2.7953 (1)	2.4566 (1)	2.2530 (1)
	5.6234 (1)	4.7713 (1)	4.3406 (1)	4.0748 (1)	5.2332 (1)	4.3227 (1)	3.8713 (1)	3.5959 (1)
	6.6874 (1)	5.8200 (1)	5.3737 (1)	5.0961 (1)	6.2233 (1)	5.2609 (1)	4.7736 (1)	4.4735 (1)
	8.4090 (1)	7.7242 (1)	7.3620 (1)	7.1346 (1)	7.8254 (1)	6.9370 (1)	6.4689 (1)	6.1755 (1)

For fixed k, P*, q and m, the above table gives the values of [illegible] satisfying

$$P\{X_k \geq [illegible] \max_{1 \leq j \leq k-1} X_j\} = \int_0^1 I^{k-1}_{[illegible]}(q,m)\, b\,(x;q,m)\, dx = P^*$$

where X_i, $i = 1,2,\ldots,k$ are i.i.d. beta random variables with parameters q and m and $I_x(q,m)$ and $b(x;q,m)$ as c.d.f. and density, respectively.

1. The four entries in each cell refer to $P^* = .99, .95, .90, .75$ from top to bottom.
2. The number in the parentheses refers to the power of 10^{-1} by which each entry should be multiplied to get [illegible].

Table 2 (continued)

q \ m	k = 4: 1	k = 4: 2	k = 4: 3	k = 4: 4	k = 5: 1	k = 5: 2	k = 5: 3	k = 5: 4
1	1.3332 (2)	9.2370 (3)	7.9410 (3)	7.3100 (3)	1.2500 (2)	8.4460 (3)	7.1820 (3)	6.5700 (3)
	6.6667 (2)	4.6728 (2)	4.0357 (2)	3.7244 (2)	6.2500 (2)	4.2707 (2)	3.6472 (2)	3.3446 (2)
	1.3333 (1)	9.4890 (2)	8.2447 (2)	7.6344 (2)	1.2500 (1)	8.6676 (2)	7.4441 (2)	6.8482 (2)
	3.3333 (1)	2.4951 (1)	2.2123 (1)	2.0720 (1)	3.1250 (1)	2.2744 (1)	1.9909 (1)	1.8510 (1)
2	1.1547 (1)	8.3040 (2)	7.0650 (2)	6.4072 (2)	1.1180 (1)	7.9052 (2)	6.6677 (2)	6.0150 (2)
	2.5820 (1)	1.9120 (1)	1.6490 (1)	1.5079 (1)	2.5000 (1)	1.8193 (1)	1.5546 (1)	1.4136 (1)
	3.6515 (1)	2.7713 (1)	2.4177 (1)	2.2264 (1)	3.5355 (1)	2.6354 (1)	2.2772 (1)	2.0845 (1)
	5.7735 (1)	4.6444 (1)	4.1659 (1)	3.9019 (1)	5.5902 (1)	4.4097 (1)	3.9131 (1)	3.6404 (1)
3	2.3712 (1)	1.7925 (1)	1.5410 (1)	1.3985 (1)	2.3208 (1)	1.7321 (1)	1.4785 (1)	1.3356 (1)
	4.0548 (1)	3.1692 (1)	2.7719 (1)	2.5434 (1)	3.9685 (1)	3.0609 (1)	2.6568 (1)	2.4256 (1)
	5.1087 (1)	4.0915 (1)	3.6243 (1)	3.3528 (1)	5.0000 (1)	3.9500 (1)	3.4710 (1)	3.1941 (1)
	6.9336 (1)	5.8501 (1)	5.3270 (1)	5.0167 (1)	6.7861 (1)	5.6411 (1)	5.0909 (1)	4.7656 (1)
4	3.3981 (1)	2.6749 (1)	2.3309 (1)	2.1257 (1)	3.3437 (1)	2.6052 (1)	2.2565 (1)	2.0495 (1)
	5.0814 (1)	4.1328 (1)	3.6673 (1)	3.3851 (1)	5.0000 (1)	4.0234 (1)	3.5471 (1)	3.2597 (1)
	6.0427 (1)	5.0259 (1)	4.5158 (1)	4.2034 (1)	5.9460 (1)	4.8913 (1)	4.3649 (1)	4.0439 (1)
	7.5984 (1)	6.6142 (1)	6.0980 (1)	5.7755 (1)	7.4768 (1)	6.4309 (1)	5.8840 (1)	5.5432 (1)

It is assumed that $\Sigma_i, \Sigma_{11}^{(i)}, \Sigma_{22}^{(i)}$ $(i=1,2,\ldots,k)$ are all positive definite.

We are interested in selecting a subset containing the population associated with the smallest $|\Sigma_i/|\Sigma_{11}^{(i)}| \equiv |\Sigma_{22}^{(i)} - \Sigma_{21}^{(i)} {\Sigma_{11}^{(i)}}^{-1} \Sigma_{12}^{(i)}| = \sigma_i$, say. In other words, if we consider for each population the conditional distribution of the q_2 set when the q_1 set is fixed, then our criterion of ranking is the conditional generalized variance. This provides a justification for the choice of the criterion. If the observations are taken on the variables of the q_2 set, holding the variables of the q_1 set fixed, then the problem reduces to selecting in terms of the generalized variance for the (conditional) normal distributions with dimensionality q_2. This problem has already been solved by Gnanadesikan and Gupta [27]. A situation may arise where we want to select in terms of the conditional generalized variance, but in practice the observations come when all the p variables are random. This section deals with a selection procedure for that case.

Let S_i be the sample covariance matrix from π_i based on n observations, $i=1,2,\ldots,k$. Further let the partition of S_i be denoted by

$$(2.3.2) \qquad S_i = \begin{pmatrix} S_{11}^{(i)} & S_{12}^{(i)} \\ & \\ S_{21}^{(i)} & S_{22}^{(i)} \end{pmatrix}, \quad i = 1,2,\ldots,k.$$

Also let $s_i = |S_i|/S_{11}^{(i)}| \equiv |S_{22}^{(i)} - S_{21}^{(i)} {S_{11}^{(i)}}^{-1} S_{12}^{(i)}|$. Then, we propose the following procedure

R: Select π_i iff $\frac{s_i}{C} \leq \min\limits_{1 \leq r \leq k} s_r$, where $C = C(k,P^*,n,q_1,q_2) > 1$ is chosen so as to satisfy the basic P*-condition.

Theorem 2.3.1. Let Ω be the space of the covariance matrices $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$ of k p-variate normal populations. Then

$$\text{(2.3.3)} \qquad \inf_{\Omega} P(CS|\Omega) = \int_0^\infty [1-G(x/C)]^{k-1}\, dG(x),$$

where $G(x)$ is the cdf of a random variable which is the product of q_2 independent chi-square variables with d.f. $n-q_1-1, n-q_1-2, \ldots, n-q_1-q_2$ respectively.

Proof. Denoting the jth largest of the σ_i by $\sigma_{[j]}$ and the s_j associated with it by $s_{(j)}$, we have

$$P(CS|\Omega) = P(s_{(1)} \leq Cs_{(r)}, \quad r = 2, \ldots, k)$$

$$= P\left(\frac{s_{(r)}}{\sigma_{[r]}} \geq \frac{1}{C}\frac{s_{(1)}}{\sigma_{[1]}}, \quad r = 2, \ldots, k\right)$$

$$\geq P\left(\frac{s_{(r)}}{\sigma_{[r]}} \geq \frac{1}{C}\frac{s_{(1)}}{\sigma_{[1]}}, \quad r = 2, \ldots, k\right)$$

$$= P\left(A_r \geq \frac{1}{C} A_1, \quad r = 2, \ldots, k\right),$$

where $A_r = \frac{s_{(r)}}{\sigma_{[r]}}$, $r=1,\ldots,k$. It is known that $S_{22}^{(i)} - S_{21}^{(i)} S_{11}^{(i)^{-1}} S_{12}^{(i)}$ has the Wishart distribution $W_{q_2}(n-1-q_1, \Sigma_{22}^{(i)} - \Sigma_{21}^{(i)} \Sigma_{21}^{(i)^{-1}} \Sigma_{12}^{(i)})$ and hence A_r is distributed as the product of q_2 independent chi-square variables with d.f. $n-q_1-1, \ldots, n-q_1-q_2$ respectively. Thus A_r, $r=1,\ldots,k$, are independent and identically distributed with cdf $G(x)$. Hence (2.3.3) follows.

Corollary 2.3.1. The constant C defining the procedure R is given by

$$\int_0^\infty [1-G(x/C)]^{k-1}\, dG(x) = P^*. \tag{2.3.4}$$

The constant C satisfying (2.3.4) is the reciprocal of the 100(1-P*) percentage point of the distribution of $\eta_{min} = \min(\frac{\eta_2}{\eta_1}, \ldots, \frac{\eta_k}{\eta_1})$, where $\eta_1, \ldots, \eta_k$ are i.i.d. with cdf $G(x)$. The exact distribution of η_i is not known for $q_2 > 2$. When $q_2 > 2$, an approximation suggested by Hoel [47] can be used. A detailed discussion of this approximation has been given in Gnanadesikan and Gupta [27]. For $q_2 = 2$, $2\eta_i^{1/2}$ is distributed as a chi-square with $2(n-q_1-2)$ d.f. and hence in this case $C^{1/2}$ is the reciprocal of the lower 100(1-P*) percentage point of $F_{min} = \min(\frac{\chi^2_{\nu,2}}{\chi^2_{\nu,1}}, \ldots, \frac{\chi^2_{\nu,k}}{\chi^2_{\nu,1}})$, where $\chi^2_{\nu,r}$, $r=1,\ldots,k$, are independent chi-square variables each with $\nu=2(n-q_1-2)$ d.f. Thus we have

Theorem 2.3.2. When $q_2 = 2$, $k > 1$,

$$\inf_{\Omega} P(CS|R) = P(F_{min} \geq \frac{1}{C^{1/2}}). \tag{2.3.5}$$

As remarked by Gnanadesikan and Gupta [27], the constant C is related to the constant of the procedure discussed by Gupta and Sobel [42], which has been tabulated by Gupta and Sobel [43] and Krishnaiah and Armitage [52].

Remark 2.3.1. Though we did not state it earlier explicitly the fact that the distribution of s is stochastically increasing in σ, it is easily verified as follows. If s_1 and s_2 have the associated σ_1, σ_2 such that $\sigma_1 < \sigma_2$, then

$$
\begin{aligned}
P(s_1 \leq x) &= P(A_1 \leq \frac{x}{\sigma_1}) \\
&= P(A_2 \leq \frac{x}{\sigma_1}) \\
&\geq P(A_2 \leq \frac{x}{\sigma_2}) \\
&= P(s_2 \leq x) \ .
\end{aligned} \tag{2.3.6}
$$

Consequently, using Theorem 1.6.1, the procedure Ω is unbiased.

CHAPTER III

SELECTION PROCEDURES FOR RESTRICTED FAMILIES OF DISTRIBUTIONS

3.1. Introduction

Barlow and Gupta [4] were the first to consider the problem of selection for restricted families of probability distributions. Let $\pi_1, \pi_2, \ldots, \pi_k$ be k populations with associated continuous distribution functions F_i, $i=1,\ldots,k$. The restriction of the distributions considered to a particular class is in the sense that these distributions are ordered in certain sense with respect to a specified continuous distribution G. The distributions F_i are, otherwise, unspecified. The order relations considered by Barlow and Gupta [4] are partial ordering relations on the space of distributions. The purpose of this ordering relation is to provide a lower bound for the probability of a correct selection over the space Ω of the k-tuples $(F_1,\ldots,F_k)$. Thus we are enabled to define distribution-free procedures in the sense that the constant defining the procedure depends on the knowledge of G and not on the specific forms of F_i. In this chapter we are concerned with selection procedures for such distributions which are partially ordered with respect to G. We define a general partial ordering of which some of the order relations considered are special cases. It is shown that some of the results obtained by Barlow and Gupta can be obtained under this general ordering. We also study the implications of some of the orderings for special choices of G.

3.2. Some Preliminary Results

We start with a few definitions.

Definition 3.2.1. A real valued function φ defined on $[0,\infty)$ is said to be star-shaped if $\varphi(\alpha x) \leq \alpha\varphi(x)$ for $x \geq 0$ and $0 \leq \alpha \leq 1$.

Definition 3.2.2. A real valued function φ defined on an interval I is said to be an r-function if, for $x \in I$ and $\alpha x \in I$ and $0 \leq \alpha \leq 1$,

$$\varphi(\alpha x) \leq \alpha\varphi(x) , \quad x \in I \cap (0,\infty),$$

$$\varphi(\alpha x) \geq \alpha\varphi(x) , \quad x \in I \cap (-\infty,0) \quad \text{and}$$

$$\varphi(0) = 0, \quad \text{if } 0 \in I$$

Definition 3.2.3. A real valued function φ defined on an interval I is said to be a t-function if $\varphi(x+b) \geq \varphi(x)+b$ whenever $x \in I$, $x+b \in I$ and $b \geq 0$.

From the above definitions the following results follow immediately. We state them as lemmas.

Lemma 3.2.1. A real valued function φ defined on $[0,\infty)$ is star-shaped iff

(i) $\varphi(x)/x$ is non-decreasing in $x > 0$ and

(ii) $\varphi(0) \leq 0$.

Lemma 3.2.2. A real valued function φ defined on an interval I is an r-function iff

(i) $\varphi(x)/x$ is non-decreasing in $x > 0$, $x \in I$

(ii) $\varphi(x)/x$ is non-increasing in $x < 0$, $x \in I$ and

(iii) $\varphi(0) = 0$, if $0 \in I$.

Lemma 3.2.3. A real-valued function φ defined on an interval I is a t-function iff $\varphi(x)-x$ is non-decreasing in $x \in I$.

As an easy consequence of Lemma 3.2.3 we obtain the following lemma.

Lemma 3.2.4. If φ is an r-function defined on an interval I and differentiable on I, (i) $x\varphi'(x) \geq \varphi(x)$ for $x > 0$, $x \in I$ and (ii) $x\varphi'(x) \leq \varphi(x)$ for $x < 0$, $x \in I$ where $\varphi'(x) \equiv \frac{d}{dx}\varphi(x)$.

Lemma 3.2.5. If φ is an r-function defined on an interval I possessing the derivative at every point of I, then $\varphi'(x) \geq \varphi'(0)$ for all $x \in I$.

Proof. We assume that the origin is an internal point of I. Suppose, for any $x_0 \in I \cap (0,\infty)$, we have $\varphi'(x_0) < \varphi'(0)$. Then

$$\begin{aligned} \varphi(x_0) &\leq x_0\, \varphi'(x_0), \text{ by lemma 3.2.4} \\ &< x_0\, \varphi'(0) \\ &= x_0 \lim_{x \to 0^+} \varphi(x)/x. \end{aligned}$$

This implies that there exists an $x_1 < x_0$ such that

$$\varphi(x_0) < x_0\, \varphi(x_1)/x_1. \tag{3.2.1}$$

Letting $\alpha = x_1/x_0 < 1$, (3.2.1) becomes $\alpha\varphi(x_0) < \varphi(\alpha x_0)$, which is a contradiction of the hypothesis that φ is an r-function. Hence $\varphi'(x) \geq \varphi'(0)$ for $x \in I \cap (0,\infty)$. By a similar argument, it can be shown that $\varphi'(x) \geq \varphi'(0)$ for $x \in I \cap (-\infty,0)$. This completes the proof of Lemma 3.2.5.

Theorem 3.2.1. If φ is an r-function on an interval I containing the origin and $\varphi'(0) \geq 1$, then φ is a t-function on I.

Proof. Since φ is an r-function, by Lemma 3.2.5, $\varphi'(x) \geq \varphi'(0)$ for all $x \in I$. Hence $\varphi'(0) \geq 1$ implies that $\varphi'(x) \geq 1$ for $x \in I$, which in turn implies that $\varphi(x)-x$ is non-decreasing in $x \in I$. Thus, by Lemma 3.2.3, φ is a t-function on I.

Remark 3.2.1. The converse of Theorem 3.2.1 is not true. If φ is a t-function on I, then $\varphi'(0) \geq 1$ necessarily, but φ need not be an r-function. The following example illustrates the fact.

Example. Let

$$(3.2.2) \qquad \varphi(x) = x^4/4-2x^3/3+x^2/2+x, \qquad x \geq 0$$
$$= -x^4/4+2x^3/3-x^2/2+x, \qquad x < 0.$$

Differentiating w.r.t. x,

$$(3.2.3) \qquad \varphi'(x) = x^3-2x^2+x+1, \qquad x \geq 0$$
$$= -x^3+2x^2-x+1, \qquad x < 0.$$

We can write (3.2.3) as

$$(3.2.4) \qquad \varphi'(x) = |x|(x-1)^2+1 \quad \text{for all } x.$$

Thus $\varphi'(x) \geq 1$ and hence φ is a t-function. To see that φ is not an r-function, set x=1 and $\alpha = \frac{1}{2}$. Then $\varphi(\alpha x) = 107/192$ and $\alpha\varphi(x)=104/192$. Hence $\alpha\varphi(x) < \varphi(\alpha x)$, which violates the definition of an r-function.

Remark 3.2.2. Incidentally, we note that $\varphi'(x) \geq \varphi'(0)$ is only a necessary condition for φ to be an r-function. In the above example we see that $\varphi'(x) \geq 1 = \varphi'(0)$, but φ is not an r-function.

Now we define a more general function.

Definition 3.2.4. Let $h \equiv h_{a,b}$, $a \geq 1$, $b \geq 0$, be a real valued function defined on the real line. A real valued function φ defined on an interval I is said to be an h-function if $\varphi(h(x)) \geq h(\varphi(x))$ for $x\epsilon I$ and $h(x)\epsilon I$ for all $a \geq 1$, $b \geq 0$.

Corollary 3.2.1. Let $h_{a,b}(x) = ax$, $a \geq 1$ and φ be a real valued function defined on $[0,\infty)$. Then, if φ is an h-function, it is star-shaped.

Proof. Since φ is an h-function with $h(x) = ax$, $\varphi(ax) \geq a\varphi(x)$, $x \geq 0$. Setting $\alpha = a^{-1}$ and $ax=y$, we have $\varphi(\alpha y) \leq \alpha\varphi(y)$ for $y \geq 0$ and $0\leq \alpha \leq 1$. Hence φ is star-shaped.

Corollary 3.2.2. Let $h_{a,b}(x) = x+b, b > 0$ and φ be a real valued function on an interval I. Then, if φ is an h-function, it is a t-function.

Proof. By the hypothesis, we have $\varphi(x+b) \geq \varphi(x)+b$, which means that φ is a t-function.

Now we rephrase our results in terms of order relations between distributions. Consider two absolutely continuous distributions F and G. Let I denote the support of the distribution F. Also let $\varphi \equiv G^{-1}F$.

Definition 3.2.5. F is said to be star-shped with respect to G (written $F \underset{*}{\leq} G$) iff $F(0) = G(0) = 0$ and φ is starshped on I.

Definition 3.2.6. F is said to be r-ordered with respect to G (written $F \underset{r}{<} G$) iff $F(0) = G(0) = \alpha$, $0 < \alpha < 1$ and φ is an r-function on I.

Definition 3.2.7. F is tail-ordered with respect to G (written $F \underset{t}{<} G$) iff $F(0) = G(0) = \alpha$, $0 < \alpha < 1$ and φ is a t-function on I.

Definition 3.2.8. Let $h_{a,b}(x)$, $a \geq 1$, $b \geq 0$, be a real valued function defined on the real line. Then F is said to be h-ordered with respect to G (written $F \underset{h}{<} G$) iff $F(0) = G(0)$ and φ is an h-function on I.

Remark 3.2.3. In Definitions 3.2.6 and 3.2.7 we have imposed the condition that $F(0) = G(0)$ which gives $\varphi(0) = 0$. It is not crucial to our discussions what that equal value of $F(0)$ and $G(0)$ is. The usual definition of r-ordering states that $F(0) = G(0) = \frac{1}{2}$ (see Barlow and Gupta [4]). We take $F(0) = G(0) = \alpha$, where $0 < \alpha < 1$.

Assuming that the distributions F and G have densities f and g respectively, we note that $\varphi'(x) \equiv \frac{d}{dx} G^{-1}F(x) = f(x)/g(G^{-1}F(x))$. Hence $\varphi'(0) = f(0)/g(0)$, if it exists. Now we state many of our earlier results in the context of the orderings defined as corollaries.

Corollary 3.2.3. Let $F \underset{r}{<} G$. Then $f(x)/g(G^{-1}F(x) \geq f(0)/ g(0)$.

Corollary 3.2.4. If $F \underset{r}{<} G$ and $f(0) \geq g(0)$, then $F \underset{t}{<} G$.

Remark 3.2.4. $F \underset{t}{<} G$ does not necessarily imply $F \underset{r}{<} G$.

Corollary 3.2.5. If $h_{a,b}(x) = ax, a \geq 1$ and $F(0) = G(0) = 0$, then $F \underset{h}{<} G \Rightarrow F \underset{t}{<} G$.

Definition 3.2.9. A relation $\lesssim$ on the space of distributions is a partial ordering if

(i) $F \lesssim F$ for all distributions F and

(ii) $F \lesssim G$ and $G \lesssim H$ imply $F \lesssim H$.

Remark 3.2.5. $F \lesssim G$ and $G \lesssim F$ do not necessarily imply $F \equiv G$.

It is known that $\underset{*}{<}$ and $\underset{r}{<}$ are partial orderings. It can be shown easily in the cases of $\underset{t}{<}$ and $\underset{h}{<}$ the conditions of partial ordering are satisfied. We will verify below the condition (ii) in the case of h-ordering. We need show that if $G^{-1}F(h(x)) \geq h(G^{-1}F(x))$ and $H^{-1}G(h(x)) \geq h(H^{-1}G(x))$ then $H^{-1}F(h(x)) \geq h(H^{-1}F(x))$. Now,

$$H^{-1}F(h(x)) = H^{-1}GG^{-1}F(h(x))$$

$$\geq H^{-1}Gh(G^{-1}F(x)), \text{ since } H^{-1}G \text{ is an increasing function}$$

$$\geq h(H^{-1}GG^{-1}F(x))$$

$$= h(H^{-1}F(x)).$$

Remark 3.2.6. If X has distribution F and Y has distribution G, then $G^{-1}F(X) \underset{st}{=} Y$, where $\underset{st}{=}$ indicates stochastic equality. If $F \underset{\sim}{<} G$, then $G^{-1}F(h(x)) \geq h(G^{-1}F(x))$ for all x on the support of F, i.e., $G^{-1}F(h(X)) \underset{st}{\geq} h(G^{-1}F(X))$ which is same as $G^{-1}F(h(X)) \underset{st}{\geq} h(Y)$.

Lemma 3.2.6. If $G^{-1}F(h(x)) \leq h(G^{-1}F(x))$, then $G \underset{h}{<} F$.

Proof. Since $G^{-1}F(h(x)) \leq h(G^{-1}F(x))$ for all x, setting $x = F^{-1}G(y)$, we get

$$(3.2.5) \qquad G^{-1}F(hF^{-1}G(y)) \leq h(y) \quad \text{for all } y.$$

Since $G^{-1}F$ is an increasing function, (3.2.5) yields

$$(3.2.6) \qquad h(F^{-1}G(y)) \leq F^{-1}G(h(y)),$$

which means $G \underset{h}{<} F$.

In the light of the above lemma, we restate Lemma 1.3.2.

Lemma 3.2.7. If $F \underset{h}{<} G$, then, for any positive integer t,

$$(3.2.7) \qquad \int F^t(h(x))\, dF(x) \geq \int G^t(h(x))\, dG(x).$$

Remark 3.2.7. If $X_1,\ldots,X_k$ are i.i.d. with distribution F and $Y_1,\ldots,Y_k$ are i.i.d. with distribution G and $F \underset{h}{<} G$, then (3.2.7) is equivalent to

(3.2.8) $$P(h(X_k) \geq X_r,\ r=1,\ldots,k-1) \geq P(h(Y_k) \geq Y_r,\ r=1,\ldots,k-1).$$

3.3. A General Selection Problem

Let $\pi_1,\pi_2,\ldots,\pi_k$ be k populations. The random variable X_i associated with π_i has a continuous distribution F_i, $i=1,\ldots,k$. We assume that there exists one among the k populations which is stochastically larger than any other. Let us denote the distribution of that population by $F_{[k]}$. Then the assumption can be expressed as

(3.3.1) $$F_i(x) \geq F_{[k]}(x) \quad \text{for } i=1,\ldots,k \text{ and all } x.$$

We also assume that there exists a continuous distribution G such that

(3.3.2) $$F_i(x) \precsim G \quad \text{for } i = 1,\ldots,k,$$

where $\precsim$ denotes any partial ordering on the space of distributions. If $\underline{X}_i = (X_{i1},X_{i2},\ldots,X_{in})$ is the observed sample from π_i, then we confine ourselves to the class of statistics $T_i = T(\underline{X}_i)$ that preserve both the ordering relations (3.3.1) and (3.3.2), i.e., we have

(3.3.3) $$P_{F_i}(T(\underline{X}) \leq x) \geq P_{F_{[k]}}(T(\underline{X}) \leq x) \quad \text{for } i=1,\ldots,k \text{ and all } x$$

and

(3.3.4) $$F_{T(\underline{X}_i)} \precsim G_{T(\underline{Y})}, \quad i = 1,\ldots,k,$$

where $F_{T(\underline{X}_i)}$ represents the cdf of $T(\underline{X}_i)$ under F_i and $G_{T(\underline{Y})}$ is the cdf of $T(\underline{Y})$ under G, $\underline{Y}=(Y_1,\ldots,Y_n)$ being a random sample from G.

Now, let us assume that the partial ordering in (3.3.2) is h-ordering, where $h \equiv h_{c,d}$, $c \geq 1$, $d \geq 0$ possesses the properties (1.1.3). Then, for selecting a subset containing the population associated with $F_{[k]}$, we propose the rule

$$\text{(3.3.5)} \qquad R: \text{Select } \pi_i \text{ iff } h(T_i) \geq \max_{1 \leq r \leq k} T_r.$$

For convenience, let $T_{(k)}$ denote the T_i associated with $F_{[k]}$ and $T_{(r)}$, $r=1,\ldots,k-1$ denote the T_i associated with the other populations. Then

$$\text{(3.3.6)} \qquad P(CS|R) = P(h(T_{(k)}) \geq \max_{1 \leq r \leq k-1} T_{(r)})$$

$$= \int_{-\infty}^{\infty} \prod_{r=1}^{k-1} F_{T_{(r)}}(h(x))\, dF_{T_{(k)}}(x)$$

$$\geq \int_{-\infty}^{\infty} F_{T_{(k)}}^{k-1}(h(x))\, dF_{T_{(k)}}(x), \text{ by (3.3.3).}$$

Now, by (3.3.4), $F_{T_{(k)}} \underset{h}{<} G_{T(\underline{Y})}$ and hence, using Lemma 3.2.7, we obtain

$$\text{(3.3.7)} \qquad P(CS|R) \geq \int_{-\infty}^{\infty} G_T^{k-1}(h(x))\, dG_T(x),$$

where $G_T \equiv G_{T(\underline{Y})}$. The constants c and d are found satisfying the condition

$$\text{(3.3.8)} \qquad \int_{-\infty}^{\infty} G_T^{k-1}(h(x))\, dG_T(x) = P^*.$$

Lemma 3.3.1. If $F \underset{h}{<} G$, then $F_j \underset{h}{<} G_j$, where F_j and G_j are the distributions of the jth order statistic in a sample of size n from F and G.

Proof. It is known that $F_j(x) = B_{j,n}(F(x)) \equiv B_{j,n}F(x)$ and $G_j(x) = B_{j,n}G(x)$ where $B_{j,n}(x) = j\binom{n}{j}\int_0^x u^{j-1}(1-u)^{n-j}du$. Hence $G_j^{-1}F_j(x) = [B_{j,n}G]^{-1}B_{j,n}F(x) = G^{-1}F(x)$, which gives the desired result.

Remark 3.3.1. If $F_i \underset{*}{<} G$ in (3.3.2), then we take $h(x) = cx$, $c \geq 1$ and the constant c of the procedure is given by

$$\int_0^\infty G_T^{k-1}(cx)\, dG_T(x) = P^*. \tag{3.3.9}$$

Remark 3.3.2. If $F_i \underset{t}{<} G$ in (3.3.2), then we use the procedure with $h(x) = x+d$, $d \geq 0$ and the constant d is found so as to satisfy

$$\int_{-\infty}^\infty G_T^{k-1}(x+d)\, dG_T(x) = P^*. \tag{3.3.10}$$

Barlow and Gupta [4] have considered procedures for selecting the population with the largest quantile of a specified order when the populations F_i are star-shaped with respect to G. Lemma 3.3.1 and Remark 3.3.1 show that their results follow from our general result. They also discuss a procedure for selecting the population with the largest median when the distributions centered at their medians are r-ordered with respect to $G(G(0) = \frac{1}{2})$ with the additional condition that $\varphi'(0) \geq 1$ where $\varphi \equiv G^{-1}F_{[k]}(x+\Delta_{[k]})$, $\Delta_{[k]}$ denoting the median of $F_{[k]}$. From Remark 3.3.2 and Corollary 3.2.4, we can see that the selection problem considered by Barlow and Gupta can be accomplished for a larger family of distributions.

3.4. Some Special Cases of Tail Ordering

In this section we will study the properties of F when $F \underset{t}{<} G$ for some special choices of G.

(a) Logistic Distribution, $G(x) = \frac{1}{1+e^{-x}}$, $x \in R$.

In this case, $G^{-1}(y) = \log(y|(1-y))$, $0 \leq y \leq 1$ and $\varphi(x) = G^{-1}F(x) = \log(F(x)/(1-F(x))$. Since $\varphi'(x) \geq 1$, we get $f(x) \geq F(x)\bar{F}(x)$, where $\bar{F}(x) = 1-F(x)$.

(b) Double Exponential, $G'(x) = \frac{1}{2} e^{-|x|}$, $x \in R$.

Here $G(x) = 1 - \frac{1}{2} e^{-x}$, $x \geq 0$

$= \frac{1}{2} e^{x}$, $x < 0$.

Hence $G^{-1}(y) = -\log 2(1-y)$, $y \geq \frac{1}{2}$

$= \log 2y$, $y < \frac{1}{2}$.

Thus $\varphi(x) = -\log 2(1-F(x))$, $x \geq 0$

$= \log 2F(x)$, $x < 0$.

This gives $\varphi'(x) = f(x)/\bar{F}(x)$, $x \geq 0$

$= f(x)/F(x)$, $x < 0$.

Hence $F \underset{t}{<} G \Rightarrow f(x) \geq \bar{F}(x)$ for $x \geq 0$ and $f(x) \geq F(x)$ for $x < 0$.

(c) Rectangular, $G(x) = \frac{x+1}{2}$, $-1 \leq x \leq 1$.

We see that $G^{-1}(y) = 2y-1$, $0 \leq y \leq 1$. Hence $\varphi(x) = 2F(x)-1$ on the support of F and $\varphi'(x) = 2f(x)$. Hence $F \underset{t}{<} G \Rightarrow f(x) \geq \frac{1}{2}$.

Definition 3.4.1. The distribution F is said to belong to the family $\mathcal{F}_\ell$ iff $F(0) = \frac{1}{2}$ and the density $f(x)$ satisfies the inequality

(3.4.1) $$f(x) \geq F(x)\,\bar{F}(x) \quad \text{for all } x \text{ on the support of } F.$$

<u>Definition 3.4.2</u>. The distribution F is said to belong to the family $\mathcal{F}_e$ iff $F(0) = \frac{1}{2}$ and the density f(x) satisfies the inequalities

$$(3.4.2) \quad \begin{cases} f(x) \geq \overline{F}(x), & x \geq 0 \quad \text{and} \\ f(x) \geq F(x), & x < 0. \end{cases}$$

The following obvious statements can now be made.

(i) $F \underset{t}{<} G$, where $G(x) = \frac{1}{1+e^{-x}} \Leftrightarrow F \in \mathcal{F}_\ell$

(ii) $F \underset{t}{<} G$, where $G'(x) = \frac{1}{2} e^{-|x|} \Leftrightarrow F \in \mathcal{F}_e$.

<u>Lemma 3.4.1</u>. $F \in \mathcal{F}_e \Rightarrow F \in \mathcal{F}_\ell$.

<u>Proof</u>. Since $0 \leq F(x), \overline{F}(x) \leq 1$, (3.4.2) implies (3.4.1). Hence the lemma.

<u>Lemma 3.4.2</u>. If F is the standard normal distribution, then $F \in \mathcal{F}_\ell$.

<u>Proof</u>. Let $\Psi(x) = f(x) - F(x)\overline{F}(x)$, where $f(x) = (2\pi)^{-1/2} \exp(-x^2/2)$, $-\infty < x < \infty$. It can be seen that $f'(x) = -xf(x)$ and hence $\Psi'(x) = f(x)(2F(x)-x-1)$. For $x > 0$,

$$\begin{aligned} 2F(x)-x-1 &= \left(\frac{2}{\pi}\right)^{1/2} \int_0^x e^{-t^2/2}\, dt - x \\ &< \left(\frac{2}{\pi}\right)^{1/2} x - x \\ &< 0. \end{aligned}$$

Hence $\Psi(x)$ strictly decreases in $x > 0$. Also $\Psi(0) > 0$ and $\lim_{x\to\infty} \Psi(x) = 0$. Thus $\Psi(x) > 0$ for any $x \geq 0$. Since $\Psi(-x) = \Psi(x)$, $\Psi(x) > 0$ for $x < 0$. This completes the proof of the lemma.

Since the double exponential belongs to $\mathcal{F}_e$, from Lemma 3.4.1, we see that double exponential $\underset{t}{<}$ logistic. It can also be verified that double exponential and normal are not tail ordered.

If we consider the logistic distribution F_θ given by

$$(3.4.3) \qquad F_\theta(x) = \frac{1}{1+e^{-x/\theta}}\,; \quad x\epsilon R,\ \theta > 0$$

$$= F(x/\theta),$$

where F(x) is the logistic distribution with $\theta = 1$. For $\theta_1, \theta_2 > 0$,

$$(3.4.4) \qquad F_{\theta_2}^{-1} F_{\theta_1}(x) = \theta_2\, F^{-1}\, F(x/\theta) = \theta_2 x/\theta_1.$$

Thus $F_{\theta_t} \underset{t}{<} F_{\theta_2}$ iff $\theta_2 > \theta_1$. Hence we say that the logistic distributions t-succeed each other as θ increases. It follows immediately that $F_\theta \epsilon\, \mathcal{F}_\ell$ for $\theta \leq 1$.

Suppose we consider the double exponential distribution G_θ given by

$$(3.4.5) \qquad G_\theta'(x) = \frac{1}{2\theta}\, e^{-|x|/\theta}; \ x\epsilon R,\ \theta > 0.$$

Then

$$G_\theta(x) = 1 - \frac{1}{2}\, e^{-x/\theta}, \quad x \geq 0$$

$$= \frac{1}{2}\, e^{x/\theta}, \quad x < 0.$$

Hence $G_\theta(x) = G(x/\theta)$, where G(x) is the double exponential with $\theta = 1$. For θ_1, $\theta_2 > 0$,

$$G_{\theta_2}^{-1} G_{\theta_1} = \theta_2 G^{-1} G(x/\theta_1) = \theta_2 x/\theta_1. \quad (3.4.6)$$

Thus $G_{\theta_1} \underset{t}{<} G_{\theta_2}$ iff $\theta_2 > \theta_1$. Hence the double exponential distributions G_θ t-succeed each other as θ increases. It follows that $G_\theta \in \mathcal{F}_e$ for $\theta \leq 1$.

3.5. Selection with Respect to Median for Distributions Tail-Ordered with Respect to a Specified G

Suppose $\pi_1,\dots,\pi_k$ are k populations with the associated continuous distributions $F_1,\dots,F_k$. Let Δ_i be the median of π_i, i=1,...,k. We assume (3.3.1) and

$$F_{[k]}(x+\Delta_{[k]}) \underset{t}{<} G(x), \quad (3.5.1)$$

where $G(0) = \frac{1}{2}$ and $\Delta_{[k]}$ is the median associated with $F_{[k]}$. We take a sample of size n from each of the k populations. In the notation of Section 3.3, we choose $T_i = T_{j,i}$. $T_{j,i}$ is the jth order statistic in the sample from π_i. To select a subset of the k populations containing the population with the largest median $\Delta_{[k]}$, we propose the rule R:

R: Select π_i iff

$$T_{j,i} \geq \max_{1\leq r\leq k} T_{j,r} - d, \quad j \leq \frac{n+1}{2} < j+1 \quad (3.5.2)$$

where $d \geq 0$ is chosen to satisfy the P* requirement. Assuming n odd, from our general discussions of Section 3.3, we know that d satisfies

$$\int_{-\infty}^{\infty} G_{j,n}^{k-1}(x+d)\, dG_{j,n}(x) = P^*, \quad (3.5.3)$$

where $G_{j,n}(x)$ is the distribution of the jth order statistic (here it is the sample median) in a sample of size n from G.

Suppose $U_0, U_1, \ldots, U_p$ are (p+1) independent and identically distributed random variables each with distribution G_j. Let $Z_i = U_i - U_0$, $i=1,\ldots,p$. Then (3.5.3) is same as

$$P\{\max_{1 \leq i \leq p} Z_i \leq d\} = P^*, \tag{3.5.4}$$

with p=k-1. Hence d is the 100 P* point of the distribution of $Z_{max} \equiv \max_{1 \leq i \leq p} Z_i$.

Distribution of Z_{max}. Let H(z) be the cdf of Z_{max}. Then

$$H(z) \equiv P\{Z_{max} \leq z\} = \int_{-\infty}^{\infty} G_{j,n}^{p}(z+x)\; g_{j,n}(x)\; dx \tag{3.5.5}$$

where

$$g_{j,n}(x) = j\binom{n}{j}\; G^{j-1}(x)\; [1-G(x)]^{n-j}\; g(x) \tag{3.5.6}$$

and

$$\begin{aligned} G_{j,n}(x) &= \int_{-\infty}^{x} g_{j,n}(t)\; dt \\ &= I_{G(x)}(j,n-j+1) \\ &= \sum_{t=j}^{n} \binom{n}{t}\; G^{t}(x)\; (1-G(x))^{n-t}, \end{aligned} \tag{3.5.7}$$

where $I_y(u,v)$ is the incomplete beta function defined in (2.2.6). Expanding $G^p_{j,n}(z+x)$ in powers of $1-G_{j,n}(z+x)$ using (3.5.7), we obtain

$$(3.5.8) \qquad H(z) = j\binom{n}{j} \sum_{r=0}^{np} \int_{-\infty}^{\infty} C(r,p;n,j)[1-G(z+x)]^r G^{j-1}(x)[1-G(x)]^{n-j} g(x)dx$$

where $C(r,\ell;n,j)$ is the coefficient of y^r in $[\sum_{t=j}^{n} \binom{n}{t}(1-y)^t y^{n-t}]^\ell$ and the coefficients are given by the recursive relations (2.2.30) and (2.2.31). We can rewrite (3.5.8) as

$$(3.5.9) \quad H(z) = 1+j\binom{n}{j} \sum_{r=n-j+1}^{np} C(r,p;n,j)\int_{-\infty}^{\infty}[1-G(z+x)]^r G^{j-1}(x)[1-G(x)]^{n-j} dG(x).$$

When $G(x) = \dfrac{1}{1+e^{-x}}$, we get

$$(3.5.10) \quad H(z)=1+j\binom{n}{j} \sum_{r=n-j+1}^{np} C(r,p;n,j)\int_{0}^{1} y^{j-1}(1-y)^{n-j+r}\{1+(e^z-1)y\}^{-r}\, dy$$

$$=1+j\binom{n}{j} \sum_{r=n-j+1}^{np} \sum_{\alpha=0}^{n-j+r} (-1)^\alpha C(r,p;n,j)\binom{n-j+r}{\alpha}\int_{0}^{1} y^{j+\alpha-1}$$

$$\{1+(e^z-1)y\}^{-r}\, dy.$$

For $z \geq 0$, letting $t = (e^z-1)y/1+(e^z-1)y$,

$$(3.5.11)\quad H(z)=1+j\binom{n}{j}\sum_{r=n-j+1}^{np}\sum_{\alpha=0}^{n-j+r}(-1)^{\alpha}C(r,p;n,j)\binom{n-j+r}{\alpha}(e^{z}-1)^{-(j+\alpha)}\int_{0}^{1-e^{-z}}t^{j+\alpha-1}(1-t)^{r-j-\alpha-1}\,dt$$

$$=1+j\binom{n}{j}\sum_{r=n-j+1}^{np}\sum_{\alpha=0}^{n-j+r}(-1)^{\alpha}\,C(r,p;n,j)\binom{n-j+r}{\alpha}(e^{z}-1)^{-(j+\alpha)}K(e^{z},0,r,\alpha,j)$$

where the function K is defined by (2.2.54). Hence the constant d of the procedure R defined by (3.5.2) is given by

$$(3.5.12)\quad j\binom{n}{j}\sum_{r=n-j+1}^{np}\sum_{\alpha=0}^{n-j+r}(-1)^{r}\,C(r,p;n,j)\binom{n-j+r}{\alpha}(e^{z}-1)^{-(j+\alpha)}K(e^{z},0,r,\alpha,j)=1-P^{*},$$

for $j=\frac{n+1}{2}$.

The problem of selecting the population which has the smallest median, assuming that population to be stochastically smaller than any of the rest can be dealt with analogously.

CHAPTER IV

SOME MODIFICATIONS OF THE GOAL AND THE PROCEDURES

4.1 The Case of t Best Populations

Suppose t of the k populations are tagged 'best'. We need $t < k$ for the problem to be non-trivial. In this case, one can ask for a procedure which will select a subset of random size $\geq s$, $1 \leq s \leq t$, such that with a minimum guaranteed probability $P^*(\binom{k}{s}^{-1} < P^* < 1)$ the selected subset will include any s of the t best populations. It is natural to look at the two extreme cases, namely, s=1 and s=t. There is very little available in the literature regarding the case s=t. The main difficulty is with regard to the distribution problem involved. There has been some work done in the case of s=1 with some modifications in the basic formulation. Under the subset selection formulation, in the notations of Chapter I, suppose we use Procedure R_h. Then we see that a correct selection fails to occur if $h(X_{(\alpha)}) < \max_{1 \leq r \leq k} X_{(r)}$ for $\alpha = k-t+1, \ldots, k-t$. Since $h(x) \geq x$ on the support of the distributions $X_{\max} = \max_{1 \leq r \leq k} X_{(r)}$ must be one of $X_{(1)}, \ldots, X_{(k-t)}$. Hence we have

$$(4.1.1) \quad P(CS|R_h) = 1 - \sum_{i=1}^{k-t} \int \prod_{\alpha=k-t+1}^{k} F_{[\alpha]}(h^{-1}(x)) \prod_{\substack{\beta=1 \\ \beta \neq i}}^{k-t} F_{[\beta]}(x) dF_{[i]}(x),$$

where h^{-1} is the inverse of h, assumed to exist uniquely. It can be seen that

$$\inf_{\Omega} P(CS|R_h) = \inf_{\Omega_1} P(CS|R_h), \tag{4.1.2}$$

where $\Omega_1 = \{\underline{\lambda}|\lambda_{[1]} \leq \cdots \leq \lambda_{[k-t]} \leq \lambda = \lambda_{[k-t+1]} = \cdots = \lambda_{[k]}\}$. In order to evaluate $\inf_{\Omega_1} P(CS|R_h)$, we want $\sup_{\Omega_1} B$, where

$$B \equiv B(\lambda_{[1]},\ldots,\lambda_{[k-t]}) \sum_{i=1}^{k-t} A_i \tag{4.1.3}$$

$$A_i \equiv A_i(\lambda_{[1]},\ldots,\lambda_{[k-t]}) = \int U(x) \prod_{\substack{\beta=1 \\ \beta \neq i}}^{k-t} F_{[\beta]}(x) dF_{[i]}(x) \tag{4.1.4}$$

and

$$U(x) = F^t_{[k-t+1]}(h^{-1}(x)). \tag{4.1.5}$$

We integrate A_1 by parts, use it in (4.1.3) and differentiate w.r.t. $\lambda_{[1]}$. Then, after simplifications, we have

$$\frac{\partial B}{\partial \lambda_{[1]}} = -\int U'(x) \frac{\partial}{\partial \lambda_{[1]}} F_{[1]}(x) \prod_{\beta=2}^{k-t} F_{[\beta]}(x)\, dx. \tag{4.1.6}$$

Since F_λ is stochastically increasing in λ, $\frac{\partial}{\partial \lambda_{[1]}} F_{[1]}(x) \leq 0$. Hence, if we assume that $h^{-1}(x)$ is non-decreasing in x on the support of F_λ, then $U'(x) \geq 0$ for any x belonging to the support and consequently $\frac{\partial B}{\partial \lambda_{[1]}} \geq 0$. It can now be shown by the methods of Chapter I that, if $\lambda_{[1]} = \cdots = \lambda_{[m]} = \lambda' \leq \lambda_{[m+1]} \leq \cdots \leq \lambda_{[k-t]}$, $1 \leq m \leq k-t$, then $B(\lambda',\ldots,\lambda'. \lambda_{[m+1]},\ldots,\lambda_{[k-t]})$ is non-decreasing in λ. Consequently, $\sup_{\Omega_1} B$ takes place when $\lambda_{[1]}=\cdots=\lambda_{[k-t]}=\lambda' = \lambda_{[k-t+1]}$. Hence

(4.1.7) $$\sup_{\Omega_1} B = (k-1) \sup_{\lambda} \int F_{\lambda}^{t} (h^{-1}(x))F_{\lambda}^{k-t-1}(x)dF_{\lambda}(x)$$

$$= (k-1) \sup_{\lambda} A(\lambda,t), \text{ where}$$

(4.1.8) $$A(\lambda,t) = \int F_{\lambda}^{t} (h^{-1}(x))\, F_{\lambda}^{k-t-1}(x)\, dF_{\lambda}(x).$$

When $t = k-1$,

(4.1.9) $$A(\lambda,t) = \int F_{\lambda}^{k-1} (h^{-1}(x))\, dF_{\lambda}(x)$$

and by Remark 1.4.4, $A(\lambda,k-1)$ is non-increasing in λ if

(4.1.10) $$\frac{\partial}{\partial\lambda} F_{\lambda}(h^{-1}(x))f_{\lambda}(x) - \frac{d}{dx} (h^{-1}(x))\frac{\partial}{\partial\lambda} F_{\lambda}(x)f_{\lambda}(h^{-1}(x)) \leq 0.$$

Letting $y = h^{-1}(x)$ and assuming that $h(y)$ increases in y, (4.1.10) becomes

(4.1.11) $$\frac{\partial}{\partial\lambda} F_{\lambda}(h(y))f_{\lambda}(y) - h'(y)\frac{\partial}{\partial\lambda} F_{\lambda}(y)f_{\lambda}(h(y)) \geq 0,$$

which is the sufficient condition needed in Chapter I for $\Psi(\lambda;c,d,k)$ to be non-decreasing in λ.

When $t < k-1$, $A(\lambda,t)$ is non-increasing in λ if

(4.1.12) $$\frac{\partial}{\partial\lambda} \varphi(x,\lambda)f_{\lambda}(x) - \frac{\partial}{\partial\lambda} F_{\lambda}(x)\, \frac{\partial}{\partial x} \varphi(x,\lambda) \leq 0,$$

where $\varphi(x,\lambda) = F_\lambda^t(h^{-1}(x))F_\lambda^{k-t-1}(x)$. We have

$$(4.1.13)\quad \frac{\partial}{\partial x}\varphi(x,\lambda) = F_\lambda^{k-t-2}(x)F_\lambda^{t-1}(h^{-1}(x))[t\frac{d}{dx}h^{-1}(x)F_\lambda(x)f_\lambda(h^{-1}(x))+ (k-t-1)F_\lambda(h^{-1}(x))f_\lambda(x)]$$

and

$$(4.1.14)\quad \frac{\partial}{\partial\lambda}\varphi(x,\lambda)=F_\lambda^{k-t-2}(x)F_\lambda^{t-1}(h^{-1}(x))[tF_\lambda(x)\frac{\partial}{\partial\lambda}F_\lambda(h^{-1}(x))+(k-t-1) F_\lambda(h^{-1}(x))\frac{\partial}{\partial\lambda}F_\lambda(x)].$$

Using (4.1.13) and (4.1.14) in (4.1.12), we see that $A(\lambda,t)$ is non-increasing in λ if

$$(4.1.15)\quad f_\lambda(x)[tF_\lambda(x)\frac{\partial}{\partial\lambda}F_\lambda(h^{-1}(x))+(k-t-1)F_\lambda(h^{-1}(x))\frac{\partial}{\partial\lambda}F_\lambda(x)] - \frac{\partial}{\partial\lambda}F_\lambda(x)[tF_\lambda(x)\frac{d}{dx}h^{-1}(x)f_\lambda(h^{-1}(x))+(k-t-1)f_\lambda(x)F_\lambda(h^{-1}(x))] \le 0.$$

The condition (4.1.15) reduces to

$$(4.1.16)\quad tF_\lambda(x)[\frac{\partial}{\partial\lambda}F_\lambda(h^{-1}(x))f_\lambda(x)-\frac{d}{dx}h^{-1}(x)\frac{\partial}{\partial\lambda}F_\lambda(x)f_\lambda(h^{-1}(x))] \le 0,$$

which is equivalent to (4.1.10) since $tF_\lambda(x) \ge 0$. Hence, if (4.1.11) is satisfied, $\inf_\Omega P(CS|R_h)$ can be evaluated. The expected size of the subset is given by

(4.4.17) $$E(S|R_h) = \sum_{i=1}^{k} \prod_{\substack{r=1 \\ r\neq i}}^{k} F_{[r]}(h(x))\, dF_{[i]}(x),$$

the expression we had earlier in Chapter I and we know that $\sup_{\Omega} E(S|R_h)$ is attained when the parameters are equal provided that (1.6.6) holds.

4.2. A Modification in the Requirements

Suppose $\pi_1,\dots,\pi_k$ are k normal populations with the unknown means $\lambda_1,\dots,\lambda_k$ respectively and the common variance σ^2. The ordered means are denoted by $\lambda_{[1]} \leq \cdots \leq \lambda_{[k]}$. The goal is to select a non-empty subset including the best, i.e. the population associated with $\lambda_{[k]}$. Let $\pi_{(i)}$ denote the population associated with $\lambda_{[i]}$. Then we impose the following requirements on the selection procedure.

(4.2.1) $$\begin{cases} \text{(a)} \quad P\{\pi_{(k)} \text{ is included when } \theta_{[k]} - \theta_{[k-1]} \geq \delta\} \geq 1-\alpha \\ \text{(b)} \quad P\{\pi_{(i)} \text{ is included when } \theta_{[k]} - \theta_{[i]} \geq \delta'\} \leq \beta, \end{cases}$$

where α and β are usually small, and $0 < \alpha,\beta < 1-\frac{1}{k}$ and $0 < \delta \leq \delta'$. We will discuss here the case where σ^2 is known. In this case we can assume without any loss of generality that $\sigma^2 = 1$. We propose the rule

R: Include π_i in the selected subset iff

(4.2.2) $$\bar{X}_i \geq \bar{X}_{max} - d/\sqrt{n},\ d > 0,$$

where $\bar{X}_i$ is the mean of a sample of size n from π_i, $i=1,\dots,k$, $\bar{X}_{max} = \max(\bar{X}_1,\dots,\bar{X}_k)$ and the constants d and n are to be determined so as to satisfy (4.2.1). For Procedure R we know that

$$(4.2.3) \qquad P(CS|R) = \int_{-\infty}^{\infty} \prod_{r=1}^{k-1} \Phi(y+d+\sqrt{n}\,\delta_{k,r})\, d\Phi(y),$$

where $\delta_{i,j} = \lambda_{[i]} - \lambda_{[j]}$ and Φ is the cdf of $N(0,1)$. The first requirement is that

$$(4.2.4) \qquad \inf_{\Omega_\delta} P\{\pi_{(k)} \text{ is included}\} \geq 1-\alpha,$$

where $\Omega_\delta = \{\underline{\lambda}: \lambda_{[1]} \leq \cdots \leq \lambda_{[k-1]} \leq \lambda_{[k]} - \delta\}$. In Ω_δ, $\delta_{k,1} \geq \delta_{k,2} \geq \cdots \geq \delta_{k,k-1} \geq \delta$. Hence the infimum takes place when $\delta_{k,1} = \delta_{k,2} = \cdots = \delta_{k,k-1} = \delta$, i.e., when $\lambda_{[1]} = \cdots = \lambda_{[k-1]} = \lambda_{[k]} - \delta$. Thus we have

$$(4.2.5) \qquad \int_{-\infty}^{\infty} \Phi^{k-1}(y+d+\sqrt{n}\,\delta)\, d\,\Phi(y) \geq 1-\alpha.$$

For $i=1,\ldots,k-1$,

$$(4.2.6) \qquad P\{\pi_{(i)} \text{ is included}\} = \int_{-\infty}^{\infty} \prod_{\substack{r=1 \\ r\neq i}}^{k} \Phi(y+d+\sqrt{n}\,\delta_{i,r})\, d\,\Phi(y).$$

Using Procedure R, $P\{\pi_{(i)} \text{ is included}\} \leq P\{\pi_{(j)} \text{ is included}\}$ for $i < j$. To see this, we note that $\delta_{i,r} \leq \delta_{j,r}$, $r \neq i,j$, $r=1,\ldots,k$ and $\delta_{i,j} \leq \delta_{j,i}$. Hence $\prod_{\substack{r=1 \\ r\neq i}}^{k} \Phi(y+d+\sqrt{n}\,\delta_{i,r}) \leq \prod_{\substack{r=1 \\ r\neq j}}^{k} \Phi(y+d+\sqrt{n}\,\delta_{j,r})$, which gives the desired result. Now we prove a lemma.

<u>Lemma 4.2.1</u>. For $i=1,\ldots,k-2$,

(4.2.7) $$P\{\pi_{(i)} \text{ is included when } \theta_{[k]} - \theta_{[i]} \geq \delta'\}$$

$$\leq P\{\pi_{(k-1)} \text{ is included when } \theta_{[k]} - \theta_{[k-1]} \geq \delta'\}.$$

<u>Proof</u>. For $i=1,\ldots,k-2$,

$$P\{\pi_{(i)} \text{ is included} | \underline{\theta}: \theta_{[k]} - \theta_{[i]} \geq \delta'\}$$

$$\leq P\{\pi_{(k-1)} \text{ is included} | \underline{\theta}: \theta_{[k]} - \theta_{[i]} \geq \delta'\}$$

$$\leq P\{\pi_{(k-1)} \text{ is included} | \underline{\theta}: \theta_{[k]} - \theta_{[k-1]} \geq \delta'\}.$$

Let $\Omega_{i,\delta'} = \{\underline{\theta}: \theta_{[k]} - \theta_{[i]} \geq \delta'\}$, $i=1,\ldots,k-1$. Then $\sup_{\Omega_{i,\delta'}} P\{\pi_{(i)} \text{ is included}\} \leq \sup_{\Omega_{k-1,\delta'}} P\{\pi_{(k-1)} \text{ is included}\}$. Hence the condition (4.2.1)-(b) is satisfied if $\sup_{\Omega_{k-1,\delta'}} P\{\pi_{(k-1)} \text{ is included}\} \leq P$, i.e.,

(4.2.8) $$\sup_{\Omega_{k-1,\delta'}} \int_{-\infty}^{\infty} \prod_{\substack{r=1 \\ r\neq k-1}}^{k} \Phi(y+d+\sqrt{n}\, \delta_{k-1,r})\, d\Phi(y) \leq P.$$

In $\Omega_{k-1,\delta'}$, $\delta_{k-1,1} \geq \delta_{k-1,2} \geq \cdots \geq \delta_{k-1,k-2}$ and $\delta_{k-1,k} \leq -\delta'$. Hence the supremum is attained when $-\infty = \lambda_{[1]} = \cdots = \lambda_{[k-2]} \leq \lambda_{[k-1]} = \lambda_{[k]} - \delta$. Thus (4.2.8) becomes

(4.2.9) $$\int_{-\infty}^{\infty} \Phi(y+d-\sqrt{n}\, \delta')\, d\Phi(y) \leq P.$$

Using the result $\int_{-\infty}^{\infty} \Phi(Ay+B)\, d\Phi(y) = \Phi(\frac{B}{\sqrt{1+A^2}})$, (4.2.9) reduces to $\Phi(2^{-1/2}(d-\sqrt{n}\,\delta')) \leq \beta$ or $d-\sqrt{n}\,\delta' \leq 2^{1/2}\,\Phi^{-1}(\rho)$. Since $\rho < \frac{1}{2}$, $d-\sqrt{n}\,\delta' < 0$ and hence $\sqrt{n} > d/\delta'$. The constants d and n of Procedure R must be determined to satisfy

$$d - \sqrt{n}\,\delta' \leq 2^{\frac{1}{2}}\,\Phi^{-1}(\rho) \tag{4.2.10}$$

and

$$\int_{-\infty}^{\infty} \Phi^{k-1}(y+d+\sqrt{n}\,\delta)\, d\Phi(y) \geq 1-\alpha. \tag{4.2.11}$$

Since the left hand side of (4.2.10) decreases in n and that of (4.2.11) increases in n, we first solve for d and n from

$$d- \sqrt{n}\,\delta' = 2^{\frac{1}{2}}\,\Phi^{-1}(\rho) \tag{4.2.12}$$

and

$$\int_{-\infty}^{\infty} \Phi^{k-1}(y+d+\sqrt{n}\,\delta)\, d\Phi(y) = 1-\alpha, \tag{4.2.13}$$

assuming that n is a positive number, not necessarily an integer. The existence of $d=d_0$ and $n=n_0$ satisfying (4.2.12) and (4.2.13) will depend upon α and ρ. Using (4.2.12) in (4.2.13) we obtain

$$A(d) \equiv \int_{-\infty}^{\infty} \Phi^{k-1}(y+d(1+\frac{\delta}{\delta'})-2^{\frac{1}{2}}\,\Phi^{-1}(\rho)\,\frac{\delta}{\delta'})\, d\Phi(y) = 1-\alpha. \tag{4.2.14}$$

As d increases from 0 to ∞, $A(d)$ increases from D to 1, where

(4.2.15) $$D = \int_{-\infty}^{\infty} \Phi^{k-1}(y - 2^{\frac{1}{2}} \Phi^{-1}(\beta) \tfrac{\delta}{\delta'}) \, d\Phi(y).$$

For $k=2$, $\beta \leq \frac{1}{2}$ and hence $\Phi^{-1}(\beta) \leq 0$. For $k > 2$, $\beta \leq 1 - \frac{1}{k}$ and so $\Phi^{-1}(\beta)$ could be of either sign. But in practice β will be small and so we can assume that $\beta < \frac{1}{2}$. Then $a = -2^{1/2} \Phi^{-1}(\beta) \frac{\delta}{\delta'} > 0$. Thus we get

(4.2.16) $$D = \int_{-\infty}^{\infty} \Phi^{k-1}(y+a) \, d\Phi(y), \quad a > 0.$$

Obviously, $D > \frac{1}{k}$. Hence we can solve for d from (4.2.14) provided $D \leq 1-\alpha$. When $k=2$, this condition reduces to $\Phi(\frac{a}{\sqrt{2}}) \leq 1-\alpha$ or equivalently, $-\frac{\delta}{\delta'} \Phi^{-1}(\beta) \leq \Phi^{-1}(1-\alpha)$. Since $\Phi^{-1}(\beta) = -\Phi^{-1}(1-\beta)$, the condition becomes

(4.2.17) $$\delta/\delta' \leq \Phi^{-1}(1-\alpha)/\Phi^{-1}(1-\beta).$$

If we specify $\alpha \leq \beta$, then (4.2.17) is satisfied because $\delta \leq \delta'$. In the above discussion, we assumed $\delta > 0$. The equation (4.2.13) can be solved for d if $\delta = 0$. Hence we can get n from (4.2.14).

4.3. Some Other Procedures

For the problem of selecting a subset containing the best, in the set-up of Chapter I, we may consider a procedure R_1 defined as follows:

R_1: Select the population π_i iff

(4.3.1) $$x_{[k]} - x_i \leq c(x_{[k]} - x_{[1]}), \quad 0 < c < 1$$

where $x_{[1]} \leq \cdots \leq x_{[k]}$ are the ordered observations. The intuitive reasoning for the procedure is obvious from its form. We can write (4.3.1) as

$$(4.3.2) \qquad x_i \geq c\, x_{[1]} + (1-c)\, x_{[k]}.$$

The distribution problem involved is difficult in this situation.

Another problem is to consider procedures of the type

R_2: Select π_i iff $x_i \geq \Psi(x_{[1]},\ldots,x_{[k]})$,

where $\Psi(x_{[1]},\ldots,x_{[k]})$ is a function symmetric in the arguments and determine the choice of Ψ under certain conditions to be imposed on the procedure.

Let $\pi_{[r]}$ denote the population from which we got the observation $x_{[r]}$. Then one can ask for a sufficient condition under which, for any s $(1 \leq s \leq k)$, the probability that the set $\{\pi_{[i_1]},\ldots,\pi_{[i_s]}\}$ includes the best population cannot exceed the probability that the set $\{\pi_{[k-s+1]},\ldots,\pi_{[k]}\}$ includes the best population. This would provide a justification to use an interval of the type $[x_{[k]}-d,\ x_{[k]}]$ or $[x_{[k]}/c,\ x_{[k]}]$ where $d > 0$, $c > 1$ and determine the constant subject to the probability requirement.

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13. ABSTRACT

Let π_i,i=1,...,k, be a continuous population with associated distribution function F_{λ_i}, $\lambda_i \epsilon \Lambda$, an interval on the real line. Chapter I defines a class of procedures for selecting a non-empty subset of the k populations, such that the probability of a correct selection (PCS), i.e. selection of a subset which includes the population with the largest (smallest) λ_i, is at least P*, a preassigned level. A generalization (Theorem 1.4.2) of a result of Lehmann is used to obtain a sufficient condition for the monotonicity of a probability integral leading to the evaluation of the infimum of PCS over the parameter space. Results concerning the supremum of the expected subset size are obtained. More specific results are obtained when the density $f_\lambda(x)$ is a convex mixture of a sequence of known density functions. The next chapter examines the selection from multivariate normal populations in terms of multiple correlation coefficient and illustrates the applications of the results of Chapter I. In Chapter III, a partial ordering (h-ordering) is defined on the space of probability distributions and selection from populations h-ordered w.r.t. a known distribution G is discussed. The last chapter briefly discusses some possible variations in the goal and the procedures.

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14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.